The Stochastic-Quantum Correspondence

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Abstract

This paper argues that every quantum system can be understood as a sufficiently general kind of stochastic process unfolding in an old-fashioned configuration space according to ordinary notions of probability. This argument is based on an exact correspondence between the class of 'indivisible' stochastic processes and quantum theory. This new stochastic-quantum correspondence demotes the wave function from a primary ontological ingredient to a secondary mathematical tool, and yields a deflationary account of exotic quantum phenomena, such asinterference, decoherence, entanglement, noncommutative observables, and wave-function collapse. At a more practical level, the stochastic-quantum correspondence leads to a novel reconstruction of quantum theory, alongside the Hilbert-space, path-integral, and quasiprobability representations, and also provides a framework for using Hilbert-space methods to formulate highly generic, non-Markovian types of stochastic dynamics, with potential applications throughout the sciences.

1 Introduction

The theory of stochastic processes describes the phenomenological behavior of systems with definite configurations that evolve probabilistically in time. Quantum theory is a comprehensive mathematical apparatus for making measurement predictions when taking into account the microscopic constituents of various kinds of physical systems, from subatomic particles to superconductors. At an empirical level, both theories involve probabilities, and at the level of formalism, both employ vectors and matrices.

The primary goal of this paper is to introduce an exact correspondence between a highly general class of stochastic processes and quantum theory, within which measuring devices and observers are incorporated as ordinary subsystems. This *stochastic-quantum correspondence* takes the form

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of a simple 'dictionary' expressing any time-dependent stochastic matrix in terms of a suitable combination of Hilbert-space ingredients.

From a practical standpoint, the stochastic-quantum correspondence provides a systematic framework for constructing highly generic forms of stochastic dynamics, much as the Lagrangian and Hamiltonian formulations of classical mechanics provide systematic frameworks for constructing deterministic dynamics. Potential applications range from turbulence to finance, to name just two examples. Importantly, the stochastic-quantum correspondence does not require assuming that the stochastic dynamics in question can be modeled as a Markov chain.

Taking a more foundational perspective, this paper also uses the stochastic-quantum correspondence to show that physical models based on old-fashioned configuration spaces and ordinary forms of probability, combined with stochastic dynamics, can replicate all the empirical predictions of textbook quantum theory—including interference, decoherence, entanglement, noncommutative observables, and wave-function collapse—without relying on the austere and metaphysically opaque Dirac-von Neumann axioms (Dirac 1930, von Neumann 1932). In this alternative approach, a given system moves stochastically along a physical trajectory in a prosaic, classical-looking configuration space. The ingredients of the Hilbert-space formulation, including the wave function, then go the way of the luminiferous aether of 19-century electromagnetism—they are no longer primary or ontological features of the theory.[1]

At the very least, this approach yields a new formulation of quantum theory, one that is based on a picture of stochastic systems evolving in configuration spaces within the framework of ordinary probability theory. This formulation therefore joins a list of ways to formulate quantum systems that include the traditional Hilbert-space formulation (Dirac 1930, von Neumann 1932), the path-integral formulation (Dirac 1933, Feynman 1942, Feynman 1948), and the quasi-probability formulation (Wigner 1932, Moyal 1949). As noted by Feynman (1948), "there is a pleasure in recognizing old things from a new point of view," and "there is always the hope that the new point of view will inspire an idea for the modification of present theories, a modification necessary to encompass present experiments."

The present work is not continuous with earlier efforts to identify a fundamental relationship that connects stochastic processes and quantum theory. The most well-known of these approaches are due to Bopp (1947, 1952, 1953), Fényes (1952), and Nelson (1967, 1985). For a contemporary example, see Friederich (2024). Altogether different are stochastic-collapse models (Ghirardi, Remini, Weber 1986; Bassi, Ghirardi 2003), in which a quantum system's wave function or density matrix is assumed to experience stochastic fluctuations through time.

Section 2 will start with the definition of an *indivisible stochastic process*, along with introducing the key distinction between *divisible* and *indivisible* dynamics. Section 3 will describe the stochastic-quantum correspondence in detail, including the notion of a *division event*. Section 4 will provide a detailed treatment of the measurement process, which will entail introducing the notion of an *emergeable*, and then turn to a larger analysis of the measurement problem and the uncertainty

[1]For a discussion of some of the outstanding problems in the philosophy of quantum theory, see Myrvold (2022). For an extensive analysis of the role of the wave function in quantum theory, see Ney, Albert (2013).

principle. Section 5 will conclude the paper with a brief discussion, which will include identifying a fundamental *category problem* in textbook versions of quantum theory, as well as describe several open questions to be addressed in future work.

2 Indivisible Stochastic Processes

2.1 Basic definitions

An *indivisible stochastic process*[2] will be defined as a model consisting of two basic ingredients: a *configuration space* $\mathcal{C}$; and a dynamical law in the form of a family of *transition maps* $\Gamma_{t \leftarrow t_0}$ that act linearly on probability distributions over $\mathcal{C}$ at times t_0 from some index set, called *conditioning times*, to yield corresponding probability distributions over $\mathcal{C}$ at times t from some possibly distinct index set, called *target times*. The configuration space $\mathcal{C}$ (the kinematics) and the transition maps $\Gamma_{t \leftarrow t_0}$ (the dynamics) will constitute the *fixed* features of the model, whereas the probability distributions will be *contingent* features allowed to vary from one physical instantiation or run of the model to another.

For the purposes of this paper, the set of target times t will usually be assumed to be isomorphic to the real line $\mathbb{R}$, up to a choice of measurement units. The set of conditioning times t_0 will be assumed to contain at least one element, which can be taken to be the "initial time" 0 without loss of generality. Note that the target time t is treated here as a real-valued variable that can be zero, positive, or negative, so there is no assumption of any fundamental breaking of time-reversal invariance. The choice of conditioning times might appear to single out the initial time 0 as a special time, but Subsection 3.7 will show that for systems in sufficiently strong contact with a repeatedly eavesdropping environment, as would be the case for generic macroscopic systems, the initial time 0 will typically be only one of many conditioning times that play a similar role.

The formalism for an indivisible stochastic process is easiest to express in the case in which the system's configuration space $\mathcal{C} \equiv \{1, \ldots, N\}$ has a finite number[3] N of configurations labeled by positive integers $1, \ldots, N$, perhaps under a suitable form of coarse-graining. In that case, the system's *standalone probabilities* at a conditioning time t_0 can be denoted by $p_j(t_0)$, the standalone probabilities at a target time t can be denoted by $p_i(t)$, and the transition maps $\Gamma_{t \leftarrow t_0}$ consist of *conditional probabilities*

$$\Gamma_{ij}(t \leftarrow t_0) \equiv p(i, t | j, t_0), \tag{1}$$

each of which is the conditional probability for the system to be in its ith configuration at the target time t, given that the system is in its jth configuration at the conditioning time t_0. Being

[2] For pedagogical treatments of the theory of stochastic processes, see the textbooks by Rosenblatt (1962), Parzen (1962), Doob (1990), or Ross (1995).

[3] All the formulas ahead can be extended to systems with continuous configuration spaces. For ease of exposition, the finite, discrete case will be assumed going forward. Bear in mind that "finite" can be *extremely* large, and "discrete" can be well below any feasible experimental sensitivity or resolution.

probabilities, these quantities satisfy the usual non-negativity conditions

$$p_j(t_0), p_i(t), \Gamma_{ij}(t \leftarrow t_0) \geq 0, \tag{2}$$

as well as the normalization conditions

$$\sum_{j=1}^{N} p_j(t_0) = \sum_{i=1}^{N} p_i(t) = \sum_{i=1}^{N} \Gamma_{ij}(t \leftarrow t_0) = 1. \tag{3}$$

Then from the law of total probability, or marginalization, $p_i(t) = \sum_{j=1}^{N} p(i,t|j,t_0)p_j(t_0)$, one has the *linear* relationship

$$p_i(t) = \sum_{j=1}^{N} \Gamma_{ij}(t \leftarrow t_0)p_j(t_0), \tag{4}$$

where the standalone probabilities $p_j(t_0)$ at the conditioning time t_0 are assumed to be arbitrary and contingent, and can therefore be freely adjusted without altering the conditional probabilities $\Gamma_{ij}(t \leftarrow t_0)$, which are regarded as fixed features of the model.

Let $p(t_0)$ denote an $N \times 1$ *probability vector* whose entries are given by the standalone probabilities $p_j(t_0)$, $p(t)$ denote the analogous $N \times 1$ probability vector with entries given by $p_i(t)$, and $\Gamma(t \leftarrow t_0)$ denote the $N \times N$ time-dependent *transition matrix* consisting of the conditional probabilities $\Gamma_{ij}(t \leftarrow t_0)$. Then one can naturally recast the linear marginalization relationship (4) in matrix form as

$$p(t) = \Gamma(t \leftarrow t_0)p(t_0). \tag{5}$$

The non-negativity and normalization conditions on the time-dependent transition matrix $\Gamma(t \leftarrow t_0)$ identify it as a *(column) stochastic matrix* for each pair of times t and t_0. On physical grounds, $\Gamma(t \leftarrow t_0)$ will be assumed to satisfy the continuity condition that in the limit $t \to t_0$, it approaches its value $\Gamma(t_0 \leftarrow t_0)$, which will be taken to be the $N \times N$ identity matrix $\mathbb{1} \equiv \mathrm{diag}(1, \ldots, 1)$.

