Intertheoretical Relationships based on Three-model Framework

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Abstract

Intertheoretical relationships have been traditionally investigated through the notions of reduction and emergence. Recently, the focus has shifted towards the relationship between models for elaborating intertheoretical relationships in physics. This article demonstrates that three, rather than two, types of models are essential for elucidating some intertheoretical relationships. Beyond the conventional higher- and lower-level models, an intermediate-level model is crucial for establishing connections between the theories. This framework is not only applicable to some practical cases but also effectively captures the characteristics of two significant intertheoretical relationships: between classical and quantum mechanics, and between thermodynamics and statistical mechanics. By applying this framework to these cases, this study highlights both the similarity and the difference in these intertheoretical relationships.

Keywords: Interthoeretical Relationship, Reduction, Emergence

1. Introduction

The notions of reduction and emergence have been central to understanding intertheoretical relationships. For instance, Nagel (1961) [28] and Schaffner (1967) [39] define the reduction as the logical relationships between propositions of the theories. However, their definitions have proven

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challenging to apply to physical sciences. In response, Batterman (2002) [2] and Butterfield (2011a, b) [7, 8] have proposed alternative definitions of reduction and emergence. Batterman proposes the definitions of reduction and emergence between a higher-level theory T_h and a lower-level theory T_l, appealing to the mathematical limits. Suppose that there is an essential parameter ϵ in T_l such that $T_h = \lim_{\epsilon \to \infty} T_l$. If the behavior that appears when taking this limit is regular, then the case is reduction. Alternatively, that is, if the behavior is singular, the case is emergence. Unlike Batterman's approach, Butterfield argues that the mathematical limit is not indispensable for the reduction and emergence, which can be defined separately. At first, Butterfield defines reduction as a logical drivability of the higher-level theorem from the lower-level theory. On the other hand, he defines emergence through the notions of novelty and robustness. If a higher-level property is indefinable from the lower-level theory and remains stable under various choices and assumptions about its components, then this property satisfies both requirements and is thus a case of emergence. In the definition provided by Butterfield, reduction and emergence are compatible, and this view is widely accepted (Crowther 2015 [10]). Both of researchers aim to define the notions of reduction and emergence as inter-theoretical relationships.

The quintessential example of this topic is the relationship between thermodynamics and statistical mechanics, as Palacios (2022) [32] summarizes. Statistical mechanics addresses thermodynamical phenomena from a microscopic perspective, typifying reduction, while its use of the thermodynamical limit in explaining phase transitions, despite the finite nature of the world, exemplifies emergence. Previously, the relationships between theories in physics have been based on the investigation of the characteristics of these theories themselves, focusing on bridging laws such as Boltzman's entropy formula and mathematical tools like the thermodynamical limit.

More recently, model-based definitions of reduction and emergence have been introduced to refine our understanding of the intertheoretical relationship. These definitions hinge on the relationships between models, with comparisons extending beyond the mathematical formalism to include properties derived from the models, thus determining whether cases are reduction or emergence. In particular, Rosaler (2015) [34] suggests that a reduction between models implies an intertheoretical reduction. While this perspective is compelling, it opens avenues for further refinement in understanding how relationships between models translate into intertheoretical relationships.

This article posits the existence of an intermediate-level model that bridges lower- and higher-level models. Consequently, some intertheoretical relationships are founded not on two but rather on three models. Applying this framework to two intertheoretical cases validates this approach and elucidates the characteristics of the intermediate-level model. Focusing on the intermediate-level model demonstrates how both reduction and emergence appear in the intertheoretical relationship in particular cases. In addition, this model facilitates a comparative analysis of the similarity and difference between the intertheoretical relationships of classical and quantum mechanics and between thermodynamics and statistical mechanics.

Section 2 of this paper delineates the perspective that defines reduction and emergence through the relationships between models, proposing a three-model framework. Sections 3 and 4 demonstrate this framework's applicability and philosophical implications through targeted case studies. Section 3 considers the relationship between quantum and classical mechanics. In particular, we focus on the derivation of quasi-classicality, illustrating how the three-model framework effectively encapsulates this case. Section 4 investigates the universality of critical phenomena within thermodynamics and statistical mechanics, which is regarded as a representative example of emergence. Here, the significance of coarse-graining is discussed to substantiate that critical phenomena exemplify emergence, thereby validating the effectiveness of the three-model framework. Section 5 refines our understanding of the proposed framework and elucidates the similarities and the differences between the two case studies.

2. Three-model framework

The notions of reduction and emergence in physics have been defined in various ways. For instance, Batterman (2002) [2] defines the intertheoretical reduction and emergence, through the lens of mathematical limits, positing that emergence signifies the failure of reduction. Conversely, Butterfield (2011a, b) [7, 8] argues that reduction and emergence are compatible. These traditional approaches conceptualize reduction and emergence as intertheoretical relationships in physics. In contrast, recent efforts have sought to redefine these notions through the prism of model relationships, addressing and resolving ambiguities inherent in the previous definitions. This shift is motivated by the practical focus in scientific research on models rather than solely on theories. Notably, Rosaler (2015, 2016, 2019) [34, 35, 36] defines the reduction as a relationship between models. De Haro (2019) [11] approaches the notion of emergence similarly.

Exploring this further, Rosaler's work ([34, 35, 36]) on the relationship between classical and quantum mechanics exemplifies an intertheoretical reduction, denoted as $reduction_T$, which is defined by a reduction between models termed $reduction_M$. Consider a higher-level theory T_h and a lower-level theory T_l, with their respective models denoted by M_h and M_l.

> **Criteria for Local Inter-Theoretic Reduction**: Theory T_h reduces$_T$ to theory T_l iff for every system K in the domain of T_h – that is, for every physical system K whose behavior is accurately represented by some model M_h of T_h – there exists a model M_l of T_l also representing S such that M_h reduces to T_l (Rosaler 2015 [34] 59).

As delineated above, the intertheoretical reduction depends on the relationship between models.

The reduction between two models, M_h and M_l, of a dynamical system is defined through the following conditions. Initially, both models, M_h and M_l, have to represent the same target system, K. The state space of M_h is represented by S_h, and the deterministic evolution of an initial state $x_h \in S_h$ over time is denoted by $D_h(x_h, t)$, where t represents time. Correspondingly, the state space and the temporal evolution in the lower-level model M_l are denoted by S_l and $\mathcal{D}_l(x_l, t)$, respectively. For a set of initial states x_h in S_h, denoted by d_h, $D_h(x_h, t)$ have to track the evolution of system K within a specified margin of error δ for a duration of time τ or longer. The specific values of δ and τ are determined by the empirical constraints necessary to fit the model to the state space. A bridge function $B(x_l)$ refers to a time-independent map $B : S_l \to S_h$, which links these two models for all $x_l \in d_l$. For reduction from the higher-level model M_h to the lower-level model M_l to occur — denoted as $reduces_m$ — the following must hold:

> iff there exists a differentiable function $B : S_l \to S_h$ that does not depend explicitly on time and nonempty subset $d_l \subset S_l$ in the domain of B such that $d_h \subset B(d_l)$, and for $x_l \in d_l$,
>
> $$|B(\mathcal{D}_l(x_l, t)) - D_h(B(x_l), t)| < 2\delta, \tag{1}$$
>
> or more concisely,
>
> $$B(\mathcal{D}_l(x_l, t)) \approx D_h(B(x_l), t), \tag{2}$$

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for all $0 \leq t \leq \tau$ (Rosaler 2015 [34] 61).

This definition implies that a reduction between models is achieved when the behavior provided by the higher-level model approximately corresponds to the behavior of the lower-level model. In essence, the approximate agreement between the trajectories provided by these models suffices for establishing local reduction between models.

