Categorification of Perspectives

Aboutorab Yaghmaie*[1] and Bijan Ahmadi Kakavandi†[2]

[1]Institute for Science and Technology Studies, Shahid Beheshti University
[2]Department of Applied Mathematics, Faculty of Mathematical Sciences, Shahid Beheshti University

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Abstract

To characterize scientific perspectivism as a realist stance, Massimi introduces her own notion of perspectival truth, which necessitates scientific theories or models from different perspectives being stitched together. Drawing on recent efforts to reify scientific theories using category theory, we propose adjunction as a formal mechanism whereby scientific theories interconnect across different perspectives. Finally, we argue that enriching perspectival realism with this foundational framework does not compromise the symmetry of scientific perspectives to which it adheres.

Keywords: perspectival realism; perspectival truth; internal approach; adjunction.

1. Introduction

Scientific communities in each historical and cultural situation first have certain knowledge claims about the world, produced through reliable methods and resources, and second certain methodological or epistemic principles that justify the reliability of these methods and resources. As the situation changes, these methodological principles also change, while the world and the correspondence relation between knowledge claims and the world remain fixed. This summarizes Massimi's (2022, pp. 5-9) *perspectival realism*, often expressed by the motto 'all scientific knowledge is contextual'. This stance is driven by, *inter alia*, two key semantic and epistemic motivations[1] which may be molded into these schematic inferences. According to the

*a_yaghmaie@sbu.ac.ir
†b_ahmadi@sbu.ac.ir

[1]Addressing the issue of inconsistent models is another challenge that perspectivists seek to tackle (see e.g. Massimi, 2022, Ch. 3).

substantialist approaches to scientific representation,[2] the epistemic virtuousness of scientific theories depends essentially on their representationality (or representational adequacy), in that theory T leads agent C to cognize the world or target W *because* T represents (adequately) W.[3] Furthermore, according to both agent-based and hybrid accounts of scientific representation,[4] whether T represents W or not is (partially) grounded in the actions of C, which are determined by the context in which C acts, or more generally by her perspective.[5] As such, the cognition T bestows, such as the understanding of W it confers, is grounded in perspectival considerations, in such a way that by varying the perspective the epistemic virtuousness of T may vary, echoing the motto of the perspectivist. With this semantic rationale, the epistemic one runs as follows. The reliability of experimental, theoretical, and technological resources for making scientific knowledge depends on the epistemic principles that justify their reliability. These principles are highly situational, as scientific communities in each historical and cultural context determine which principles may ground the reliability of the resources. Given this, whether a claim is epistemically acceptable cannot be answered from *nowhere*, but only by standing *somewhere* and considering the epistemic principles pertaining to the perspective in question (Massimi, 2022, pp. 7-8).

In furtherance of her project, however, Massimi imposes a constraint on perspectival realism, which may be termed 'cross-perspectival assessment':

> What is different in my treatment is the role played by scientific perspectives in assessing reliability. I see reliability not as the sort of thing that individual epistemic communities can sanction or ratify on their own. My case for perspectival realism rests ultimately on the ability to assess reliable scientific knowledge claims *across a plurality of scientific perspectives* (Massimi, 2022, p. 5, emphasis in original).

As we see later, the constraint of cross-perspectival assessment assures her that perspectival realism is ultimately a realist stance committed to truth as a concept independent of any specific perspective. Indeed, her perspectival realism must not be so strong as to allow the world is portrayed from a God's-eye perspective, i.e. decontextualizing scientific knowledge. On the other hand, it must not be so liberal as to lead to relativism, thereby rendering the way the world

[2]The substantialist approaches to scientific representation maintain that this conception is constituted by more fundamental factors, such as agents' actions (e.g. interpreting the components of scientific models), objective relations (e.g. structural similarity between models and the world), or both, in such a way that these factors explain why scientific models represent their targets. Deflationist approaches deny such a constitution, while agreeing that the representational status can be explained through other notions, such as the inferential status of scientific models. For more on this, see Yaghmaie (2023).

[3]This resembles the *representational assumption* of Massimi (2022, p. 44).

[4]Those views on scientific representation that hold it to be *explanatorily constituted* solely by the agent's actions in dealing with representation are agent-based, such as the DEKI account (Frigg and Nguyen, 2020). Those that add objective relations between the source and target of representation to these constituents are hybrid, including Bueno and Colyvan's (2011) account. Ground-theoretically speaking, whereas in the former scientific representation is fully grounded in the agent's actions, in the latter it is partially grounded in each kind of constituent.

[5]Massimis's (2022, p. 44) *perspectival assumption* and her analysis of this assumption undergird such a claim.

is meaningless. In other words, perspectival realism should establish itself as a *middle ground* between standard scientific realism and relativism. Given this, I will refer to this challenge as the 'middle ground problem'. By proposing the notion of *perspectival truth* (Massimi, 2018) and characterizing different perspectives not as isolated patches but as *intersecting* with one another, Massimi attempts to address the problem. For her, the notion of intersecting is a methodological feature, realized "whenever more than one perspective is required to refine the reliability of the claims of knowledge advanced" (Massimi, 2022, p. 340). In addition to this methodologically construed notion, originally proposed in her paper (Massimi, 2018), she introduces in her book another notion, *interlacing*, capturing "how historically a number of situated scientific perspectives have encountered and traded with one another some of their tools, instruments and techniques" (Massimi, 2022, p. 340). Although methodologically intersecting and historically interlacing scientific perspectives form the basis of Massimi's bottom-up, phenomena-based perspectival realism, this can be further coordinated by a third dimension showing how *theoretically* different scientific theories from different perspectives *interconnect*.

Assuming that the theoretical knowledge claims of physical theories can be reconstructed within category theory, as many philosophers have recently proposed (see e.g. Halvorson and Tsementzis, 2017; Weatherall, 2017; Feintzeig, 2024),[6] this article introduces a novel category-theoretic tool for examining how scientific theories interconnect across different perspectives, even though the formal treatment of scientific perspectivism within category theory is not unprecedented (Karakostas and Zafiris, 2022). This proposal, which characterizes scientific theories as categories connected by a functorial machinery, i.e. adjunction, is based on an assumption and has a significant implication. First, we show that such characterization of scientific theories allows for perspectival realism. Drawing on discussions of the equivalence of scientific theories and their relativity (Barrett, 2022), i.e. the fact that the equivalence depends on non-formal considerations such as the subject's choices in characterizing these theories and their relations, we argue that this effectively accommodates the perspectival view of scientific theories. Secondly, addressing Stemeroff's (2022) recent critique of Massimi's perspectival realism, we show how this stance can be molded into a higher-order form of structuralism. As an upshot, readers more interested in engaging with philosophical issues rather than formal technicalities will see how category-theoretic tools can contribute to the development of philosophical stances, having already proved their value in addressing long-standing debates such as the rivalry between relationalism and substantivalism in the philosophy of spacetime (Dougherty, 2020; Ladyman and Presnell, 2020).

The paper is structured as follows. The next section discusses Massimi's strategy for addressing the middle ground problem. We will argue that her perspectival notion of truth, analyzed through a historical-cultural approach, can be further elaborated with the help of a

[6]The use of category-theoretic tools in the literature on the foundations of physics has a longer pedigree (see e.g. Isham and Butterfield, 2000; Döring and Isham, 2008a,b,c,d).

theoretical apparatus. Section 3 shows how the category-theoretic literature on equivalence of scientific theories, specifically the relativity of equivalence, supports the core idea behind perspectival realism. This is by the claim that cognizing the world via a scientific theory is always contextualized by an agential degree of freedom. This freedom is induced by the scope of options afforded by the subject's choices, and we model it using a functorial language. Section 4 introduces the notion of adjunction in category theory as a transitional tool to move from a perspective to another. This section is supplemented by a case in general relativity, which has been recently discussed by Wu and Weatherall (2024), to show how the existence of an adjunction between the categories purporting to represent spacetime structures provides what perspectival realism needs. Section 5 takes up the critique Stemeroff (2022) has recently presented against Massimi's perspectival realism and outlines an outlook for how perspectival realism may develop in light of this objection.

