Primitive Spectra: Order-Completion and the Arithmetic Structure of Scale

Alexander Yiannopoulos*

A|Ω⟩ Institute, New Orleans

December 12, 2025

Abstract

We propose a structuralist reconstruction of the real continuum, arguing that it is not a primitive ontological stage for physics but an emergent completion of the prime-exponent lattice $\mathcal{M}$. By identifying $\mathbb{R}$ as the Dedekind completion $\hat{\mathcal{M}}$, we demonstrate that the natural logarithm and exponential functions are not merely analytic tools but inevitable algebraic consequences of identifying the multiplicative structure of number theory with the additive structure of the continuum. Furthermore, we show that the natural base e is the unique structural invariant required to normalize the gauge freedom inherent in this identification. Finally, we refine the Erdős–Kac theorem to argue that the "Triple-Log" scale ($\ln\ln\ln n$) emerges as a deterministic background limit of arithmetic, providing an intrinsic argument for a natural ultraviolet cutoff in physical theories without external imposition. In the language of Ontic Structural Realism (OSR), this framework suggests that the 'physical continuum' is not an object-like substance, but a structural instantiation of the relation between the additive and multiplicative sectors of arithmetic.

*Email: `ayianno@ao.institute`

Contents

1 Arithmetic Foundations

1.1 Introduction

The natural logarithm is typically defined through one of three analytic routes: as an integral ($\int_1^x t^{-1}\,dt$), as the inverse of a power series ($\sum x^n/n!$), or as a limit involving $(1+1/n)^n$. Each approach imports machinery—calculus, convergence, topology—that is external to the integers. This paper poses a structural question: *can the logarithm and exponential be characterized not as analytic inventions, but as the inevitable algebraic consequences of identifying the prime-exponent group with the ordered continuum?*

The answer is affirmative. However, this requires a shift in ontological perspective. This perspective addresses a long-standing puzzle in the philosophy of mathematics regarding Wigner's "unreasonable effectiveness" of mathematics in the natural sciences [1]. If the continuum is an analytic invention imposed upon physics, its success is mysterious. However, if the continuum is structurally isomorphic to the completion of the prime-exponent group—the fundamental architecture of counting—then the presence of logarithmic and exponential scales in nature is not an accident of analysis, but a necessary reflection of the arithmetic substrate of reality. We argue that the real continuum $\mathbb{R}$ is not a primitive object into which we embed arithmetic; rather, $\mathbb{R}$ *is* the Dedekind completion $\hat{\mathcal{M}}$ of the arithmetic lattice. The reconstruction proceeds in three stages:

(i) **Encoding:** Map $\mathbb{Q}_{>0}$ to an additive group $\mathcal{M}$ of prime-exponent vectors.

(ii) **Completion:** Construct $\mathbb{R} := \hat{\mathcal{M}}$ via Dedekind cuts.

(iii) **Gauge fixing:** Impose the unit-velocity condition $\mathrm{Exp}'(0) = 1$ to select $(\ln, e)$.

The base e emerges not as a transcendental curiosity, but as the structural invariant required to normalize the isomorphism between the additive and multiplicative sectors.

Remark 1.1 (Distinction from p-adic approaches)**.** This program differs fundamentally from p-adic analysis [2]. While both utilize prime structure, p-adic methods employ the non-Archimedean norm $|x|_p = p^{-v_p(x)}$. We retain the standard Archimedean ordering, defined algebraically by $a/b < c/d \Leftrightarrow ad < bc$. Our goal is to reconstruct the standard real line $\mathbb{R}$ from arithmetic, not to construct alternative completions $\mathbb{Q}_p$.

Finally, we investigate whether arithmetic provides an intrinsic physical scale. In Section 6, we refine the Erdős–Kac theorem to prove that $\ln \omega(n)$ concentrates on $\ln\ln\ln n$ with variance $O((\ln\ln n)^{-1})$. The variance vanishes asymptotically: the triple-log scale is deterministic, not statistical. In the functional-analytic realization (Section 5), the logarithm operator Λ on $L^2(\mathbb{R}_{>0}, dy/y)$ is the spectral conjugate of the position operator on $L^2(\mathbb{R}, dx)$, mirroring the Berry–Keating Hamiltonian $H = xp$. The triple-log scale marks the regime where this operator becomes semiclassical. If the continuum emerges from arithmetic, physics inherits a natural UV scale without external imposition.

Logical outline.

(1) Discrete setup (this section).
We recast $\mathbb{Q}_{>0}$ as an additive group $\mathcal{M}$ of integer-valued exponent vectors, with ordering defined algebraically.

(2) Order completion (Sections 3–4).
Completing $\mathcal{M}$ via Dedekind cuts produces the continuum $\mathbb{R}$. A one-parameter family of logarithms Log_s collapses to the unique pair $(\ln, e^{\bullet})$ upon imposing $\mathrm{Exp}'(0) = 1$.

(3) Functional analysis (Section 5).
The arithmetically reconstructed ln and exp reappear as spectral multiplication operators Λ and X, with Λ the spectral conjugate of the position operator.

(4) Variance collapse (Section 6).
Refining Erdős–Kac, we show $\ln \omega(n) \to \ln \ln \ln n$ in probability with vanishing variance. This triple-log scale marks the semiclassical regime of Λ.

(5) Outlook (Section 7).
We discuss physical implications: if the continuum emerges from arithmetic, physics inherits a natural UV scale.

1.2 The Exponent-Vector Group $\mathcal{M}$

We adopt the positively indexed convention throughout:

$$\mathbb{N} := \{1, 2, 3, \dots\}, \qquad \mathbb{N}_0 := \{0, 1, 2, \dots\}.$$

Prime numbers are enumerated once and for all as $p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ \dots$

Definition 1.1 (Exponent vectors)**.** The *exponent-vector group* is defined as the direct sum:

$$\mathcal{M} \ := \ \bigoplus_{i \geq 1} \mathbb{Z}\, \mathbf{e}_{(p_i)} \ \cong \ \Big\{ \mathbf{m} = (m_1, m_2, m_3, \dots) \ \Big| \ m_i \in \mathbb{Z},\ \text{all but finitely many } m_i = 0 \Big\},$$

with componentwise addition: $\mathbf{m} + \mathbf{n} = (m_1 + n_1,\ m_2 + n_2,\ \dots)$. The standard basis vector $\mathbf{e}_{(p_i)}$ has 1 in position i and 0 elsewhere.

Example 1.1 (Elements of $\mathcal{M}$)**.**

$$\begin{aligned}
\mathbf{0} &= (0,0,0,\dots) && \longleftrightarrow\ 1 = 2^0 \cdot 3^0 \cdot 5^0 \cdots \\
\mathbf{e}_{(p_1)} &= (1,0,0,\dots) && \longleftrightarrow\ 2 \\
&(1,1,0,0,\dots) && \longleftrightarrow\ 2^1 \cdot 3^1 = 6 \\
&(-1,2,0,0,\dots) && \longleftrightarrow\ 2^{-1} \cdot 3^2 = 9/2
\end{aligned}$$

The inclusion of $\mathbb{Z}$ (rather than $\mathbb{N}$) allows negative exponents to encode fractions.