An illustrative case of local reduction is the relationship between classical and quantum mechanics as discussed by Rosaler (2019 [36] 296–297). In this instance, the higher-level model is represented by a classical mechanical model on the $6N$-dimensional phase space governed by the Hamiltonian equation. Conversely, the lower-level model is a non-relativistic quantum mechanical model defined within the non-relativistic quantum mechanical space, that is, the N-particle Hilbert space. Its dynamics is given by the Shrödinger equation. The mapping $B : S_l \to S_h$ facilitates a connection between these state spaces. Furthermore, the Ehrenfest theorem, which describes how the expected values in quantum mechanics exhibit classical-like behaviors, demonstrates how the behaviors of the lower-level model approximately mirror those of the higher-level model. Therefore, this case satisfies the criteria for local reduction, noting that classical properties per se do not appear from the quantum models, but rather that quasi-classical properties exhibiting approximately classical characteristics appear (see Section 3.2).

Further exploring the concept of reduction, Palacios (2019) [31] examines phase transitions, revealing that this phenomenon meets the condition of reduction as she defines. Like Rosaler, Palacios incorporates the perspectives about approximations into her definition of reduction. She carefully investigates differences between two different types of phase transitions — first-order and continuous — and shows that both entail reduction, albeit of different types. This paper concentrates on critical phenomena specifically. Palacios articulates the notion of Limiting reduction as follows:

> **Limiting reduction** Let Q^l denote a relevant quantity of T_l, and Q^h, a relevant quantity of T_h. Then a quantity Q^h of T_h limiting reduces to a corresponding quantity Q^l of T_l iff (i) $\lim_{x\to\infty} Q^l_x = Q^h$ (where x represents a parameter appearing in T_l) and (ii) the limiting operation makes physical sense (Palacios 2019 [31] 625).

Here the indices l and h represent the lower- and higher-levels, respectively. In this definition, for instance, the finite statistical mechanics is the lower-level theory (T_l), and the infinite statistical mechanics is the higher-level

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theory (T_h). The first condition, (i), posits that taking mathematical limits gives a correspondence between the variables of different theories. Condition (ii) is satisfied when the limiting operation is justified in any sense in physics[1].

Rosaler and Palacios consider the concept of reduction to involve an approximate correspondence between properties derived from models; this type of reduction is termed "approximate reduction" in this article. In parallel, the notion of emergence is expected to be defined in relation to the inter-model relationships. De Haro (2019) [11] offers a definition of emergence in terms of a linkage map not only between theories but between models. Note that De Haro's characterization of emergence does not imply an incompatibility between emergence and reduction as Butterfield argues.

Consider two theoretical constructs: a higher-level theory (or model) denoted by T_t (or M_t) and a lower-level theory (or model) denoted by T_b (or M_b). The interpretation of these theories establish a link between theory and our empirical world:

> We have emergence iff two bare theories, T_b and T_t, are related by a linkage map, and if in addition the interpreted top theory has novel aspects relative to the interpreted bottom theory (De Haro 2019 [11] 10).

The distinction in empirical facts derived from these theories suggests emergence. Although this definition pertains to theories, the terms "theories" and T can equivalently be substituted with "models" and M. Indeed, the relationship between microscopic models and the coarse-grained model, as explained by the renormalization group (RG) method, exemplifies emergence within this framework. This is because although the microscopic models do

[1]One of the justifications for taking mathematical limits is the Butterfield Principle outlined by Landsman (2013) [20] such that

> a limit is justified as being mathematically convenient and empirically adequate, if the values of the quantities evaluated in the limit at least approximate the values of the quantities "on the way to the limit," that is, for large but finite values of the parameter x_0: $\lim_{x\to\infty} Q_x \approx Q_{x_0}$ (Palacios 2019 [31] 626).

When the behavior of a system in the limit is approximately the same as the behavior of a system whose parameters are large but finite, the mathematical operation to take the limit is justified in physics. In this case, the condition of limiting reduction is satisfied, as in Rosaler's approach.

not exhibit the universal properties of critical exponents, the coarse-grained model, particularly at the fixed point, demonstrates this property. Consequently, the domains of these models differ, indicating that the relationship concerning critical phenomena implies emergence.

More generally, the notion of a minimal model is helpful for understanding the feature of the higher-level model such as the case of RG method, which implies emergence. Batterman and Rice (2014) [6] argue that, in some cases, the details of target systems are neglected to derive a model, and the neglected details are not only irrelevant but also prevent us from understanding phenomena. This type of models, which deliberately dismisses the details, is referred to as the minimal model. An instance of a minimal model is one derived using the RG method. The RG method includes a coarse-graining procedure to dismiss the details of the microscopic structure by lowering the resolution of the description. The model produced through the RG method provides the explanation of the critical phenomena, which the detailed model fails to provide. Indeed, as explained above, De Haro's definition supports the view that the case of RG method is emergence, because the model before coarse graining does not demonstrate the property of critical phenomena but the model after that does. In this case, the model provided by dismissing the irrelevant details demonstrate why the critical phenomenon is a case of emergence. In sum, one case is emergence when the higher-level model successfully explains the phenomena, whereas the more detailed lower-level model fails to do so (Morita 2023 [24]).

These characterizations of emergence highlight the importance of novelty because the existence of a novel property in the higher-level model, compared with the lower-level model, implies emergence. The significance of novelty for emergence is evident in the inter-theoretical definitions by Batterman and Butterfield. In Batterman's definition, when something novel appears in the limit, the case is emergence. Similarly, in Butterfield's definition of emergence, the novelty is the condition of emergence. Furthermore, like the notion of approximate reduction, this definition of emergence also relies on comparing properties provided by the lower- and higher-level theories or models. The existence of such novel properties entail emergence.

Rosaler (2015; 2016; 2019) [34, 35, 36] posits that once a reductive relationship between models is established, the corresponding intertheoretical relationships are consequently affirmed. However, on closer examination, it becomes apparent that three types of models are necessary to establish these intertheoretical relationships. Thus, the intertheoretical relationship is fun-

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damentally based on these three models. This insight — although grounded in the typical example of intertheoretical relationship discussed by Rosaler — has not been explicitly articulated in the previous analyses. In the case examined by Rosaler (2015) [34], the Ehrenfest theorem facilitates the linkage between classical and quantum mechanical models. It is true that some quantum models representing the quantum systems exhibit behaviors that are approximately classical (quasi-classical) and that can thus be categorized as cases of approximate reduction. However, not all quantum systems demonstrate quasi-classical properties. To derive models that demonstrate quasi-classicality, certain idealizations or operations to omit details of quantum models are required. This idealized model demonstrates the quasi-classical property, similar to the property derived from the classical model. This additional model is essential to bridge the lower- and higher-level models. In this paper, this type of model will be referred to as the "intermediate-level model"[2]. The specifics and validity of this concept will be explored in the subsequent section.

Before we apply this framework to specific cases in physics, let us summarize the framework of this idea: the intertheoretical relationship relies on the existence of three models.

Three-model framework: We designate the lower- and higher-level theories denoted by T_l and T_h, respectively. The term "lower" refers to more fundamental or microscopic theories, while the "higher" indicate more derivative or macroscopic theories. For instance, T_l might be quantum mechanics, with the corresponding T_h being classical mechanics. These theories provide models — the lower-level model (M_l) and the higher-level models (M_l). To bridge these models, an intermediate-

[2] A preliminary version of some ideas presented here — particularly concerning the notion of the intermediate-level models in the relationships between quantum and classical mechanics, and between thermodynamics and statistical mechanics — appeared in Morita (2024) [25]. However, the focus of that work was on emergence, and it did not provide a framework for analyzing inter-theoretical relationships. In particular, the structure introduced in this paper (Three-model framework) and its application to the comparative analysis of the two cases were not developed in the earlier account. The present article thus offers a new perspective by articulating the conceptual structure underlying these relations and by systematically comparing theories within a unified framework. Besides, since Morita (2024) was published in Japanese, there is a merit for readers in repeating some of the discussion in the book in English here.

level model (M_i) is introduced. The comparison focuses on properties derived from each model. In particular, comparisons are made between the intermediate-level and higher-level model, and between the intermediate-level and lower-level model. These comparisons serve to determine whether the relationship between these models is reductive, emergent, both, or neither.