2. Massimi Finding a Middle Ground

Massimi's (2018) key means for addressing the middle ground problem is perspectival truth, purporting to hinge *perspective-dependent knowledge claims* with *perspective-independent worldly states of affairs*. Should this attempt succeed, perspectival realism would prove itself to be the best of both worlds, i.e. as a stance embodying both the metaphysical tenet of realism and the epistemic tenet of relativism (Massimi, 2018, p. 345, fn. 8). However, can one hold that scientific representations, which ground knowledge claims, vary by history and culture in essence yet target the way the world is independent of any contexts? Is it reasonable to claim that a model represents the world scientifically within a perspective but still "gets things right"? Answering these questions positively, Massimi first puts the available options on the table to construe the apt concepts of truth for perspectival realism and then excludes them one by one to get the desideratum. Let's see how her project moves forward.

To capture more easily the scene Massimi sets,[7] let S denotes a scientific representation, T its target, SC its semantic content, and P_i the i^{th} perspective within which the representation relation in question, R, is framed.[8] According to the *indexical notion of perspectival truth* or

[7]In what follows, I couch Massimi's linguistic notions of knowledge claim, truth-maker, truth-value and truth-conditions in terms of model-theoretic terminology of scientific representation (model), target, representational status and fitness or representation-conditions, though these are simply two flavours of speaking here.

[8]We separate a scientific representation from its semantic content to allow it to carry distinct semantic contents across different vantage points while retaining its identity. As a further terminological clarification, we define a model as comprising a representation alongside its semantic content. Moreover, by M represents T, we mean that *M fits T*, *effectively* or *adequately* representing T. Here, we do not commit ourselves to any specific account of scientific representation but assume its schematic form within Massimi's framework, which involves the following components: (1) a scientific representation comprising a carrier and a semantic content (e.g. the equation of state for an ideal gas, where the carrier is its mathematical formulation and the semantic content is the interpretation of the equation as an ideal gas); (2) a target (e.g. the thermodynamic features of an actual gas); and (3) a representation relation between the scientific representation and the target, analyzed differently across various accounts of scientific representation.

representation (Massimi, 2018, p. 348), the semantic content of a scientific representation and also its target are historical-cultural dependent in such way that the knowledge claims grounded in the representation relation from different vantage points have their own *truth-makers*. In Figure 1, there exists a representation S that, in two different perspectives P_1 and P_2, has two distinct semantic contents SC_{P_1} and SC_{P_2}. This sort of indexicality, however, extends beyond semantic contents and also contextualizes targets. Therefore, S with SC_{P_1} and S with SC_{P_2} are both representational, each having its own targets, i.e. T_{P_1} and T_{P_2}, respectively.

$$S \underset{P_2}{\overset{P_1}{\rightrightarrows}} \begin{matrix} SC_{P_1} \\ SC_{P_2} \end{matrix} \underset{R_{P_2}}{\overset{R_{P_1}}{\rightrightarrows}} \begin{matrix} T_{P_1} \\ T_{P_2} \end{matrix}$$

Figure 1: indexical notion of perspectival representation

It can be shown that perspectival realism so spelled out inevitably collapses into either standard scientific realism or fact-relativism, leaving the middle ground problem unresolved (Massimi, 2018, p. 351). Consider a model, a representation and its semantic content, such as the caloric model. While it is no longer regarded as representational from a contemporary perspective, it was deemed representational from a vantage point of the 18th century. This raises a pressing question: what, precisely, was the target of that earlier representation? To claim that it was not truly representational but merely considered so by an 18th-century community of scientists reduces this notion of truth to one favored by standard scientific realists. On the other hand, asserting that the target was constructed by that community positions perspectival realism as a fact-relativist stance.

Upon the *relativist notion of perspectival representation* (Massimi, 2018, pp. 348-349), the semantic content of a scientific representation does not change across varying perspectives, but whether it fits the world is perspective-dependent. Equivalently, the content of knowledge claims is context-independent, whereas their *truth-value* depends on the perspective in which they are exercised. As shown in Figure 2, S with content SC is representational within perspective P_1 (denoted by R_{P_1}), but not within perspective P_2 (denoted by $\neg R_{P_2}$).

$$S \underset{P_2}{\overset{P_1}{\rightrightarrows}} SC \underset{\neg R_{P_2}}{\overset{R_{P_1}}{\rightrightarrows}} T$$

Figure 2: relativist notion of perspectival representation

For perspectival realism to qualify as a realist stance, this strategy also fails (Massimi, 2018, p. 352). To see this, let M_1 and M_2 be two incompatible models, each fitting its own target system within its respective perspective P_1 and P_2, i.e. representing T_1 and T_2, respectively. Since these models are about the same part of the world, the realist faces a challenge in attributing the right thing to the world. After all, T_1 and T_2 consist of incompatible facts or state of affairs, that both cannot be incorporated into the world . Thus understood, what is called by Massimi the 'normative dimension' of realism', that the realist should get things right, is violated.

The third conception, i.e. the *contextual notion of perspectival representation*, allows both the semantic content of a representation and the ways in which it is determined to fit the world, or equivalently both the semantic contents of grounded knowledge claims and their *truth-conditions*, to be perspectival (Massimi, 2018, pp. 349-350). Indeed, the *context of use* in which a scientific representation is made shapes both the meaning of the elements constituting the representation and the associated representation-conditions. Given a representation S, for instance, suppose it has two semantic contents SC_{P_1} and SC_{P_2} within two perspectives P_1 and P_2. Besides this kind of perspectivality, whether S with SC_{P_i} fits T depends on the conditions under which the fitness is defined. More precisely, while S with SC_{P_i} fits T with respect to the representation-conditions P_i imposes, it does not fit according the conditions P_j determines. Figure 3 illustrates such a situation in greater detail.

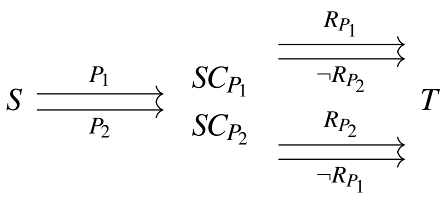

Figure 3: contextual representation

For example, while a model of water is representational in a context of use in which water is conceived as a viscous entity, i.e. in hydrodynamics, the model is not so in a context of use treating water as a discrete matter, i.e. in statistical mechanics. Thus, it is the context of use that determines both the appropriate interpretation of a representation and the representation-conditions of its resulting model. There are some worries about this conception, however. First, as discussed by Massimi (2018, p. 353) herself, this notion reduces perspectival realism as a realist stance to a Kantian one. Finally, "water-as-an-object-in-itself is not amenable to scientific knowledge", but just "water-as-an-object-of-experience is subject to different scientific models". Worse than this is the assurance that the targets of two models, i.e. two experiential appearances, pertain to the same thing, i.e. water *per se*. In such a case, we are effectively presented with two models, each with its own semantic content, representing two different targets. Figure 4 schematically illustrates this situation, where AP_{P_i} denotes the appearance corresponding to the perspective P_i.

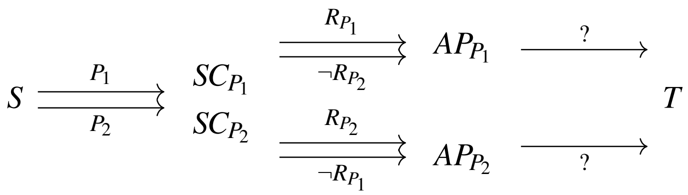

Figure 4: problems of contextual representation

Confronting these shortcomings, however, can we connect the two models from the two perspectives in a way that assures us they are representing the same thing? By adopting an *epistemic strategy*, Massimi (2018, p. 353-358) adeptly addresses this realist demand. Indeed, she suggests that the representational status of each model should not only be *assessable within* its own perspective by determining whether epistemic criteria or the "standards of performance-adequacy" (Massimi, 2018, p. 354) defined within that perspective are satisfied, but also *assessable from* another perspective by examining whether those criteria or standards are still met when model's representational status is evaluated from this new perspective. If a scientist holds that her model respects accuracy (as an epistemic criterion or a standard of performance-adequacy) in her perspective, meets a minimal condition of accuracy (as a representation-condition), and is thereby deemed representational (as a representational status), another scientist from her own perspective should also assess whether the model meets this standard, though the satisfaction of this standard is defined within the latter perspective. Metaphorically speaking, if a football player believes she has scored a goal because she has seen the ball cross the goal line, a spectator should also assess her score by determining whether the ball has crossed the line. However, what the player sees as crossing the line may differ from what the spectator perceives, as they view the field from different vantage points.