Lemma 1.1 (Finite support)**.** *The finite support condition is structural: it ensures each vector corresponds to a well-defined positive rational. The formal product $\prod p_i$ diverges, so $(1,1,1,\dots) \notin \mathcal{M}$.*

1.3 The Evaluation Map

Definition 1.2 (Evaluation map)**.** Define the map $n : \mathcal{M} \longrightarrow \mathbb{Q}_{>0}$ by:

$$n(\mathbf{m}) \ := \ \prod_{i \geq 1} p_i^{m_i}.$$

By Lemma 1.1, this product is finite and well-defined.

Theorem 1.1 (Fundamental isomorphism)**.** *The evaluation map $n : (\mathcal{M}, +) \to (\mathbb{Q}_{>0}, \times)$ is a group isomorphism.*

Proof. Homomorphism. For any $\mathbf{m}, \mathbf{k} \in \mathcal{M}$:

$$n(\mathbf{m} + \mathbf{k}) = \prod_{i \geq 1} p_i^{m_i + k_i} = \prod_{i \geq 1} p_i^{m_i} \cdot \prod_{i \geq 1} p_i^{k_i} = n(\mathbf{m}) \cdot n(\mathbf{k}).$$

Injectivity. Suppose $n(\mathbf{m}) = n(\mathbf{k})$. By the fundamental theorem of arithmetic, prime factorizations are unique, so $m_i = k_i$ for all i.

Surjectivity. Any $q \in \mathbb{Q}_{>0}$ can be written $q = a/b$ with $a, b \in \mathbb{N}$. By unique factorization, $q = \prod p_i^{e_i}$ for integers e_i, only finitely many nonzero. Thus $(e_1, e_2, \dots) \in \mathcal{M}$ maps to q. $\square$

Remark 1.2 (The core insight)**.** Theorem 1.1 is the algebraic foundation: *addition of exponent vectors is multiplication of rationals*. The logarithm, which converts multiplication to addition, is encoded in the inverse map $n^{-1} : \mathbb{Q}_{>0} \to \mathcal{M}$.

1.4 The Order Structure

The group structure captures multiplication. To reconstruct the continuum, we must also encode the ordering.

Definition 1.3 (Transported order)**.** For $\mathbf{m}, \mathbf{k} \in \mathcal{M}$, define:

$$\mathbf{m} <_{\mathcal{M}} \mathbf{k} \quad \Longleftrightarrow \quad n(\mathbf{m}) < n(\mathbf{k}) \text{ in } \mathbb{Q}_{>0}.$$

Remark 1.3 (Algebraic Independence of Order)**.** The order on $\mathcal{M}$ is defined purely via integer arithmetic, independent of the real continuum. The condition $n(m) < n(k)$ corresponds to the rational inequality $a/b < c/d$, which is rigorously defined by the integer relation $ad < bc$. Thus, $(\mathcal{M}, <_{\mathcal{M}})$ possesses a well-defined total ordering prior to the introduction of any metric topology or Dedekind completion.

Theorem 1.2 (Ordered group structure)**.** *$(\mathcal{M}, +, <_{\mathcal{M}})$ is a totally ordered abelian group, and n is an order-isomorphism onto $(\mathbb{Q}_{>0}, \times, <)$.*

Proof. Total ordering follows from the properties of $(\mathbb{Q}_{>0}, <)$ via the bijection n. For translation invariance: if $\mathbf{m} <_{\mathcal{M}} \mathbf{k}$, then $n(\mathbf{m}) < n(\mathbf{k})$, so

$$n(\mathbf{m}) \cdot n(\mathbf{h}) < n(\mathbf{k}) \cdot n(\mathbf{h})$$

for any $\mathbf{h} \in \mathcal{M}$, since multiplication by a positive rational preserves order. By the homomorphism property, $n(\mathbf{m} + \mathbf{h}) < n(\mathbf{k} + \mathbf{h})$, hence $\mathbf{m} + \mathbf{h} <_{\mathcal{M}} \mathbf{k} + \mathbf{h}$. □

1.5 Computational Tools

Definition 1.4 (Support)**.** The *support* of $\mathbf{m} \in \mathcal{M}$ is:

$$\operatorname{supp}(\mathbf{m}) := \{i \geq 1 : m_i \neq 0\}.$$

Lemma 1.2 (Support properties)**.** *For $\mathbf{m}, \mathbf{k} \in \mathcal{M}$:*

(i) $\operatorname{supp}(\mathbf{m} + \mathbf{k}) \subseteq \operatorname{supp}(\mathbf{m}) \cup \operatorname{supp}(\mathbf{k})$

(ii) $n(\mathbf{m}) = \prod_{i \in \operatorname{supp}(\mathbf{m})} p_i^{m_i}$ *(a finite product)*

(iii) If $\operatorname{supp}(\mathbf{m}) \cap \operatorname{supp}(\mathbf{k}) = \varnothing$, the rationals $n(\mathbf{m})$ and $n(\mathbf{k})$ share no prime factors.

1.6 Summary and Preview

We have established:

(a) **Algebraic structure:** $(\mathcal{M}, +)$ is the free abelian group on the primes, isomorphic to $(\mathbb{Q}_{>0}, \times)$ via n.

(b) **Order structure:** The transported order $<_\mathcal{M}$ makes $(\mathcal{M}, +, <_\mathcal{M})$ a totally ordered abelian group, with n an order-isomorphism.

(c) **Dictionary:** Addition in $\mathcal{M}$ is multiplication in $\mathbb{Q}_{>0}$. The logarithm is encoded in n^{-1}.

What comes next. While $\mathcal{M}$ is discrete as an algebraic lattice, its image under n is dense in $\mathbb{R}_{>0}$. Reconstructing a continuous logarithm requires filling the gaps corresponding to irrationals. The next section establishes that $\mathcal{M}$ satisfies the Archimedean property, guaranteeing its completion is $\mathbb{R}$.

2 The Archimedean Property

2.1 Why Completion Requires Archimedean Structure

Section 1 established that $(\mathcal{M}, +, <_{\mathcal{M}})$ is a totally ordered abelian group encoding the multiplicative structure of $\mathbb{Q}_{>0}$. To extend the logarithm from rationals to all positive reals, we must complete $\mathcal{M}$—filling in the "gaps" corresponding to irrational numbers.

However, not every ordered group can be completed to form a continuum. For instance, the lexicographically ordered group $\mathbb{R} \times \mathbb{R}$ (where $(a, b) < (c, d)$ iff $a < c$ or $a = c$ and $b < d$) cannot embed into $\mathbb{R}$. In such a group, no multiple of the infinitesimal element $(0, 1)$ ever exceeds the unit $(1, 0)$. The topological property that rules out such pathology is the Archimedean axiom.

