Review of "Closing the Hole Argument", by Hans Halvorson and J.B. Manchak (*British Journal for the Philosophy of Science* 76, 2025)

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For *Mathematical Reviews*

This article attempts to crack the toughest of tough nuts—*viz.*, the hole argument of general relativity (GR).[1] Here are its two central claims:

1. Weatherall (2018) has "convincingly argued" (p. 1) that the hole argument isn't generated simply by dint of the existence of distinct but isomorphic (i.e. isometric) models of GR.

2. There is nevertheless a worry that, were there to exist multiple distinct isometries relating two isomorphic models of GR (that restrict to the same isometry on a proper subset of their domain), the hole argument could re-arise. But in fact a theorem proved by Geroch (1969) rules this out, so the hole argument is thereby "closed".

Ad (1): that the hole argument isn't generated simply by virtue of the existence of distinct but isomorphic models of GR should be uncontroversial—to say only this much would be to elide crucial aspects of the set-up of the argument, *inter alia* the diffeomorphism invariance of the dynamics (see Menon and Read (2023) and Pooley (2021)). But what the authors have in mind, I take it, is what Pooley and Read (2025) call Weatherall's 'argument from mathematical structuralism'—namely, that it is mandatory to use only maps which witness the isometry of two isometric models of GR as standards of cross-model identification of mathematical points; if one does this, then (the claim goes) GR does not 'generate' a philosophical problem (of hole indeterminism).

As one might expect, I beg to differ that Weatherall (2018) has "convincingly argued" this point; in particular, I think one can push back in at least the following ways:

i. Given that, on standard set-theoretic foundations, models of GR—i.e. Lorentzian manifolds[2]—just are structured sets, it's not *prima facie* ob-

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[1] I won't repeat the details of the argument here, but see Norton et al. (2023) and Pooley (2021) for recent reviews.

[2] At least if we set aside the stress-energy of material fields as implicitly definable from the metric via the Einstein equation, as authors typically do in these discussions.

vious (or at least requires some more sustained and systematic argumentation) that using maps between isometric models of GR which are not isometries is to be implicated in "semantic ascent" (p. 25), to transcend the "language of general relativity" (p. 25), etc.[3]

ii. When one looks to mathematical and/or physical practice, it's also not obvious that practitioners don't avail themselves of such resources, which makes such stringent restrictions on what counts as GR look specious. (See Cudek (forthcoming, §5), Gomes and Butterfield (2023, §3.2), and Landsman (2023, §1).)

iii. Even if one accepts the argument from mathematical structuralism, it's not a given that the hole argument is blocked, if one has other reasons to be believe in metaphysical differences which 'cut finer' than isometries. (See Pooley and Read (2025, §5).)[4]

I would like to see from 'formalists' such as Weatherall (2018) and the authors of the present article a more thorough engagement with the above points; at least, it seems to me that it's here that the most pressing outstanding loci of disagreement can be found. On the 'category-first' view taken in the article under review, "the structure [is] defined in terms of the morphism" (p. 8)[5]—so, when we're dealing with Lorentzian manifolds, their structure is "defined" by their being objects in the category of Lorentizan manifolds; as such, it makes no sense to compare them using maps which are not morphisms in that category, including (in general) the identity map. Fair enough, and in that case the authors of the present piece will of course be unmoved by passages from Menon and Read (2023) such as the following:[6]

> There is (at this point, at least) no prohibition on comparing any two isometric models of general relativity using diffeomorphisms which do not witnesses those models' being isometric—in the above case, assuming that ψ is non-trivial, one could for example compare those models using the identity map 1_M, which [...] does not witness their being isometric [...]. (Menon and Read 2023, p. 9)

[3]Cf. Cheng and Read (2025) in response to Bradley and Weatherall (2022); separately, cf. Cudek (forthcoming). I take it that both Weatherall (2018) and the authors of the present piece endorse some kind of 'categorico-structuralism' as against what one might call the above 'set-theoretic constructivism'. But then I think it's helpful to be explicit that most of the authors' points are conditional on this categorico-structuralism, and I also think that it would be helpful for the authors (at some point, if not in this article) to argue against set-theoretic constructivism on its own terms (i.e., by not simply insisting—as the authors do later their article—that to talk of set theory when doing (what looks like) GR is in fact not to work with GR but rather with some augmented theory 'GR+ZF_m'), and moreover I further think that it would be helpful to give more explicit articulation of and argumentation for categorico-structuralism.