This cross-perspectival assessment proceeds for science as well. For example, suppose that a model of water in hydrodynamics, as a perspective, is representational with respect to the representation-conditions determined by this perspective. This internal adequacy, Massimi (2018, p. 355) suggests, is not enough; it must also be assessable from another perspective, i.e. statistical mechanics as another perspective. For statistical mechanics to serve not only as a context of use but also as a "context of assessment", we should first translate the hydrodynamic properties of water involved in the model into statistical features and then determine whether its representational status is preserved once the hydrodynamic representation-conditions are transformed into statistical representation-conditions. If so, the two kinds of *internal* and *external* adequacy are shown to hold. A similar mechanism should work for a model of water in statistical mechanics, ensuring that the two models, within the two different perspectives, represent the same part of the world. Figure 5 illustrates this process schematically, showing the movement back and forth between the two perspectives. Here, $SC_{P_i}^{P_j}$ denotes the semantic content of S within perspective i, as translated into perspective j, and $R_{P_i}^{P_j}$ denotes the representational status of the model with respect to the representation-conditions of perspective P_i, transformed into those of perspective P_j.

$$S \overset{P_1}{\underset{P_2}{\rightrightarrows}} \begin{matrix} SC_{P_1} \\ \\ SC_{P_2} \end{matrix} \overset{P_2}{\underset{P_1}{\rightrightarrows}} \begin{matrix} SC_{P_1}^{P_2} \\ \\ SC_{P_2}^{P_1} \end{matrix} \overset{R_{P_1}^{P_2}}{\underset{R_{P_2}^{P_1}}{\rightrightarrows}} T$$

Figure 5: perspectival representation

Massimi (2018, p. 357) mentions in passing some theoretical mechanisms for moving back and forth between different perspectives, e.g. limiting procedures and inter-theoretic relations. However, her book (Massimi, 2022) primarily focuses on discussing how perspectives *intersect* and *interlace* across historical and cultural contexts, rather than characterizing how they *interconnect* through theoretical machineries. Indeed, Massimi follows a form of historical naturalism, i.e. identifying cases in which scientific perspectives from different historical and cultural contexts exchange their tools and techniques with one another, while inferences from their associated models are cross-perspectivally assessed. By considering Massimi's (2022, pp. 94-109) detailed and nuanced discussion of the development of nuclear models between the 1930s and 1950s, for instance, we find not only how the reliability of model-based inferences is checked across different perspectives (i.e. how they intersect), but also how knowledge of the nucleus is produced in an evolving manner through the trading of experimental techniques and the interchanging of theoretical assumptions (i.e. how they interlace):

> ...[T]hese models show the *collaborative and social nature* of scientific knowledge production, the seamless flow through which model-based knowledge claims are historically put forward, modified, corrected, and reenacted. To what extent was the Nobel Prize-winning 1949 shell model an evolution of (instead of an abrupt shift from) the 1934 shell models? How to classify Schmidt's odd-particle model in this lineage? ... What about the relation between Gamow's 1929 liquid drop model and Bohr's 1936 compound model? How to locate the Rainwater-Bohr-Mottelson 'unified model' – with its combination of liquid drop model and shell model – in this model genealogy?
> ... These models are *dynamic evolving tools* with a *history of their own*, which is often intertwined with the history of other scientific models (Massimi, 2022, pp.106-107, emphases in original).

Although this bottom-up philosophical methodology for connecting different scientific perspectives *indirectly* reveals how the more abstract and theoretical components of scientific perspectives interconnect, we propose here a theoretical framework that *directly* articulates the process of stitching together scientific theories from different vantage points. Put differently, while Massimi accounts for how theories intersect and interlace across perspectives, we aim to supplement her view by introducing a theoretical tool that explicates how they interconnect when conceived as category-theoretic constructions. Our theoretical apparatus for this approach is adjunction, but before using it, we must first show that reconstructing scientific theories category-theoretically makes room for perspectival realism. The next section deals with this task.

3. From the Relativity of Equivalence to Perspectival Vantage Points

As outlined in the introduction, the role of agents in constituting scientific representations and, in turn, representation-based knowledge claims makes perspectivism a defensible view. What these agential roles are and how they determine a scientific representation have been extensively discussed by various agent-based accounts (see e.g. Boesch, 2017; Frigg and Nguyen, 2020) and hybrid accounts (see e.g. van Fraassen, 2008; Bueno and Colyvan, 2011) of scientific representation. Thus, let us suffice with a paradigmatic example introduced by van Fraassen (2014, p. 279), which shows why the representational nature of scientific theories cannot be fully captured by formal features alone:

> A scientific model is a mathematical structure offered as representation for certain phenomena. A representation has content. A representation of gas diffusion is not the same thing as a representation of temperature distribution, even if the math is the same.

Thus understood, while from the perspective of, for example, a chemist an equation may represent the diffusion flux, the same equation may be used by a physicist to study the heat flux, as they *interpret* the elements involved in the equation from their own vantage points. As such, the need for interpretation paves the way for perspectivism. In what follows, however, we will argue that the relativity inherent in the (in)equivalence of scientific theories further supports a form of perspectivism.[9] To reach this conclusion, we must take two steps. Drawing on the internal approach to interpreting scientific theories recently advanced by Dewar (2023; 2024), we will first discuss how interpretation is grounded in equivalence. Then, using the category-theoretic approach to characterizing scientific theories (see e.g. Rosenstock et al., 2015; Halvorson and Tsementzis, 2017; James et al., 2018; Barrett, 2019; Feintzeig, 2024) and the relativity of (in)equivalence explored by Barrett (2022), we will conclude that perspectivism is a permissible stance.

[9]There is a long-standing debate among philosophers of physics regarding the necessary and sufficient conditions for the equivalence of scientific theories. We do not want to address this issue here, but will simply assume that theoretical equivalence goes beyond empirical equivalence and necessitates some sort of interpretational or semantic equivalence. For an overview of different notions of equivalence in physical theories, see Weatherall (2019a; 2019b). Analyses provided by Coffey (2014) and Teitel (2021) have had significant impacts on the literature.

3.1 Equivalence Determining Interpretation

Dewar (2023) has recently outlined two approaches to interpreting scientific theories, though his framework may embody other forms of scientific representation, e.g. scientific models. According to the *external approach*, on which most theories of scientific representation have developed, interpreting a scientific representation involves assigning content or meaning to the appropriate parts of the representation by using a map. Both the content and the map, as the agent-based and hybrid accounts suggest, are determined by the agent dealing with the representation and the context in which she uses the model. How this map is defined and how the fit between a model, i.e. a scientific representation together with its content, and its target is determined depends on the account of representation in question, which are not our concerns here. The point at issue here is that the interpretation of a scientific representation is determined by a relation or map in which one relata is external to the representation. Equipped with this notion of interpretation, we find a relatively simple answer for van Fraassen's conundrum mentioned above: two models with the same math may not be identical or theoretically equivalent due to their possibly different interpretations. As such, *whether or not two scientific models or theories are equivalent is grounded in interpretational considerations construed externally*. Beyond this basic characterization of the external approach, it may also be enriched by embedding an *indirect* form of assigning meaning, according to which first the empirical part of a scientific representation receives content through the mechanism discussed above, and then its theoretical or non-empirical part becomes meaningful by establishing a new map between the components of this latter and those of the former part. It is because of this feature that Dewar (2023, p. 1) points to the external approach by saying "interpreting a theory is like translating from one language to another".