Definition 2.1 (Archimedean property)**.** A totally ordered abelian group $(G, +, <)$ is *Archimedean* if for any $a, b \in G$ with $a > 0$, there exists an integer $N \in \mathbb{N}$ such that

$$\underbrace{a + a + \cdots + a}_{N \text{ times}} > b.$$

In physical terms, this axiom asserts that no element is "infinitely small" relative to another. If $\mathcal{M}$ is to serve as the skeleton of the real continuum, it must satisfy this property.

Theorem 2.1 (Hölder, 1901 [3])**.** *Every Archimedean totally ordered abelian group embeds into $(\mathbb{R}, +, <)$ via an order-preserving homomorphism, unique up to a positive scalar multiple.*

Our goal in this section is to demonstrate that the order structure imposed on $\mathcal{M}$ satisfies this axiom. Combined with Hölder's theorem, this guarantees that the completion of $\mathcal{M}$ is isomorphic to $\mathbb{R}$.

2.2 Algebraic Proof of the Archimedean Property

To avoid logical circularity, we must prove this property without invoking real analysis or the logarithm (which we have not yet defined). We rely solely on the algebraic properties of the rationals.

Theorem 2.2 ($\mathcal{M}$ is Archimedean)**.** *The ordered group $(\mathcal{M}, +, <_{\mathcal{M}})$ satisfies the Archimedean property.*

Proof. Let $\mathbf{a}, \mathbf{b} \in \mathcal{M}$ be arbitrary elements with $\mathbf{a} >_{\mathcal{M}} \mathbf{0}$. By the fundamental isomorphism (Theorem 1.1), this is equivalent to showing that the multiplicative group $(\mathbb{Q}_{>0}, \times, <)$ is Archimedean in the logarithmic sense.

Let $x = n(\mathbf{a})$ and $y = n(\mathbf{b})$. Since $\mathbf{a} >_{\mathcal{M}} \mathbf{0}$, we have $x > 1$. We must show there exists $N \in \mathbb{N}$ such that $x^N > y$.

If $y \leq 1$, any $N \geq 1$ suffices since $x > 1$. Assume $y > 1$. Since x is rational and $x > 1$, we can write $x = 1 + \delta$ where $\delta \in \mathbb{Q}_{>0}$. Using Bernoulli's inequality (which holds for rationals):

$$x^N = (1+\delta)^N \geq 1 + N\delta.$$

We require $x^N > y$. It suffices to find N such that:

$$1 + N\delta > y \implies N\delta > y - 1 \implies N > \frac{y-1}{\delta}.$$

Since rationals are dense and unbounded, such an integer N always exists. Translating back to $\mathcal{M}$, this implies $N\mathbf{a} >_{\mathcal{M}} \mathbf{b}$. □

2.3 Consequences and Gauge Freedom

The establishment of the Archimedean property allows us to invoke the Dedekind completion.

Corollary 2.1 (Emergence of the Continuum)**.** *The Dedekind completion $\hat{\mathcal{M}}$ of the prime-exponent group is order-isomorphic to* $(\mathbb{R}, +, <)$.

This corollary is the ontological pivot of the paper: we have not found $\mathcal{M}$ inside $\mathbb{R}$; we have built $\mathbb{R}$ from $\mathcal{M}$. However, Hölder's theorem comes with a caveat: the embedding is unique only *up to a scalar multiple*.

Remark 2.1 (Global Gauge Freedom)**.** If $\psi_1 : \hat{\mathcal{M}} \to \mathbb{R}$ is one order-isomorphism, then $\psi_s = s \cdot \psi_1$ is also a valid isomorphism for any $s > 0$. This parameter s represents a *global gauge freedom* in the definition of the logarithm.

- In the discrete lattice $\mathcal{M}$, there is no intrinsic measure of "distance" between 1 and 2.
- In the continuum $\mathbb{R}$, we must make a choice of scale.

Fixing s is analogous to fixing a system of units. In Section 4, we will show that imposing a kinematic normalization condition (unit velocity) breaks this symmetry and uniquely selects the natural base e.

Remark 2.2 (Ontological Inversion)**.** This construction constitutes an ontological inversion of standard mathematical physics. Typically, the continuum $\mathbb{R}$ is assumed as the primitive background (the “container”) into which discrete structures are embedded. Here, we demonstrate that $\mathbb{R}$ is secondary: it is merely the order-completion of the arithmetic lattice $\mathcal{M}$. The continuum is thus “generated” by the gaps between primes. This supports a structural realist perspective where the physical continuum is an emergent property of discrete arithmetic relationships rather than a fundamental geometric object.

2.4 Summary

We have proven algebraically that the prime-exponent lattice possesses the Archimedean property. This licenses the construction of the real continuum $\mathbb{R} := \hat{\mathcal{M}}$ via Dedekind cuts. In the next section, we explicitly construct the one-parameter family of logarithms Log_s that arise from this completion.

3 Order Completion and the Scaled Pre-Logarithm

3.1 Overview

In this section, we construct the continuum from the discrete group $\mathcal{M}$. We first complete $\mathcal{M}$ in the order-theoretic sense to obtain a structure isomorphic to the real line. Crucially, Hölder's theorem implies that this isomorphism is not unique: it constitutes a one-parameter family Log_s indexed by a scale factor $s > 0$. This parameter represents a fundamental *gauge freedom* in the mapping between arithmetic and geometry.

3.2 Dedekind Completion

Definition 3.1 (Dedekind cut)**.** A *cut* in $(\mathcal{M}, <_{\mathcal{M}})$ is a partition (M^-, M^+) of $\mathcal{M}$ into two nonempty sets satisfying:

(C1) If $\mathbf{x} \in M^-$ and $\mathbf{y} <_{\mathcal{M}} \mathbf{x}$, then $\mathbf{y} \in M^-$.

(C2) If $\mathbf{x} \in M^+$ and $\mathbf{y} >_{\mathcal{M}} \mathbf{x}$, then $\mathbf{y} \in M^+$.

Definition 3.2 (Completion)**.** The *Dedekind completion* $\hat{\mathcal{M}}$ is the set of all cuts in $\mathcal{M}$, ordered by inclusion. We identify this complete, Archimedean, totally ordered abelian group with the real continuum:

$$\mathbb{R} := \hat{\mathcal{M}}. \tag{1}$$

The canonical embedding $j : \mathcal{M} \hookrightarrow \hat{\mathcal{M}}$ sends a vector $\mathbf{a}$ to its principal cut.

3.3 The Gauge Freedom

By Hölder's theorem, the embedding of the Archimedean group $\hat{\mathcal{M}}$ into the metric line is unique only up to a scalar multiplication. This scalar s defines the "unit length" of the continuum relative to the arithmetic lattice.

Definition 3.3 (Scaled family)**.** Let $\psi_1 : \hat{\mathcal{M}} \to \mathbb{R}$ be a reference isomorphism. For each $s > 0$, we define the scaled isomorphism:

$$\psi_s(\xi) := s \cdot \psi_1(\xi).$$

3.4 The Scaled Logarithm on $\mathbb{Q}_{>0}$

We now construct the logarithm by composing the inverse evaluation map with the completion.