Crucially, the transition matrix $\Gamma(t \leftarrow t_0)$ will *not* be assumed to be 'divisible,' a term that seems to have originated in the research literature in a 2008 paper by Wolf and Cirac (2008) in the context of quantum channels.[4] That is, $\Gamma(t \leftarrow t_0)$ will generically be *indivisible* (Milz, Modi 2021), meaning that for intermediate times t' satisfying $t > t' > t_0$, there will not generally exist a genuinely stochastic matrix $\tilde{\Gamma}(t \leftarrow t')$ satisfying the composition law or *divisibility condition*

$$\Gamma(t \leftarrow t_0) = \tilde{\Gamma}\big(t \leftarrow t'\big)\Gamma\big(t' \leftarrow t_0\big). \tag{6}$$

In particular, the stochastic process based on the transition matrix $\Gamma(t \leftarrow t_0)$ will generically fail to be Markovian, so its dynamical laws will not be iterative over time in the sense of repeated matrix multiplication $\Gamma\Gamma\cdots\Gamma$, and the model will also lack specific dynamical laws describing transitions

[4]Note that this notion of divisibility is conceptually distinct from the much older concept of *infinite divisibility*, which refers to a probability distribution that can be expressed as the probability distribution of a sum of any integer number of independent and identically distributed random variables.

between arbitrarily chosen intermediate times.

For small configuration spaces, it is easy to devise smooth, time-dependent, non-Markovian, indivisible transition matrices. Examples include 2×2 transition matrices of the form

$$\Gamma(t \leftarrow 0) \equiv \begin{pmatrix} f(t) & 1 - f(t) \\ 1 - f(t) & f(t) \end{pmatrix} \tag{7}$$

for $f(t) \equiv \exp\bigl(-t^2/\tau^2\bigr)$, with τ a constant with units of time, or for $f(t) \equiv \cos^2 \omega t$, with ω a constant with units of inverse-time. These two time-dependent transition matrices are provably indivisible, because any matrix $\tilde{\Gamma}(t \leftarrow t')$ satisfying the divisibility condition above would need to have negative entries for at least some pairs of times t and t', and would therefore not be a genuine stochastic matrix.

Next, consider a *random variable* $A(t)$ with (not necessarily distinct) real-valued *magnitudes* $a_1(t), \ldots, a_N(t)$ determined by the system's configuration $i = 1, \ldots, N$ and possibly also depending explicitly on the time t. The *expectation value* $\langle A(t) \rangle$ is then defined as the statistical average of $A(t)$ over the system's standalone probability distribution at t:

$$\langle A(t) \rangle \equiv \sum_{i=1}^{N} a_i(t) p_i(t). \tag{8}$$

One can define various statistical moments of $A(t)$ by appropriate generalizations of this basic definition.

2.2 Markovian and non-Markovian stochastic processes

In general, the dynamical laws of a *non-Markovian stochastic process* consist of a tower of conditional probabilities of arbitrary order:

$$\begin{aligned} &p(i,t) && \text{(zeroth order)}, \\ &p(i,t|j_1,t_1) && \text{(first order)}, \\ &p(i,t|j_1,t_1;j_2,t_2) && \text{(second order)}, \\ &p(i,t|j_1,t_1;j_2,t_2;j_3,t_3) && \text{(third order)}, \end{aligned} \tag{9}$$

and so forth. From these conditional probabilities, one can use the basic rules of probability theory to construct all joint and standalone probabilities at all choices of times, such as, say, three-time joint probabilities

$$p(i_1,t_1;i_2,t_2;i_3,t_3) = p(i_1,t_1|i_2,t_2;i_3,t_3) p(i_2,t_2|i_3,t_3) p(i_3,t_3). \tag{10}$$

Specifying a particular non-Markovian stochastic process uniquely would therefore require providing an infinite amount of information in the form of the tower of arbitrary-order conditional probabilities (9). Moreover, all the joint probabilities that are definable from this tower of arbitrary-

order conditional probabilities would then need to be related to each other by an intricate web of marginalization operations, such as

$$p(i_1, t_1; i_3, t_3) = \sum_{i_2} p(i_1, t_1; i_2, t_2; i_3, t_3). \tag{11}$$

One traditional approach for avoiding these difficulties is to make the *Markov approximation*, which leads to a *Markov process* or *Markovian stochastic process*. According to the Markov approximation, one assumes that any higher-order conditional probability of the form $p(i, t|j_1, t_1; j_2, t_2; \dots)$ is equal to the first-order conditional probability $p(i, t|j_k, t_k)$ for which t_k is the closest conditioning time to the target time t satisfying $t_k < t$.

An indivisible stochastic process represents an alternative approach in which one avoids making the Markov approximation but instead works with *equivalence classes* of non-Markovian processes. In detail, one considers the entire equivalence class of non-Markovian stochastic processes that may differ in their higher-order conditional probabilities but share the same first-order conditional probabilities $p(i, t|j, t_0)$ defined in (1). As will be shown in the work ahead, these first-order conditional probabilities will be enough to give agreement with all the empirical predictions of quantum theory.

To the extent that quantum theory is empirically adequate, the higher-order conditional probabilities are then unobservable in experiments and will be left unspecified in this paper. In particular, probabilities assigned to *whole trajectories*, as constructed from higher-order conditional probabilities in the sense of (10), are then left unspecified as well. The higher-order conditional probabilities of an indivisible stochastic process could, in principle, vary contingently from one set of instantiations or runs of the process to another. Whether there exists some theoretical principle that picks out one set of higher-order or whole-trajectory probabilities from all the various possibilities is a question that will be left to future work.

3 The Stochastic-Quantum Correspondence

3.1 The dictionary

One of the goals of this paper will be to introduce a new and highly general framework for formulating time-dependent transition matrices $\Gamma(t \leftarrow t_0)$, conceptually akin to the Lagrangian or Hamiltonian frameworks for formulating deterministic dynamics for mechanical systems.

For purposes of notational simplicity, the conditioning time t_0 will now be taken to be the "initial time" 0. The starting place will then be to 'solve' the non-negativity condition $\Gamma_{ij}(t \leftarrow 0) \geq 0$ on the individual entries of the transition matrix $\Gamma(t \leftarrow 0)$ by expressing them in the following way:

$$\Gamma_{ij}(t \leftarrow 0) = |\Theta_{ij}(t \leftarrow 0)|^2. \tag{12}$$

Keep in mind that this equation is an entry-by-entry by statement and does not involve the standard

rule for matrix multiplication.[5] Note also that this equation is not a postulate—it is a mathematical identity.

The $N \times N$ matrix $\Theta(t \leftarrow 0)$ introduced here is guaranteed to exist, although it is not unique, so one can view it as metaphorically akin to a "potential" for $\Gamma(t \leftarrow 0)$, in analogy with the relationship between a potential energy and a Newtonian force.[6] Its entries $\Theta_{ij}(t \leftarrow 0)$ could be taken to be the real square roots of the corresponding quantities $\Gamma_{ij}(t \leftarrow 0)$, but they could also include complex numbers, quaternions, or even the elements of a more general algebra. With the eventual goal of reproducing the usual Hilbert-space formalism of quantum theory, this paper will choose $\Theta_{ij}(t \leftarrow 0)$ to involve only the complex numbers at most.

Due to the normalization condition on the transition matrix $\Gamma(t \leftarrow 0)$, the matrix $\Theta(t \leftarrow 0)$ must satisfy the *summation condition*

$$\sum_{i=1}^{N} |\Theta_{ij}(t \leftarrow 0)|^2 = 1. \tag{13}$$

For now, no further conditions, such as unitarity, will be imposed on $\Theta(t \leftarrow 0)$, whose significance will soon become more clear.

Let $e_1, \ldots, e_N$ define the system's *configuration basis*, where e_i has a 1 in its ith entry and 0s in all its other entries. With $\dagger$ as the standard adjoint operation, let

$$P_i \equiv e_i e_i^\dagger = \mathrm{diag}(0, \ldots, 0, \underset{\substack{\uparrow \\ i\text{th entry}}}{1}, 0, \ldots, 0), \tag{14}$$

denote a *configuration projector*, which is an $N \times N$ matrix consisting of a single 1 in its ith row and ith column, and 0s in all its other entries. Letting $\mathrm{tr}(\cdots)$ denote the usual trace, one can then recast the identity $\Gamma_{ij}(t \leftarrow 0) = |\Theta_{ij}(t \leftarrow 0)|^2$ relating the entries of $\Gamma(t \leftarrow 0)$ with the entries of $\Theta(t \leftarrow 0)$ as

$$\boxed{\Gamma_{ij}(t \leftarrow 0) = \mathrm{tr}(\Theta^\dagger(t \leftarrow 0) P_i \Theta(t \leftarrow 0) P_j).} \tag{15}$$

This equation is a new result and will turn out to serve as the basic *dictionary* of the *stochastic-quantum correspondence*. This dictionary translates between the formalism of indivisible stochastic processes, as symbolized by $\Gamma_{ij}(t \leftarrow 0)$ on the left-hand side, and an expansive set of mathematical tools for constructing stochastic dynamics, as embodied by the right-hand side.[7]

[5] This expression for the transition matrix $\Gamma(t \leftarrow 0)$ can be regarded as a factorization of the form $\Gamma(t) = \overline{\Theta(t)} \odot \Theta(t)$, where the overbar denotes complex conjugation and where $\odot$ is the *Schur-Hadamard product* defined for arbitrary $N \times N$ matrices X and Y as entry-wise multiplication: $(X \odot Y)_{ij} \equiv X_{ij} Y_{ij}$ (Shur 1911, Horn 1990).

[6] This nonuniqueness implies a previously unrecognized form of gauge invariance for all quantum systems, in which one changes the individual entries $\Theta_{ij}(t \leftarrow 0)$ by arbitrary, time-dependent phase factors: $\Theta_{ij}(t \leftarrow 0) \mapsto \exp(\theta_{ij}(t)) \Theta_{ij}(t \leftarrow 0)$. These gauge transformations then alter the structure of the resulting Hilbert-space representation ahead, including the dynamics, in such a way that all empirical results remain unchanged. Due to space limitations, nothing more will be discussed in this paper about this novel form of gauge invariance, which is distinct from an altogether different form of highly general gauge invariance introduced by Brown (1999).