This framework assists in identifying similarities and differences between cases of intertheoretical relationships[3]. In addition, to establish the connection between the higher- and lower-level theories, some models demonstrate a *substitute property*. The term "substitute property" refers to a property similar but distinct from the empirically confirmed property expected to be explained. This property makes it possible to understand the intertheoretical relationships. The following sections will examine the relationships between classical and quantum mechanics, as well as between thermodynamics and statistical mechanics.

3. Classical and quantum mechanics

One way to establish the link between classical and quantum mechanics is the classical limits such as $\hbar \to 0$. As Batterman argues, the mathematical limit plays the important role in connecting the theories. In fact, Batterman suggests that the relationship between classical and quantum mechanics is a case of emergence, based on his limit-based definitions of reduction and emergence. More recently, Feintzeig and Steeger have investigated the features and implications of the classical limit (Feintzeig 2022 [15]; Steeger and Feintzeig 2021 [41]). The authors point out that, even if the classical limit is singular and this is regarded as emergence in Batterman's definition, it does not suffice to determine the full theoretical structure of classical mechanics from quantum mechanics.

Admittedly, the classical limit is an important means by which to connect quantum mechanics with classical mechanics. However, the meaning of taking the limit of the constant $\hbar$ remains controversial. It has already been pointed out that the mathematical limit is conceptually distinct from

[3]This three-model framework, in particular the existence of the intermediate-level model, reflects the characteristics of actual cases in physics, offering a conceptual tool for understanding and clarifying intertheoretical relationships.

the notions of reduction and emergence (Bangu 2015 [1]; Franklin and Knox 2018 [17]; Rosaler 2015 [34]). In addition, Landsman argues that "[c]lassical physics emerges from quantum theory in the limit $\hbar \to 0$ or $N \to \infty$ *provided that the system is in certain "classical" states and is monitored with "classical" observables only*" (Landsman 2007 [21], 422, *italics* are original). He points out that in order to demonstrate the classical property from quantum mechanics in the limit, the state has to be classical at some extent. So what is this quantum state that is classical at some extent? Wallace and Rosaler consider the partially classical property, called quasi-classicality in the Ehrenfest theorem. Sections 3.1 and 3.2 consider the Ehrenfest theorem and quasi-classicality to reveal the relationship between quantum and classical mechanics from the

3.1. Ehrenfest theorem and classical mechanics

One of the representative methods for connecting classical and quantum mechanics is the Ehrenfest theorem. This theorem, which is derived from quantum mechanics, demonstrates the relationship between expecting values like classical mechanics.

To outline this theorem, let $\hat{P}$ denote the momentum operator, V the potential, and $\hat{X}$ the position operator of the particle. Consider a quantum mechanical particle with mass m moving in a potential $V(x)$, represented by the Hamiltonian $\hat{H} = (\hat{P}^2/2m) + V(\hat{X})$. In this case, the Ehrenfest theorem is given by

$$m\frac{d^2}{dt^2}\langle\hat{X}\rangle = -\langle\nabla V(\hat{X})\rangle, \tag{3}$$

where $\langle\cdot\rangle$ denotes the quantum expectation value. On the other hand, the classical equation of motion for a particle of mass m moving in the potential $V(x)$ is

$$m\frac{d^2}{dt^2}x = -\nabla V(x). \tag{4}$$

At first glance, these equations look similar. However, while the quantum equation (Eq. 3) represents a relationship between expectation values, the classical equation (Eq. 4) represents a relationship between physical quantities themselves. Thus, these two equations cannot be regarded as the same.

The Ehrenfest theorem alone does not establish the relationship between quantum and classical mechanics; to establish the link,

$$\langle \nabla V(\hat{X}) \rangle = \nabla V(x = \langle \hat{X} \rangle) \tag{5}$$

is required. This equality is satisfied for particular quantum states. In particular, if the wave packet $|\psi\rangle$ has a spatial width that is narrower than the length scale of the potential V, then

$$\frac{d\langle \hat{P} \rangle}{dt} \sim - \left. \frac{\partial V(X)}{\partial X} \right|_{\langle \hat{X} \rangle}, \tag{6}$$

where $d\langle \hat{X} \rangle / dt = \langle \hat{P} \rangle / m$ holds. Does this case imply the reduction of classical mechanics to quantum mechanics?

This case does not straightforwardly imply reduction. Landsman points out that "Ehrenfest's Theorem by no means suffices to have classical behavior, since it gives no guarantee whatsoever that $\langle \hat{X} \rangle$ behaves like a point particle" (Landsman 207 [21], 476). He suggests that the equation (Eq. 3) alone does not establish the link between quantum and classical mechanics. Similarly, Hartle (2011) [18] offers some reasons why the Ehrenfest theorem cannot be regarded as a derivation of classical mechanics from quantum mechanics. For instance, the Ehrenfest theorem merely provides the relationship between the expectation values. Thus, even if the behavior of the Moon can be described through this theorem in quantum mechanics, it merely reveals that the Newtonian trajectory is most probable among several possible trajectories of the Moon. Quantum mechanics demonstrates that the classical mechanical time-evolution is probabilistically more plausible than others, but this does not mean that the quantum mechanical time-evolution is the same as the classical time-evolution.

However, as Rosaler (2015) [34] points out, the approximate correspondence implies the approximate reduction. In fact, the relationship between quantum and classical mechanics is established in the case of the particular state such as the narrowly peaked wave packet. Consider a classical system such as the N particles system whose state is represented in the $6N$-dimensional phase space Γ_N. The time-evolution of this state is described by the Hamilton's canonical equations such that

$$\frac{dX}{dt} = \frac{\partial H}{\partial P}, \qquad \frac{dP}{dt} = -\frac{\partial H}{\partial X}. \tag{7}$$

When X denotes the position, P the momentum, and V the potential, H is called as the Hamiltonian satisfying $H = (P^2/2M) + V(X)$. This equation demonstrates the trajectory of the system when an initial state $x_h = (X, P)$ is given.

On the other hand, in the case of quantum mechanics, sates are represented by points on the Hilbert space. The Hilbert space of quantum states of N-particles systems is denoted by $\mathcal{H}_N$ and the state by $|\psi\rangle \in \mathcal{H}_N$. The time evolution is given by the Schrödinger equation

$$\hat{H} |\psi\rangle = i \frac{\partial}{\partial t} |\psi\rangle , \tag{8}$$

where $\hat{H} = (\hat{P}^2/2M) + V(\hat{X})$ is called the Hamiltonian operator.

Rosaler defines the map B that establishes a link between the state spaces of classical N-particle state and the quantum N-particle state as $B : \mathcal{H}_N \to \Gamma_N$. For an initial state $x_l \in \mathcal{H}_N$ of quantum mechanical system, B is defined as follows;

$$B(x_l) \equiv (\langle\psi|\hat{X}|\psi\rangle, \langle\psi|\hat{P}|\psi\rangle) = (\langle\hat{X}\rangle, \langle\hat{P}\rangle). \tag{9}$$

In other words, B is defined as a function that, provided with an initial quantum mechanical state x_l, yields the expectation values of position and momentum $\langle\hat{X}\rangle$ and $\langle\hat{P}\rangle$ associated with this state.