Criticizing the external approach for its non-naturalistic nature and its reliance on the relationship between two theories to delimit the meaning of one (Dewar, 2023, pp. 10-11), Dewar advocates and defends an *equivalence-first approach* to interpretation. This method might trace its roots to Cantor's and Frege's practice of attributing abstract properties to objects or sets of objects, guided by the so-called Hume's principle (Linnebo, 2017, p. 30). The basic idea is that, to define a collection of properties (e.g. redness, blueness, etc.), we first define a relation (e.g. being co-colored) on all objects that may have these properties. We then determine the relevant equivalence classes that partition the objects. In the final step, we assign a property to each class, asserting that object A and object B possess the same property if they are related (e.g. co-colored) or, equivalently, belong to the same class. For instance, they possess blueness if they belong to the class to which blueness is attributed.

Following Hume's principle, we can also identify the interpretation of a theory by staying within the space of all theories, without appealing to external apparatuses such as interpretation mapping. More precisely, by defining an equivalence relation on this space, which here is theoretical or semantic equivalence, theories T and T' have the same semantic content if they belong to the same class defined by the relation of theoretical equivalence, i.e. if they are theoretically equivalent. Using an intrinsic style of speaking, we can say that theory T has interpretation I if T belongs to the class to which interpretation I is attributed. In this manner, the internal approach overturns the conventional priority of interpretation over equivalence, grounding the former in the latter.

Be that as it may, one may voice the worry that the internal approach does not provide a sufficiently fine-grained tool for interpreting different parts of a theory, but merely determines the meaning of the theory as a whole. By using the notion of invariance and following a simple instruction, we may address this concern: to determine the representational content of part A involved in theory T, vary it to obtain an inequivalent theory, T'. With two different theories at hand, we may identify the content of the varied part, i.e. A. Furthermore, if changing A does not result in an inequivalent theory but only an equivalent one, then A is *artefactual* and lacks *representational* content. Stated otherwise, "the theory is committed to whatever is invariant across equivalence, i.e. to all and only that which is shared by equivalent models" (Dewar, 2023, p. 13).

Having introduced the basic components of the internal approach, we are now in a position to see whether it, like the external one, makes room for the agential degree of freedom in interpreting scientific theories, whereby perspectival dimensions are implemented. After all, equivalence appears to be a formal notion inherently independent of an agent's actions. As such, interpretation, when grounded in equivalence, would also be unaffected by any perspectival consideration. This raises the question: how does the internal approach deal with van Fraassen's conundrum? Although Dewar (2023, pp. 16-17) uses a straightforward inferential maneuver to settle this problem, he allows an agent's actions to be incorporated within his framework by acknowledging the way an agent ascribes a category to a theory or its models. Characterizing scientific theories as categories, the question of whether two theories are categorically equivalent would depend on what categories an agent ascribes to theories in light of contextual aspects. This is the only room for the agential freedom in interpreting theories that Dewar (2023, p. 14) recognizes in his internal approach. In the next section, however, we show that the interpretation based on equivalence is influenced by contextual-perspectival factors in a far more substantial way, i.e. by the relativity generated through the choice of the functor intended to compare two theories.

3.2 The Relativity of (In)Equivalence

Fielding philosophical inquiries about physical theories, several philosophers of physics in recent years have made it a practice to present these theories or their models within the framework of category theory. For example, Halvorson and Manchak (2022) articulate the models of the general theory of relativity using a category whose objects are Lorentzian manifolds and whose morphisms are isometries, aiming to put an end to the quarrel surrounding the Hole argument concerning whether the theory represents spacetime points or relations. Similarly, Weatherall (2016) employs this framework to examine whether Newtonian gravitation and geometrized Newtonian gravitation are equivalent. Having reified physical theories in this manner, scrutinizing their equivalence requires a relatively straightforward methodology: theories T and T' are equivalent if their associated categories of models, $\mathbf{Mod}_T$ and $\mathbf{Mod}_{T'}$, are *categorically equivalent*. Given two categories $\mathbf{C}$ and $\mathbf{D}$, they are *equivalent* if there exist (covariant) functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ in such a way that $F \circ G$ is naturally isomorphic to $\mathrm{id}_{\mathbf{D}}$ and $G \circ F$ is naturally isomorphic to $\mathrm{id}_{\mathbf{C}}$.[10] There exists another characterization of equivalence, which will be more frequently invoked in the following discussions. It states that a functor $F : \mathbf{C} \to \mathbf{D}$ induces the equivalence between $\mathbf{C}$ and $\mathbf{D}$ if and only if it is full (i.e. for every objects C, C' of $\mathrm{Obj}_{\mathbf{C}}$ and every morphism $g : F(C) \to F(C')$ of $\mathrm{Mor}_{\mathbf{D}}$, there exists a morphism $f : C \to C'$ of $\mathrm{Mor}_{\mathbf{C}}$ such that $F(f) = g$), faithful (i.e. for every objects C, C' of $\mathrm{Obj}_{\mathbf{C}}$ and every morphisms $f, f' : C \to C'$ of $\mathrm{Mor}_{\mathbf{C}}$, if $F(f) = F(f')$, then $f = f'$) and essentially surjective (i.e. for every object D of $\mathrm{Obj}_{\mathbf{D}}$, there exists an object C of $\mathrm{Obj}_{\mathbf{C}}$ such that $F(C)$ is isomorphic to D).

As we see, equivalence between two categories depends on the comparing functor that also guarantees the two categories are representationally equivalent. However, it is always possible to compare even two equivalent categories using a functor that enable a transition from one category to another by either losing or adding categorical information. Thus understood, whether two given categories are conceived of as representationally equivalent or not turns on the choice of an agent for a *relevant* functor that is supposed to compare them. If an equivalence-inducing functor is chosen, then they have the same content. On the other hand, if forgetful functors (those that lose some information) or free functors (those that add some information) are considered, then they are not representationally equivalent.[11] This form of agential freedom in choosing the appropriate functor, which may be dubbed 'functorial relativity', is the very point that has already been insightfully highlighted by Barrett (2022, p. 17) in his discussion about the amount of categorical structure embedded in categories:

> And in general, there are many functors between two categories. The question is then the following: Which (if any) are the 'right' functors to consider when we compare the structure of objects in category $\mathscr{C}$ to objects in category $\mathscr{D}$?

[10]We have assumed here that the reader is familiar with the basics of category theory. For a terse introduction to it, see Awodey (2006). For a more accessible book, see Perrone (2024).

[11]Some words to explicate these conceptions are worth discussing. Drawing on Baez and Shulman's (2007) analysis, we may conceive of mathematical constructions as comprising (at most) three components: *stuff* (e.g. the underlying set of a group), *structure* defined on the stuff (e.g. the group operation), and *properties* (e.g. the associativity of the group operation). Thus construed, losing information may pertain to any of these components, or to any combination of them. The functor from the category of groups to the category of sets, $F : \mathbf{Grp} \to \mathbf{Set}$, which sends every group to its underlying set and every homomorphism to the corresponding function, forgets structure (i.e. the group operation). This functor is *not full*. The functor from the category of abelian groups to the category of groups, $G : \mathbf{Ab} \to \mathbf{Grp}$, which is an inclusion functor, forgets a property (i.e. being abelian). This functor is *not essentially surjective*. The functor from the category of small categories to the category of sets, $H : \mathbf{Cat} \to \mathbf{Set}$, which sends every category to its set of objects and every functor to the corresponding function between those sets, forgets stuff (i.e. the set of morphisms). This functor is *not faithful*.

Given these observations, it becomes clear that the agential freedom in the internal approach is not restricted to assigning the *relevant* categories to (models of) theories but also extends to choosing the *appropriate* functors for comparing these categories. Both kinds of freedom are guided and shaped by perspectival considerations. That said, one might question whether considering functors that do not induce equivalence between equivalent categories has any applicable or practical significance, or whether it is merely a mathematical game. To see that this is not the case, it is insightful to examine an example from a much more robust field, namely mathematics, where equivalent categories may be treated as inequivalent due to contextual concerns. We present it here in a category-theoretic style, though one may refer to Sakai (1971) for a more rigorous treatment in the standard manner of mathematical practice.