[7] Similar-looking formulas appear incidentally in the equations (3)–(6) of Auffeves and Gragnier (2017) as an intermediate step in proving a lemma that the authors use for conceptually different purposes.

3.2 The Hilbert-space representation

The matrix $\Theta(t \leftarrow 0)$ belongs to the set of operators acting on a *Hilbert space*, meaning a complete inner-product space over the complex numbers. More explicitly, $\Theta(t \leftarrow 0)$ picks out a Hilbert space $\mathcal{H} \cong \mathbb{C}^N$ that is isomorphic to the vector space $\mathbb{C}^N$ of $N \times 1$ column vectors $v, w, \ldots$ with complex-valued entries, under the inner product $v^\dagger w$. One therefore arrives at a *Hilbert-space formulation* for constructing highly generic forms of stochastic dynamics.

The linear marginalization relationship (4), $p_i(t) = \sum_{j=1}^N \Gamma_{ij}(t \leftarrow 0) p_j(0)$, between the system's standalone probabilities $p_j(0)$ at the initial time 0 and the standalone probabilities $p_i(t)$ at the target time t can now be recast as

$$p_i(t) = \mathrm{tr}(P_i \rho(t)). \tag{16}$$

Here

$$\rho(t) \equiv \Theta(t \leftarrow 0) \left[\sum_{j=1}^N p_j(0) P_j \right] \Theta^\dagger(t \leftarrow 0) = \Theta(t) \mathrm{diag}(\ldots, p_j(0), \ldots) \Theta^\dagger(t) \tag{17}$$

is an $N \times N$ time-dependent matrix that is positive semidefinite, $\rho(t) \geq 0$, is self-adjoint, $\rho^\dagger(t) = \rho(t)$, has unit trace, $\mathrm{tr}(\rho(t)) = 1$, and is generically non-diagonal. Crucially, notice how the linearity of the marginalization relationship (4) is ultimately responsible for the linearity of the relationship between the matrix $\rho(t)$ and its value $\rho(0)$ at the initial time 0.

Similarly, by substituting $p_i(t) = \mathrm{tr}(P_i \rho(t))$ from (16) into the definition $\langle A(t) \rangle \equiv \sum_{i=1}^N a_i(t) p_i(t)$ of the expectation value of a random variable $A(t)$, one obtains

$$\langle A(t) \rangle = \mathrm{tr}(A(t) \rho(t)). \tag{18}$$

Here $A(t)$ is now understood to be the $N \times N$ time-dependent, diagonal matrix whose entries are the random variable's individual magnitudes $a_1(t), \ldots, a_N(t)$:

$$A(t) \equiv \sum_{i=1}^N a_i(t) P_i = \mathrm{diag}(\ldots, a_i(t), \ldots). \tag{19}$$

In the special case in which the system's standalone probability distribution at the initial time 0 is pure, meaning that one of the system's configurations j is occupied with probability 1, the system's probability vector at the initial time 0 is equal to the jth configuration basis vector e_j, which again has a 1 in its jth entry and 0s in all its other entries. One can then define an $N \times 1$ column vector

$$\Psi(t) \equiv \Theta(t \leftarrow 0) e_j, \tag{20}$$

which is ultimately just the jth column of $\Theta(t \leftarrow 0)$. Due to the summation condition $\sum_{i=1}^N |\Theta_{ij}(t \leftarrow 0)|^2 = 1$ from (13), this column vector $\Psi(t)$ automatically has unit norm according to

$$\sqrt{\Psi^\dagger(t) \Psi(t)} = 1. \tag{21}$$

Moreover, the ith component $\Psi_i(t)$ of $\Psi(t)$ is equal to the specific complex-valued matrix entry $\Theta_{ij}(t \leftarrow 0)$. This component $\Psi_i(t)$ is a purely law-like quantity, in the sense of being just another name for a part of $\Theta(t \leftarrow 0)$, which is itself just a way of encoding the system's dynamical law, as embodied by the system's transition matrix $\Gamma(t \leftarrow 0)$.

It follows from a short calculation that when the purity condition $\Psi(0) = e_j$ holds at the initial time 0, the self-adjoint matrix $\rho(t)$ defined above is rank-one and has factorization

$$\rho(t) = \Psi(t)\Psi^\dagger(t). \tag{22}$$

The probability formula $p_i(t) = \mathrm{tr}(P_i\rho(t))$ from (16) then simplifies to

$$p_i(t) = |\Psi_i(t)|^2, \tag{23}$$

and the formula $\langle A(t)\rangle = \mathrm{tr}(A(t)\rho(t))$ from (18) for the expectation value of a random variable $A(t)$ becomes

$$\langle A(t)\rangle = \Psi^\dagger(t)A(t)\Psi(t). \tag{24}$$

Looking at all these results, one notices a striking resemblance to mathematical objects and formulas that are familiar from textbook quantum theory.[8] Specifically, one sees that $\Theta(t \leftarrow 0)$ plays the role of a *time-evolution operator*, $\rho(t)$ is a *density matrix,* $\Psi(t)$ is a *state vector* or *wave function*, and $A(t)$ represents an *observable*.[9] The probability formulas $p_i(t) = \mathrm{tr}(P_i\rho(t))$ and $p_i(t) = |\Psi_i(t)|^2$ coincide with the *Born rule*, and $\langle A(t)\rangle = \mathrm{tr}(A(t)\rho(t))$ and $\langle A(t)\rangle = \Psi^\dagger(t)A(t)\Psi(t)$ have the same form as the standard expressions for quantum expectation values.

Despite the similarity to expressions found in quantum theory, as well as the appearance of non-diagonal matrices, it is important to keep in mind that the system under investigation here is always fundamentally in a specific configuration $i = 1, \ldots, N$ in its configuration space $\mathcal{C}$ at any given time and that the system's dynamics is completely captured by the transition matrix $\Gamma(t \leftarrow 0)$, whose entries are conditional probabilities $p(i, t|j, 0)$. The mathematical objects $\Theta(t \leftarrow 0)$, $\rho(t)$, $\Psi(t)$, and $A(t)$, despite being extremely useful, do not naturally have direct physical meanings, in part because they are not uniquely defined by $\mathcal{C}$ or by $\Gamma(t \leftarrow 0)$.

3.3 Kraus decompositions

In the most general case, a time-evolution operator $\Theta(t \leftarrow 0)$ may not satisfy any nontrivial constraints apart from the summation condition $\sum_{i=1}^N |\Theta_{ij}(t \leftarrow 0)|^2 = 1$ from (13). It will turn out to be helpful to find alternative ways of representing the $N \times N$ matrix $\Theta(t \leftarrow 0)$ in terms of more tightly constrained mathematical objects.

For $\beta = 1, \ldots, N$, and with P_β the corresponding configuration projector defined in (14), let

[8] For pedagogical treatments of quantum theory, see the textbooks by Griffiths and Schroeter (2018); Townsend (2012); Shankar (1994); Sakurai and Napolitano (2010); and Schumacher and Westmoreland (2010).

[9] Note that for the purposes of this paper, the terms 'operator' and 'matrix' will be used interchangeably, as will the terms 'state vector' and 'wave function.'

$K_\beta(t \leftarrow 0) \equiv \Theta(t \leftarrow 0)P_\beta$ be the $N \times N$ matrix defined to share its βth column with $\Theta(t \leftarrow 0)$, but with 0s in all its other entries:

$$K_{\beta,ij}(t \leftarrow 0) \equiv (\Theta(t \leftarrow 0)P_\beta)_{ij} \equiv \delta_{\beta j}\Theta_{ij}(t \leftarrow 0) = \begin{cases} \Theta_{ij}(t \leftarrow 0) & \text{for } \beta = j, \\ 0 & \text{for } \beta \neq j, \end{cases} \tag{25}$$

where $\delta_{\beta j}$ is the usual Kronecker delta. The summation condition on $\Theta(t \leftarrow 0)$ then becomes the statement that the matrices $K_1(t \leftarrow 0), \ldots, K_N(t \leftarrow 0)$ satisfy the *Kraus identity*:

$$\sum_{\beta=1}^{N} K_\beta^\dagger(t \leftarrow 0)K_\beta(t \leftarrow 0) = \mathbb{1}. \tag{26}$$

These matrices are therefore called *Kraus operators* (Kraus 1971). One can then write the basic relationship $\Gamma_{ij}(t \leftarrow 0) = |\Theta_{ij}(t \leftarrow 0)|^2$ from (12) in an alternative form called a *Kraus decomposition*:

$$\Gamma_{ij}(t \leftarrow 0) = \sum_{\beta=1}^{N} |K_{\beta,ij}(t \leftarrow 0)|^2. \tag{27}$$

Kraus decompositions play a key role in quantum information theory. They provide (non-unique) generalizations of unitary time evolution known as *quantum channels*, or *completely positive trace-preserving (CPTP) maps*, that are needed for some kinds of open quantum systems.

3.4 Unistochastic processes

The existence of a Kraus decomposition for the time-evolution operator $\Theta(t \leftarrow 0)$ is a crucial new result, and it has an important corollary. Specifically, if $\Theta(t \leftarrow 0)$ is not *already* a unitary matrix, then one can *turn it into* a unitary matrix by enlarging or *dilating* the original N-element configuration space $\mathcal{C}$ to one containing at most N^3 configurations. One can then formally regard the original indivisible stochastic process as a subsystem of this dilated stochastic process.