[4]But—fair enough—if one squints hard enough one can probably convince oneself to agree with Weatherall (2018) that these commitments are not 'generated' by GR.

[5]Cf. footnote 3.

[6]Similar remarks are to be found in Pooley and Read (2025).

But again, what one would really like to see (if not in this paper then elsewhere) is more sustained and sympathetic argument for the categorico-structuralist outlook.[7] The authors proceed as if the position is uncontroversial and obvious—but it's important to remember that the view is still surely a minority one, and so (in my view, at least), the ball is in its proponents' court when it comes to rendering it compelling.[8]

In any case, let me now also for the sake of argument adopt categorico-structuralism and proceed to (2). The thought here is that, if (again, following injunctions from Weatherall (2018)) we're only allowed to consider maps which witness isometries between the models of GR under consideration, then if (as it turns out *per impossibile*) there were multiple such maps between two hole-diffeomorphic models of general relativity (that restrict to the same isometry on a proper subset of their domain), one could use one of those maps as a standard of cross-model identification of points, and the other to generate a hole diffeomorphism—in other words, two such distinct maps could, it seems, permit the hole argument to re-arise.

Good news, then, that the authors of the present piece invoke a modification of a result from Geroch (1969) in order to demonstrate that, in the case of the hole argument, there can be no two such distinct isometries. The relevant theorem is this (p. 17):

> Theorem 1: Let (M, g) and (M', g') be relativistic spacetimes. If φ and ψ are isometries from (M, g) to (M', g') such that $\varphi|_O = \psi|_O$ for some non-empty open subset O of M, then $\varphi = \psi$.

(In the article, a puzzling thing now happens. In their Corollary 2, the authors claim the 'non-existence of hole isomorphisms', in the sense that these must be the identity map for the underlying manifold, 1_M. But this is only because they restrict to map from (M, g) to itself, so of course the identity map is an isometry in this case! See Menon and Read (2023, pp. 11–12). I return to this below.)

Now, what is the significance for the hole argument of Theorem 1? The charitable read on it presented by Menon and Read (2023) is that it can be used to 'plug a hole' is the formalist response to the hole argument offered by Weatherall (2018). Although such a reading is not presented very explicitly in the article currently under review, Manchak (for one) seems to endorse this understanding of the significance of this result from Geroch (1969), claiming that "[W]eatherall acknowledges that his argument rests on the [G]eroch (1969) rigidity result in this way", and endorsing this claim attributed to Weatherall.[9]

But does the claim that arguments presented by Weatherall (2018) *vis-à-vis* the hole argument require recourse to the rigidity result of Geroch (1969) in fact

[7] In an ideal world, one would also like to see a defence of the view that such considerations about mathematical objects have *any* bearing on the hole argument, *pace* arguments by Teitel (2022). (My thanks to Frank Cudek for raising this point.)

[8] Very likely, defending categorical-structuralism will involve engaging with older criticisms of (positions akin to) the approach such as those from Winnie (1986, §2.5)—criticisms which, in my view, deserve more attention than they have yet received.

