Category Theory as an Explanatory Foundation*

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Abstract

Can category theory be a foundation of mathematics? While category theory is taken to organize the body of mathematical knowledge and practice, its status as a foundational theory has been disputed. I argue that category theory can serve as a foundation of mathematics especially in an explanatory sense. The explanatory sense of foundation is both historically situated and philosophically motivated; I examine its historical uses by some of the pioneering figures as well as its theoretical ramifications across several philosophical topics. Then what makes category theory explanatory in the relevant sense? I argue that the explanatory nature of category theory can be understood as its capacity to offer abstract explanation. To support this, the case studies on mathematical explanation through Galois theory and category theory will be considered, which will be tied in with more traditional accounts of category theory's foundational status. Thus, I argue that category theory's foundational status can be supported through the lens of mathematical explanation, which demonstrates a successful interplay between the two traditions in philosophy of mathematics, i.e., the foundation-oriented tradition and the practice-oriented tradition.

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1 Introduction

When William Lawvere ([1964] 2005) first developed a categorical set theory labeled 'An Elementary Theory of the Category of Sets (ETCS)', he already had its foundational role in mind. That is, he foresaw that category theory (CT) can offer an account that rivals other conventional accounts of "foundation of mathematics".

> [ETCS] provides a foundation for mathematics which is quite different from the usual set theories in the sense that much of number theory, elementary analysis, and algebra can apparently be developed within it even though no relation with the usual properties of ∈ can be defined. […] Philosophically, it may be said that these developments partially support the thesis that even in set theory and elementary mathematics it is also true as has long been felt in advanced algebra and topology, namely that the substance of mathematics resides not in Substance […] but in Form (Lawvere [1964] 2005, 7)

Lawvere's quote suggests that (i) his theory is intended to be an alternative to conventional set theories, e.g., Zermelo-Fraenkel set theory (ZF), and (ii) it partially supports the philosophical view that mathematics is about "Form" not "Substance", which modern philosophers interpret as a commitment to mathematical structuralism. His foundational motivation was even more explicitly expressed in his theory of the category of categories aptly labeled 'The Category of Categories as a Foundation for Mathematics (CCAF)' (Lawvere 1966).

The thesis that CT-based frameworks (e.g., ETCS, CCAF) can suitably answer foundational questions continues to win support from many logicians, mathematicians, and philosophers (see Landry and Marquis 2005). The question of identifying CT's distinctive foundational characteristics remains a different question though. That is, we need to ask what makes CT especially attractive as a foundation of mathematics.

One tempting answer is that CT stands out from other foundations since it aligns better with mathematical structuralism; CT is to be preferred in that it serves as a vehicle for the philosophical view that "the substance of mathematics resides […] in Form", i.e., in structure. This has given rise to the debate, among structuralists, about whether CT can indeed adequately play such a role.[1] However, this answer presupposes the preferability of structuralism as a philosophical position, which can be disputed by non-structuralists. Moreover, Lawvere's above quote suggests that his development of CT-based frameworks *supports* structuralism; it does not *presuppose* structuralism as the doctrine that foundational accounts need to align with. Hence, accepting CT-foundations based on a preference for structuralism can be question-begging. Even when granting structuralism, we still want to ask what makes CT special and how this supports structuralism.

In this paper, I argue that CT is foundational in an explanatory sense. That is, one of the reasons why we find CT attractive as a foundational account is that CT explains mathematical truths. First, I discuss the organizational character of CT, which many CT-foundationalists boast as an attractive feature of CT (Section 2). I consider how this organizational character is closely linked to the notion of explanation, which has been, in turn, recognized as a "foundational" quality (Section 3). And I argue that CT is indeed explanatory, which makes it suitable for an explanatory foundation (Section 4).

2 Category Theory is Organizational

In this section, I consider the organizational character of CT. First, I consider various characterizations of CT as an organizational theory, which proponents of CT-foundation boast as a foundational feature (Section 2.1). Nevertheless, while CT's organizational character is hardly disputed, its foundational significance has been contested. I discuss Kreisel's (1971) influential critique of CT-foundation that is based on his distinction of 'foundation' from 'organization' (Section 2.2).

2.1 Organization as Foundation

According to one version of the history of mathematics, the "foundational crisis" in the late 19th and the early 20th century arose because many mathematicians began to worry about the *certainty*

1 See McLarty (1993; 2004), Awodey (1996; 2004), and Landry (1999; 2011) for some influential suggestions that CT can serve as a vehicle of structuralism. For some influential objections, see Hellman (2003) and Shapiro (2005).

of mathematics. In the wake of non-Euclidean geometry, mathematical truths could no longer be considered the hallmark of infallible knowledge as they were once believed to be.[2] Whether this historical narrative is correct or not, some believe that certainty is, at least, not the only ideal that a foundation of mathematics should aim to offer. For example, Saunders Mac Lane, one of the originators of CT, stated as follows:

> in one sense a foundation is a security blanket: If you meticulously follow the rules laid down, no paradoxes or contradictions will arise. In reality there is now no guarantee of this sort of security […] Alternatively, set theory and category theory may be viewed as proposals for the organization of Mathematics. (Mac Lane 1986, 406)

According to Mac Lane, set theory and CT are foundational in that they help organize mathematics. Both proponents and critics of CT-foundations generally agree that CT provides more "organization" than set theory in at least two relevant senses. First, CT is "organizational" in the sense that it is taken to provide practical benefits to working mathematicians.