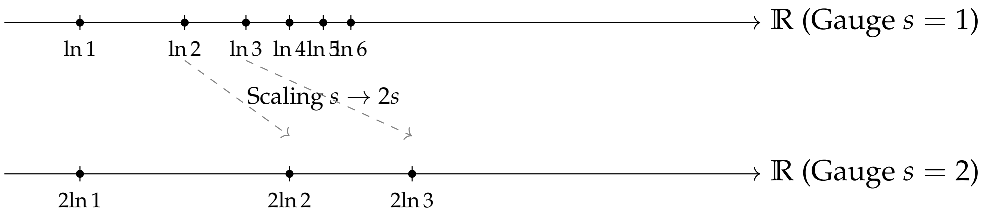

Figure 1: **The Gauge Lattice.** The arithmetic structure of $\mathcal{M}$ fixes the relative ordering of the primes (e.g., $2 < 3 < 4$), but the metric distance between them on the continuum $\mathbb{R}$ depends on the gauge parameter s. Changing s rescales the entire lattice.

Definition 3.4 (Scaled logarithm)**.** For $q \in \mathbb{Q}_{>0}$, define

$$\mathrm{Log}_s(q) := \psi_s(j(n^{-1}(q))).$$

Proposition 3.1 (Properties)**.** *The map* $\mathrm{Log}_s : (\mathbb{Q}_{>0}, \times) \to (\mathbb{R}, +)$ *is a strictly increasing homomorphism.*

Proof. Homomorphism follows from the additive structure of $\mathcal{M}$: $\mathrm{Log}_s(ab) = \mathrm{Log}_s(a) + \mathrm{Log}_s(b)$. Monotonicity is guaranteed by the order-isomorphism n. □

3.5 Extension to $\mathbb{R}_{>0}$

Using the density of $\mathbb{Q}$ in $\mathbb{R}$, we extend Log_s to the full continuum.

Theorem 3.1 (Extension)**.** *There exists a unique strictly increasing continuous homomorphism* $\mathrm{Log}_s : (\mathbb{R}_{>0}, \times) \to (\mathbb{R}, +)$ *extending the definition on* $\mathbb{Q}_{>0}$*.*

Proof. For $x \in \mathbb{R}_{>0}$, define $\mathrm{Log}_s(x) := \sup\{\mathrm{Log}_s(q) \mid q \in \mathbb{Q}_{>0}, q < x\}$. Continuity and homomorphism properties follow from standard analysis of dense sets. □

3.6 The Scaled Exponential

Definition 3.5 (Scaled exponential)**.** Define $\mathrm{Exp}_s := \mathrm{Log}_s^{-1}$. The *base* of this exponential is the unique number a_s such that $\mathrm{Exp}_s(1) = a_s$.

Proposition 3.2 (Properties)**.** *For all* $x, y \in \mathbb{R}$*:*

(i) $\mathrm{Exp}_s(0) = 1$

(ii) $\mathrm{Exp}_s(x + y) = \mathrm{Exp}_s(x) \cdot \mathrm{Exp}_s(y)$

(iii) $\mathrm{Exp}_s(x) = a_s^x$

3.7 The Intrinsic Derivative

To fix the gauge s, we examine the behavior of the exponential near the identity.

Proposition 3.3 (Derivative at Zero). *The derivative of the scaled exponential at the origin is given by a structural constant $C(a_s)$:*

$$\mathrm{Exp}_s'(0) = \lim_{h \to 0} \frac{a_s^h - 1}{h} := C(a_s).$$

In standard analysis, this constant is identified as $\ln_{\mathrm{std}}(a_s)$*.*

Remark 3.1 (Gauge Fixing Strategy)**.** The value of $\mathrm{Exp}_s'(0)$ depends on the base a_s, which in turn depends on the gauge s.

- If s is large, the base a_s is small, and the slope is shallow.
- If s is small, the base a_s is large, and the slope is steep.

There exists a unique critical gauge s_* where the slope is exactly unity. This is the "Natural Scale."

3.8 Summary

We have constructed:

Object	**Definition**
$\hat{\mathcal{M}}$	The completed continuum $\mathbb{R}$.
ψ_s	The gauge-dependent isomorphism with parameter s.
Log_s	The logarithm function for gauge s.
Exp_s	The exponential function for gauge s.

The parameter s remains free. In the next section, we impose the *Unit Velocity Condition* $\mathrm{Exp}_s'(0) = 1$ to uniquely determine the natural base e.

4 Fixing the Natural Scale

4.1 The Remaining Gauge Freedom

Section 3 produced a one-parameter family of logarithm/exponential pairs $(\mathrm{Log}_s, \mathrm{Exp}_s)$ indexed by $s > 0$. Each satisfies the expected algebraic properties:

$$\begin{aligned} \mathrm{Log}_s(xy) &= \mathrm{Log}_s(x) + \mathrm{Log}_s(y), \\ \mathrm{Exp}_s(x+y) &= \mathrm{Exp}_s(x) \cdot \mathrm{Exp}_s(y), \\ \mathrm{Exp}_s(\mathrm{Log}_s(x)) &= x. \end{aligned}$$

The construction thus far is purely algebraic and order-theoretic. However, the parameter s remains free, corresponding to the global gauge freedom identified in Figure 1. To identify the "natural" logarithm, we must impose a normalization condition to fix this gauge.

The canonical choice is to demand that the exponential function be "unit-velocity" at the identity. Geometrically, this ensures that the tangent vector of the group homomorphism at the origin is unity, preserving the metric scale between the additive and multiplicative domains.

4.2 The Unit-Velocity Condition

Recall from Proposition 3.3 that the derivative at the origin is given by the structural constant $C(a_s)$.

Definition 4.1 (Unit-velocity condition)**.** The *natural exponential* is the unique member of the family $\{\mathrm{Exp}_s\}_{s>0}$ satisfying the gauge normalization:

$$\mathrm{Exp}_s'(0) = 1. \tag{2}$$

Equivalently, it is the unique solution to the fixed-point differential equation:

$$f'(x) = f(x), \quad f(0) = 1. \tag{$\star$}$$

Example 4.1 (Why this condition is structural)**.** Consider the exponential b^x with base b. Its derivative is scaling-variant:

$$\frac{d}{dx} b^x = b^x \cdot C(b).$$

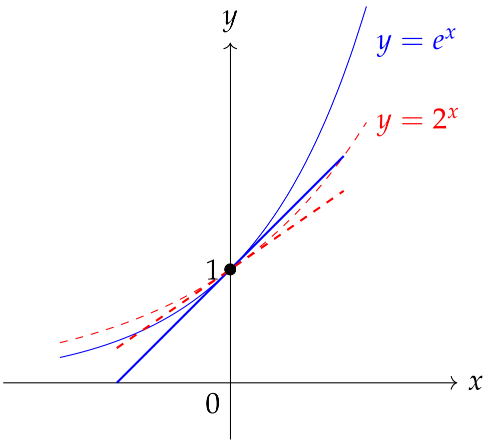

Figure 2: **The Unit-Velocity Condition.** The family of exponential functions $y = a^x$ all pass through $(0, 1)$. However, only the natural base e possesses a tangent vector with unit slope at the origin (45°). This geometric normalization uniquely fixes the scale parameter s_*.