Example 1. Let $\mathbf{Top_{cH}}$ be the category of compact Hausdorff topological spaces (as its objects) and continuous maps (as its morphisms), and let $\mathbf{C^\star\text{-}Alg_{uc}}$ be the category of unital commutative $C^\star$-algebras (as its objects) and *-homomorphisms (as its morphisms). There exists a well-known (contravariant) functor $G : \mathbf{Top_{cH}} \to \mathbf{C^\star\text{-}Alg_{uc}}$, called 'Gelfand representation', which maps any compact Hausdorff topological space X to the unital commutative $C^\star$-algebra $GX := C(X)$, consisting of all complex-valued continuous functions, and any continuous map $f : X \to Y$ to *-homomorphism $Gf : C(Y) \to C(X)$, defined by $Gf(\varphi) := \varphi \circ f$ for all $\varphi \in C(Y)$. This functor is full, faithful, and essentially surjective, and so induces an equivalence between the two categories.[12] Particularly, it preserves the topological structure of $\mathbf{Top_{cH}}$, since for any *-homomorphism $g : C(Y) \to C(X)$, there is a continuous map $f : X \to Y$ such that $Gf = g$.

In spite of this equivalence, we may consider another functor $V : \mathbf{Top_{cH}} \to \mathbf{C^\star\text{-}Alg_{uc}}$ that is not full and therefore does not induce any equivalence between the two categories. It sends any compact Hausdorff topological space X to Abelian von Neumann algebra $VX := \ell^\infty(X)$, consisting of all complex-valued bounded functions, and any continuous map $f : X \to Y$ to *-homomorphism $Vf : \ell^\infty(Y) \to \ell^\infty(X)$, defined by $Vf(\varphi) := \varphi \circ f$ for all $\varphi \in \ell^\infty(Y)$. This functor forgets the topological structure of $\mathbf{Top_{cH}}$, since there exists a *-homomorphism $g : \ell^\infty(Y) \to \ell^\infty(X)$ for which there is no continuous map $f : X \to Y$ such that $Vf = g$. Now we have two categories $\mathbf{Top_{cH}}$ and $\mathbf{C^\star\text{-}Alg_{uc}}$ which may be seen as *equivalent* when studying complex-valued continuous functions and therefore considering the functor G, but as *inequivalent* when investigating complex-valued bounded functions and deploying the functor V. This reveals that analyzing functors that do not induce equivalence between equivalent categories is not merely a mathematical exercise but a routine aspect of mathematics.

Let us recap the main ideas covered in this section. The aim of the article is to provide a theoretical tool furnished by category theory to show how different theories from different perspectives interconnect. This goal is defensible only if characterizing scientific theories within category theory permits the kind of relativism that perspectival realism demands. This section

[12] More precisely, the equivalence holds between $\mathbf{Top_{cH}}$ and the opposite of $\mathbf{C^\star\text{-}Alg_{uc}}$ because G is contravariant.

concludes that the equivalence-first approach to interpreting scientific theories, along with the relativity inherent in comparing category-theoretically reified theories, makes perspectival realism a viable stance which can be further enriched by a new category-theoretic apparatus for explicating the way in which they interconnect.

4. Stitching Perspectives Together

Before starting our somewhat tedious journey to the mathematics of adjunction and its philosophical fruits, let's read a literarily bland but philosophically inspiring short story. Once upon a time, three friends - a philosopher and two photographers - were walking through a desert when they spotted a beautiful but distant tree. The photographers decided to capture it. Photographer 1 set up the tripod, mounted the camera, and took a picture. Then Photographer 2 stood behind the same tripod and took her own shot. Photographer 1's image turned out slightly blurred, while Photographer 2's photo had intense micro-contrast. This sparked a disagreement. Photographer 1 argued that the second photo was too sharp, unnaturally so, claiming it had subtle contrasts the tree's true texture didn't have. Photographer 2 disagreed, saying the first photo was overly blurred, failing to capture the distinct edges and boundaries of the tree. They couldn't settle the debate by comparing the photos to the real tree - it was simply too far away. So, they turned to the philosopher for guidance. After some thought, the philosopher offered a solution: "Increase the contrast in the first photo to see if it becomes like the second, and blur the second photo to see if it resembles the first". However, the outcome wasn't what they expected. The altered photos were not exactly identical, so they could not be attributed to the real tree. But something surprising happened: the two transformed images became very similar, somewhere between the blurred and the hyper-sharp photos. Though the philosopher couldn't tell them which photo was more realistic, he assured them that both photos were of the same tree, seen through different lenses - perspectives intertwined by the very acts of blurring and sharpening.

Now the question is: if our theories are like photos captured from different perspectives, is there any mathematical machinery, such as blurring or increasing contrast, that can help us determine whether the theories are representations of the same thing? We believe that adjunction technology in category theory guides us towards this aim. In what follows, we will first define adjunction and its essential features. Then, we will discuss a toy model in algebraic topology to illustrate how this category-theoretic tool allows us to represent the same mathematical entity from different perspectives, with each perspective revealing some interesting features of the entity. Finally, we apply adjunction to a case in general relativity, showing how it equips perspectival realism with a new formal apparatus to relate different theoretical perspectives.

Before moving on to our proposal, a brief remark is in order. As previously mentioned, using category-theoretic techniques to concretize perspectivism is not new, particularly as proposed by Karakostas and Zafiris (2022) in developing a perspectivist picture of the way quantum mechanics probes the world. Their motivation for adopting a perspectivist stance towards quantum mechanics stems from the implications of the Kochen-Specker theorem, which precludes any *global* picture of a quantum system where the truth values of all relevant propositions are definite. In fact, quantum mechanics offers only fragmented glimpses of such a system. These sets of *local* claims, however, should be glued together to provide a representation of the quantum system in question. At this stage, the notions of adjunction and presheaves in category theory are used to link the category of quantum event algebras, which serves as the structure assigned to the target system, to the category of presheaves of Boolean event algebras, which serves as the structure representing the target system.

Although both our proposal and theirs may be conceived of as a theoretical enrichment of perspectival realism, there are some subtle and deep differences between them. For instance, their suggestion is fundamentally defined within the external approach, as they ascribe a certain structure to the target system that is supposed to be represented, whereas such an assignment is entirely absent in our framework. We think that following the internal approach is better aligned with a perspectivist view because, in their external proposal, epistemic access to the worldly system (from nowhere) is achieved by ascribing a category-theoretic structure to it. From this perspective, adjunction provides a cross-categorical tool for connecting categories as representations and categories as the represented, while in our framework, adjunction is an endo-theoretical tool for stitching together the categories that are to represent the target system. Whether Karakostas and Zafiris's proposal ultimately tilts more towards a structural realist stance rather than a perspectival realist one is an interesting question left for another time.

4.1 Adjunction Defined

There are different ways to define adjunction in category theory (see e.g. Agore, 2023, Ch. 3). In what follows, we introduce it in a manner that best fits the framework of perspectivism (Simmons, 2011, Ch. 5). Consider two objects C and D which belong to categories $\mathbf{C}$ and $\mathbf{D}$, respectively. What is the relationship between these two objects?[13] The answer is that they are *incomparable* as they 'live' in different worlds, realms or *perspectives*. Despite this incomparability, however, we can use a trick: translating an object (e.g. C) into another category (e.g. $\mathbf{D}$) through a functor (e.g. $F : \mathbf{C} \to \mathbf{D}$) and then comparing the translated object (e.g. $F(C)$) with another object that remains in that category through a morphism (e.g. $g : F(C) \to D$). There is an issue here, however. For any comparison or relationship $g : F(C) \to D$, there exists another relationship $f : C \to G(D)$, generated by another comparing functor $G : \mathbf{D} \to \mathbf{C}$. To remedy this concern, we may impose a constraint by considering a one-to-one correspondence between any two relationships, represented by two inverse functions $()^{\#} : \mathrm{Hom}_{\mathbf{C}}(C, G(D)) \to \mathrm{Hom}_{\mathbf{D}}(F(C), D)$ and $()_{\flat} : \mathrm{Hom}_{\mathbf{D}}(F(C), D)) \to \mathrm{Hom}_{\mathbf{C}}(C, G(D)$ such that $(f)^{\#} = g$ and $(g)_{\flat} = f$.[14] This condition

[13]Recall that, according to the category-theoretic conception of scientific theories, a model of a theory is represented by an object of a category.