According to the Ehrenfest theorem explained above, the narrowly peaked wave packet enables us to connect the quantum mechanical relationship between the position and momentum operators with the classical mechanical relationship between position and momentum. Let us denote $|\psi\rangle$ as the narrowly peaked wave packet. Based on the Ehrenfest theorem, $|\psi\rangle$ satisfies the following relationship;

$$\begin{aligned} \frac{d}{dt}(\langle\psi|\hat{X}|\psi\rangle, \langle\psi|\hat{P}|\psi\rangle) &\approx \left(\left.\frac{\partial H}{\partial P}\right|_{\langle\hat{X}\rangle,\langle\hat{P}\rangle}, -\left.\frac{\partial H}{\partial X}\right|_{\langle\hat{X}\rangle,\langle\hat{P}\rangle} \right) \\ &= \left(\left.\frac{P}{M}\right|_{\langle\hat{P}\rangle}, -\left.\frac{\partial V(X)}{\partial X}\right|_{\langle\hat{X}\rangle} \right) \end{aligned} \tag{10}$$

This formula implies that the expectation values of position and momentum in quantum mechanics approximately follow the equations of motion in classical mechanics. When the wave function represents a narrowly-peaked wave packet, the trajectory of the top of it approximately corresponds to the trajectory given by classical mechanics. Therefore, this case is the approximate reduction, as Rosaler defines.

This situation demonstrates that at least in some particular states, the quantum mechanical trajectory is approximately classical. This approximately classical mechanical property derived from quantum mechanics is called quasi-classicality. This property reveals how quantum and classical mechanics are connected through the three-model framework.

3.2. Emergence of classicality

Rosaler (2015, 2016, 2019) [34, 35, 36] argues that the relationship between quantum and classical mechanics satisfies the requirements for local reduction. Classical mechanical properties themselves cannot be derived from quantum mechanics; however, the quasi-classical property can be derived. Quasi-classicality is defined as follows:

> Quasi-classicality: The system in question must possess determinate or approximately determinate values for position and momentum simultaneously (Rosaler 2016 [35] 57).

Quasi-classicality differs from classicality, because quasi-classicality does not always require that position and momentum have determinate values at the same time. In classical mechanics, models of a point particle demonstrate the classicality. In quantum mechanics, models of quantum states such as the narrowly peaked wave packet show that the behavior of the top of the wave function is quasi-classical similar to point particles. Thus, the relationship between classical and quantum mechanics is considered one case of approximate reduction.

Wallace (2012) [43] elucidates how quasi-classicality is derived from the quantum mechanical model. Specifically, he describes the behavior of the quantum state of a narrowly peaked wave packet as being approximately isomorphic to that of a classical point particle system. This approximate correspondence can be summarized by the following framework (Wallace 2012 [43] 66–67).

$$
\begin{aligned}
&|\langle \mathbf{x}|\psi(t)\rangle|^2 \simeq 0 \ \text{ unless } \mathbf{x} \simeq \mathbf{q}(t) \\
&\leftrightarrow \text{Wave packet is centered at } \mathbf{q}(t) \\
&\leftrightarrow |\psi(t)\rangle \text{ instantiates classical particle with trajectory } \mathbf{q}(t)
\end{aligned}
$$

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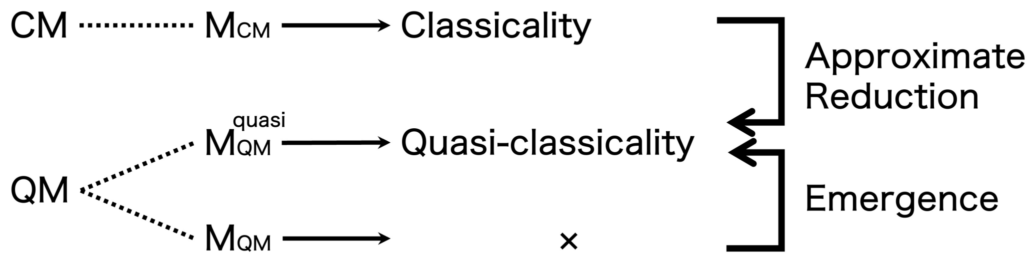

Figure 1: The intertheoretical relationship between quantum mechanics (QM) and classical mechanics (CM) is based on the three kinds of models. Comparing the properties derived from these models, the relationship between M_{QM}^{quasi} and M_{QM} is emergence, whereas the relationship between M_{CM} and M_{QM}^{quasi} is approximate reduction.

In this framework, the approximate isomorphism between the behaviors of a classical point particle and the center of narrowly peaked wave packet is established. This analysis suggests that quantum states do not exhibit classical behavior but rather quasi-classical behavior.

Considering the intertheoretical relationship based on three models, the models in this scenario are categorized as follows:

1. **Lower-level model** M_{QM}: This is the quantum model accurately representing the entire narrowly peaked wave packet, not just the peak. M_{QM} demonstrates quantum behaviors as obeying the Schrödinger equation.
2. **Intermediate-level model** M_{QM}^{quasi}: This model represents only the peak of the wave packet, discarding other factors deemed irrelevant for the classical property. M_{QM}^{quasi} exhibits quasi-classical behavior approximating that of a classical point particle.
3. **Higher-level model** M_{CM}: This is the classical mechanical model and represents the classical point particle. This model demonstrates classicality.

In this configuration, M_{QM}, as the lower-level model, displays distinct quantum characteristics, but not the quasi-classical features. However, as demonstrated by the Ehrenfest theorem, the model M_{QM}^{quasi}, the intermediate-level model, exhibits quasi-classicality. The move from M_{QM} to M_{QM}^{quasi} requires the abstraction or dismissal of certain details of the wave packet, suggesting that this case is emergence. This is because without applying the Ehrenfest theorem and the idealization to omit irrelevant details, the lower-level model would not display even quasi-classical properties and the quasi-classicality would thus be a novel property compared with M_{QM}. So, this is a case of emergence.

Regarding the relationship between M_{QM}^{quasi} and M_{QM}, it seems that if the mere idealization to omit the details is regarded as emergence, then any simplified model brings about emergence. In fact, this sort of idealization is a pervasive method in science. However, the idealization used in this case does not merely disregard the details. As explained above, the notion of the minimal model is deeply related to emergence (Batterman and Rice 2014 [6]). Consider the relationship between M_{QM}^{quasi} and M_{QM}, again. The M_{QM} represents the narrowly peaked wave packet faithfully, and the M_{QM}^{quasi} represents the same wave packet but neglects the irrelevant factors for deriving the quasi-classicality. As explained above, the behavior of the top of the narrowly peaked wave packet is approximately classical. In other words, the details of the wave packet would have to be dismissed to show quasi-classicality. Without this idealization, the behavior of the wave packet represented by M_{QM} is purely quantum and its behavior is based on the Schrödinger equation. In this sense, the M_{QM}^{quasi} is a minimal model to derive the substitute property, quasi-classicality. Therefore, the relationship between between M_{QM}^{quasi} and M_{QM} is emergence.

In contrast, the relationship between M_{CM} and M_{QM}^{quasi} is approximate reduction, as the higher-level model (M_{CM}) demonstrates classicality, which is approximately the same property exhibited by M_{QM}^{quasi}. This approximate correspondence between classicality and quasi-classicality underpins the approximate reduction between these models. In summary, the intertheoretical relationship between classical and quantum mechanics in the context of the Ehrenfest theorem demonstrates both reduction and emergence. The relationships among three models are visually represented in Fig. 1.