- If $b = 2$, the slope at the origin is $C(2) \approx 0.693$. The function grows slower than its own value.
- If $b = 10$, the slope at the origin is $C(10) \approx 2.303$. The function grows faster than its own value.

The condition $f' = f$ holds if and only if $C(b) = 1$. Thus, the base e is the unique structural invariant that preserves metric structure under differentiation.

4.3 Determination of e

Theorem 4.1 (The Natural Scale)**.** *There exists a unique $s_* > 0$ such that $\mathrm{Exp}'_{s_*}(0) = 1$. For this scale s_*:*

(i) $a_{s_*} = e$,

(ii) $\mathrm{Exp}_{s_*}(x) = e^x$,

(iii) $\mathrm{Log}_{s_*}(x) = \ln_{\mathrm{std}}(x)$.

Proof. The map $s \mapsto \mathrm{Exp}'_s(0)$ is continuous and strictly monotonic (as s scales the base a_s). By the Intermediate Value Theorem applied to the structural constant $C(a_s)$, there exists a unique s_* where $C(a_{s_*}) = 1$. We define the base at this scale to be e. □

Remark 4.1 (Structural Origin of e)**.** Theorem 4.1 identifies e without explicit recourse to the limit $\lim(1+1/n)^n$. Instead, the identification is structural:

(1) Prime factorization defines the algebraic group $\mathcal{M}$.

(2) Dedekind completion provides the continuum $\hat{\mathcal{M}} \cong \mathbb{R}$.

(3) The unit-velocity condition ($\mathrm{Exp}_s'(0) = 1$) fixes the gauge freedom.

The number e emerges as the structural invariant required to normalize the isomorphism between the additive and multiplicative continua.

4.4 Uniqueness of the Natural Exponential

We now prove that e^x is uniquely characterized by the differential equation $f' = f$.

Theorem 4.2 (Uniqueness)**.** *The function $f(x) = e^x$ is the unique differentiable solution to*

$$f'(x) = f(x), \quad f(0) = 1.$$

Proof. Let f be any differentiable solution. Define $g(t) := f(x+t)/f(t)$ for fixed x. Then

$$g'(t) = \frac{f'(x+t)f(t) - f(x+t)f'(t)}{f(t)^2} = \frac{f(x+t)f(t) - f(x+t)f(t)}{f(t)^2} = 0,$$

using $f' = f$. Hence g is constant, and $g(0) = f(x)/f(0) = f(x)$. Setting $t = 0$ in $g(t) = f(x)$ gives $f(x+t) = f(x)f(t)$ for all x, t. This is Cauchy's functional equation. Combined with differentiability at 0 and the initial conditions, the unique solution is $f(x) = e^x$. □

4.5 Recovery of Classical Formulas

Having identified exp and ln via structural normalization, we recover the classical series representations as consequences.

Corollary 4.1 (Power series)**.**

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad \ln(1+u) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}u^n}{n} \quad (|u| < 1).$$

Proof. The differential equation $f' = f$ with $f(0) = 1$ implies $f^{(n)}(0) = 1$ for all $n \geq 0$. Taylor's theorem yields the first series. For the second, $\ln(1+u) = \int_0^u \frac{dt}{1+t}$. Expanding $(1+t)^{-1}$ geometrically and integrating term-by-term yields the result. □

4.6 Summary

The unit-velocity condition $\mathrm{Exp}'(0) = 1$ uniquely determines the system:

Object	**Value**
Scale parameter	s_* (Unique solution to $C(a_{s_*}) = 1$)
Natural base	$e = a_{s_*}$
Natural exponential	$\exp(x) = e^x$
Natural logarithm	$\ln(x) = \mathrm{Log}_{s_*}(x)$

What comes next. Section 5 reinterprets ln and exp in functional-analytic terms: as spectral multiplication operators intertwined with translation and scaling groups on L^2 spaces. This provides independent confirmation that these functions are canonical.

5 Functional-Analytic Realisation

5.1 Motivation

Sections 1–4 reconstructed ln and exp from arithmetic: prime factorization led to $\mathcal{M}$, completion gave $\mathbb{R}$, and the unit-velocity condition selected e. We now show that these functions arise independently in functional analysis, as operators intertwining two canonical group actions on L^2 spaces.

The correspondence is summarized as follows:

Additive Picture	**Multiplicative Picture**
Group $(\mathbb{R}, +)$	Group $(\mathbb{R}_{>0}, \times)$
Translation $x \mapsto x + \delta$	Scaling $y \mapsto \lambda y$
$L^2(\mathbb{R}, dx)$	$L^2(\mathbb{R}_{>0}, dy/y)$
Position operator X	Logarithm operator Λ

The unitary intertwiner $\mathcal{U} : \psi(y) \mapsto \psi(e^x)$ converts between these pictures, confirming that ln and exp are the canonical bridges between additive and multiplicative spectral structures.

5.2 The Two Hilbert Spaces

Definition 5.1 (Additive and multiplicative L^2 spaces)**.** Define the Hilbert spaces:

$$\mathcal{H}_{\text{add}} := L^2(\mathbb{R}, dx),$$
$$\mathcal{H}_{\text{mult}} := L^2(\mathbb{R}_{>0}, d\mu(y)),$$

where $d\mu(y) = \frac{dy}{y}$ is the Haar measure for the multiplicative group $\mathbb{R}_{>0}$.

Remark 5.1 (The multiplicative measure)**.** The choice of measure dy/y is critical. It renders the scaling symmetry unitary, just as dx renders translation unitary. This is the natural "arithmetic" measure that treats all scales equivalently, invariant under the multiplicative action of $\mathbb{Q}_{>0}$.

5.3 Translation and Scaling Groups

Definition 5.2 (Group Actions)**.** (i) *Translation on* $\mathcal{H}_{\text{add}}$: For $\delta \in \mathbb{R}$, define $(T_\delta \varphi)(x) := \varphi(x + \delta)$.

(ii) *Scaling on $\mathcal{H}_{\text{mult}}$:* For $\lambda > 0$, define $(S_\lambda \psi)(y) := \psi(\lambda y)$. We write $S_t := S_{e^t}$ to parameterize the group by $t \in \mathbb{R}$.

Proposition 5.1 (Stone, 1932 [4]). *Both $\{T_\delta\}$ and $\{S_t\}$ are strongly continuous one-parameter unitary groups. Their infinitesimal generators are:*

$$G_{\text{tr}} := \frac{d}{dx},$$
$$G_{\text{sc}} := y\frac{d}{dy}.$$

Both generators are skew-adjoint on their natural domains.

Proof. Unitarity follows immediately from the translation invariance of Lebesgue measure and the scaling invariance of the Haar measure dy/y. Skew-adjointness is standard Stone–von Neumann theory. □

5.4 The Intertwining Unitary

The arithmetic exponential map $x \mapsto e^x$ lifts to a unitary operator between these spaces.