[14]Given category $\mathbf{C}$ and $C, C' \in \mathrm{Obj}_{\mathbf{C}}$, we define $\mathrm{Hom}_{\mathbf{C}}(C, C') := \{f \in \mathrm{Mor}_{\mathbf{C}} \mid f : C \to C'\}$. If this collection forms a set, it is called a *hom-set*.

is called the *bijection requirement* (Simmons, 2011, p. 164).

Now suppose that C relates to C' through a morphism $h : C' \to C$ in $\mathbf{C}$ and D relates to D' through a morphism $k : D \to D'$ in $\mathbf{D}$. Imposing once again the above constraint on the new objects C' and D' implies the identities:

$$(G(k) \circ f \circ h)^{\#} = k \circ (f)^{\#} \circ F(h)$$

and

$$(k \circ g \circ F(h))_{\flat} = G(k) \circ (g)_{\flat} \circ h$$

This condition is named the *naturality requirement* (Simmons, 2011, p. 164). With this in place, we can give a formal definition of adjunction.

Adjunction: Given two categories $\mathbf{C}$ and $\mathbf{D}$, an adjunction is $\langle F, G, ()^{\#}, ()_{\flat} \rangle$, where $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ are two (covariant) functors, and for each $C \in \mathrm{Obj}_{\mathbf{C}}$ and $D \in \mathrm{Obj}_{\mathbf{D}}$, the functions

$$()^{\#} : \mathrm{Hom}_{\mathbf{C}}(C, G(D)) \to \mathrm{Hom}_{\mathbf{D}}(F(C), D)$$

and

$$()_{\flat} : \mathrm{Hom}_{\mathbf{D}}(F(C), D)) \to \mathrm{Hom}_{\mathbf{C}}(C, G(D)$$

are an inverse pair of bijections, natural in C and D. F and G are called the *left adjoint* and the *right adjoint*, respectively. Also, the adjunction is denoted by $F \dashv G$.

Although, among the three categorical conceptions for comparing categories, namely isomorphism, equivalence, and relation through an adjunction, adjunction is the weakest form, it is strong enough to ensure that the two categories connected through an adjunction may be used to represent the same thing. The most intuitive case demonstrating this feature is the two processes of dilation and erosion in mathematical morphology, which are characterized by two adjoint functors (Rosiak, 2022, pp. 195-205). Indeed, blurring and increasing contrast, as suggested by the philosopher in our story, were nothing but an appeal to the category-theoretic apparatus of adjunction to resolve the dispute, guaranteeing that two photos are just two pictorial representations of the same tree. In mathematics, too, adjunctions are applied, among other things, to represent different features of the same thing. In the following, we provide a warm-up mathematical example to illustrate this application of adjunction.

Example 2. Consider the so-called *figure eight* (Figure 6) as a topological space, denoted by $\mathbb{S}^1 \vee \mathbb{S}^1$, and let us explore its different topological features, particularly the number of its holes and loops. To count its loops, we may use the fundamental group $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)$, while to count its holes, we may use the first homology group $H_1(\mathbb{S}^1 \vee \mathbb{S}^1)$. These two mathematically

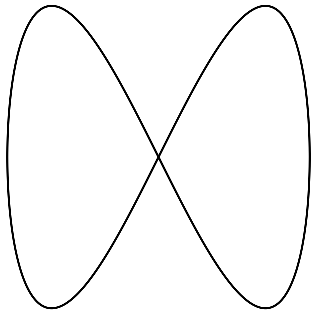

Figure 6: the figure eight

representational tools, however, are incomparable, since the first is an object of the category of groups, **Grp**, while the second is an object of the category of Abelian groups, **Ab**. Thus understood, it is not at all certain that they represent two features of the *same* thing unless we find that **Grp** and **Ab** are two categories connected through an adjunction.

To achieve this, we should first construct two processes of translation, one from **Grp** to **Ab** and another from the latter to the former, in such a way that the bijection and naturality requirements are satisfied. The translation from **Grp** to **Ab** is achieved through the process of Abelianization and its associated functor $\mathrm{Ab} : \mathbf{Grp} \to \mathbf{Ab}$, sending $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)$ to $\mathrm{Ab}(\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)) = \pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)/[\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1), \pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)]$, which is the quotient of $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)$ by its commutator subgroup. The other translation is also provided by the inclusion functor $I : \mathbf{Ab} \to \mathbf{Grp}$. By the Hurewicz theorem (May, 1999, p. 118), which states that for any path-connected space X and positive integer n, there exists a group homomorphism from n-th homotopy group $\pi_n(X)$ to n-th homology group $H_n(X)$, the translated object $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)/[\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1), \pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)]$ and the object $H_1(\mathbb{S}^1 \vee \mathbb{S}^1)$ are isomorphic in **Ab**. Therefore, for the relation between $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)/[\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1), \pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)]$ and $H_1(\mathbb{S}^1 \vee \mathbb{S}^1)$ in **Ab** induced by Abelianization, there would be one and only one relation between $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)$ and $I(H_1(\mathbb{S}^1 \vee \mathbb{S}^1)) = H_1(\mathbb{S}^1 \vee \mathbb{S}^1)$ in **Grp**, where the correspondence satisfies the naturality condition. As such, the Hurewicz theorem provides a topological realization of the adjunction between the categories **Grp** and **Ab**, with Ab and I as two adjoint functors (Agore, 2023, pp. 163-164).

As we see, there exists a form of coordination between the perspectival realist's conception of scientific theories depicted from different perspectives, which purport to be representations of the same part of the world, and their category-theoretic reifications, which are connected via adjunction. Just as two scientific theories from different vantage points have their own semantic contents while their elements can be translated into each other, two categories forming an adjunction are completely different yet their objects can be translated into each other. Moreover, just as the relation between the translated elements of the scientific theory in one perspective and the corresponding elements of the scientific theory in another perspective ensures the representation of the same part of the world, the relation between the translated objects of one category and those of another category through adjunction ensures that the two categories represent the same thing. Thus, the machinery of adjunction is well-suited to fulfill the desideratum of the perspectival realist in delineating her stance as a middle ground between relativism and traditional realism. *Two categories forming an adjunction represent different targets because they are two different categories (the relativist dimension), but they represent different aspects of the same thing because they are connected through an adjunction (the realist dimension).* This feature is manifested in our mathematical example. In the next section, we will scrutinize a physical case in which adjunction effectively satisfies the stipulation of the perspectival realist.

4.2 Representing Spacetime Adjunctively

For those relationalists about spacetime who are nauseated by any trace of spacetime points in their ontology and admit only spatio-temporal relations, there is nothing better than abolishing the physico-mathematical entities that purport to represent such individual things. This is the project developed by Earman in 1970s and 1980s (see e.g. Earman, 1977, 1989), inspired by Geroch's (1972) introduction of an algebraic alternative to the differential geometric framework for articulating the general theory of relativity. After all, the latter relies on manifolds constituted by sets whose elements possess primitive identity. For us who view the affair through the lens of perspectivism, it *seems* that Einstein algebras and Lorentzian manifolds define two distinct perspectives for depicting regions of possible worlds, and that the choice of which perspective is preferable may hinge on perspectival considerations, including metaphysical parsimony, theoretical simplicity, unifying power, and so on.

Yet, do these two theoretical frames determine two *genuinely* different perspectives? Recalling the internal approach, we cannot simply conclude that they do by saying that one of them is inherently a substantivalist vantage point, since it represents first spacetime points and then fields on them or properties thereof, while the other is relationalist by placing priority on fields. In fact, some decades later, Rosenstock, Barrett, and Weatherall (2015) show that if we reify the models of the theory in these two mathematical frameworks via two categories, they prove to be dually equivalent, possessing the same representational content. In terms of perspectivism augmented with the internal approach, the two categories of Einstein algebras and Lorentzian manifolds define the same perspective because they belong to the same equivalence class, even if someone initially reads off two *apparently* different ontologies from them.