In more detail, one starts by combining the given system's N-element configuration space $\mathcal{C}$ with an ancillary configuration space $\mathcal{C}'$ of some size $N' \leq N^2$ to yield a dilated configuration space given by the Cartesian product $\tilde{\mathcal{C}} = \mathcal{C} \times \mathcal{C}'$, which then has size $\tilde{N} \leq N^3$. By definition, the elements of this dilated configuration space $\tilde{\mathcal{C}}$ take the form of ordered pairs (i, i'), where $i \in \mathcal{C}$ labels the original system's configurations and $i' \in \mathcal{C}'$ labels the configurations of the ancillary system, or *ancilla*, which need not be regarded as physical. The *Stinespring dilation theorem* (Stinespring 1955, Keyl 2002) then implies the existence of an $\tilde{N} \times \tilde{N}$ *unitary* time-evolution operator $\tilde{\Theta}(t \leftarrow 0) = \tilde{U}(t \leftarrow 0)$ whose corresponding $\tilde{N} \times \tilde{N}$ transition matrix $\tilde{\Gamma}(t \leftarrow 0)$ yields the original $N \times N$ transition matrix $\Gamma(t \leftarrow 0)$ by marginalization over the ancilla's configuration i' at time t, for at least some choices of the ancilla's configuration j' at the initial time 0: $\Gamma_{ij}(t \leftarrow 0) = \sum_{i'=1}^{N'} \tilde{\Gamma}_{(i,i')(j,j')}(t \leftarrow 0)$. This fact establishes the inevitability of unitary time evolution in quantum theory.

Again, the ancilla here need not be treated as a physical subsystem in its own right. It is important to keep in mind that any empirical patterns in the observed behavior of the dilated

system that become manifest after formally including the ancilla were already present, if implicitly, before the formal dilation step.

Without any real loss of generality, the preceding arguments imply that one can focus on the case in which the time-evolution operator is unitary,

$$\Theta(t \leftarrow 0) = U(t \leftarrow 0), \tag{28}$$

meaning that

$$U^{\dagger}(t \leftarrow 0) = U^{-1}(t \leftarrow 0). \tag{29}$$

The basic relationship (12) between the system's transition matrix $\Gamma(t \leftarrow 0)$ and the time-evolution operator $\Theta(t \leftarrow 0)$ then becomes

$$\Gamma_{ij}(t \leftarrow 0) = |U_{ij}(t \leftarrow 0)|^2. \tag{30}$$

Equivalently, in dictionary form (15), one has

$$\Gamma_{ij}(t \leftarrow 0) = \mathrm{tr}(U^{\dagger}(t \leftarrow 0) P_i U(t \leftarrow 0) P_j). \tag{31}$$

The system's transition matrix $\Gamma(t \leftarrow 0)$ is then said to be a *unistochastic matrix*. That is, a unistochastic matrix is a square matrix whose individual entries are the modulus-squares of the corresponding entries of a unitary matrix.

Unistochastic matrices were first introduced in 1954 by Horn (1954) , who originally called them 'ortho-stochastic matrices.' The modern term 'unistochastic matrix' was introduced by Thompson in 1989 (Thompson 1989; Nylen, Tam, Uhlig 1993). The term *orthostochastic matrix* now refers to a square matrix whose entries are the modulus-squares of the corresponding entries of a *real orthogonal* matrix.

Every orthostochastic matrix is unistochastic. Importantly, however, the reverse is not generally true, meaning that the complex numbers generically play a necessary role in formulating a unistochastic transition matrix $\Gamma(t \leftarrow 0)$ in terms of a unitary time-evolution operator $U(t \leftarrow 0)$. Even when the complex numbers are not strictly necessary for writing down a unitary time-evolution operator $U(t \leftarrow 0)$, such as if the time-evolution operator can be taken to be real and orthogonal, it is still very convenient to employ the complex numbers for a given Hilbert-space representation, so that one can take advantage of the many useful constructs that show up in standard treatments of quantum theory, like spectral decompositions and self-adjoint symmetry generators.[10]

It follows immediately from the dictionary formula (31) relating $\Gamma(t \leftarrow 0)$ and $U(t \leftarrow 0)$ that every unistochastic transition matrix is *doubly stochastic*, or *bistochastic*, which means that summing

[10]Intriguingly, time-reversal operators include a complex-conjugation operator K that *anticommutes* with i, meaning that $Ki = -iK$, so the three mathematical objects i, K, and iK satisfy $-i^2 = K^2 = (iK)^2 = iK(iK) = 1$. They therefore generate a Clifford algebra isomorphic to the *pseudo-quaternions* (Stueckelberg 1960). In a sense, then, the Hilbert spaces of quantum systems are actually defined not over the complex numbers alone, but over the pseudo-quaternions, although K is not usually used in the definition of observables.

over any of its rows *or* any of its columns always yields 1:

$$\sum_{i=1}^{N} \Gamma_{ij}(t \leftarrow 0) = \sum_{j=1}^{N} \Gamma_{ij}(t \leftarrow 0) = 1. \tag{32}$$

An indivisible stochastic process whose transition matrix $\Gamma(t)$ is a unistochastic matrix will be called a *unistochastic process*.

To provide a simple example, note that every permutation matrix Σ is, in particular, a unitary matrix. Moreover, because the entries Σ_{ij} of a permutation matrix Σ are all 1s and 0s, they are individually invariant when one computes their modulus-squares, so every permutation matrix is *also* a unistochastic matrix. It follows that a discrete, deterministic system whose dynamics is defined by a permutation matrix Σ is a special case of a unistochastic process.[11]

Importantly, one can go in the other direction by expressing any unitary time-evolution operator $U(t \leftarrow 0)$ in terms of a time-dependent transition matrix $\Gamma(t \leftarrow 0)$ on an underlying configuration space $\mathcal{C}$, as noted, for example, in a paper by Korzekwa and Lostaglio (2021). The analysis ahead will explain how to extend this observation into a new and comprehensive correspondence between indivisible stochastic processes and quantum systems, going beyond more elementary approaches that merely embed stochastic processes into a proper subclass of quantum systems.[12]

Assuming a unistochastic process based on a unitary time-evolution operator $U(t \leftarrow 0)$ that is a differentiable function of the time t, one can define a corresponding self-adjoint generator $H(t) = H^{\dagger}(t)$, called the system's *Hamiltonian*, according to

$$H(t) \equiv i\hbar \frac{\partial U(t \leftarrow 0)}{\partial t} U^{\dagger}(t \leftarrow 0). \tag{33}$$

Here, the factor of i ensures that the $N \times N$ matrix $H(t)$ is self-adjoint, and, for present purposes, the *reduced Planck constant* $\hbar$ is a fixed quantity introduced for purposes of measurement units. Ultimately, the specific numerical value of $\hbar$ in any given set of units must be determined empirically by comparison with experiments.

In terms of the Hamiltonian, the system's density matrix $\rho(t)$ then evolves in time according to the *von Neumann equation*,

$$i\hbar \frac{\partial \rho(t)}{\partial t} = [H(t), \rho(t)], \tag{34}$$

its state vector $\Psi(t)$ (if it exists) evolves according to the *Schrödinger equation*,

$$i\hbar \frac{\partial \Psi(t)}{\partial t} = H(t)\Psi(t), \tag{35}$$

[11] Moreover, if δt denotes each discrete time step, n denotes the integer number of time steps, and t denotes a smooth time parameter, then because real-valued powers of a permutation matrix Σ are guaranteed to be unitary, the formula $\Gamma_{ij}(n\,\delta t + t \leftarrow n\,\delta t) \equiv |(\Sigma^{t/\delta t})_{ij}|^2$ defines a unistochastic matrix that analytically interpolates the original discrete, deterministic process to a smooth, unistochastic process.

[12] For instance, the *classical-to-classical channels* defined in the treatment by Wilde (2017) consist of turning stochastic matrices into a proper subclass of quantum channels that map diagonal density matrices into diagonal density matrices.

and its expectation values $\langle A(t)\rangle$ evolve according to the *Ehrenfest equation*,

$$\frac{d\langle A(t)\rangle}{dt} = \frac{i}{\hbar}\mathrm{tr}([H(t), A(t)]\rho(t)) + \left\langle \frac{\partial A(t)}{\partial t} \right\rangle. \tag{36}$$

Note that the brackets $[X, Y]$ that naturally show up in these equations are genuine *commutators* $XY - YX$, not Poisson brackets, and involve products of non-diagonal matrices that do not generally commute with each other under matrix multiplication.

The emergence of these famous equations from a physical model based on a stochastically evolving trajectory in a configuration space $\mathcal{C}$ is a surprising new result.

3.5 Interference

The appearance of the Schrödinger equation in the previous section is an important signal that the dictionary (15) is more than just a tool for using Hilbert-space methods to craft highly general forms of stochastic dynamics. It also suggests that indivisible stochastic processes have the resources to replicate the features of quantum theory more broadly.

As another hint pointing in this direction, one starts by noting again that an arbitrary time-dependent transition matrix $\Gamma(t \leftarrow 0)$ is generically *indivisible*, in the sense that it does not satisfy the divisibility condition discussed in Section 2 at arbitrary times. To see what goes wrong with divisibility, suppose that at some time t', the transition matrix $\Gamma(t' \leftarrow 0)$ has a matrix inverse $\Gamma^{-1}(t' \leftarrow 0)$, and define a new $N \times N$ matrix $\tilde{\Gamma}(t \leftarrow t')$ according to

$$\tilde{\Gamma}\big(t \leftarrow t'\big) \equiv \Gamma(t \leftarrow 0)\Gamma^{-1}\big(t' \leftarrow 0\big). \tag{37}$$

As an immediate consequence, one then has

$$\Gamma(t \leftarrow 0) = \tilde{\Gamma}\big(t \leftarrow t'\big)\Gamma\big(t' \leftarrow 0\big), \tag{38}$$

which resembles the divisibility condition (6). However, it follows from an elementary theorem of linear algebra that the inverse of a stochastic matrix can only be stochastic if both matrices are permutation matrices and, therefore, do not involve nontrivial probabilities. Hence, the matrix $\tilde{\Gamma}(t \leftarrow t')$ defined above is not generically stochastic, so one does not obtain a genuine form of divisibility.