> [Mac Lane] took a foundation of mathematics to be a body of truths which organize mathematics as do I here. More specifically, that is truths which actually serve in practice to define the concepts of mathematics and prove the theorems. I do not merely mean truths which could in some principled sense possibly organize the practice but truths actually used in textbooks and journal articles, and discussed in seminar rooms and over beer, so their notions do occur in practice. (McLarty 2013, 81)

> A foundation makes explicit the essential general features, ingredients, and operations of a science as well as its origins and general laws of development. The purpose of making these explicit is to provide a guide to the learning, use, and further development of the science. A "pure" foundation that forgets this purpose and

2 For a popular account of such a historical perspective, see, e.g., Kline (1982), but cf. e.g., Ayoub (1982) and Stewart (1982). For a more philosophically dense account, see Giaquinto (2002).

> pursues a speculative "foundation" for its own sake is clearly a nonfoundation. (Lawvere and Rosebrugh 2003, 235)

For instance, ETCS is taken to be organizational in the sense that it accords with how mathematicians conceive of sets. ZF, on the other hand, is taken to exhibit some features that "mathematicians never work with", e.g., that "sets are legitimate objects by Leibniz's law" (McLarty 1993, 496–97).

Second, CT is taken to be "organizational" in a more theoretical sense as well. Following the Hilbertian idea of "conceptual foundation" for mathematics, Landry (2011) suggests that CT axioms can provide a conceptual analysis of the respective axiomatized domains. For example, the ETCS axioms are taken to provide a conceptual analysis of the set-theoretic concepts.

> When *conceptually analysing* the concepts of the branches of mathematics that are themselves organized set-theoretically, the category theorist can take the ETCS axioms as a *mathematical* conceptual scheme for organizing, in category-theoretic terms, what we say about the mathematical or logical structure of these set-structured objects as, cat-structured, concepts, i.e., what can be asserted in terms of anything that satisfies the ETCS axioms. (Landry 2011, 449)

Hence, the proponents for CT's foundational use typically boast of its "organizational" character either in the practical sense or the conceptual sense. I will not ask how these two senses hang together; instead, suffice it to say that both senses of 'organization' buttress the powerful metaphor that CT can be seen as *the language of mathematics*.

> Just as mathematics can be seen as providing the language for the world – it allows us to talk about physical objects in structural terms without having to be about any particular object – so category theory can be seen as providing the language for mathematics (Landry 1999, S26)

Some suspect that CT's organization of mathematics is limited, e.g., it cannot easily account for some aspects of analysis (Mathias 2001), but not every critic of CT-foundation pursues this line of thought. More problematically, as we will see in the next section, some argue that "organization", in whatever sense, has little to do with "foundation".

2.2 Organization vs. Foundation

Granting CT's organizational character, does this make CT foundational? Kreisel (1971) famously disagreed (cf. Feferman and Kreisel 1969): organization and foundation serve different purposes, so there is little reason to believe that CT is foundational.

> Above all, there is the distinction between logical and mathematical foundations or, better, between the *foundation* and *organization* of mathematics. The distinction is useful for analyzing the nature of the problems presented by (existing) category theory, and, more generally, for analyzing the role of foundations for working mathematicians. (Kreisel 1971, 189)

For Kreisel, foundation *precedes* mathematical practice; it provides what the practice is grounded in, which the practicing mathematicians need to care little about. Furthermore, it also explains why foundation has to go against mathematical practice; if foundation were to appeal to the practice, then it would be putting the cart before the horse.

> Foundations are concerned with the *validity*, constructive or nonconstructive, of mathematical principles; in practice one hardly looks at principles unless one is convinced of their validity, and one's principal interest is to make them *efficient*. Put differently, foundations provide reasons *for* axioms […] Foundations provide an *analysis* of practice. To deserve this name, foundations must be expected to introduce notions which do *not* occur in practice. (Kreisel 1971, 191–92)

In contrast, Kreisel observes that organization inherently involves mathematical practice. Unlike foundation, organization can be an interesting feature for practicing mathematicians since a better organization of the concepts can help better present mathematical findings to their audience.

> practice is concerned with deductions *from* axioms, and the organization of practice wants to make these deductions as intelligible as possible. Some practical consequences stare one in the face. (Kreisel 1971, 191)

Kreisel is on board with the metaphor that CT can be a new organizational language of mathematics. He believes, however, that this may indeed go against the idea of CT-foundation.

> Foundations and organization are similar in that both provide some sort of more systematic exposition. But a step in this direction may be crucial for organization, yet foundationally trivial, for instance a new *choice of language* when (i) old theorems are simpler to state but (ii) the primitive notions of the new language are defined in terms of the old, that is if they are logically dependent on the latter. Quite often, (i) will be achieved by using new notions with more 'structure', that is less analyzed notions, which is a step in the *opposite* direction to a foundational analysis. In short, foundational and organizational aims are liable to be actually contradictory. (Kreisel 1971, 192)

Kreisel observes that CT's organization does not offer a proper foundational analysis of the existing mathematical theories. That is, CT may be a new language of mathematics, but it is a language that either presupposes highly complex mathematics itself or can be explicated only by using unanalyzed concepts such as 'structure'. Hence, CT is not adequate as an alternative foundation compared to conventional set theories, e.g., ZF.

> existing practice of category theory definitely raises organizational problems, above all a good choice of language for the formulation of category theory. It is then a separate matter to what extent a proposed theory of categories, formulated in this hypothetical language, raises foundational problems for an adequate reduction to set theory […] Existing formulations, e.g. by Lawvere in this volume,[3] patently do not raise such a problem. (Kreisel 1971, 192)

3 Lawvere's paper cannot be found in this volume. I conjecture that Kreisel is likely referring to Mac Lane's (1971) introduction to ETCS and CCAF in the same volume.