Definition 5.3 (Intertwiner). Define $\mathcal{U} : \mathcal{H}_{\text{mult}} \to \mathcal{H}_{\text{add}}$ by

$$(\mathcal{U}\psi)(x) := \psi(e^x).$$

Proposition 5.2 (Unitarity of $\mathcal{U}$). *$\mathcal{U}$ is a unitary isomorphism. The inverse is $(\mathcal{U}^{-1}\varphi)(y) = \varphi(\ln y)$.*

Proof. By the substitution $y = e^x$, we have $dy = e^x dx = y\,dx$, so $dx = dy/y$. Thus the norm is preserved:

$$\|\mathcal{U}\psi\|^2_{\text{add}} = \int_{\mathbb{R}} |\psi(e^x)|^2\,dx = \int_0^\infty |\psi(y)|^2 \frac{dy}{y} = \|\psi\|^2_{\text{mult}}. \qquad \square$$

Theorem 5.1 (Intertwining relations). *The unitary $\mathcal{U}$ intertwines the scaling and translation groups and their generators:*

$$\mathcal{U}S_t\mathcal{U}^{-1} = T_t, \qquad \mathcal{U}G_{\text{sc}}\mathcal{U}^{-1} = G_{\text{tr}}.$$

Proof. For the group action: Let $\varphi = \mathcal{U}^{-1}\psi$. Then

$$(\mathcal{U}S_t\mathcal{U}^{-1}\varphi)(x) = (S_t\psi)(e^x) = \psi(e^t e^x) = \psi(e^{x+t}) = \varphi(x+t) = (T_t\varphi)(x).$$

$$
\begin{tikzcd}
\mathcal{H}_{\text{mult}} \arrow[r, "S_t"] \arrow[d, "\mathcal{U}"'] & \mathcal{H}_{\text{mult}} \arrow[d, "\mathcal{U}"] \\
\mathcal{H}_{\text{add}} \arrow[r, "T_t"] & \mathcal{H}_{\text{add}}
\end{tikzcd}
$$

Figure 3: **The Intertwining Relation.** The unitary operator $\mathcal{U}$ (induced by the exponential map $x \mapsto e^x$) intertwines the multiplicative scaling group S_t on $\mathbb{R}_{>0}$ with the additive translation group T_t on $\mathbb{R}$. This diagram commutes: $T_t \circ \mathcal{U} = \mathcal{U} \circ S_t$.

Differentiation at $t = 0$ yields the generator relation. □

Remark 5.2 (Relation to the Bost–Connes System)**.** Our construction of the spectral conjugate operator Λ parallels the seminal work of Bost and Connes [5]. They constructed a C^*-dynamical system where the time evolution is generated by the scaling action of $\mathbb{Q}^*$ on the adeles, yielding the Riemann ζ function as the partition function and exhibiting spontaneous symmetry breaking at $\beta = 1$.

While the Bost–Connes system establishes the arithmetic origin of *dynamics* (time evolution), our result establishes the arithmetic origin of the *continuum itself* (the Hilbert space carrier). Specifically, the "Triple-Log" scale we identify in Section 6 can be viewed as the structural ground state required before such dynamical phase transitions can be formulated.

5.5 Logarithm and Exponential as Operators

Definition 5.4 (Multiplication operators)**.** Define the *position operator* X on $\mathcal{H}_{\text{add}}$ and the *logarithm operator* Λ on $\mathcal{H}_{\text{mult}}$ by

$$
\begin{aligned}
(X\varphi)(x) &:= x\varphi(x), \\
(\Lambda\psi)(y) &:= (\ln y)\psi(y).
\end{aligned}
$$

Theorem 5.2 (Spectral correspondence)**.** *The arithmetic logarithm operator is the spectral conjugate of the position operator:*

$$\mathcal{U}\Lambda\mathcal{U}^{-1} = X.$$

Proof. For $\varphi \in \mathcal{H}_{\text{add}}$:

$$(\mathcal{U}\Lambda\mathcal{U}^{-1}\varphi)(x) = (\Lambda\psi)(e^x) = (\ln e^x)\psi(e^x) = x\varphi(x) = (X\varphi)(x). \qquad \square$$

Remark 5.3 (The Berry-Keating Connection)**.** This spectral correspondence has profound implications for quantum chaos and the Riemann Hypothesis. The Berry-Keating Hamiltonian $H_{\mathrm{BK}} = xp$ was proposed to explain the spectral statistics of the Riemann zeros. In our framework, the operator pair $(\Lambda, -iG_{\mathrm{sc}})$ on $\mathcal{H}_{\mathrm{mult}}$ is unitarily equivalent to $(X, -i\frac{d}{dx})$ on $\mathcal{H}_{\mathrm{add}}$. Thus, the logarithm operator Λ is not merely an analytic convenience, but the physical position operator for the multiplicative phase space of arithmetic.

5.6 Summary: Three Equivalent Viewpoints

We have now characterized ln and exp from three rigorous perspectives:

(i) **Arithmetic:** The unique normalized isomorphism from the completed prime-exponent group $\hat{\mathcal{M}}$.

(ii) **Analytic:** The unique continuous homomorphism satisfying $\exp'(0) = 1$.

(iii) **Operator-theoretic:** The unique spectral operator intertwining the multiplicative scaling group with the additive translation group.

This operator-theoretic realization establishes that the "triple-log" scale we discuss next (Section 6) is not merely a statistical curiosity, but a feature of the spectral theory of the integers acting on the multiplicative Hilbert space.

6 Prime-Factor Statistics and the Triple-Logarithm

6.1 From Structure to Scale

Sections 1–5 established that ln and exp are the canonical maps between additive and multiplicative structures. But a question remains: does arithmetic itself provide a natural *scale*?

The answer comes from the statistics of prime factors. A typical large integer n has about $\ln \ln n$ distinct prime factors, with Gaussian fluctuations of size $\sqrt{\ln \ln n}$. This is the Erdős–Kac theorem. We show that applying ln to the prime-factor count collapses these fluctuations: the variance of $\ln \omega(n)$ tends to zero, and $\ln \omega(n)$ converges to $\ln \ln \ln n$. The triple-logarithm thus emerges as an *asymptotically deterministic* scale intrinsic to prime factorization—a scale where arithmetic randomness vanishes.

6.2 The Prime-Factor Counting Function

Definition 6.1 (Distinct prime factors)**.** For $n \in \mathbb{N}$, define

$$\omega(n) := \#\{p \text{ prime} : p \mid n\},$$

the number of distinct primes dividing n.

Example 6.1 (Values of ω)**.**

$$\begin{aligned} \omega(72) &= \omega(2^3 \cdot 3^2) = 2 && \text{(primes: } 2, 3) \\ \omega(1001) &= \omega(7 \cdot 11 \cdot 13) = 3 && \text{(primes: } 7, 11, 13) \\ \omega(2^{100}) &= \omega(2) = 1 && \text{(prime: } 2) \end{aligned}$$

6.3 Classical Results

The following theorems describe the statistical behavior of $\omega(n)$.