There is an alternative—and far-reaching—way to understand the generic indivisibility of a time-dependent transition matrix $\Gamma(t \leftarrow 0)$. To this end, suppose that $\Gamma(t \leftarrow 0)$ is unistochastic, with unitary time-evolution operator $U(t \leftarrow 0)$. Then, for any two times t and t', one can define a *relative* time-evolution operator

$$U\big(t \leftarrow t'\big) \equiv U(t \leftarrow 0)U^{\dagger}\big(t' \leftarrow 0\big), \tag{39}$$

which is guaranteed to be unitary and which yields the composition law

$$U(t \leftarrow 0) = U\big(t \leftarrow t'\big)U\big(t' \leftarrow 0\big). \tag{40}$$

Note that this composition law does not extend to the transition matrix $\Gamma(t \leftarrow 0)$ due to cross terms.

With

$$\Gamma_{kj}\big(t' \leftarrow 0\big) \equiv |U_{kj}\big(t' \leftarrow 0\big)|^2 \tag{41}$$

defined as usual, and defining

$$\Gamma_{ik}\big(t \leftarrow t'\big) \equiv |U_{ik}\big(t \leftarrow t'\big)|^2, \tag{42}$$

which is manifestly unistochastic, one sees that the discrepancy between the true transition matrix $\Gamma(t \leftarrow 0)$ and its would-be division $\Gamma(t \leftarrow t')\Gamma(t' \leftarrow 0)$ is given by

$$\Gamma_{ij}(t \leftarrow 0) - \big[\Gamma\big(t \leftarrow t'\big)\Gamma\big(t' \leftarrow 0\big)\big]_{ij} = \sum_{k \neq l} \overline{U_{ik}(t \leftarrow t')\Psi_k(t')}U_{il}\big(t \leftarrow t'\big)\Psi_l\big(t'\big), \tag{43}$$

where $\Psi(t') \equiv U(t' \leftarrow 0)e_j$ is the system's state vector at the time t', in keeping with the general definition of state vectors discussed in Subsection 3.2, and where the overbar notation denotes complex conjugation.

Remarkably, the right-hand side of (43) gives the general mathematical formula for quantum interference, despite the absence of manifestly quantum-theoretic assumptions. One sees from this analysis that interference is a direct consequence of the stochastic dynamics not generally being divisible. More precisely, interference is nothing more than a generic discrepancy between the *actual* indivisible stochastic dynamics and a *heuristic-approximate* divisible stochastic dynamics. Interference encodes the fact that the underlying stochastic dynamics is indivisible, despite the way that unitary time-evolution operators look *superficially* divisible.

In particular, quantum-mechanical interference does not imply that matter has a physically wavelike nature, contrary to frequent claims in textbook treatments (Feynman et al. 1965). Indeed, from the perspective of the present discussion, the notion that quantum-mechanical interference ever necessitated assigning matter a physically wavelike quality was merely an unfortunate accident of history, arising from the fact that many early empirical examples of interference in quantum systems happened to resemble the behavior of interfering waves propagating in three-dimensional physical space.

These historical examples were clearly special cases. *Multiparticle* systems have Schrödinger waves that propagate through high-dimensional configuration spaces, as Schrödinger himself noted in his early work on wave mechanics (Schrödinger 1926). For more abstract systems, like *qubits*, there fail to exist continuous configuration spaces for Schrödinger waves altogether.

This new way of thinking about quantum-mechanical interference has implications for the interpretation of the famous *double-slit experiment*. Recall that in the double-slit experiment, an emitter sends one particle at a time toward a wall with two slits in it, and a detection screen on

the other side of the wall records the particle's eventual landing site. In the usual 'classical' description of the experiment, one asks first which slit the particle enters, and then, *conditioning on the answer*, one then *restarts* the dynamics with that slit as the new initial condition. Over many repetitions of the experiment, the detection screen records a statistical blend from the landing sites of particles passing through the *upper* slit and particles passing through the *lower* slit. In the case of quantum-mechanical particles like electrons, however, one instead finds that the landing sites form a 'wavelike' interference pattern, and the conclusion is supposedly that each particle is really a Schrödinger wave of some kind or that the particle fails to go through one slit or the other.[13]

According to the approach laid out in this paper, the particle always fundamentally has a single location and is never in both holes simultaneously. The final interference pattern on the detection screen is not due to any purported physical reality of Schrödinger waves, but due to the generic indivisibility of time evolution for quantum systems. One cannot divide up the particle's evolution into, firstly, its transit from the emitter to the slits, and then, secondly, conditioned on which slit the particle enters, the particle's transit from the slits to the detection screen. The interference that shows up in the double-slit experiment may be surprising, but that is only because indivisible stochastic dynamics can be highly unintuitive. In the historical absence of a sufficiently comprehensive framework for describing indivisible stochastic dynamics, it was difficult to recognize just how unintuitive such dynamics could be or what sorts of empirical appearances it could produce.

In response to this last point, one might suggest that Schrödinger waves nonetheless offer a superior means of explaining why the double-slit experiment yields the results that it does. Unfortunately, such hopes are dashed as soon as one considers sending in *two* particles on each run of the experiment. A two-particle system's Schrödinger wave evolves in a *six-dimensional* configuration space, which is arguably not more physically transparent than indivisible stochastic dynamics. Indeed, where are the slits supposed to be *located* in this six-dimensional configuration space?

Of course, if one regards the quantum-mechanical particles that make up matter as arising more fundamentally from underlying *quantum fields*, then the wavelike properties of those quantum fields ensure that particles of matter have wavelike properties as well and therefore exhibit a *wave-particle duality*. That said, there is nothing about the analysis of the double-slit experiment alone that calls for positing quantum fields. The necessity of quantum field theory comes from other theoretical and empirical considerations.[14] One should also keep in mind that quantum fields are conceptually distinct from Schrödinger waves.

3.6 Implications of interference

The fact that interference shows up in a sufficiently generic stochastic model means that relative phase factors in state vectors have clear empirical signatures, even in the absence of the usual axioms

[13]The exposition by Feynman, Leighton, and Sands (1965) ends up at precisely such a conclusion: "It is *not* true that the electrons go *either* through hole 1 or hole 2." [Emphasis in the original.] This conclusion, however, does not logically follow from the empirical appearance of interference effects, but also implicitly depends on the hidden assumption that the behavior of an electron in a double-slit experiment can be described by divisible dynamics.

[14]For a modern motivation, see the textbook by Weinberg (1996).

of textbook quantum theory. These empirical manifestations of relative phases are strong evidence that it should be possible to carry out measurements on a much wider set of observables than those that are represented by diagonal matrices in an indivisible stochastic process's configuration basis. Indeed, Subsection 4.2 will show that non-diagonal, self-adjoint matrices will turn out to be candidate observables as well.

Thinking more broadly, this overall analysis means that if one is given an indivisible stochastic process, then there will generically be a quantitative discrepancy between the system's actual behavior—as predicted theoretically or measured empirically—and predictions made for the system based on a heuristic-approximate divisible or Markovian approximation to the system's stochastic dynamics. Again, this discrepancy is precisely interference.

One way to understand this discrepancy is to note that under a divisibility approximation, one can assign definite probabilities to each of the system's possible trajectories by iteratively applying transition matrices, according to the composition law $\Gamma(t \leftarrow 0) = \Gamma(t \leftarrow t')\Gamma(t' \leftarrow 0)$ from (6). Iteratively applying transition matrices is not generically possible for indivisible stochastic processes, which do not assign unique probabilities to whole trajectories, as explained in Subsection 2.2.

In the Hilbert-space formulation of an indivisible stochastic process, one can nonetheless assign complex-valued quantities called *amplitudes* to the system's possible trajectories, using the fact that unitary time-evolution operators *can* be composed iteratively, $U(t \leftarrow 0) = U(t \leftarrow t')U(t' \leftarrow 0)$, as in (40). These amplitudes form the conceptual basis for the *path-integral formulation* of quantum theory (Dirac 1933, Feynman 1942, Feynman 1948). From the standpoint of the stochastic-quantum correspondence, which gives an alternative formulation of quantum theory, the fact that these amplitudes 'interfere' with each other does not mean that they all physically occur in some sort of literal superposition or that the system simultaneously takes all such paths in reality, but is merely an artifact of the indivisible dynamics of the underlying indivisible stochastic process.

Collectively, the foregoing observations imply that interference is not unique to quantum systems but should arise in a much broader set of physical circumstances. Indeed, given any probabilistically evolving system with indivisible or non-Markovian dynamics, one should now be able to interpret any discrepancies between the behavior of such a system and the behavior of a heuristic-approximate divisible or Markovian approximation as manifestations of interference. As a concrete prediction, one could therefore imagine experimentally measuring interference effects for essentially any system that can be modeled using indivisible or non-Markovian stochastic dynamics.

3.7 Division events and the Markov approximation

Why do discrete-time Markov chains provide such a good approximation to so many stochastic processes in the real world? One intuitively reasonable explanation is that when a system is not isolated from a noisy and intrusive environment, delicate correlations from one time to another 'wash out' over short time scales as those correlations leak out into the environment.

Deriving this intuitive picture from first principles in a more precise way might appear to be a difficult task. Indeed, such a derivation would seem to require finding a more general framework for

describing a non-Markovian process and then showing that such a process becomes approximately Markovian in the appropriate physical circumstances. Fortunately, this paper provides just such a framework.