What do we make of Kreisel's critique that tears organization apart from foundation? Notably, his critique is based on his interesting notion of 'foundation'.[4] Fifty years later, some suspect that there may not be a single sense of 'foundation', so one should be wary of talking past each other when talking about foundation.[5]

In the next section, I consider a historically well-recognized notion of foundation that contradicts Kreisel's notion (cf. Marquis 2009, sec. 5.5). While giving a counterexample to Kreisel may not be significant per se, this will serve as an opportunity to highlight the connection between the conception of a CT-foundation and the wealth of literature on what is meant by the term 'foundation'.

3 Explanatory Foundation

In this section, I consider the explanatory notion of foundation, which will be closely associated with the organizational character of CT discussed in the previous section. First, I give an overview of the positions of some well-known figures in the history of foundation of mathematics; this will show that the sense of 'foundation' relevant to CT has been long recognized (Section 3.1). Second, I consider why this particular sense of 'foundation' can be of philosophical interest; not only does this account for organization, but also for certain epistemic and ontological implications (Section 3.2).

3.1 Historical Overview

When Kreisel characterized 'foundation' as what provides "reasons *for* axioms", he sharply distinguished it from "deductions *from* axioms" that have to do with practice. Historically, however, many have argued otherwise. For example, Bertrand Russell ([1907] 2014) famously

4 Awodey (2004) separates his "top-down" view of CT from the "bottom-up"/"foundationalist" perspective (cf. Landry and Marquis 2005; Reck and Schiemer 2020, sec. 3.3), which has therefore been interpreted as an "anti-foundational" view as well (McDonald 2012). One may ask whether this "anti-foundational" view of CT aligns with Kreisel's notion of "foundation", which I do not address here.

5 See, e.g., Marquis (1995), Feferman (1998), Shapiro (2004; 2011), Maddy (2017; 2019), and Rodin (2021) for pluralistic expositions of the notion of "foundation of mathematics".

suggested that what follows from axioms can also give reasons for axioms, which he compares to an inductive method in science:

> in mathematics, except in the earliest parts, the propositions from which a given proposition is deduced generally give the reason why we believe the given proposition. But in dealing with the principles of mathematics, this relation is reversed. […] we tend to believe the premises because we can see that their consequences are true, instead of believing the consequences because we know the premises to be true. But the inferring of premises from consequences is the essence of induction; thus the method in investigating the principles of mathematics is really an inductive method, and is substantially the same as the method of discovering general laws in any other science. (Russell [1907] 2014, 573)

Scientists draw empirical consequences from general laws and, at the same time, use such empirical consequences for both discovering and validating the laws. The same is true of mathematics; a foundation of arithmetic can be yielded through "taking the ordinary propositions of arithmetic as our empirical premises" that we can deduce from there the supposed foundation (Russell [1907] 2014, 574).

What are the theoretical benefits of such an inductive methodology? It is worth noting that Russell, among many, mentions organization as one of the benefits of its use.

> In the first place, when a number of facts are shown to follow from a few premises, this is not only a new truth in itself, but also an organization of our knowledge, making it more manageable and more interesting. (Russell [1907] 2014, 580)

Mathematical axioms, inductively justified, organize the knowledge of the axiomatized domain. Dedekind-Peano axioms, for example, organize those arithmetic truths that had already been known. As such, Russell seemed to have believed that a foundation of arithmetic bestows organization.

Russell believed that such axioms *qua* laws are more than mere enumerative summaries of "empirical" consequences; more importantly, such an aspect of scientific laws is conventionally taken to be *explanatory*. Following Russell's analogy between science and mathematics, many embraced the idea that mathematical axioms, or those that are foundational, need to be explanatory:

Kurt Gödel, a renowned logician inclined to Platonism, famously followed Russell's inductive methodology in his discussion of which set-theoretic axioms we need to accept:

> a decision about its truth is possible also in another way, namely, inductively by studying its "success," that is, its fruitfulness in consequences and in particular in "verifiable" consequences, i.e., consequences demonstrable without the new axiom, whose proofs by means of the new axiom, however, are considerably simpler and easier to discover, and make it possible to condense into one proof many different proofs. (Gödel 1947, 520–21)

Also, it has been reported that Gödel's explanatory view about axioms was not limited to set theory.

> Professor Gödel suggests that so-called logical or set-theoretical 'foundations' for number-theory, or any other well established mathematical theory, is explanatory, rather than really foundational, exactly as in physics. (Mehlberg 1962, 86) [6]

Hence, Gödel seemed to have believed that foundation and explanation go together. Interestingly, a similar view could be found in an essay by Nicolas Bourbaki, a mathematical group that explicitly sought to avoid philosophical issues (see Corry 2004, sec. 7.2):

> What the axiomatic method sets as its essential aim, is exactly that which logical formalism by itself can not supply, namely the profound intelligibility of mathematics. Just as the experimental method starts from the *a priori* belief in the permanence of natural laws, so the axiomatic method has its cornerstone in the conviction that, not only is mathematics not a randomly developing concatenation of syllogisms, but neither is it a collection of more or less "astute" tricks, arrived at by

6 Mehlberg reports that this account of Gödel's view was informally conveyed to him "some years ago" during a discussion he had with Gödel at Princeton.

> lucky combinations, in which purely technical cleverness wins the day. Where the superficial observer sees only two, or several, quite distinct theories, lending one another "unexpected support" […] through the intervention of a mathematician of genius, the axiomatic method teaches us to look for the deep-lying reasons for such a discovery, to find the common ideas of these theories, buried under the accumulation of details properly belonging to each of them, to bring these ideas forward and to put them in their proper light. (Bourbaki 1950, 223)

As a groundwork for their grand project of the architecture of mathematics, Bourbaki emphasized how the axiomatic method allows us to see the "common ideas" shared by seemingly disparate theories, which gives "the deep-lying reasons" *why* such theories are "lending one another "unexpected support"". Through this, Bourbaki pursued "[b]etter organization, better understanding," and "more efficiency" through their groundwork (Marquis 2020, 47). Their idea about the axiomatic method as a solution to the *why*-questions, leading to better organization and understanding, can be reasonably labeled as "explanatory" in hindsight.