This is not the end of story, however. Wu and Weatherall (2024) have recently shown that the (dual) equivalence between the two formulations of the general theory of relativity holds only if the Hausdorff condition is satisfied, i.e. distinct points of manifolds have disjoint neighborhoods. More specifically, they demonstrate that non-diffeomorphic (i.e. not smoothly equivalent) non-necessarily Hausdorff manifolds may correspond to isomorphic Einstein algebras. This implies that the algebraic structure alone may not uniquely determine the underlying manifold's topology and differentiable structure in the non-necessarily Hausdorff setting. It is worth noting that tinkering with a topological feature to see how the dual equivalence breaks down is not merely out of mathematical curiosity but has representational implications. Indeed, non-Hausdorff manifolds allow for certain types of indeterminism, as they enable bifurcating geodesics (where particle trajectories can split into multiple paths) and branching spacetimes (where alternative possible spacetimes coexist). While the lack of unique geodesics makes the emergence of closed timelike curves more likely and may potentially violate strong causality, non-Hausdorff manifolds should not be outright rejected in physics. Instead, they are better understood *modally*, where they are interpreted as representing bundles of alternative spacetimes rather than a single spacetime, as argued recently by Joanna Luc (2020).

When dealing with non-necessarily Hausdorff manifolds, we find ourselves outside the equivalence class in which the categories of Lorentzian manifolds and Einstein algebras exist. This situation can be interpreted as presenting two distinct perspectives: a perspective with the category of Lorentzian manifold as its representative element and another with the category of non-necessarily Hausdorff manifolds. Now the question is: do these two representatives stand for the same target, i.e. spacetime, but from different perspectives? According to our proposed methodological criterion, they do just in case there exists an adjunction between the two categories. In what follows, we show that this is indeed the case, thereby establishing that spacetime can be represented from both perspectives.

Let us first delineate the meaning of Hausdorff property for a topological space and its implications. A topological space X is called *Hausdorff* if any two distinct points have disjoint neighborhoods, i.e. for every $x,y \in X$ such that $x \neq y$, there exist two open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. The Hausdorff property is closely intertwined with the uniqueness of limits in the sense that a topological space is Hausdorff if and only if every net within it converges to at most one point, thereby guarantying the uniqueness of the limit. In contrast, if X is not Hausdorff, then the algebra of real- or complex-valued continuous functions on X does not separate the points of X. This means that there exist at least two points $x,y \in X$ such that $x \neq y$ but $f(x) = f(y)$ for every continuous map $f : X \to \mathbb{R}$. That being said, there is a natural way to render any topological space Hausdorff (McDuff, 2006, p. 272), which is called the 'Hausdorffization' of a topological space and is established by the theorem below.

Theorem 4.1. *For each topological space X, there exists a Hausdorff space $H(X)$ and a continuous map $q_X : X \to H(X)$ such that for every Hausdorff topological space Y and every continuous function $f : X \to Y$, there exists a unique continuous map $\bar{f} : H(X) \to Y$ satisfying $f = \bar{f} \circ q_X$. In other words, the following diagram commutes:*

$$\begin{array}{ccc} X & \overset{f}{\searrow} & \\ {\scriptstyle q_X}\downarrow & & \\ H(X) & \underset{\bar{f}}{\longrightarrow} & Y \end{array}$$

Moreover, the topological space $H(X)$ with the above property is unique up to homeomorphism.

Proof. We first introduce a relation R on X, defined by stating that for any $x,y \in X$, xRy if and only if for every Hausdorff topological space Y and every continuous function $f: X \to Y$, $f(x) = f(y)$.[15] It is straightforward to verify that R is an equivalence relation. Now, let $H(X) = X/R$ be endowed with the quotient topology, and let $q_X : X \to H(X)$ be the standard quotient map that sends each $x \in X$ to its equivalence class $R[x]$. Now, for every Hausdorff topological space Y and every continuous function $f: X \to Y$, the function $\bar{f}: H(X) \to Y$, defined by $\bar{f}(R[x]) = f(x)$, is precisely the unique map required in the statement.

To see that the topological space $H(X)$ is Hausdorff, let $R[x]$ and $R[y]$ be two distinct points in $H(X)$, where $x, y \in X$. Since it is not the case that xRy, it follows from the definition that there exists a Hausdorff topological space Y and a continuous map $f: X \to Y$ such that $f(x) \neq f(y)$. Since Y is Hausdorff, there exist two open subsets U and V of Y such that $f(x) \in U$, $f(y) \in V$, and $U \cap V = \emptyset$. Now, $\bar{f}^{-1}(U)$ and $\bar{f}^{-1}(V)$ are disjoint open neighborhoods of $R[x]$ and $R[y]$ in $H(X)$, respectively. Thus, $H(X)$ is Hausdorff, completing the proof.

For uniqueness, let $H_1(X)$ and $H_2(X)$ be two Hausdorff topological spaces, and let $q_X^1 : X \to H_1(X)$ and $q_X^2 : X \to H_2(X)$ be two continuous maps such that both satisfy the statement. By the first part of the theorem, there exist two continuous maps $\overline{q_X^1} : H_2(X) \to H_1(X)$ and $\overline{q_X^2} : H_1(X) \to H_2(X)$ such that $q_X^2 = \overline{q_X^2} \circ q_X^1$ and $q_X^1 = \overline{q_X^1} \circ q_X^2$. Thus, we obtain $q_X^2 = (\overline{q_X^2} \circ \overline{q_X^1}) \circ q_X^2$ and $q_X^1 = (\overline{q_X^1} \circ \overline{q_X^2}) \circ q_X^1$. Since q_X^1 and q_X^2 are surjective, they admit right inverses. Hence, $\overline{q_X^2} \circ \overline{q_X^1} = \mathrm{Id}_{H_2(X)}$ and $\overline{q_X^1} \circ \overline{q_X^2} = \mathrm{Id}_{H_1(X)}$, showing that these maps are mutual inverses. Therefore, $H_1(X)$ and $H_2(X)$ are homeomorphic. □

Considering the above theorem, we are now ready to establish how an adjunction holds between the category of all topological spaces and the category of Hausdorff topological spaces, with continuous functions as morphisms, denoted by **Top** and $\mathbf{Top_H}$, respectively. Clearly, $\mathbf{Top_H}$ is a full subcategory of **Top**, and we have the inclusion (forgetful!) functor $I : \mathbf{Top_H} \to \mathbf{Top}$. Now, using Theorem 4.1, one can define the so-called *Hausdorffization* functor $H : \mathbf{Top} \to \mathbf{Top_H}$. In fact, given any continuous map $f : X \to Y$ between two topological spaces X and Y, one can define $H(f) : H(X) \to H(Y)$ by $H(f)(q_X(x)) = q_Y(f(x))$ for all $x \in X$. One can readily verify that H is a well-defined functor between two categories.

Theorem 4.2. *The Haudsorffization functor is left adjoint to the inclusion functor, i.e.* $H \dashv I$.

Proof. Let X be a topological space and Y be a Hausdorff topological space. We aim to show that there is a natural bijection $\mathrm{Hom}(H(X), Y) \cong \mathrm{Hom}(X, Y)$, which is simply a reformulation of Theorem 4.1. In fact, each $f : X \to Y$ corresponds to $\bar{f} : H(X) \to Y$, and each $\bar{f} : H(X) \to Y$ corresponds to $f = \bar{f} \circ q_X$, as stated in Theorem 4.1. The naturality requirement is immediate. □

Now let us turn to the different categories representing spacetime, using the notations of Wu and Weatherall (2024), wherein **HMan** and **nnHMan** denote the category of all Hausdorff

[15] An alternative but equivalent way to introduce R is to define it as the intersection of all equivalence relations $S \subset X \times X$ such that X/S is a Hausdorff space.

smooth manifolds and the category of all non-necessarily Hausdorff manifolds, respectively. Clearly, the restriction of the Hausdorffization functor H defines a functor H_1 : **nnHMan** $\to$ **HMan**, which is the left adjoint of the inclusion functor I_1 : **HMan** $\to$ **nnHMan** once again. Note that, as Wu and Weatherall discuss, the category **HMan** of all Hausdorff smooth manifolds and the category **HAlg** of all algebras of smooth functions on objects of **HMan** are equivalent, though this is not the case for the category **nnHMan** of all non-necessarily Hausdorff smooth manifolds and the category **nnHAlg** of all algebras of smooth functions on objects of **nnHMan**.