To set things up, one starts by introducing a composite system $\mathcal{SE}$ consisting of a *subject system* $\mathcal{S}$ together with an *environment* $\mathcal{E}$. The configurations of the subject system's configuration space $\mathcal{C}_{\mathcal{S}}$ will be labeled by $i = 1, \ldots, N$, and the configurations of the environment's configuration space $\mathcal{C}_{\mathcal{E}}$ will be labeled by $e = 1, \ldots, M$, where $M \geq N$. The configuration space of the composite system is then the Cartesian product[15] $\mathcal{C}_{\mathcal{SE}} = \mathcal{C}_{\mathcal{S}} \times \mathcal{C}_{\mathcal{E}}$, meaning that each element of $\mathcal{C}_{\mathcal{SE}}$ is a simple ordered pair of the form (i, e). One then singles out N configurations of the environment by labeling them as $e(1), \ldots, e(N)$.

For the dynamics, suppose for simplicity that the composite system evolves according to an overall unistochastic transition matrix with individual entries

$$\Gamma^{\mathcal{SE}}_{ie,i_0e_0}(t \leftarrow 0) = |U^{\mathcal{SE}}_{ie,i_0e_0}(t \leftarrow 0)|^2. \tag{44}$$

Furthermore, suppose that the subject system and the environment interact up to a time $t' > 0$ in such a way that they end up with joint probabilities of the form

$$p^{\mathcal{SE}}_{i'e'}(t') = p^{\mathcal{S}}_{i'}(t')\delta_{e'e(i')}. \tag{45}$$

(Note the appearance of a Kronecker delta here.) This formula describes an idealized *statistical correlation* between the configuration i' of the subject system at t' and the corresponding configuration $e(i')$ of the environment.

If there is to be any possibility of the two subsystems evolving independently for times $t > t'$ after the interaction has concluded, then it should be possible to factorize the composite system's relative time-evolution operator $U^{\mathcal{SE}}(t \leftarrow t')$ between the two subsystems for $t > t'$ as the following tensor product:[16]

$$U^{\mathcal{SE}}_{ie,i'e'}(t \leftarrow t') = U^{\mathcal{S}}_{ii'}(t \leftarrow t')U^{\mathcal{E}}_{ee'}(t \leftarrow t') \quad \text{for } t > t'. \tag{46}$$

In light of the Born rule, as derived in Subsection 3.2, the joint probabilities $p^{\mathcal{SE}}_{i'e'}(t')$ correspond to a wave function[17]

$$\Psi^{\mathcal{SE}}_{i'e'}(t') = \Psi^{\mathcal{S}}_{i'}(t')\delta_{e'e(i')}. \tag{47}$$

The composite system's wave function at later times $t > t'$ after the interaction is therefore given in terms of the relative time-evolution operator $U^{\mathcal{SE}}(t \leftarrow t')$ according to

[15] The right-hand side of this equation is indeed a Cartesian product, not a tensor product, because this equation is a statement about the composite system's configuration space, not about its dynamics or Hilbert-space representation.

[16] Note the natural appearance of a tensor product here, $U^{\mathcal{SE}}(t \leftarrow t') = U^{\mathcal{S}}(t \leftarrow t') \otimes U^{\mathcal{E}}(t \leftarrow t')$, because this statement refers to the composite system's dynamics in the system's Hilbert-space representation.

[17] If necessary, one can easily write down idealized examples of unitary time-evolution operators for the composite system that produce this wave function. For instance, one could use $U^{\mathcal{SE}}(t' \leftarrow 0) \equiv \sum_{i'} P^{\mathcal{S}}_{i'} \otimes R^{\mathcal{E}}_{e(i')}$, where $P^{\mathcal{S}}_{i'}$ is the i'th configuration projector for the subject system and where $R^{\mathcal{E}}_{e(i')}$ is a unitary transformation that takes the environment's initial configuration to the configuration $e(i')$.

$$\Psi^{\mathcal{SE}}_{ie}(t) = \sum_{i',e'} U^{\mathcal{SE}}_{ie,i'e'}\big(t \leftarrow t'\big)\Psi^{\mathcal{SE}}_{i'e'}\big(t'\big) = \sum_{i'} U^{\mathcal{S}}_{ii'}\big(t \leftarrow t'\big)\Psi^{\mathcal{S}}_{i'}\big(t'\big)U^{\mathcal{E}}_{ee(i')}\big(t \leftarrow t'\big). \tag{48}$$

From the Born rule (23), one sees that the joint probabilities for $t > t'$ are given by

$$p^{\mathcal{SE}}_{ie}(t) = |\Psi^{\mathcal{SE}}_{ie}(t)|^2. \tag{49}$$

Carrying out an ordinary marginalization over the configuration e of the environment and invoking the unitarity of the environment's relative time-evolution operator $U^{\mathcal{E}}(t \leftarrow t')$, one obtains the standalone probabilities $p^{\mathcal{S}}_i(t)$ for the subject system alone for $t > t'$:

$$\begin{aligned} p^{\mathcal{S}}_i(t) &= \sum_e p^{\mathcal{SE}}_{ie}(t) = \sum_{i'_1,i'_2} \overline{U^{\mathcal{S}}_{ii'_1}(t \leftarrow t')\Psi^{\mathcal{S}}_{i'_1}(t')}U^{\mathcal{S}}_{ii'_2}\big(t \leftarrow t'\big)\Psi^{\mathcal{S}}_{i'_2}\big(t'\big)\sum_e \overline{U^{\mathcal{E}}_{ee(i'_1)}(t \leftarrow t')}U^{\mathcal{E}}_{ee(i'_2)}\big(t \leftarrow t'\big) \\ &= \sum_{i'} |U^{\mathcal{S}}_{ii'}\big(t \leftarrow t'\big)|^2|\Psi^{\mathcal{S}}_{i'}\big(t'\big)|^2, \end{aligned} \tag{50}$$

where, again, the overbar notation denotes complex conjugation.

Taking the limit $t \to t'$ and referring back to the Born rule again, one sees that the subject system's standalone probabilities at the time t' are

$$p^{\mathcal{S}}_{i'}\big(t'\big) = |\Psi^{\mathcal{S}}_{i'}\big(t'\big)|^2. \tag{51}$$

One also sees from the last line of the calculation above that, as in Subsection 3.5, one can identify

$$\Gamma^{\mathcal{S}}_{ii'}\big(t \leftarrow t'\big) \equiv |U^{\mathcal{S}}_{ii'}\big(t \leftarrow t'\big)|^2. \tag{52}$$

Hence, one ends up with a genuinely linear relationship that precisely mirrors the linear marginalization formula (4) introduced in Section 2, with t' now effectively serving as a new 'initial time':

$$p^{\mathcal{S}}_i(t) = \sum_{i'} \Gamma^{\mathcal{S}}_{ii'}\big(t \leftarrow t'\big)p^{\mathcal{S}}_{i'}\big(t'\big). \tag{53}$$

Applying the original linear marginalization formula (4) from the actual initial time 0 to the time t', one also has the equation

$$p^{\mathcal{S}}_{i'}\big(t'\big) = \sum_j \Gamma^{\mathcal{S}}_{i'j}\big(t' \leftarrow 0\big)p^{\mathcal{S}}_j(0). \tag{54}$$

Combining these results immediately yields

$$p^{\mathcal{S}}_i(t) = \sum_j \Gamma^{\mathcal{S}}_{ij}(t \leftarrow 0)p^{\mathcal{S}}_j(0), \tag{55}$$

where $\Gamma^{\mathcal{S}}(t \leftarrow 0)$ is a transition matrix that is manifestly *divisible* at t':

$$\Gamma^{\mathcal{S}}(t \leftarrow 0) \equiv \Gamma^{\mathcal{S}}\big(t \leftarrow t'\big)\Gamma^{\mathcal{S}}\big(t' \leftarrow 0\big). \tag{56}$$

Thus, the interaction between the subject system $\mathcal{S}$ and the environment $\mathcal{E}$ up to the time t' has led to a transition matrix $\Gamma^{\mathcal{S}}(t \leftarrow 0)$ for the subject system that is divisible at t', which has become a valid conditioning time.

It is therefore natural to refer to the new conditioning time t' as a *division event*. An important corollary is that the initial time 0 is not a unique time but is instead only one of many division events inevitably experienced by a system in sufficiently strong contact with a repeatedly eavesdropping environment, in the sense that the interactions with the environment lead to correlations that look approximately like those in the formula (45) for $p^{\mathcal{S}\mathcal{E}}_{i'e'}(t')$ above.[18] Division events will play a crucial role going forward.

Suppose that these kinds of division events can be approximated as occurring regularly over a characteristic time scale δt. Suppose, moreover, that the unistochastic dynamics is homogeneous in time, in the sense that $U^{\mathcal{S}}(t + \delta t \leftarrow t) = U^{\mathcal{S}}(\delta t \leftarrow 0)$ for all times t. Then the subject system's transition matrix after any integer number $n \geq 1$ of time steps δt is given by

$$\Gamma^{\mathcal{S}}(n\,\delta t \leftarrow 0) = (\Gamma^{\mathcal{S}})^n, \tag{57}$$

where

$$\Gamma^{\mathcal{S}}_{ij} \equiv |U^{\mathcal{S}}_{ij}(\delta t \leftarrow 0)|^2. \tag{58}$$

The stochastic dynamics therefore takes the form of a discrete-time Markov chain. This analysis provides a theoretical explanation for the ubiquity of Markovian stochastic dynamics in so many real-world cases, and represents another new result.