These figures just considered, by-and-large, represent the "Euclidean" camp in the foundation of mathematics broadly construed, focusing on the formal aspect of mathematics. In contrast, Imre Lakatos is often interpreted as a forerunner of the "maverick" tradition that emphasized more hermeneutic approaches (Kitcher and Aspray 1988). Nonetheless, when it comes to the inductive or explanatory aspect in mathematics, he was in full agreement.

> The methodology of a science is heavily dependent on whether it aims at a Euclidean or at a quasi-empirical ideal. […] The basic rule of the latter is to search for bold, imaginative hypotheses with high explanatory and 'heuristic' power (Lakatos 1976, 207)

> The battle between rival mathematical theories is most frequently decided also by their relative explanatory power. (Lakatos 1976, 218)

As a “quasi-empiricist”, Lakatos observed a parallel between science and pure mathematics, so he viewed that consideration of explanatory power in science naturally applies to that of mathematics as well.

The above authors are diverse in their theoretical background: Gödel was avowedly philosophical in his approach, whereas Bourbaki was not. Lakatos was critical of the “Euclidean” tradition altogether. Nonetheless, they all attest to the view that a proper foundation should be explanatory. Hence, contrary to Kreisel, it is hardly the case that foundation has little to do with organization or “deduction *from* axioms”. Many, at least, use the term ‘foundation’ that does not align with Kreisel’s usage. I further argue that this explanatory notion of ‘foundation’ has many philosophically interesting features.

3.2 Why an Explanatory Foundation Matters

What makes the explanatory use of the term ‘foundation’ more interesting than Kreisel’s? I consider three philosophical virtues that make the explanatory notion interesting.

First, explanation provides an account of the organization of the explained domain in the sense discussed in Section 2.1. As Russell suggested, explanation through axiomatization provides “an organization of our knowledge, making it more manageable and more interesting”. Bourbaki described it as “a considerable economy of thought” (1950, 227); mathematical facts, which may seem demonstrable only by “lucky combinations”, can be explained by identifying “the common ideas […] buried under the accumulation of details” (1950, 223). When the explanandum is organized, we can make sense of what seemed to be a mere coincidence or accidental truth in mathematics (see Baker 2009).

Second, explanation bestows more reasons to believe; you are more justified to accept a foundational account if it turns out to explain the mathematical theory that is founded upon it. Hence, as Russell and others pointed out, explanation is taken to provide inductive support. In other words, we can approach foundational questions through the use of an inference to the best explanation.[7]

Lastly, explanation may bring with it ontological implications as well. For instance, as with scientific explanation, mathematical explanation seems closely related to the notion of *reality*:

> the only workable criterion of reality [...] is the *explanatory criterion*: something is real if its positing plays an indispensable role in the explanation of well-founded phenomena. (Psillos 2005, 398–99)

Explanation has been widely discussed in the recent literature on the ontology of mathematics: the "enhanced" indispensability arguments for mathematical realism appeal to the explanatory role of mathematics in sciences (Baker 2005) and, even within pure mathematics, the use of inference to the best explanation in mathematics has been linked to the problem of ontological commitment (Lange 2022).

For these reasons, the notion of an explanatory foundation seems to have interesting philosophical ramifications; if CT turns out to be foundational in an explanatory sense, then it may have both epistemological and ontological implications. What needs to be shown, then, is that CT is indeed explanatory in a relevant sense. We can agree that CT is organizational and that explanation goes along with organization, but does this mean that CT itself is explanatory? Unfortunately, the literature that considers both CT-foundations and mathematical explanation together is sparse. In the next section, to overcome this gap, I link CT to a relatively well-discussed case study of mathematical explanation.

7 Heron (2021) recently argued that the focus on explanation in a foundational context is misguided for various reasons; the focus should instead be on theoretical virtues. I conjecture that the talk of theoretical virtues instead of explanation will make little difference in the context of Galois theory and CT that will be discussed in Section 4.

4 Abstract Explanation: From Galois Theory to Category Theory

The literature on explanation in mathematics often distinguishes mathematical explanation in science from explanation within mathematics (Mancosu 2018; D'Alessandro 2019). I focus on a specific type of explanation within pure mathematics, which is known as *abstract explanation*. First, I introduce Christopher Pincock's (2015) account of how Galois theory explains the unsolvability of the quintic in radicals, which serves as an exemplar of abstract explanation (Section 4.1). Based on three case studies in CT, I argue that CT also offers abstract explanation (Section 4.2). Finally, we consider how CT's abstract explanation ties in with the idea of explanatory foundation (Section 4.3).

4.1 Abstract Explanation in Galois Theory

Mathematicians have for centuries known how to solve polynomial equations with the degree under 5; as long as the degree of the highest exponent is 4 or lower, we have general formulas allowing us to find the roots of the equations given the coefficients. For the quintic, i.e., fifth-degree polynomial equations, however, it was unclear whether such a general solution is available. Moreover, even when accepting the impossibility of such a general solution, i.e., the unsolvability in radicals, it remained mysterious *why* that is the case. In fact, it was Niels Henrik Abel who first proved the unsolvability of the quintic in radicals, but it remained unclear if his "heavily computational" proof adequately addressed the *why*-question (see Kiernan 1971, sec. 6). Galois theory emerged from this historical background. Evariste Galois' distinct approach from the theory that was posthumously named after him, i.e., Galois theory, is taken to have shed light on the *why*-question. That is, "Not only does [Galois's idea] prove that the general quintic has no radical solutions, it also explains why the general quadratic, cubic and quartic *do* have radical solutions and tells us roughly what they look like" (Stewart 2008, 116).[8]