4. Thermodynamics and statistical Mechanics

The intertheoretical relationship between thermodynamics and statistical mechanics is another intriguing example of this issue. Statistical mechanics has been employed to explain thermodynamic phenomena from the perspective of the behavior of components (particles), thereby suggesting reductive relationships between these theories. However, scholars such as Liu (1999) [22] argue that phase transitions pose challenges to the reductive view and imply emergence. The argument to derive the view that phase transitions are emergence can be classified into two parts. First, explaining phase transitions necessitates some form of infinite idealizations within the statistical mechanical framework despite the finite nature of our world. Secondly, the mathematical limits employed (such as $N \to \infty$, where N is the number of particles) to explain phase transitions in thermodynamics shows the singularity, while the corresponding function (partial function) in statistical mechanics is analytic. On the other hand, Callender (2001) [9] challenges this view of the anti-reductive view, proposing that these phenomena might still be explainable through reductive ways. Presently, critical phenomena serve as a focal case study for exploring both reduction and emergence within the field of physics (Palacios 2022 [32]).

Critical phenomena are considered emergence, owing to the characteristics of renormalization group methods as discussed by Batterman (2010, 2011) [3, 4] and Morrison (2012, 2015) [26, 27]. Morrison argues that while the RG method derives the fixed point for explaining universality, the microscopic models cannot fully explain the fixed point. In this context, universality refers to the fact that microscopically distinct systems demonstrate the same macroscopic properties. The infinitely iterative transformations of the RG method map different systems onto the same fixed point, and analysing this fixed point provides the accounts for the universality of critical phenomena. Furthermore, these transformations include a coarse-graining procedure that discards certain details. Consequently, the critical phenomena demonstrating universality are partially independent of their constituents, which supports the view that this case is emergence.

Several scholars argue that critical phenomena should not be classified as emergence but rather as examples of reduction (Menon and Callender 2013, Reutlinger 2017, Reutlinger and Saatsi 2018, Palacios 2019) [23, 33, 38, 31]. In particular, Palacios (2019) [31] maintains that the explanation of critical phenomena through the RG method meets the criteria for the limiting reduction, whereas Batterman and Morrison regard critical phenomena as emergence. As mentioned above, Batterman argues that the indispensability of the RG methods, which involves disregarding irrelevant microscopic details, indicates the failure of reduction. Within the RG framework, there are two different mathematical limits considered: the thermodynamic limit ($N \to \infty$) and the infinitely iterative renormalization transformation ($n \to \infty$), where N and n denote the number of particles and the number of the renormalization transformations, respectively. SM denotes the finite version of statistical mechanics, and the infinite counterpart derived through these mathematical limits is represented by SM^{∞}. When an approximate correspondence exists between the values of quantities provided by SM and SM^{∞}, the relationship between the models of SM and SM^{∞} fulfills the condition of limiting reduction. If the behavior explained by the fixed point through the two mathematical limits corresponds approximately to the behavior provided by statistical mechanics without such mathematical limits, then this case exemplifies the limiting reduction.

Although fulfilling the requirements for the limiting reduction initially appears challenging, Palacios (2019) [31] effectively demonstrates how the relationship between SM and SM^{∞} regarding critical phenomena constitutes a limiting reduction. Supposing that K represents a fixed point,

$$K^{\mathrm{SM}}_{N_0,n_0} \approx \lim_{n\to\infty}\lim_{N\to\infty} K^{\mathrm{SM}}_{N,n}, \tag{11}$$

and the mathematical limits in this case are physically meaningful (Palacios 2019 [31] 636). This means that the infinite limits result in a limiting reduction between SM and SM^{∞}.

While the mathematical limits do not always suggest the emergence of critical phenomena, another aspect of the RG method — the coarse-graining procedure — might imply emergence. Section 4.1 and 4.2 will examine how the coarse-graining procedure relates to emergence, applying the three-model framework to this situation.

4.1. Scaling theory

The view that critical phenomena represent cases of emergence is grounded in the essential roles of mathematical limits and the coarse-graining procedure in the RG framework. This raises the following question: if the mathematical limits alone do not imply emergence, can the coarse-graining procedure by itself imply emergence? Morita (2023) [24] asserts that coarse graining in the RG method does imply emergence, but he does not address the intertheoretical relationship between thermodynamics and statistical mechanics. Originally, the coarse-graining procedure regarding critical phenomena was theoretically developed by Kadanoff (1966) [19] as the scaling theory in explaining experimental and calculative results. In particular, a notable achievement of this approach is the derivation of the Rushbrooke equality, which exemplifies the practical application of scaling theory and coarse graining.

At first, to understand the universality of critical phenomena such as the Rushbrooke equality, we define critical exponents; α, β, and γ, which characterize the singularity at the critical points. c denotes specific heat (where $c \sim C/V$, C is the heat capacity, and V is the volume), T denotes the temperature, and T_c denotes the transition temperature. The exponent α is defined by the behavior of c when the magnetic field $h = 0$ and $T \sim T_c$;

$$c \sim \frac{1}{|T - T_c|^{\alpha}}.$$

β is characterized by the behavior of magnetization m as follows:

$$m \sim (T_c - T)^{\beta}.$$

Similarly, magnetic susceptibility χ defines γ, mirroring the definition for α:

$$\chi \sim \frac{1}{|T - T_c|^{\gamma}}.$$

According to thermodynamics, the Rushbrooke inequality

$$\alpha + 2\beta + \gamma \geq 2 \tag{12}$$

was derived (Rushbrooke 1963 [37]). On the other hand, an equation

$$\alpha + 2\beta + \gamma = 2 \tag{13}$$

was empirically established before the theoretical derivation of the Rushbrooke inequality[4]. In essence, while a simple model calculation could demonstrate this equation (referred to Eq. (13)), thermodynamics alone was unable

[4]This section considers the Rushbrooke equality, but this is not the only formula about critical exponents that shows the universality. For instance, these following equations are common among several different materials and cannot be derived from microscopic models although these are empirically confirmed facts.

$$\beta(1+\delta) \geq 2 - \alpha$$
$$\nu(2-\eta) \geq \gamma$$
$$\nu\delta \geq 2 - \alpha.$$

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to account for this empirical fact about critical exponents (Essam and Fisher 1963 [14] 809).

Widom (1965) [45] introduced a scaling function, to derive the Rushbrooke equality from statistical mechanics. He merely presupposed the scaling function, and his idea is called now the scaling hypothesis. Actually, by assuming the scaling hypothesis, the Rushbrooke equality could indeed be derived from statistical mechanical models. Following this, Kadanoff (1966) [19] proposed a theoretical framework to justify the existence of the scaling function, known as the scaling theory, which incorporates a procedure for coarse graining the description of system. These historical developments mean that the coarse-graining procedure is essential for deriving universal properties from the microscopic models in statistical mechanics[5].

While the thermodynamic model falls short of demonstrating the equality, the statistical mechanical model, refined through coarse graining, succeeds in demonstrating it. It is from this coarse-grained model that the phenomenological property is derived. If the coarse-graining procedure is the primary factor supporting the view that the RG implies emergence, rather than the mathematical limits, then assessing the scaling theory is crucial to determine whether the coarse-graining procedure implies emergence[6].

4.2. Emergence in the scaling theory

In the scaling theory-based explanation outlined so far, three types of models are involved. Let M_{SM} represent a model of statistical mechanics, $M_{SM}^{c.g.}$ represent a model provided through coarse graining M_{SM}, and M_{TD} represent a mode of thermodynamics.

1. **Lower-level model** M_{SM}: The two-dimensional Ising model is one example here. This model offers a detailed microscopic description but

[5]For details on deriving the Rushbrooke equality from the scaling theory and the inequality from thermodynamics, see Stanley (1971) [40] and Takahashi and Nishimori (2017) [42].