[9] See `https://www.youtube.com/watch?v=lFbfhISreFY&t=825s`.

stand up to scrutiny? Menon and Read (2023, p. 14) go on to question whether it's really necessary to invoke Geroch's theorem (or adaptations thereof) in order to secure this result, writing:[10]

> [W]e in fact think that the above line—that Halvorson and Manchak close a hole in Weatherall's argument by proving the negation of `Distinct isometries`—concedes too much to Halvorson and Manchak, and not enough to Weatherall. For in fact, the negation of `Distinct isometries` is unnecessary for Weatherall's argument (i.e., what Pooley and Read (2025) dub the 'argument from mathematical structuralism', as presented in Weatherall (2018)) to proceed as intended (of course, whether Weatherall's argument is ultimately successful is another matter, to which we turn below). For even if there were to exist multiple diffeomorphisms witnessing the isometry between models $\mathcal{M} = \langle M, g_{ab} \rangle$ and $\mathcal{M}' = \langle M, \psi^* g_{ab} \rangle$, these maps would differ at most by a transformation which leaves the metric invariant (i.e., an automorphism of the metric)—in which case, a multiplicity of such maps would still not imply indeterminism. To see this, suppose that there are two pull-backs of the metric which coincide: $\psi_1^* g_{ab}(p) = \psi_2^* g_{ab}(p)$. From this, it follows that $(\psi_1 \circ \psi_2^{-1})^* g_{ab}(p) = g_{ab}(p)$—so $\psi_1 \circ \psi_2^{-1}$ is an $\texttt{Isometry}_1$ of g_{ab}. For a generic metric, these isometries are just the identity, so $\psi_1 = \psi_2$. And in the case in which g_{ab} has non-trivial isometries (in the sense of $\texttt{Isometry}_1$), $\psi_1 \circ \psi_2^{-1}$ is *still* an automorphism of the metric, and so does not shift fields on the manifold in such a way as to lead to the possibility of the Hole Argument re-arising. Given this, the above reconstruction of the contribution of Halvorson and Manchak's results to Weatherall's argument does not seem compelling: Weatherall's arguments needed nothing like such results to begin with; the denial of `Distinct isometries` is not a crucial-but-implicit element of his reasoning. (Menon and Read 2023, p. 14)

Here, however, I do not in fact think that what Menon and Read (2023) say is quite right, so let me try to clear up how things stand. Begin by recalling from Menon and Read (2023) the following definitions of different species of isometry:

$\texttt{Isometry}_1$: For all elements $p \in M$, $d^* g_{ab}(p) = g_{ab}(p)$.

$\texttt{Isometry}_2$: For all elements $p, q \in M$, if $d(p) = q$, then, $d^* g_{ab}(q) = g_{ab}(p)$.

It is only isometries in the sense of $\texttt{Isometry}_1$ which are associated with Killing vector fields, conserved quantities via Noether's first theorem, etc.

Let's now reconstruct the reasoning of Menon and Read (2023) as presented in the above paragraph. The strategy is to proceed by exhaustion: first consider the case where the model (M, g) of GR under consideration has no non-trivial

[10] Citations have been updated in this passage.

isometries in the sense of $\texttt{Isometry}_1$, and then consider the case in which it does have non-trivial isometries in the sense of $\texttt{Isometry}_1$. The reasoning in the first case proceeds as follows:

P1. Consider a model $\mathcal{M}_1 = (M, g)$ and some isometric model $\mathcal{M}_2 = (M, \tilde{g})$.[11]

P2. Suppose that there are multiple maps which witness the isometry between $\mathcal{M}_1$ and $\mathcal{M}_2$.

C1. Consider two such maps, call them ψ_1 and ψ_2. Then $\tilde{g} = \psi_1^* g = \psi_2^* g$, so $(\psi_2 \circ \psi_1^{-1})^* g = g$.

P3. *Ex hypothesi*, there are no non-trivial isometries (in the sense of $\texttt{Isometry}_1$).

C2. So $\psi_2 \circ \psi_1^{-1} = \text{Id}$, so $\psi_1 = \psi_2$.