Mathematical details aside, Galois' approach reconceptualizes the unsolvability of the quintic using the notions of 'field', 'field extension', 'group', etc. Instead of directly dealing with polynomial equations, it focuses on the field extension determined by the equation and the group

8 Also see Pincock (2015, 1–2) for more quotes from non-philosophical sources on the strength of the Galois-theoretic proof compared to that of Abel's proof.

of automorphisms thereof, which is a Galois group. It turns out that we can ascribe a certain property to the Galois group when the corresponding general equation is solvable in radicals. This property of the Galois group is aptly called 'solvability'. That is, the polynomial is solvable in radicals if and only if it has a solvable Galois group; it allows you to determine whether a polynomial is solvable in radicals without actually finding the solutions. And you can show that the Galois group is not solvable exactly when the corresponding equation is quintic or higher; the general quintic, for example, corresponds to the Galois group S_5, which is unsolvable. Hence, we prove the intended result that the general polynomial equation with the degree of 5 or higher is unsolvable in radicals.[9][10]

Many philosophers of mathematics have observed that working mathematicians tend to consider Galois' approach more explanatory than Abel's earlier proof, therefore treating it as an established example of mathematical explanation. But there remains another philosophical question: what makes the Galois-theoretic proof explanatory?

Among many possible answers, we focus on Pincock's (2015) influential account, which focuses on the above fact that the solvability of polynomials corresponds to the solvability of the corresponding Galois group. What this fact is taken to show, according to Pincock, is that "What makes a given polynomial equation solvable, we should say, is that its Galois group is solvable" (2008, 11). That is, the relation between the solvability of polynomials and the solvability of Galois groups is asymmetric; the former is taken to depend on the latter, and not vice versa. Then what justifies this dependence claim? Pincock's view is based on the Galois group's *abstractness* relative to polynomials.

> I claim that the Galois theory proof makes an appeal to entities that are more abstract than the entities involved in the theorem. Each Galois group is more abstract than the

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9 You can find a more rigorous exposition of the Galois-theoretic approach to the unsolvability of the quintic in most introductory textbooks of abstract algebra (e.g., Fraleigh 2013). For a self-contained exposition, see Stillwell (1994).

10 This result should *not* be conflated with the claim that every quintic (or higher formula) is unsolvable in radicals; instead, the result means that there *exist* quintics (and higher) that cannot be solved in radicals. I appreciate an anonymous reviewer for sharing this concern.

collection of automorphisms of a field extension, and the theorem is about these field extensions. (Pincock 2015, 13)

For the criterion of abstractness, he draws "on the broadly structuralist thought that some objects can have other objects as instances." (Pincock 2015, 12) A specific group of automorphisms is an instance of a Galois group, which makes the facts about Galois groups more abstract than the facts about automorphisms and, likewise, about polynomial equations.[11] Hence, given the abstractness of Galois groups relative to polynomials, we have a ground to claim that the solvability of polynomials depends on the solvability of the Galois group. Pincock argues that this dependence relation is exactly what backs the mathematical explanation in the Galois-theoretic proof:

> My proposal is that the Galois theory proof explains because there is a special sort of dependence relation between facts about groups and facts about polynomial equations. […] In our case the phenomenon is the distribution of solvability properties among polynomial equations. The explanation picks out the right groups and the appropriate group-theoretic property and shows how these facts are responsible for the features of the equations. (Pincock 2015, 7)

The unsolvability of the quintic is a phenomenon to be explained; while its truth was already verified by Abel's proof, an adequate explanation of this truth was arguably given by the Galois-theoretic proof. What makes an appeal to the Galois group, which is apparently foreign to the concept of polynomials, explanatory? Being more abstract than polynomials, the Galois group "permits the determination of a necessary and sufficient condition for the central property mentioned in the theorem" (Pincock 2015, 10). That is, you can explain why the phenomenon has to hold by appealing to a more abstract entity. Hence, Pincock takes the case of Galois theory to be an exemplar of "abstract mathematical explanation", which "explain[s] by appeal to an entity that is more abstract than the subject-matter or topic of the theorem" (Pincock 2015, 2).[12]

11 Cf. Marquis (2014; 2016) for an analysis of 'abstract' and 'abstraction', which correspond to their usage in mathematical practice.

12 Is abstract explanation Kitcher's (1989) "explanatory unification"? While Pincock takes unification to be "evidence that a dependence relation obtains" (Pincock 2015, 15), it is not taken to be constitutive of the

Admittedly, abstract explanation may not be the only type of mathematical explanation available, and moreover, it has been questioned whether Pincock's abstract explanation account is indeed faithful to some of the details of Galois' approach.[13] Nonetheless, Pincock's account can still be understood as presenting what is needed for abstract explanation to take place; whether the details of Galois' approach really satisfy the conditions of abstract explanation or not, if there is some case that satisfies such conditions, it will give us a reason to believe that we have a case of abstract explanation. Indeed, CT seems to offer many cases amenable to the present account of abstract explanation, and abstract explanation seems to be the type of explanation needed for "explanatory foundation".[14] In the next section, we will examine such cases in CT, which can be understood as providing abstract explanation.

4.2 Abstract Explanation in Category Theory

I argue that abstract explanations, the exemplar of which we discussed in the previous section, can be found in CT as well.[15] Recall that, when explaining a certain phenomenon, abstract explanation appeals to more abstract entities on which the phenomenon depends. We will examine three case studies from the elementary-level CT, which, I argue, offer abstract explanation. Each case contrasts a pair of proofs for the same theorem, one proof only using the phenomenon-level notions and another proof using CT. I argue that, in each case, the CT-based proof can be understood as offering an abstract explanation of the theorem.

dependence relation itself. That is, unification follows from abstract explanation, not vice versa.