In a sense, division events represent a kind of spontaneous breaking of time-translation symmetry due to interactions between a given system and its environment. Like other forms of spontaneous symmetry breaking, division events therefore transcend the line between what are fixed aspects of the laws and what are contingencies.[19]

3.8 Decoherence

Had the environment not interacted with the subject system, then the subject system's density matrix $\rho^{\mathcal{S}}(t')$ at the time t' would have generically been non-diagonal, in accordance with the

[18] Although generically always approximate, division events will become nearly exact when the environment is sufficiently macroscopic, for precisely the same reasons that decoherence becomes nearly exact in such cases. Any resulting discrepancies in the effective stochastic laws will therefore be minuscule in real-world cases. These tiny discrepancies in the effective laws for subsystems are inevitable in all no-collapse formulations or interpretations of quantum theory.

[19] The author would like to thank an anonymous reviewer for requesting clarification on this point.

general definition (17) provided in Subsection 3.2:

$$\rho^{\mathcal{S}}\big(t'\big) = U^{\mathcal{S}}\big(t' \leftarrow 0\big) \left[\sum_j p_j(0) P_j \right] U^{\mathcal{S}\dagger}\big(t' \leftarrow 0\big). \tag{59}$$

By contrast, suppose that the environment indeed interacts with the subject system to produce a division event at t', as defined in the previous subsection. In that case, the standalone probability $p_i^{\mathcal{S}}(t)$ for the subject system to occupy its ith configuration at $t > t'$ is given by the linear marginalization relationship (53), which can be written instead as

$$p_i^{\mathcal{S}}(t) = \mathrm{tr}(P_i \rho^{\mathcal{S}}(t)), \tag{60}$$

where

$$\rho^{\mathcal{S}}(t) \equiv U^{\mathcal{S}}\big(t \leftarrow t'\big) \rho^{\mathcal{S}}\big(t'\big) U^{\mathcal{S}\dagger}\big(t \leftarrow t'\big), \tag{61}$$

and where, in turn,

$$\rho^{\mathcal{S}}\big(t'\big) \equiv \sum_{i'} p_{i'}^{\mathcal{S}}\big(t'\big) P_{i'}^{\mathcal{S}} = \mathrm{diag}(\dots, p_{i'}^{\mathcal{S}}\big(t'\big), \dots), \tag{62}$$

which is diagonal.

On comparing the two expressions above for the subject system's density matrix $\rho(t')$ at t', one sees that the interaction with the environment has effectively eliminated the off-diagonal entries, or *coherences*, in the subject system's density matrix. This phenomenon is called *decoherence*, and the foregoing analysis makes clear that decoherence is nothing more than the mundane leakage of correlations into the environment when viewed through the lens of the Hilbert-space formulation.

This analysis also sheds new light on the meaning of coherences in density matrices, as well as on *superpositions* in state vectors, where superpositions are related to coherences in the case of a rank-one density matrix through the formula $\rho_{i_1 i_2}(t) = \Psi_{i_1}(t)\overline{\Psi_{i_2}(t)}$, in accordance with the relationship (22) between state vectors and density matrices discussed in Subsection 3.2. From the standpoint of this analysis, superpositions and coherences are merely indications that one is catching a given system when it is in the midst of an indivisible stochastic process, between division events, rather than implying that the system is literally in 'multiple states at once.' In other words, coherences and superpositions are mathematical artifacts of the fundamental indivisibility of the underlying stochastic process, when represented using a *superficially* divisible unitary time-evolution operator.

These results may also help explain why the precise connection between quantum theory and stochastic processes has historically remained unclear for so long. If one assumes a Markov approximation, as is often the case in the research literature on stochastic processes, then coherences and superposition do not show up, meaning that density matrices remain diagonal, state vectors remain trivial, and nontrivial unistochastic dynamics cannot arise.

3.9 Entanglement

Consider next a composite system $\mathcal{AB}$ consisting of a pair of subsystems $\mathcal{A}$ and $\mathcal{B}$. Suppose that the two subsystems do not interact from the initial time 0 up to some later time $t' > 0$, but then begin interacting at t'.

For times t between 0 and t', the absence of interactions means that the composite system's transition matrix $\Gamma^{\mathcal{AB}}(t \leftarrow 0)$ factorizes into the tensor product of a transition matrix $\Gamma^{\mathcal{A}}(t \leftarrow 0)$ for $\mathcal{A}$ and a separate transition matrix $\Gamma^{\mathcal{B}}(t \leftarrow 0)$ for $\mathcal{B}$:

$$\Gamma^{\mathcal{AB}}(t \leftarrow 0) = \Gamma^{\mathcal{A}}(t \leftarrow 0) \otimes \Gamma^{\mathcal{B}}(t \leftarrow 0) \quad \text{for } 0 \leq t < t'. \tag{63}$$

Starting at the time t', however, the composite system's transition matrix $\Gamma^{\mathcal{AB}}(t \leftarrow 0)$, which encodes cumulative statistical information and therefore correlations, will fail to tensor-factorize between the two subsystems, in the sense that

$$\Gamma^{\mathcal{AB}}(t \leftarrow 0) \neq \Gamma^{\mathcal{A}}(t \leftarrow 0) \otimes \Gamma^{\mathcal{B}}(t \leftarrow 0) \quad \text{for } t > t' \tag{64}$$

for any possible transition matrices $\Gamma^{\mathcal{A}}(t \leftarrow 0)$ and $\Gamma^{\mathcal{B}}(t \leftarrow 0)$ that properly capture the respective dynamics of the two subsystems. (It is worth noting that this loss of tensor-factorization gives a highly general, model-independent way to *define* an interaction.) Even if the two subsystems have a notion of localizability in space and are eventually placed at a large separation distance at some time $t > t'$, the composite system's transition matrix will still fail to tensor-factorize between the two subsystems, thereby leading to the appearance of what looks like nonlocal stochastic dynamics across that separation distance.

However, if the composite system exhibits a division event at some later time $t'' > t'$, perhaps due to interactions between one of the subsystems and the larger environment, as spelled out in Subsection 3.7, then the composite system's transition matrix $\Gamma^{\mathcal{AB}}(t \leftarrow 0)$ will divide at t'':

$$\Gamma^{\mathcal{AB}}(t \leftarrow 0) = \Gamma^{\mathcal{AB}}\big(t \leftarrow t''\big)\Gamma^{\mathcal{AB}}\big(t'' \leftarrow 0\big) \quad \text{for } t > t'' > t'. \tag{65}$$

If the two subsystems $\mathcal{A}$ and $\mathcal{B}$ do not interact with each other after t', then the *relative* transition matrix $\Gamma^{\mathcal{AB}}(t \leftarrow t'')$ appearing here will tensor-factorize between them,

$$\Gamma^{\mathcal{AB}}\big(t \leftarrow t''\big) = \Gamma^{\mathcal{A}}\big(t \leftarrow t''\big) \otimes \Gamma^{\mathcal{B}}\big(t \leftarrow t''\big), \tag{66}$$

so the two subsystems will cease exhibiting what had looked like nonlocal stochastic dynamics.

This analysis precisely captures the quantum-theoretic notion of *entanglement*, without any invocation of a Hilbert-space picture. Systems that interact with each other start to exhibit what superficially appears to be a nonlocal kind of stochastic dynamics, even if the systems are moved far apart in physical space, and decoherence by the environment effectively causes a breakdown in that apparent dynamical nonlocality.

Due to the stochastic and non-Markovian nature of the laws in this new formulation of quantum

theory, the precise nature of this apparent dynamical nonlocality is an extremely subtle matter. Relevant questions concerning locality and causation will be treated in detail in future work.

4 Measurements

4.1 Emergeables

The preceding sections have shown that an indivisible stochastic process—that is, a physical model with kinematics based on a configuration space and dynamics based on a suitably non-Markovian stochastic law—is capable of accounting for signature features of quantum theory like superposition, interference, decoherence, and entanglement. In addition, the Hilbert-space side of the dictionary (15) contains many expressions and equations that are identical to those found in quantum theory.

However, an actual quantum system also includes observables beyond the random variables introduced in Section 2—that is, beyond the narrow class of observables that are represented by diagonal matrices. Indeed, the existence of noncommuting observables represented by self-adjoint matrices that are non-diagonal is another hallmark feature of quantum theory.

Remarkably, an indivisible stochastic process will generically contain such observables as well. Specifically, the next subsection will establish that non-diagonal, self-adjoint matrices represent candidate observables that naturally satisfy the usual probabilistic rules of quantum theory, including the Born rule, all without the need to introduce any new fundamental axioms. In so doing, the analysis ahead will demonstrate that the dictionary (15) is not merely a tool for studying a broad class of stochastic processes, but defines a comprehensive stochastic-quantum correspondence.

These non-diagonal observables therefore resemble random variables in some ways but represent emergent patterns in the overall stochastic dynamics for measurement processes and do not have a transparent interpretation at the level of the underlying configuration space $\mathcal{C}$. These observables will therefore be called *emergeables*. This terminology is intended for contrast with the system's genuine random variables, which could be called *beables*—that is, 'be-ables'—to invoke a term coined by Bell (1973) to refer to observables that express how a system can physically *be*, ontologically speaking.

There is a sense in which emergeables are not an entirely new idea, but are similar to emergent physical properties like temperatures or pressures that likewise do not have a clear meaning at the level of a system's fine-grained states. A somewhat more closely related notion appears in Niels Bohr's famous reply (Bohr 1935, Bell 1971) to the *Einstein-Podolsky-Rosen paradox* (Einstein, Podolsky, Rosen 1935), in which Bohr describes emergent observables that show up in measurement interactions. These sorts of emergent observables also play a key role in the *de Broglie-Bohm formulation*, or *Bohmian mechanics* (Bell 1982, Daumer et al. 1996), in which they account for observables other than particle positions.