What one might be tempted to claim on the basis of this reasoning—and what Menon and Read (2023) surely have in mind—is that when the spacetime (M, g) under consideration has no non-trivial isometries in the sense of $\texttt{Isometry}_1$, we needn't invoke Geroch (1969) in order to rule out there being distinct isometries from (M, g) to $(M, \tilde{g})$. However, one has to be a little careful here, for one can prove that if a model has no non-trivial isometries in the sense of $\texttt{Isometry}_1$, then it is rigid in the sense of Geroch (1969)—in other words, rigidity is a necessary condition for the first horn of the argument of Menon and Read (2023) anyway! So, in a clear sense, the rigidity result is still 'under the hood' here, even if it is not invoked explicitly in the above argument.[12]

The case in which the spacetime *does* have non-trivial isometries in the sense of $\texttt{Isometry}_1$ is somewhat harder to deal with—and it is here where the problems for the argument given by Menon and Read (2023) are especially acute. In this case, we can derive (as before) that $(\psi_2 \circ \psi_1^{-1})^* g = g$ at p for all $p \in M$, but now (unlike before) it might be the case that $\psi_2 \circ \psi_1^{-1} \neq \text{Id}$. Now, the reasoning given by Menon and Read (2023) is that *even if* we have $(\psi_2 \circ \psi_1^{-1})^* g = g$ with $\psi_2 \circ \psi_1^{-1} \neq \text{Id}$, this composition of the maps yields the same metrical value anyway (for whichever point $p \in M$ is under consideration), and so cannot lead to any underdetermination of metrical values at p which is paradigmatic of the hole argument. Note that there is nothing particular about GR here: the thought is that *whenever* we have (a) a spacetime (M, g) with non-trivial isometries in the sense of $\texttt{Isometry}_1$, and (b) distinct isometries from (M, g) to some $(M, \tilde{g})$, we are still not going to be confronted with the hole argument.

Unfortunately, the inference from (a) to (b) fails in general. To see this, consider the following example.[13] Consider non-Hausdorff GR (i.e., a version

[11] Throughout, when I say 'isometric models' without further clarification, I mean isometry in the sense of $\texttt{Isometry}_2$.

[12] I am grateful to JB Manchak for showing me the proof to this effect. I leave reconstructing the proof as an exercise for the reader, but one way to see this is that the 'giraffe' condition is higher up than rigidity in the hierarchy of asymmetry conditions presented by Manchak and Barrett (forthcoming). (In that article, the authors consider only the collection of isometries from the given Lorentzian manifold to itself.)

[13] I owe the example to JB Manchak; I am very grateful to him for presenting it to me.

of GR but with the Hausdorff condition on manifold topology dropped). Let (M, g) be Minkowski spacetime with two points p and q which are not separable by neighbourhoods. Consider a worldline γ running through p. Let $\psi_1 = \mathrm{Id}$ and let ψ_2 be the map which acts as the identity everywhere except that it exchanges p and q. Both maps count as isometries from (M, g) to $(M, \psi_1^* g) = (M, \psi_2^* g) = (M, g)$. Given that there are multiple distinct isometries from (M, g) to itself here, the rigidity result of Geroch (1969) fails. Note, moreover, that there is a clear sense of indeterminism here, given that $(\psi_2 \circ \psi_1^{-1})[\gamma] = \psi_2[\gamma] \neq \gamma$.[14]

The upshot of this example is that, *pace* Menon and Read (2023), it isn't *invariably* true that one doesn't have hole-type indeterminism when one has $(\psi_2 \circ \psi_1^{-1})^* g = g$, even when $\psi_2 \circ \psi_1^{-1} \neq \mathrm{Id}$; as a result, this second step of their argument doesn't give them what they're after. Of course, invoking the rigidity result of Geroch (1969) does secure that there is no hole indeterminism even in the second horn of the argument of Menon and Read (2023), because the rigidity result automatically rules out the existence of distinct isometries! So here, it seems, is where explicitly invoking the Geroch rigidity result suffices to shore up the argument from Weatherall (2018).