13 See D'Alessandro (2016; 2020) for critiques of Pincock's account of Galois theory.

14 For example, Galois theory organizes the existing mathematical concepts in the sense discussed in Section 2; the "formulation of the Galois theory proof involves the discovery of a novel and informative classification of solvable polynomial equations", which is "evidence of an underlying dependence relation" (Pincock 2015, 13). This aligns with the practical benefits of "organization" as well, since Galois, too, seems to have been motivated by his frustrations over the practice of his contemporary mathematics, which focused too much on complicated computational procedures (Kiernan 1971, sec. 9).

15 Also see Pincock (2015, sec. 1) and Corry (2004, 64–68) for the "conceptual" approach (which is contrasted with the "computational" or "algorithmic" approach) in the 19th century mathematics, which some may interpret as a part of the historical lineage that stretches from Galois theory to the Göttingen school's "structural" approach (Corry 2004, 253–58), eventually giving rise to CT.

Our first case study features Cayley's theorem, a basic yet important result in group theory. For any group G, we can construct its symmetric group, i.e., the group of all permutations of the underlying set of G. Cayley's theorem says that, for any G, G is isomorphic to some subgroup of the symmetric group of G.

How do we prove Cayley's theorem? First, a conventional group-theoretic proof of the theorem essentially relies on constructing the isomorphic group of G within the symmetric group. That is, for each element of G, you construct a corresponding permutation (i.e., left/right multiplication), and by putting all these permutations together, you form a group $\varphi[G]$ such that $\varphi[G]$ is a subgroup of the symmetric group and $\varphi[G]$ is isomorphic to G. Thus, you have a proof of Cayley's theorem through a concrete construction.[16]

The CT proof of Cayley's theorem goes opposite; instead of starting with group-theoretic notions, it builds off of the Yoneda Lemma, a fundamental result in CT holds across any (locally small) category **C**. Instead of going through the entire proof, we focus on the very last step of the proof, which draws Cayley's theorem from the interim result known as the Yoneda Embedding.[17]

Here's a rough sketch: Given any category **C**, we can conceive of its dual category $\mathbf{C}^{op}$ (i.e., **C** with all of its arrows reversed). In CT, a functor is a mapping from one category to another, which allows us to think of a functor from $\mathbf{C}^{op}$ to **Set**, the category of sets. Such functors (also known as presheaves) form their own category, which we write as $[\mathbf{C}^{op}, \mathbf{Set}]$. The Yoneda Embedding states the following: There is a functor Y: $\mathbf{C} \rightarrow [\mathbf{C}^{op}, \mathbf{Set}]$ such that the image of **C** under Y is isomorphic to **C**.

What does this have to do with Cayley's theorem, a group-theoretic result? The key idea is that, for any group G, you also have a corresponding category, which we may just as well call by the same name 'G'. The category G has only one object, but every arrow corresponds to an element of the group G. Now, what happens when we plug this category G into the Yoneda Embedding result? It tells us that the functor Y takes the category G to its isomorphic image. And we can show that this isomorphic image of G also forms a group-*qua*-category, the arrows of

16 See, e.g., Fraleigh (2013, sec. 8) for a full presentation of the group-theoretic proof of Cayley's theorem.

17 See, e.g., Smith (2025, chap. 37) and Riehl (2017, sec. 2.2) for full presentations of the proof of Cayley's theorem from the Yoneda Lemma.

which correspond to certain permutations of the group *G*'s underlying set. Thus, we yield the conclusion that any group *G* is isomorphic to a permutation group, which is Cayley's theorem.

Hence, we have roughly sketched two different ways to prove Cayley's theorem, one being group-theoretic and the other using CT. Then what makes the "detour" through CT, which appeals to the notions foreign to the subject matter of Cayley's theorem, worth it? Smith (2025) offers the following answer:

> So what new insight might we get out of the categorial [*sic*] detour? [. . .] There are a variety of pre-categorial results that are intuitively in the same ballpark as Cayley's theorem, telling us about how structures of one kind can be isomorphically embedded into other structures. There are some algebraic cousins, such as e.g. the result that a ring can be embedded into the endomorphism ring of its underlying abelian group. Then there are results such as that a partial ordering of objects can be mirrored in a collection of subsets of those objects ordered by inclusion. It turns out that the Yoneda embedding theorem applied to relevant categories reveals such results as again instances of the very same construction. And that will be an insight worth having. (Smith 2025, 339–40)

Cayley's theorem is a truth in group theory, which you can prove by constructing a permutation group. Nonetheless, it also has many "cousins" in different domains, i.e., ring theory and set theory. Such a commonality between these results calls out for an explanation; what makes these different results, which take place in different domains yet bear striking similarities, all true? As Smith points out, CT shows that such results are "instances of the very same construction"; not only does it demonstrate their truth but also explains *why* they are bound to be true. Cayley's theorem is, therefore, not an accident. Hence, I argue that the "insight" we get from the CT proof has an explanatory nature.

Furthermore, we argue that what the CT proof provides is an abstract explanation. Recall that abstract explanation "explain[s] by appeal to an entity that is more abstract than the subject-matter or topic of the theorem". For example, the Galois-theoretic proof of the unsolvability of the quintic appealed to the Galois groups and the (un)solvability thereof to explain the subject matter of the theorem, i.e., the (un)solvability of polynomials. In the same way, the CT proof of Cayley's theorem appeals to the category *G* to explain the property of the concrete group *G*. Just as the (un)solvability of polynomials was taken to depend on the (un)solvability of the corresponding Galois group, the group *G* depends on the corresponding category *G*, which allows the Yoneda Embedding result to give rise to Cayley's theorem. Thus, by parity of reasoning, we can find an abstract explanation in the CT proof just as Pincock found an abstract explanation in the Galois-theoretic proof.[18]

This case study, which featured an abstract explanation in the CT proof of Cayley's theorem, is not an isolated case. We will also examine Lehet (2021) and Colyvan, Cusbert, and McQueen's (2018) case studies on the CT results that (i) left adjoint functors preserve colimits and (ii) under certain conditions, the converse of (i) holds as well.[19] In both cases the philosophers have pointed out that the CT results play an explanatory role in proving mathematical facts.