[6]One might assume that models before and after coarse graining are the same, and it is meaningless to compare these models with each other. It seems that the difference between these models lies in mere descriptions. For example, even when one model is described by Hamiltonian and another by Lagrangian methods in classical mechanics, they apparently are different but equivalent (or at least have the same explanatory powers). By contrast, the models before and after coarse graining are not equivalent. Because there is a property that the model after coarse graining shows and the model before does not, these models are not equivalent.

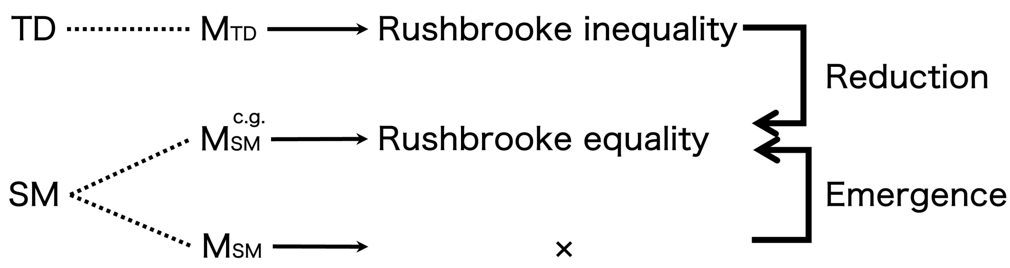

Figure 2: The intertheoretical relationship between thermodynamics (TD) and statistical mechanics (SM) is based on the three types of models. Comparing the properties derived from these models, the relationship between $M_{SM}^{c.g.}$ and M_{SM} is emergence and the relationship between M_{TD} and $M_{SM}^{c.g.}$ is reduction.

fails to demonstrates the inequality.

2. **Intermediate-level model** $M_{SM}^{c.g.}$: Derived from the scaling theory formulated by Kadanoff (1966) [19], this model successfully demonstrates the Rushbrooke equality.
3. **Higher-level model** M_{TD}: Explored by Rushbrooke (1963) [37], this model illustrates the Rushbrooke inequality.

The microscopic model (M_{SM}) fails to capture the universality of phenomena such as the Rushbrooke equality, whereas the coarse-grained statistical mechanical model ($M_{SM}^{c.g.}$) successfully exhibits this expected property. This discrepancy indicates that, without coarse graining, the statistical mechanical model cannot provide accounts for the universal phenomena. According to De Haro (2019) [11] and Morita (2023) [24], this scenario meets the criteria for emergence. They argue that when models representing the same target system are linked by a specific map and exhibit different empirical properties, this case implies emergence. Therefore, the relationship between the statistical mechanical models (M_{SM} and $M_{SM}^{c.g.}$) is regarded as emergence. Furthermore, this is the case of the minimal model (Batterman and Rice 2014 [6]). As is the case in the relationship between quantum and classical models, the intermediate-level model is required to dismiss the details of the lower-level model through the coarse-graining procedure to explain the critical phenomena. Thus, the relationship between the statistical mechanical model before coarse graining (M_{SM}) and after coarse graining ($M_{SM}^{c.g.}$) is emergence.

$M_{SM}^{c.g.}$ successfully demonstrates the Rushbrooke equality, which the thermodynamic model (M_{TD}) cannot show. The relationship between $M_{SM}^{c.g.}$ and M_{TD} is a case of reduction, but not merely the approximate reduction, detailed as follows. At first, the definition of approximate reduction between models, as introduced in Section 2, appeals to the approximate correspondence in behaviors provided by the models. In this case, the behaviors do not align sufficiently closely, because the derived inequality lacks an upper bound. Therefore, this situation does not meet the criteria for approximate reduction. However, this case is reduction in these two senses. First, the Rushbrooke inequality can be logically derived from the Rushbrooke equality. Since an equality entails a corresponding inequality as a logically weaker statement, the property exhibited by M_{TD} is logically derivable from that of $M_{SM}^{c.g.}$. This implies that the relationship in this case is reductive. Second, $M_{SM}^{c.g.}$ provides a more sophisticated understanding of the critical phenomena than M_{TD}, establishing a clear reduction from M_{TD} to $M_{SM}^{c.g.}$. While certain experiments such as those by Essam and Fisher (1963) [14] reveal a strict relationship between critical exponents, M_{TD} alone does not elucidate this equation. In contrast, the statistical mechanics with the scaling theory provides an explanation for the observation and equation. On the whole, the statistical mechanical models provide better explanations of the thermodynamics property. In this sense, this case is reduction even if this is not approximate reduction.

In statistical mechanics with scaling theory, both emergence and reduction appear. On the one hand, the relationship between the statistical mechanical models after and before coarse graining ($M_{SM}^{c.g.}$ and M_{SM}) is emergence, as illustrated in Fig. 2. On the other hand, the relationship between models in statistical mechanics and thermodynamics (M_{TD} and $M_{SM}^{c.g.}$) is characterized as reduction. The higher-level theory, namely thermodynamics (TD), provides the higher-level model M_{TD}, which demonstrates the Rushbrooke inequality. The lower-level theory statistical mechanics (SM) offers two models: the lower-level model M_{SM}, which is a microscopic model failing to show the Rushbrooke equality, and the intermediate-level model $M_{SM}^{c.g.}$, a coarse-grained model that successfully demonstrates the Rushbrooke equality. As a whole, the intertheoretical relationships between thermodynamics and statistical mechanics encompass both emergence and reduction[7].

[7]While Butterfield (2011a) [7] points out that the compatibility between reduction and emergence appears through the mathematical limits, this case demonstrates that

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Palacios (2019, 2022) [31, 32] also investigates the relationship between thermodynamics and statistical mechanics in the context of critical phenomena, based on the relationship between models. She points out that the relationship between the coarse-grained model in statistical mechanics and the thermodynamic models of phase transitions constitutes a Nagelian reduction [8]. In addition, she argues that the relationship between statistical mechanical models and the coarse-grained model is the limiting reduction and emergence. In contrast, this article views the relationship between the statistical mechanical models before and after coarse graining as emergence only. Why, then, this difference appears?

Palacios's argument clarifies that using mathematical limits does not necessarily imply the requirement for infinite systems or justify anti-reductionism. Palacios provides a topological explanation of how this case exemplifies the limiting reduction (Palaicos 2019 [31] 635–636). She considers a system on the critical surface denoted by p, where the correlation length $\xi \to \infty$, and s represents another system infinitesimally close to p but not on the critical surface. The RG flows of p and s are represented by R and D, and the fixed points of p and s are p^* and p_0, respectively. $p^* \neq p_0$, indicating that the behavior of the finite system does not even approximately correspond to that of the infinite system. However, the trajectory D of s is close to the critical trajectory R of p until it reaches the vicinity of U at the fixed point p^*. Therefore, the behavior of the finite system is approximately the same as that of the infinite system, satisfying the criteria for limiting reduction as demonstrated in Eq. (11).

The key distinction between our investigation and Palacios's argument lies in our respective choices of what is the lower-level model. In Palacios's analysis, the lower-level model corresponds to the finite system, and she examines the behaviors of p and s — where p represents a system at the critical surface and s a system near but not on the critical surface. In contrast, our study identifies the lower-level model as the one before coarse graining. In

mathematical limits are not apparently required for compatibility in the intertheoretical relationship.

[8]She defines the Nagelian reduction as follows;

> T_2 reduces$_{Nag}$ to T_1 iff the laws of T_2 can be logically deduced from the laws of T_1 along with bridge laws that connect the terms of T_1 and T_2 (Palacios 2022 [32] 54).