Having established the adjunction between the two categories, let us outline our discussion. We are given two categories, **HMan** and **nnHMan**, which determine two representational perspectives, where the latter incorporates modal considerations regarding its own target. Moreover, these two categories are connected through the adjunction, thereby standing for a single wordly target, which enables the perspectivist to adopt realist stances towards them. What is of significance here is that we have not invoked any external machinery, e.g. different but related mappings from different perspectives to a single target, to preserve that notion of truth to which Massimi refer. Indeed, the two perspectives, being stitched together adjunctively, suffice for adopting perspectival realist attitudes.

5. From a Critique to an Outlook

The concluding chapter of this article engages with a critique advanced by Stemeroff (2022), who confronts the perspectival realist with a dilemma: she must either commit to a Kantian position or inevitably gravitate toward a neo-Pythagorean one. Following his objection, one might worry that our category-theoretically enriched perspectival realism faces a similar concern, especially given that such a proposal appears to break the symmetry of scientific perspectives to which perspectival realism is committed. That said, we will argue that his attempt to corner the advocate of perspectival realism is doomed to fail, even though it prompts her to endow her stance with a higher-order structuralist flavour, an outlook for perspectival realism whose details await development in future inquiries. Let us first examine how his argument unfurls.

As discussed in Chapter 2, Massimi's preferred notion of perspectival truth requires scientific theories from different vantage points to be cross-perspectivally assessable. More precisely, we are justified in holding that a theory T from perspective P has some grip on reality just in case: first, T satisfies the representation-conditions defined within P, making it internally adequate, and second, this satisfaction is endorsed from within another scientific perspective P', rendering T externally adequate. Given that our physical theories are mathematically molded, this form of external assessment calls for a shared mathematical framework within which the knowledge claims of both perspectives can be construed and compared. This, however, brings up a vexing question: what role should we assign to the mathematical frameworks which required to stitch together different theories from different perspectives?

Stemeroff presents various options on the table, eliminating them one by one until he retains just one that is worth scrutinizing. On this option, which he calls 'epistemological Pythagoreanism' (Stemeroff, 2022, p. 515):

> The perspectivist could suggest that the mathematical methodology of modern physics is truth-conducive in the sense that the shared mathematical formalism of physics could provide a unified framework through which the structural features of a scientific perspective that are retained through the perspectival series of scientific theories can be said to have some grip on reality.

Perspectival realism, however, is a universal thesis stating that any knowledge claim is made from a perspective. As such, epistemological Pythagoreanism, when combined with this feature, implies that the shared mathematical formalism is itself a perspective, providing us the scientific claims about structural features that are preserved across the series of scientific theories from different perspectives. Thus understood, the shared mathematical framework is granted a privileged status that other perspectives lack. This asymmetry between different perspectives is at odds with the core idea of perspectivism, collapsing it "into some undesirable form of neo-Pythagoreanism" (Stemeroff, 2022, p. 516). As another fork of the dilemma, someone may argue that the shared mathematical formalism is merely one vantage point among others, without any preeminent status. This response, however, amounts to a form of Kantianism that clashes with the realist dimension of perspectival realism (Stemeroff, 2022, p. 516). Therefore, the advocate of perspectival realism is driven to adopt either neo-Pythagoreanism or Kantianism.

To clarify his argument, Stemeroff (2022, pp. 510-514) uses the case of the retention of energy-conservation laws, showing how perspectival realism requires a shared mathematical formalism to connect the different perspectives of classical physics and general relativity as a dynamical spacetime theory. Indeed, the principle of the conservation of energy is a good candidate to be taken as a claim having perspectival truth, as it not only satisfies the epistemic criteria within the perspective of classical physics but also holds in the perspective of general relativity, albeit in a local form. This local retention, however, proves to be valid only if we remain within the framework of modern differential geometry and employ the tools it introduces, such as the notions of Riemannian manifolds, Lie dragging, Killing vector fields, and so on:

> The regular nature of the Riemannian spacetimes of modern physics ensures that the typical conservation laws apply "locally" in the neighborhood of all points in a spacetime, at least approximately, and that these symmetries will hold on the global scale in a certain special class of spacetime structures. But can we give a viable perspectival account of this notion of retention and continued success? It is only from the perspective of modern differential geometry that we can define the sense in which the laws of conservation of energy found in static spacetime theories are retained as local laws in a dynamical spacetime theory, such as general relativity... However, in a perspectival account, it is not immediately clear what role we should assign to this mathematical framework (Stemeroff, 2022, p. 514).

Indeed, any worthwhile role we assign to modern differential geometry leads us to adopt either a neo-Pythagorean or a Kantian stance toward it, as evidenced by his argument for such a dilemma. It seems that, however, this conclusion is too hasty. In fact, Stemeroff is correct in arguing that the shared mathematical formalism, in this case modern differential geometry, is a human knowledge system that defines a perspective, but he is mistaken in assuming that this framework is inevitable. To put it more precisely, modern differential geometry is one of several mathematical systems that enable us to formulate the principle of the conservation of energy in general relativity. For example, general relativity might be presented within the diffeology framework as well, that is a more generalized formalism than the frame of smooth manifolds. Roughly speaking, a *diffeology* $\mathscr{D}$ on a set X is a presheaf on the category of Euclidean spaces whose morphisms are smooth maps, assigning a subset of smooth functions from $U \subseteq \mathbb{R}^n$ to X. The pair $\langle \mathscr{D}, X \rangle$ is called a *diffeological space* (nLab, 2025). In the discussion of Stemeroff's case, the diffeology program is of particular significance specially because within it Noether's first theorem, on which the conservation laws are based, is re-derived (Iglesias-Zemmour, 2013, pp. 391-392).

Mathematical physics witnesses the development of several alternative programs to articulate physical theories. For instance, the transition from classical to quantum mechanics may be reified by different formal frameworks, including canonical, geometric, and deformation quantization. For someone who has just dealt with the first two programs, it may seem that the notion of Hilbert space and the perspective it defines are inevitable for bridging classical to quantum mechanics. However, we know that the formalism of Hilbert spaces is not essential because the alternative program of deformation quantization exists and does not rely on such a notion (Yaghmaie, 2020).

Having said that, someone sympathetic to Stemeroff's concern may argue that, to secure the perspectival truth regarding the above case of theory change, some sort of linkage between alternative mathematical frameworks is needed. Otherwise, we are faced with a plurality of retentions that undermine the realist dimension of perspectival realism. Given that, the dilemma occurs once again, but this time it targets the proposed framework supposed to connect these alternatives. We think that this worry is legitimate, though suspect its domain. Indeed, the dilemma is not here applied to a physico-mathematical formalism, but to a meta-framework that overarches different physico-mathematical formalisms. Within the meta-framework of category theory, for instance, we are given that the category of smooth manifolds is embedded into the category of diffeological spaces (nLab, 2025), thereby linking the two mathematical technologies together. As a result, it seems that perspectival realism should concede to a form of *higher-order structuralism* whose domain is not defined in an object-level formalism, e.g. differential geometry, but within a meta-level framework, such as category theory.

6. Conclusion

To ascertain the objectivity of truth, Massimi incorporates an epistemic maxim into her perspectivism, according to which perspectival scientific knowledge claims not only should be assessed from within the perspective to which they belong, but also from other perspectives that have their own knowledge claims, epistemic principles, and semantic rules. This form of epistemic integration between different perspectives, which assures us that all knowledge claims represent objective truths, contrasts with the standard form of semantic integration to which realism typically appeals, where the burden of ascertaining objectivity falls on different representational mappings between scientific theories and the world. In sum, Massimi invokes an internal epistemic maneuver to preserve the objectivity of truth. As she herself reasons, however, cross-perspectival assessability of knowledge claims necessitates their cross-perspectival translatability. Given the category-theoretic characterization of scientific theories, we suggested a theoretical framework for transferring scientific models between perspectives, in such a way that Massimi's internal approach is maintained, but this time at a semantic level. To do so, we first used the internal approach to interpretation suggested by Dewar, where interpretation is grounded in equivalence, and then proposed adjunction as an endo-theoretic tool for stitching together theories from different perspectives. Finally, we argued that incorporating scientific theories into the category-theoretic framework, or any meta-level framework, imparts a higher-order structuralist aspect to perspectival realism.

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