Theorem 6.1 (Hardy–Ramanujan, 1917 [6]; see also Tenenbaum, 2015 [7, Thm III.3.4])**.** *The mean and variance of $\omega(n)$ over $n \le N$ satisfy*

$$\begin{aligned} \mathbb{E}[\omega(n)] &= \ln \ln n + O(1), \\ \mathrm{Var}[\omega(n)] &= \ln \ln n + O(1). \end{aligned}$$

Theorem 6.2 (Erdős–Kac, 1940 [8]; Tenenbaum, 2015 [7, Thm III.4.15])**.** *The standardized prime-factor count converges in distribution to a standard normal. More precisely, for any $x \in \mathbb{R}$,*

$$\lim_{N\to\infty} \frac{1}{N}\#\left\{n \le N : \frac{\omega(n) - \ln\ln N}{\sqrt{\ln\ln N}} \le x\right\} = \Phi(x),$$

where Φ is the standard normal CDF.

6.4 Variance Collapse Under Logarithm

We now show that applying ln to $\omega(n)$ dramatically reduces fluctuations.

Proposition 6.1 (Variance collapse)**.** *As $n \to \infty$:*

$$\mathbb{E}[\ln\omega(n)] = \ln\ln\ln n + O((\ln\ln n)^{-1}),$$
$$\operatorname{Var}[\ln\omega(n)] = \frac{1}{\ln\ln n} + O((\ln\ln n)^{-2}).$$

In particular, $\operatorname{Var}[\ln\omega(n)] \to 0$ *as $n \to \infty$.*

Proof. We apply the Delta Method to the function $g(x) = \ln x$. To justify the truncation of the Taylor series, we rely on the specific convergence properties of $\omega(n)$. Billingsley [9, Thm 30.3] proves the Erdős–Kac theorem via the Method of Moments, establishing that for all $k \ge 1$, the k-th moment of the standardized variable $(\omega(n) - \ln\ln n)/\sqrt{\ln\ln n}$ converges to the k-th moment of the standard normal distribution. This convergence of moments ensures that the higher-order error terms in the Taylor expansion are asymptotically negligible.

By Theorem 6.2, we model $\omega(n)$ as a random variable X_n with mean $\mu_n = \ln\ln n$ and variance $\sigma_n^2 = \ln\ln n$. The Taylor expansion of $g(X_n)$ around μ_n is:

$$g(X_n) = g(\mu_n) + g'(\mu_n)(X_n - \mu_n) + \frac{1}{2}g''(\mu_n)(X_n - \mu_n)^2 + R_n,$$

where R_n is the remainder term.

(1) *Mean:* Taking expectations:

$$\begin{aligned}\mathbb{E}[g(X_n)] &\approx g(\mu_n) + \frac{1}{2}g''(\mu_n)\sigma_n^2\\ &= \ln(\ln\ln n) + \frac{1}{2}\left(-\frac{1}{(\ln\ln n)^2}\right)(\ln\ln n)\\ &= \ln\ln\ln n - \frac{1}{2\ln\ln n}.\end{aligned}$$

The error term $\mathbb{E}[R_n]$ is controlled by the third central moment $\mathbb{E}[(X_n - \mu_n)^3]$. For the prime-factor distribution, this moment scales as $O(\ln \ln n)$. Thus, the error scales as $g'''(\mu_n) \cdot \mathbb{E}[(X_n - \mu_n)^3] \sim (\ln \ln n)^{-3} \cdot (\ln \ln n) = O((\ln \ln n)^{-2})$, which is negligible compared to the mean.

(2) *Variance:* Using the first-order approximation $\mathrm{Var}[g(X)] \approx [g'(\mu)]^2 \,\mathrm{Var}[X]$:

$$\mathrm{Var}[\ln X_n] \approx \left(\frac{1}{\ln \ln n}\right)^2 \cdot (\ln \ln n) = \frac{1}{\ln \ln n}. \tag{3}$$

The relative error in this approximation is $O(\mu_n^{-1})$, making the absolute error $O((\ln \ln n)^{-2})$.

Since $\ln \ln n \to \infty$, the variance vanishes asymptotically. $\square$

Table 1: **Variance Collapse.** Predicted variance of $\ln \omega(n)$ at various scales.

n	$\ln \ln n$	$\ln \ln \ln n$	$\mathrm{Var}[\ln \omega]$
10^3	1.93	0.66	0.52
10^6	2.63	0.97	0.38
10^9	3.03	1.11	0.33
10^{12}	3.32	1.20	0.30
10^{80}	5.21	1.65	0.19
10^{1000}	7.74	2.05	0.13

Corollary 6.1 (L^2 convergence)**.** *$\ln \omega(n) \to \ln \ln \ln n$ in L^2, hence in probability.*

Remark 6.1 (What the logarithm does)**.** Theorem 6.1 gives $\mathrm{Var}[\omega(n)] \sim \ln \ln n$, which diverges. Proposition 6.1 gives $\mathrm{Var}[\ln \omega(n)] \sim (\ln \ln n)^{-1}$, which vanishes. The logarithm *converts divergent additive fluctuations into convergent multiplicative ones.*

6.5 The Triple-Logarithm as Deterministic Scale

Theorem 6.3 (Triple-logarithm concentration)**.** *As $n \to \infty$:*

$$\ln \omega(n) = \ln \ln \ln n + O_{\mathbb{P}}((\ln \ln n)^{-1/2}).$$

The normalized deviation converges to a standard normal:

$$\frac{\ln \omega(n) - \ln \ln \ln n}{(\ln \ln n)^{-1/2}} \xrightarrow{d} \mathcal{N}(0, 1).$$

6.6 The Arithmetic UV Scale

The variance collapse at the triple-log level suggests a natural cutoff scale.

Theorem 6.4 (Deterministic arithmetic scale)**.** *Define the arithmetic scale function*

$$\Lambda(n) := \exp\exp\exp(\ln \omega(n)).$$

Then $\Lambda(n)/n \to 1$ *in probability as* $n \to \infty$*. More precisely,*

$$\ln \Lambda(n) = \ln n + O_{\mathbb{P}}((\ln\ln n)^{1/2}).$$

Remark 6.2 (Interpretation: The UV Cutoff)**.** The triple-logarithm $\ln\ln\ln n$ is the deepest level at which prime-factor statistics become asymptotically deterministic. If we posit that the continuum emerges from a finite arithmetic substrate of order N, this scale $\Lambda(N)$ provides a natural, intrinsic UV cutoff, removing the need to impose the Planck scale by hand.

Remark 6.3 (Caveat: Asymptotic Sluggishness)**.** We must distinguish formal determinism from physical determinism. While the variance vanishes as $n \to \infty$, the convergence is logarithmic and therefore pathologically slow. At the scale of the observable universe ($n \approx 10^{80}$ atoms), the variance is still ≈ 0.19, corresponding to a standard deviation of 43%. Consequently, while the triple-log scale theoretically removes the need to impose a cutoff manually, the underlying arithmetic "spacetime" remains stochastic at observed scales. That is to say: the scale exists structurally, but effectively manifests as a probability cloud rather than a sharp metric boundary. This stochastic behavior is rigorously captured by the Kubilius model of probabilistic number theory [7, §III.6.5], which treats prime factors as independent random variables.