Palacios's notation, we compare the behaviors of p and p^*, where p^* represents the fixed point achieved through the coarse-graining procedure. This fundamental difference stems from divergent objectives; Palacios's work primarily seeks to ascertain whether the mathematical limits used in the RG method inherently imply reductionism or anti-reductionism, while our study focuses on demonstrating how the coarse-graining procedure implies emergence. As the scaling theory investigated earlier reveals, the model without coarse graining fails to explain the critical phenomena adequately.

In conclusion, within the case of the critical phenomena, the scaling theory — as the pioneering theory of RG — exemplifies both emergence and reduction. This compatibility is articulated through the relationship among three distinct models, as depicted in Fig. 2. The next section moves on to discuss the philosophical implications of this perspective, emphasizing that the intertheoretical relationship hinges on three models.

5. Implications of three-model framework

Sections 3 and 4 explores how the three-model framework captures the intertheoretical relationships in physics. This section will consider the philosophical implications of this framework and synthesize the case studies. The case studies highlight that the intermediate-level models play crucial roles in linking the theories. Initially, we shall scrutinize the characteristics and features of the intermediate-level models. Furthermore, differences will be articulated as these cases are analyzed within the same framework.

5.1. Characteristics of the intermediate-level model

These case studies — examining the relationships between thermodynamics and statistical mechanics and between classical and quantum mechanics — illustrate that some intertheoretical relationships are founded on the relationships among three models, exhibiting both reduction and emergence. These models elucidate how different theories interconnect and how both reduction and emergence manifest. In particular, intermediate-level models (M_I) are vital in linking the higher- and lower-level models (M_H and M_L). As the intermediate-level model plays an important role in the three-model framework to connect models of two different levels, the aim of this section is to reveal the characteristics of the intermediate-level model. The way for introducing the intermediate-level model, idealization, reveals its features such

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that the intermediate-level model describes the target in the lower-level theoretical terms but demonstrates the (at least approximately) the property of the higher-level model.

The intermediate-level models play crucial roles in establishing the links between theories. In the case of the Rushbrooke equality, without the coarse-grained model, the relationship between thermodynamics and statistical mechanics remains unclear. The intermediate-level model connects these disparate theories by omitting the irrelevant factors from the lower-level model; this role of abstraction is fulfilled by the scaling theory, including the coarse-graining procedure. Similarly, the derivation of the quasi-classical model involves excluding non-essential aspects of the wave packet. These processes, which generate intermediate-level models, represent a form of idealization.

Weisberg (2013) [44] provides a well-known characterization of idealization within philosophy of science. He classifies idealizations into three types, of which Galilean idealization and minimalist idealization are pertinent to our cases[9]. First, let us consider the case of the scaling theory: from the statistical mechanical model M_{SM}, the scaling theory, which includes the coarse-graining procedure, introduces the coarse-grained model $M_{SM}^{c.g.}$. This procedure is not the Galilean rather the minimalist idealization. While the Galilean idealization is expected to be de-idealized in the future according to Weisberg, the minimalist idealization, which aims to abstract only the essential elements of the target phenomena, does not expect such de-idealization, as detailed models are already well established. The scaling theory exemplifies this approach. Similarly, in the case of quasi-classicality, the intermediate-level model abstracts the essential factors to exhibit the approximately classical behavior, where a detailed quantum model is already in place. Therefore, the idealization that introduces the M_i also falls under the minimalist idealization.

In philosophy of physics, Norton (2012) [29] provides a distinction between idealization and approximation. Norton argues that idealization is a real or imaginary system, whereas approximation pertains to an inexact description of the target system (Norton 2012 [29] 209). Norton points out that the coarse-graining procedure qualifies an approximation because it merely

[9]The third idealization is the multiple modeling idealization (MMI). MMI is required to deal with complex phenomena. It is true that some thermodynamical phenomena are so complex that the MMI is required. However, regarding the intertheoretical relationship, the multiple models such as climate science are not relevant, at least not in this case.

distorts the original systems. Similarly, the quasi-classical model provides an imprecise representation of the wave packet and does not create any new target system. Therefore, according to Norton's framework, the intermediate-level model is the approximation, demonstrating the desired property but not providing new systems.

As a whole, the intermediate-level model M_I is characterized as the approximation defined by Norton, and the minimalist idealization defined by Weisberg[10]. Specifically, M_I represents the target phenomena inexactly using the lower-level theoretical terms but captures the essential factors necessary for explaining the higher-level property. For instance, the coarse-grained model pertaining to the Rushbrooke equality operates within the theoretical framework of statistical mechanics, and yet exhibits the physical property associated with thermodynamics. In the same way, the quasi-classical model is articulated in quantum mechanical terms and manifests the approximately classical property. In this way, the intermediate-level model, M_I, serves as a pivotal bridge between two different theories (T_H and T_L). Consequently, this model demonstrates, at least approximately, a higher-level property in the lower-level terms, reflecting both lower- and higher-level theoretical features.

Although the role of M_I seems to mirrors that of Nagelian bridge law, it does not replace it. Instead, the bridge law connects terms and notions across the lower- and higher-level theories and is presupposed to facilitate the linkage between the lower- and higher-level models through the intermediate-level model. The intermediate-level model's function is to, at least ap-

[10]In addition to the minimalist idealization and approximation, as argued above, these intermediate-level models are characterized as the minimal model. This is because, in order to demonstrate the properties such as the quasi-classicality and universality, the details of the lower-level models have to be omitted. However, the relationships among these notions are complex. For instance, the minimalist idealization does not always result in the minimal model. If both the detailed model and the idealized model (in the minimalist sense) exhibit the same property, the case qualify as the minimalist idealization but not as the minimal model. On the other hand, some minimal models may give rise to a new target system; in such cases, they are an instance of the minimal model, but not of approximation. The intermediate-level models explored in this article satisfy the conditions for all three categories — minimal model, minimalist idealization, and approximation — but we do not claim that all intermediate-level models necessarily do so. Whether any intermediate model meets these criteria remains a matter for future investigation. More broadly, the conceptual relationships among minimal model, minimalist idealization, and approximation also require further philosophical analysis.

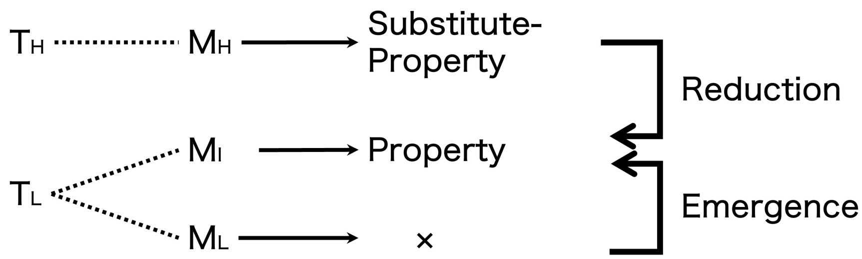

Figure 3: Generalized description of SMTD.

proximately, demonstrate the higher-level property using lower-level terms grounded on the bridge law. This model elucidates how different models are interconnected, and the bridge law is essential to this integrative role.

5.2. Differences and similarities between the case studies

The case studies within the three-model framework effectively capture the intertheoretical relationships and, furthermore, reveal the differences between them. The roles of the intermediate-level model explicate the differences between the relationship of statistical mechanics and thermodynamics (SMTD) and that between classical and quantum mechanics (QMCM). In SMTD, the intermediate-level model successfully explains the critical phenomena, specifically the Rushbrooke equality — a property cannot be derived from a lower-level statistical mechanical model or a higher-level thermodynamic model. As illustrated in Fig. 3, the lower-level theory (statistical mechanics, T_L) provides two models (M_L and M_I), with the intermediate-level model M_I demonstrating the expected property (Rushbrooke equality). In contrast, the higher-level theory (thermodynamics, T_H) produces the higher-level model M_H, which exhibits only a *substitute property* (Rushbrooke inequality), which is a property similar to but distinct from the empirically confirmed property expected to be explained (e.g., Rusubrooke inequality versus Rusbrooke equality, quasi-classicality versus classicality). In SMTD, the intermediate-level model accurately shows the property confirmed by the empirical experiments, a property that the higher-level model fails to show. Therefore, the relationship between M_I and M_H is not the approximate reduction. As a whole, in SMTD, the relationship between M_H and M_I is reduction and that between M_I and M_L is emergence.