Lehet (2021) presents (i) as an example demonstrating that CT, which allows a transfer between various results in different areas of mathematics, can offer an explanatory proof.[20] That is, (i) *explains* these various results in that it unifies these various results and tells us why they hold.

Consider, for example, the set-theoretic truth that the direct image of a function preserves unions whereas the inverse image of a function preserves both intersections and unions. Lehet observes that (i) not only leads to a proof of this set-theoretic result as a corollary but also

18 Note that the abstract explanation of Cayley's theorem via CT does not contradict other types of explanation. For example, given the highly abstract nature of CT, it is more natural to use Cayley's theorem to explain the Yoneda Lemma in a pedagogical setting, even though this sense of 'explain' is different from abstract explanation. I appreciate an anonymous reviewer for sharing this point.

19 The dual of (i) is that right adjoint functors preserve limits. (ii) is also known as the "general adjoint functor theorem". See Leinster (2016, sec. 6.3) and Riehl (2017, sec. 4.5-4.6) for mathematical expositions of these results.

20 Lehet presents the CT proof as an example of an "impure" proof (Detlefsen and Arana 2011), which supports the idea that impurity in contemporary mathematics does not compromise the ideal of mathematical explanation.

explains it. This is largely in virtue of "the fact that the direct image, inverse image, intersections and unions have this category theoretic structure" (Lehet 2021, 77), e.g., the domain and the codomain of the direct and the indirect images are categories themselves. Just as we construed the group *G* as a single-object category *G* in proving Cayley's theorem, (i) allows us to reconceptualize the set-theoretic problem in more abstract CT terms. The same can be said, *mutatis mutandis*, for other results following from (i); "the reliance on category theoretic structure" plays a key role in proving and explaining them.

Lastly, we will consider Colyvan, Cusbert, and McQueen's (2018) work on the group-theoretic theorem that free groups exist, which is known to follow from (ii). They contrast two proofs of the theorem, one being more "constructive" and the other being "abstract" that explicitly follows CT. They found through an informal survey that at least some mathematicians and physicists find the "abstract" proof more explanatory, which bears an interesting resemblance to how CT textbooks describe the group-theoretic corollary of (ii):

> This left adjoint *F*: **Set→Grp** assigns to each set *X* the free group *FX* generated by *X*, so [(ii)] has produced this free group without entering into the usual (rather fussy) explicit construction of the elements of *FX* as equivalence classes of words in letters of *X*. (Mac Lane 1998, 123)

> So [(ii)] tells us that, for instance, the free group functor exists. In [earlier examples], we began to see the trickiness of explicitly constructing the free group on a generating set *A*. […] But using [(ii)], we can avoid these complications entirely. (Leinster 2016, 164)

Both descriptions emphasize how (ii) offers an advantage by skipping an "explicit" construction of the free group; the CT result (ii) tells us that the free group exists without having to know its elements. On the one hand, what they seem to find advantageous about (ii) seems just pragmatic; you have a more straightforward proof by avoiding "fussy" "complications". On the other hand, "the real reason for the existence of free groups is found at the more abstract structural level", which can be well-described by CT (Colyvan et al. 2018, 245–46). (ii) shows that we can characterize free groups in the following way: free group's existence can be viewed as "just a special case of a more general result" and we can remain non-committal about what specifically constitutes them. If we interpret this "advantage" as an explanatory advantage, then it tells us how the abstract nature of CT aligns with what we find explanatory about (ii).

Thus, the second and the third case studies, which feature (i) and (ii) respectively, show that CT can offer mathematical explanation. I argue that, just like the first case study, they can be understood as offering abstract explanation.[21] In each case, the explanandum (i.e., a set-theoretic result, the existence of a free group) does not involve any CT notion, but the explanation succeeds in virtue of appealing to CT notions, which are far more abstract than the explanandum. That is, just as the CT explanation of Cayley's theorem relied on reconceptualizing the notion of group in CT terms, (i) and (ii) explain the set-theoretic and the group-theoretic results respectively in virtue of reconceptualizing them in CT terms; CT portrays an "abstract structural level" that the explanandum depends on. Thus, I argue that these two case studies, which have been independently judged explanatory in the literature, can be understood as involving abstract explanation.

Hence, we considered three case studies on CT proofs, which we analyzed as instances of abstract explanation. This shows that the explanatory nature of CT can be well-corroborated following the Galois theory exemplar we considered in Section 4.1. In the next section, we consider how this explanatory nature of CT ties in with the idea of explanatory foundation.

4.3 Abstract Explanation and Explanatory Foundation

Through the previous sections, we discussed the explanatory capacity of CT, which makes CT suitable for an explanatory foundation as discussed in Section 3. To discuss how CT works as an explanatory foundation, however, we need to settle one point: What should CT explain to count as an explanatory foundation? Among many possible explananda of CT, is there a distinguished subset that deserves to be called "foundational"? According to one perspective, there is such a distinguished subset, namely, sets and natural numbers, the topics traditionally considered "foundational". That is, among many CT frameworks that offer explanation, only the one that

21 Lehet (2021) and Colyvan et al. (2018) describe their cases in terms of "unification". As discussed earlier (footnote 12), we can take this to be an "evidence" for abstract explanation.

explains such "foundational" topics deserves to be called an explanatory foundation. This is, indeed, close to the traditional sense of 'foundational' that pertains to the dispute between CT and conventional set theories (see Section 1). From another perspective, however, there is no distinguished subset of mathematical topics that should be deemed "foundational"; the role of CT qua explanatory foundation, it may be argued, is orthogonal to the original debate between CT and orthodox set theories.[22]

While we can take both perspectives to be viable, here we focus on the former perspective that prioritizes a distinguished subset, i.e., sets and natural numbers, as the primary explananda of a CT framework. For our initial goal, introduced in Section 1, was to understand the appeal of CT in the traditional sense of "foundation"; our question was what makes CT attractive as a foundation of mathematics *contra* conventional set theories. Hence, going back to our initial question, we ask how CT can better explain sets and natural numbers than conventional set theories.