Charitably, this entire discussion helps to pinpoint *exactly* where the rigidity result of Geroch (1969) is used in order to 'close' the hole argument. But there's also one final issue here which one has to be careful about. The quote from Manchak which I presented above—that "[W]eatherall acknowledges that his argument rests on the [G]eroch (1969) rigidity result in this way"—implies that the Geroch result is *necessary* to block the hole argument. But, in fact, one could say that what we've shown in the above is that it is *sufficient* to do so, not that it's necessary. Now one might respond to this by saying: well, *in GR*, provably we always have rigidity (see Manchak and Barrett (forthcoming, Proposition 1)), so *qua* the consequent of the relevant conditional, rigidity *is* in fact necessary for Weatherall's argument. Within the context of GR, this is also of course true—but what I mean is that in the context of more general spacetime theories, rigidity is sufficient for underwriting Weatherall's point, but it is not known whether it is necessary. So one has to be careful here: there is both a sense in which rigidity *is* necessary for underwriting Weatherall's argument (namely, in the context of GR), and a sense in which it is not currently known to be so (namely, in a broader context of spacetime theories where rigidity might fail).[15]

To repeat and summarise, then, here is how things stand. In the case in which the model (M, g) under consideration has no non-trivial isometries in the sense of $\texttt{Isometry}_1$, one can deploy the first 'horn' of the above argument in order to show that there cannot be distinct isometries between (M, g) and some isometric $(M, \tilde{g})$. While this reasoning does not *explicitly* invoke the rigidity result, it's nevertheless the case that rigidity is a necessary condition to have a model of this kind, and so rigidity is in a clear sense still 'under the hood' here. In the case in which the model (M, g) under consideration does have non-trivial

[14]For a proof that no non-Hausdorff model is rigid, see Manchak and Barrett (forthcoming, Proposition 2).

[15]Again, my thanks to JB Manchak for helpful discussions on this paragraph.

isometries in the sense of $\texttt{Isometry}_1$, it suffices to invoke the Geroch rigidity result to show that there cannot be distinct isometries between (M, g) and some isometric $(M, \tilde{g})$; what has not yet been shown, however, is that in a broader class of spacetime theories than just GR rigidity is *necessary* to show this.

(To wrap things up here, let me return now to Corollary 2, on which I made a parenthetical remark above. What the invocation of the result of Geroch (1969) shows is that the only isometry from (M, g) to itself is indeed the identity map; when we consider isometries from (M, g) to some isometric but distinct $(M, \tilde{g})$, the result tells us that there is a unique isometry, but it needn't be the identity map. If by "non-existence" the authors mean that the isometry is the identity map, then it isn't true that hole isometries from (M, g) to $(M, \tilde{g})$ don't exist; of course, though, the authors can still maintain that the hole argument is closed insofar as the map witnessing the isometry of these models is unique.)

Stepping back, here is how I would summarise the article under review. The article is written from a still-contentious, categorico-structuralist perspective, and doesn't direct its efforts towards defending that perspective. This is fair enough—but its critiques of other authors who have written on the hole argument but who do not adopt the selfsame perspective might thereby come across as unfair. Going forward, I would like to see more explicit and sustained responses from categorico-structuralists to concerns such as (i)–(iii) presented above.

Turning to the question of whether the rigidity result of Geroch (1969) is *necessary* in order to underwrite the arguments of Weatherall (2018), we see that arguments from Menon and Read (2023) to the effect that this is not necessary do not quite hold up; on the other hand, all we have in fact seen—at least when we move to a broader class of spacetime theories than just GR—is that the result is *sufficient* to underwrite the arguments of Weatherall (2018), not that it is necessary—and as such, there is still more work to be done here, and interesting questions remain to be asked and (with any luck) answered.[16]

Acknowledgments

First and foremost, I'm grateful to JB Manchak for generous and patient exchanges on this topic. I also thank Franciszek Cudek for helpful feedback on a previous draft, and Henrique Gomes, Eleanor March, and Tushar Menon for valuable discussions. Finally, I thank Oliver Pooley for drawing my attention to Winnie's article.

[16]In later sections of their article, the authors also discuss *inter alia* different conceptions of determinism. Since those discussions are commented upon by Menon and Read (2023), and significantly expanded upon in Halvorson et al. (2025) and Manchak et al. (2025), I won't discuss them further now.

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