In contrast, in the case of QMCM, as depicted in Fig. 4, the intermediate-level model exhibits only quasi-classicality. Regarding classicality, the higher-

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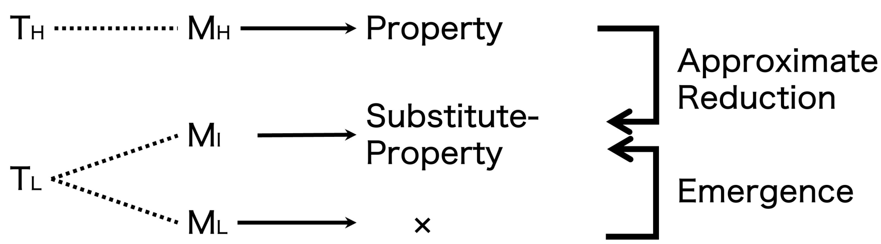

Figure 4: Generalized description of QMCM.

level model provides a more accurate explanation. The intermediate-level model of quantum mechanics demonstrates quasi-classicality but does not fully capture the classical property itself. Thus, the reduction from classical mechanics to quantum mechanics is based on an approximate correspondence and achieves only the approximate reduction, as shown by Rosaler, unlike in SMTD. In this scenario, the classicality as the expected property is exhibited by M_H, while the substitute property is shown by M_I.

The distinct nature of the intertheoretical relationships in QMCM and SMTD stems from the foundational motivations and historical contexts of the lower-level theories involved — quantum mechanics and statistical mechanics. On the one hand, quantum mechanics originated around 1900 and was developed primarily to account for phenomena such as black body radiation that classical physics could not explain. Thus, quantum mechanics is not inherently expected to replicate classical mechanical properties but rather to explain non-classical phenomena. This historical backdrop sets the stage for why, in practical terms, deriving the quasi-classical properties from quantum mechanics suffices to bridge these theoretical frameworks[11].

On the other hand, statistical mechanics was conceived with the aim of explaining thermodynamic phenomena from the perspectives of mechanics. In fact, Oono (2013) [30] argues that statistical mechanics does *not* justify thermodynamics; rather, the consistency with thermodynamics justifies statistical mechanics. Any thermodynamic phenomenon that statistical mechanics fails to explain, such as irreversibility, is inevitably a significant issue for statistical mechanics. In fact, as mentioned, statistical mechanics initially

[11]It is true that to derive classical mechanics from quantum mechanics is an important topic for both philosophy and physics (see Steeger and Feintzeig (2021) [41]). However, this article aims to show how this case implies reduction from descriptive perspectives.

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failed to account for critical phenomena. Thus, this topic was an important topic for statistical mechanics and, finally Kadanoff and Wilson introduced the new approach to address this shortcoming, as seen in Section 4[12].

The three-model framework illuminates the fundamental differences between QMCM and SMTD shown in Fig. 3 and Fig. 4. In QMCM, quantum mechanics is not primarily designed to accurately explain higher-level classical mechanical phenomena. These theories maintain a degree of autonomy, and only the approximate reduction characterizes their relationship. Rosaler's definition is descriptive, highlighting that quantum mechanics is not obligated to explain classical mechanical systems without approximations. In contrast, in SMTD, the lower-level theory (statistical mechanics) is explicitly expected to explain thermodynamic phenomena. This expectation underpins a stronger form of reduction in SMTD compared with QMCM, where the properties at the higher-level (thermodynamic) are indeed derivable from the intermediate-level model (statistical mechanics). This relationship does not rely on approximate reduction but rather on a finer correspondence[13].

[12]This understanding of statistical mechanics could evolve with future developments, particularly as quantum mechanical perspectives provide new insights. This article focuses on the relationship between classical statistical mechanics and thermodynamics. Thermodynamics generally concerns the transition from non-equilibrium to equilibrium states. However, classical statistical mechanics primarily addresses equilibrium states without explaining the macroscopic transitions from non-equilibrium to equilibrium states. To fill this gap, the eigenstate thermalization hypothesis (ETH) offers a quantum perspective into statistical mechanics on how the initial states evolve towards equilibrium states, as discussed in works such Deutsch (2018) [12]. Notably, many non-integrable systems adhere to the ETH, suggesting its potential as a foundational justification for thermodynamics from the standpoint of quantum statistical mechanics. If this perspective gains acceptance, the inertheoretical relationship in SMTD could align more closely with QMCM. The philosophical implications of ETH have not yet been thoroughly explored, with Drossel (2017) [13] being a notable exception. Future philosophical investigations will need to delve into the implications of ETH in terms of reduction and emergence, which remain open areas of research.

[13]Another typical topic about the intertheoretical relationship in physics is the relationship between relativity theory and Newtonian mechanics. This case is analogous to the relationship between thermodynamics and statistical mechanics, where the relativity theory is expected to give accounts for Newtonian spacetime in a similar way that statistical mechanics is expected to explain thermodynamic phenomena. However, the relativity theory not only explains the Newtonian spacetime but also unveils more fundamental structures of the world, much like quantum mechanics does. Therefore, this relationship is arguably more complicated than those discussed in this article. An in-depth exploration

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6. Conclusion

The three-model framework effectively captures intertheoretical relationships by providing insights into how the relationships between models imply intertheoretical connections. It also elucidates the similarities and differences between two specific intertheoretical relationships: QMCM and SMTD. A key similarity is that both cases are structured around three models, exhibiting both reduction and emergence. However, the purposes of underlying the lower-level theories — quantum mechanics and statistical mechanics — shape the specifics of these intertheoretical relationships, leading to different meanings of reduction in each cases.

This framework could be extended to more complex cases where an entity is represented in various descriptive forms. In such cases, intermediate-level models are anticipated to bridge the different descriptions, underscoring the need for more than a Nagelian bridge law to articulate intertheoretical relationships fully. As demonstrated, the existence of an intermediate-level model offers vital insights into understanding these relationships, moving beyond traditional approaches. While in the cases discussed in this article, the intermediate-level models are the minimal models, all intermediate-level models do not have to be so. What kind of idealization or transformation brings about emergence is the important future work. Other future work will explore additional complex cases, such as biophysics. This extension aims to validate further and refine the application of the three-model framework across different fields of study.

Another important issue about the intermediate level, which has not been addressed in this article, is the role of mesoscale in intertheoretical relationships. Batterman (2021) [5] has recently explored the philosophical implications of the more complicated physics cases, such as hydrodynamics and continuum mechanics. In this book, he highlights the importance of the intermediate-level structure such as mesoscopic structure. To examine the

of this relationship is a task for future research. In fact, an anonymous reviewer points out that the relationship between Newtonian gravitation and General Relativity does not require such an intermediate-level model (Fletcher 2019 [16]). This issue is not a drawback; rather, it serves to deepen our understanding of intertheoretical relationships. This is because it allows us to classify such relationships into those that can and cannot be captured within the three-model framework. Through this classification, our comprehension of intertheoretical relationships in physics can be significantly enriched. However, a detailed investigation into this matter will be left for future work.

work explored by Batterman from the perspective of the framework proposed in this article forms another topic for future work.

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