I argue that Benacerraf's (1965) classic work offers us a useful platform for answering this question. Benacerraf considers two set-theoretic accounts of natural numbers – the Zermelo ordinals and the Von Neumann ordinals –, which offer different set-theoretic characterizations of individual natural numbers and yet recover the same known facts about natural numbers. We can frame this as an explanation question: What makes these different set-theoretic accounts all adequately capture the same arithmetic facts? A Benacerrafian answer is the following: Both accounts capture the same arithmetic facts *because* they share the same "abstract structure". That is, we explain the phenomenon of the set-theoretic accounts of natural numbers by appealing to the abstract structure on which the set-theoretic accounts depend. Thus, we have an instance of abstract explanation.

What exactly is that "abstract structure", though? Is there a theory for such an abstract structure? "There is no need and no place for a further theory of abstract structures", argues McLarty (1993, 487). For ETCS, the categorical set theory (Section 오류: 참조 소스를 찾을 수 없습니다), offers exactly what the theory of "abstract structures" should offer. More specifically, a *natural numbers object* in **Set**, the category of sets, exactly serves the role of the "abstract

22 Cf. the "top-down" vs. "bottom-up" perspective in the context of CT (see footnote 4).

structure" needed for the Benacerrafian answer. Hence, ETCS, a CT framework for sets, serves as an explanatory foundation that explains the "foundational" topics in the traditional sense.[23]

Thus, we have a case for CT as an explanatory foundation in the traditional sense of 'foundational'. Also, I argue that such an explanatory nature of CT in the foundational context is not a brand-new discovery. In her survey of various foundations of mathematics, Maddy (2017) claims that CT offers "Essential Guidance" that set theory fails to provide, by which she means that:

> such a foundation is to reveal the fundamental features – the essence, in practice – of the mathematics being founded, without irrelevant distractions; and it's to guide the progress of mathematics along the lines of those fundamental features and away from false alleyways. (Maddy 2017, 305)

Maddy's description of "essential guidance" aligns with what abstract explanation offers. Consider the above Benacerrafian case again. The given phenomenon involves the set-theoretic disagreement between different accounts of natural numbers, which cries for explanation. To this, we witnessed how CT gives an explanation by revealing "the fundamental features", i.e., the "abstract structure" responsible for the different accounts of natural numbers. The set-theoretic disagreement is an "irrelevant distraction", at least, from the abstract explanation perspective. Thus, Maddy's account of CT's role of offering "Essential Guidance" corroborates its distinctive virtue as an explanatory foundation.

To come full circle, let us return to our original question: What makes CT foundational? As we saw, the champions of CT have long boasted how "organizational" CT is, but the critics such as Kreisel questioned whether organization has much to do with the notion of foundation. To meet this objection, I considered the explanatory sense of 'foundation' that is both historically placed and philosophically considered, arguing that CT is indeed explanatory via abstract explanation. Hence, we can situate our claim that CT is foundational. Some critics may still insist

23 Can we draw a further ontological conclusion about the nature of natural numbers from this answer to the why-question? While both Benacerraf and McLarty do draw their ontological conclusion (in terms of "abstract structure" and (categorical) abstract sets respectively), we may remain neutral about this question in this paper.

that what they mean by ‘foundation’ is different (cf. Section 2.2), so CT fails to be a “foundation” in a significant sense. To reiterate the gist of Section 3, however, I argue that the onus is now on them to show that the explanatory sense of ‘foundation’ ascribed to CT is philosophically less significant than their preferred sense of ‘foundation’.

5 Concluding Remarks

The main thesis of this paper was that CT enjoys a foundational status in an explanatory sense. As mentioned earlier, the term ‘foundation’ has many different senses. I have argued elsewhere that CT can be ontologically and semantically autonomous; CT need not be founded upon another mathematical theory in both ontological and semantic senses. If the main thesis of this paper is correct, then there is another interesting sense in which CT is foundational.

A key aspect of my approach was bridging two traditions in philosophy of mathematics, one tradition emphasizing the foundation of mathematics and another tradition emphasizing mathematical practice. The scholarship on mathematical explanation, for example, is often taken to belong to the latter tradition. By pointing out the explanatory of sense of ‘foundation’, I aimed to show that these two traditions are not as separate as many portray them to be. Mathematical explanation, a subject matter in the philosophy of mathematical practice, informs the foundational debate as well. As such, I conjecture that more interaction between these two camps can lead to more fruitful inquiries.

In this vein, one follow-up question concerns whether the present point about the explanatory value of CT can be generalized to mathematical structuralism. In Section 1, I briefly mentioned the view that CT serves as an adequate vehicle for structuralism. Also, the explanatory aspect of CT can be interpreted as closely associated with the “structural” approach that started to flourish at Göttingen in the early 20th century (cf. footnote 15), which has also been referred to as “methodological structuralism” (Reck and Price 2000). Given these considerations, we can ask whether mathematical explanation itself can motivate structuralism *qua* a general philosophical doctrine, which is left for future works.

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