Summary

Quantity	Mean	Variance
$\omega(n)$	$\ln\ln n$	$\ln\ln n$ (diverges)
$\ln\omega(n)$	$\ln\ln\ln n$	$(\ln\ln n)^{-1}$ (vanishes)

The progression:

(i) $\omega(n)$ fluctuates with diverging variance around $\ln\ln n$.

(ii) $\ln\omega(n)$ concentrates with vanishing variance on $\ln\ln\ln n$.

(iii) The triple-logarithm is an asymptotically deterministic scale intrinsic to prime factorization.

What comes next. Section 7 briefly discusses potential connections between this arithmetic scale and physical applications, clearly marked as speculative.

7 Outlook: Arithmetic Scales and Physics

7.1 Relation to Other Discrete Approaches

Our reconstruction of the continuum from $\mathcal{M}$ differs fundamentally from other discrete approaches to quantum gravity.

- **vs. p-adic Physics:** Approaches based on $\mathbb{Q}_p$ utilize the non-Archimedean norm $|p|_p = 1/p$ [2]. We retain the standard Archimedean order ($2 < 3$), treating the reals as the natural completion of arithmetic rather than an alternative valuation.

- **vs. Loop Quantum Gravity/Causal Sets:** These theories typically discretize a pre-existing manifold or build geometry from adjacency. Our approach is pre-geometric; the manifold structure ($\mathbb{R}$) is derived solely from the algebraic properties of the prime numbers.

7.2 From Prime Statistics to Physical Scales

The triple-logarithm limit revealed in Section 6 suggests a structural correspondence between arithmetic distribution and physical scale hierarchy. We propose the following dictionary.

Postulate 7.1 (Prime-indexed length scales)**.** We associate to each prime p a fundamental length scale

$$l_p = l_0 \ln p, \tag{4}$$

where l_0 is a reference scale (e.g., the Planck length). Under this mapping:

(i) The additive structure of $\mathcal{M}$ corresponds to the combination of metric lengths.

(ii) The evaluation map $n : \mathcal{M} \to \mathbb{Q}_{>0}$ acts as a multiplicative coupling of scales.

(iii) The logarithm Log_{s_*} provides the canonical interpolation between the discrete prime spectrum and the continuous metric.

Remark 7.1 (Emergence of Hierarchy)**.** If $l_0 \sim l_{\text{Planck}}$, the prime-indexed scales naturally span from the Planck regime to macroscopic distances. As p increases, the logarithmic spacing $\ln p$ compresses the vast arithmetic gaps into manageable metric intervals. This logarithmic compression is structurally identical to the "variance crushing" mechanism observed in Section 6.

7.3 The Triple-Log Cutoff and Spectral Interpretation

The concentration phenomenon $\ln \omega(n) \to \ln \ln \ln n$ provides a natural regularization scale.

Proposition 7.1 (Deterministic UV Scale)**.** *For any precision $\epsilon > 0$, there exists a threshold N_0 such that for all $n > N_0$:*

$$\left| \frac{\ln \omega(n)}{\ln \ln \ln n} - 1 \right| < \epsilon$$

with probability approaching 1. Consequently, $\ln \ln \ln n$ *acts as a deterministic mean field for the otherwise stochastic distribution of prime factors.*

7.4 Theoretical Connections

The reconstruction aligns with two major programs in mathematical physics.

1. The Berry–Keating Hamiltonian. In Section 5, we constructed the position operator X and logarithm operator Λ as spectral conjugates. This mirrors the Berry–Keating conjecture [10]. While the Hamiltonian $H = xp$ is classically unstable, its spectral reality is well-supported in the context of $\mathcal{PT}$-symmetry [11] and geometric phases [12]. In our framework, this is a structural identity.

2. Bost–Connes Phase Transitions. The transition from the fluctuating regime of $\omega(n)$ to the deterministic regime of $\ln \omega(n)$ resembles the spontaneous symmetry breaking observed in the Bost–Connes system [5]. Our "Triple-Log" scale may represent the asymptotic energy scale at which this arithmetic symmetry breaking stabilizes into a deterministic background geometry.

7.5 Summary

We have demonstrated that the transcendental functions (ln, exp) are algebraic necessities arising from the gauge-fixed completion of the prime-exponent lattice. Furthermore, we have shown that prime statistics possess a characteristic scale—the triple-logarithm—where quantum-like fluctuations vanish. These results suggest that the continuum is not a primitive stage for physics, but an emergent property of a discrete arithmetic substrate. In the language of Ontic Structural Realism (OSR), this framework suggests that the 'physical continuum' is not an object-like substance, but a structural instantiation of the relation between the additive and multiplicative sectors of arithmetic.

References

[1] Eugene P. Wigner. The unreasonable effectiveness of mathematics in the natural sciences. *Communications on Pure and Applied Mathematics*, 13(1):1–14, 1960. doi:10.1002/cpa.3160130102.

[2] Igor V Volovich. p-adic string. *Classical and Quantum Gravity*, 4(4):L83, 1987. doi:10.1088/0264-9381/4/4/003.

[3] Otto Hölder. Die axiome der quantität und die lehre vom mass. *Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig, Mathematisch-Physische Klasse*, 53:1–64, 1901. URL `https://archiv.saw-leipzig.de/7bda004b249a497187314a4e8061341d`.

[4] Marshall H Stone. On one-parameter unitary groups in hilbert space. *Annals of Mathematics*, pages 643–648, 1932. doi:10.2307/1968538.

[5] Jean-Benoît Bost and Alain Connes. Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory. *Selecta Mathematica, New Series*, 1(3):411–457, 1995. doi:10.1007/BF01589495.

[6] Godfrey H. Hardy and Srinivasa Ramanujan. The normal number of prime factors of a number n. *Quarterly Journal of Mathematics*, 48:76–92, 1917. URL `https://zbmath.org/?q=an:46.0262.03`.

[7] Gérald Tenenbaum. *Introduction to Analytic and Probabilistic Number Theory*, volume 163 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, third edition, 2015. doi:10.1090/gsm/163.

[8] Paul Erdős and Mark Kac. The gaussian law of errors in the theory of additive number theoretic functions. *American Journal of Mathematics*, 62(1):738–742, 1940. doi:10.2307/2371483.

[9] Patrick Billingsley. *Probability and Measure*. John Wiley & Sons, anniversary edition, 2012. ISBN 978-1-118-12237-2.

[10] Michael V Berry and Jon P Keating. The riemann zeros and eigenvalue asymptotics. *SIAM Review*, 41(2):236–266, 1999. doi:10.1137/S0036144598347497.

[11] Carl M Bender. Making sense of non-hermitian hamiltonians. *Reports on Progress in Physics*, 70(6):947, 2007. doi:10.1088/0034-4885/70/6/R03.

[12] Michael V Berry. Quantal phase factors accompanying adiabatic changes. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, 392(1802): 45–57, 1984. doi:10.1098/rspa.1984.0023.