4.2 The measurement process

With all the requisite conceptual background now in place, one can proceed to model the measurement of a generic observable as a physical process. To start, consider a composite system $\mathcal{SDE}$ consisting of three subsystems that will be called a *subject system* $\mathcal{S}$, a *measuring device* $\mathcal{D}$, and an *environment* $\mathcal{E}$. Note that one of the additional goals ahead will be to identify the criteria for a subsystem like $\mathcal{D}$ to be regarded as a genuine measuring device in the first place.

Focusing momentarily on the subject system $\mathcal{S}$, consider an $N \times N$ self-adjoint matrix

$$\tilde{A}^{\mathcal{S}} = \tilde{A}^{\mathcal{S}\dagger}, \tag{67}$$

which may or may not be one of the subject system's diagonal random variables.[20] For example, $\tilde{A}^{\mathcal{S}}$ could be an emergeable like those introduced in the previous subsection.

By the *spectral theorem*, $\tilde{A}^{\mathcal{S}}$ has a *spectral decomposition* of the form

$$\tilde{A}^{\mathcal{S}} = \sum_{\alpha} \tilde{a}_{\alpha} \tilde{P}^{\mathcal{S}}_{\alpha}, \tag{68}$$

where $\tilde{a}_{\alpha}$ are the eigenvalues of $\tilde{A}^{\mathcal{S}}$ and where $\tilde{P}^{\mathcal{S}}_{\alpha}$ are its eigenprojectors. These eigenprojectors $\tilde{P}^{\mathcal{S}}_{\alpha}$ are not generically diagonal, but they nonetheless satisfy the mutual exclusivity condition

$$\tilde{P}^{\mathcal{S}}_{\alpha} \tilde{P}^{\mathcal{S}}_{\alpha'} = \delta_{\alpha\alpha'} \tilde{P}^{\mathcal{S}}_{\alpha} \tag{69}$$

and the completeness relation

$$\sum_{\alpha} \tilde{P}^{\mathcal{S}}_{\alpha} = \mathbb{1}^{\mathcal{S}}, \tag{70}$$

where $\mathbb{1}^{\mathcal{S}}$ is the identity matrix for the subject system. These eigenprojectors therefore constitute a *projection-valued measure (PVM)*. Letting $\tilde{e}^{\mathcal{S}}_{\alpha}$ be the corresponding orthonormal basis, one has

$$\tilde{e}^{\mathcal{S}\dagger}_{\alpha} \tilde{e}^{\mathcal{S}}_{\alpha'} = \delta_{\alpha\alpha'}, \qquad \tilde{e}^{\mathcal{S}}_{\alpha} \tilde{e}^{\mathcal{S}\dagger}_{\alpha} = \tilde{P}^{\mathcal{S}}_{\alpha}. \tag{71}$$

If $\tilde{A}^{\mathcal{S}}$ happens to be one of the subject system's random variables, or beables, then the eigenvalues $\tilde{a}_{\alpha}$ are its magnitudes, and the eigenprojectors $\tilde{P}^{\mathcal{S}}_{\alpha}$ are the system's configuration projectors. If $\tilde{A}^{\mathcal{S}}$ is instead an emergeable, then $\tilde{a}_{\alpha}$ and $\tilde{P}^{\mathcal{S}}_{\alpha}$ do not *yet* have obvious physical meanings.

Suppose that the measuring device $\mathcal{D}$ has configurations $d(\alpha)$ that can be labeled by the same index α that appears in the spectral decomposition for $\tilde{A}^{\mathcal{S}}$. Similarly, suppose that the environment $\mathcal{E}$ has configurations $e(\alpha)$ that can also be labeled by α.

Generalizing the unistochastic matrix (44) from the earlier analysis of the decoherence process discussed in Subsection 3.7, suppose, moreover, that the composite system $\mathcal{SDE}$ evolves according

[20] More generally, one could take $\tilde{A}^{\mathcal{S}}$ to be a *normal matrix*, meaning a matrix that commutes with its adjoint $\tilde{A}^{\mathcal{S}\dagger}$.

to an overall unistochastic transition matrix

$$\Gamma^{\mathcal{SDE}}_{ide,i_0d_0e_0}(t \leftarrow 0) = |U^{\mathcal{SDE}}_{ide,i_0d_0e_0}(t \leftarrow 0)|^2. \tag{72}$$

Generalizing also the composite-system wave function (47) from Subsection 3.7, and letting $\tilde{e}^{\mathcal{S}}_{\alpha',i'}$ denote the i'th component of the basis vector $\tilde{e}^{\mathcal{S}}_{\alpha'}$ with respect to the subject system's configuration basis $e^{\mathcal{S}}_{i'}$, suppose that the three subsystems interact up to a time $t' > 0$ in such a way that they end up with the overall wave function[21]

$$\Psi^{\mathcal{SDE}}_{i'd'e'}\big(t'\big) = \sum_{\alpha'} \tilde{\Psi}^{\mathcal{S}}_{\alpha'}\big(t'\big)\tilde{e}^{\mathcal{S}}_{\alpha',i'}\delta_{d'd(\alpha')}\delta_{e'e(\alpha')}. \tag{73}$$

Mirroring the analogous formula (46) in Subsection 3.7, the composite system's relative time-evolution operator factorizes between the three subsystems for later times $t > t'$:

$$U^{\mathcal{SDE}}\big(t \leftarrow t'\big) = U^{\mathcal{S}}\big(t \leftarrow t'\big) \otimes U^{\mathcal{D}}\big(t \leftarrow t'\big) \otimes U^{\mathcal{E}}\big(t \leftarrow t'\big). \tag{74}$$

Then the composite system's wave function for times $t > t'$ after the interaction is

$$\begin{aligned} \Psi^{\mathcal{SDE}}_{ide}(t) &= \sum_{i',e',d'} U^{\mathcal{SDE}}_{ide,i'd'e'}\big(t \leftarrow t'\big)\Psi^{\mathcal{SDE}}_{i'd'e'}\big(t'\big) \\ &= \sum_{i'}\sum_{\alpha'} U^{\mathcal{S}}_{ii'}\big(t \leftarrow t'\big)\tilde{\Psi}^{\mathcal{S}}_{\alpha'}\big(t'\big)\tilde{e}^{\mathcal{S}}_{\alpha',i'}U^{\mathcal{D}}_{dd(\alpha')}\big(t \leftarrow t'\big)U^{\mathcal{E}}_{ee(\alpha')}\big(t \leftarrow t'\big). \end{aligned} \tag{75}$$

Invoking the Born rule (23) in Subsection 3.2, it follows from this explicit expression for the composite system's wave function that the joint probabilities for $t > t'$ are given by

$$p^{\mathcal{SDE}}_{ide}(t) = |\Psi^{\mathcal{SDE}}_{ide}(t)|^2. \tag{76}$$

Marginalizing over the configuration i of the subject system $\mathcal{S}$ as well as the configuration e of the environment $\mathcal{E}$, and invoking the unitarity of both the subject system's relative time-evolution operator $U^{\mathcal{S}}(t \leftarrow t')$ and the environment's relative time-evolution operator $U^{\mathcal{E}}(t \leftarrow t')$, if follows from a short calculation that the standalone probabilities $p^{\mathcal{D}}_d(t)$ for the measuring device $\mathcal{D}$ alone for $t > t'$ are given by

$$p^{\mathcal{D}}_d(t) = \sum_{\alpha'} |U^{\mathcal{D}}_{dd(\alpha')}\big(t \leftarrow t'\big)|^2 |\tilde{\Psi}^{\mathcal{S}}_{\alpha'}\big(t'\big)|^2. \tag{77}$$

In the limit $t \to t'$, this last result implies that

$$p^{\mathcal{D}}_{d(\alpha')}\big(t'\big) = |\tilde{\Psi}^{\mathcal{S}}_{\alpha'}\big(t'\big)|^2. \tag{78}$$

[21] It is straightforward to write down idealized examples of suitable unitary time-evolution operators for the composite system. One choice is $U^{\mathcal{SDE}}(t') \equiv \sum_{\alpha'} \tilde{P}^{\mathcal{S}}_{\alpha'} \otimes R^{\mathcal{D}}_{d(\alpha')} \otimes R^{\mathcal{E}}_{e(\alpha')}$, where $\tilde{P}^{\mathcal{S}}_{\alpha'}$ is the α'th eigenprojector appearing in the spectral decomposition for $\tilde{A}^{\mathcal{S}}$, and where $R^{\mathcal{D}}_{d(\alpha')}$ and $R^{\mathcal{E}}_{e(\alpha')}$ are unitary transformations for the measuring device and the environment, respectively, that put them in the configurations $d(\alpha')$ and $e(\alpha')$.

Hence, the measuring device $\mathcal{D}$ has a standalone probability $|\tilde{\Psi}^{\mathcal{S}}_{\alpha'}(t')|^2$ of ending up in its configuration $d(\alpha')$, exactly as predicted by the textbook version of the Born rule. One can then naturally define an expectation value $\langle \tilde{A}^{\mathcal{S}}(t')\rangle$ for $\tilde{A}^{\mathcal{S}}$ at t' as the usual kind of statistical average over device readings:

$$\langle \tilde{A}^{\mathcal{S}}(t')\rangle \equiv \sum_{\alpha} \tilde{a}_{\alpha} p^{\mathcal{D}}_{d(\alpha')}(t'). \tag{79}$$

This analysis establishes that as long as there exists a form of unistochastic time evolution for the composite system $\mathcal{SDE}$ that arrives at the appropriate wave function, the matrix $\tilde{A}^{\mathcal{S}}$ represents a genuine observable, in the sense that the time evolution leads to the measuring device ending up in the correct outcome-configuration with the correct Born-rule probability.

For times $t > t'$ after the interaction, (77) implies that the time t' is a division event for the measuring device, as defined in Subsection 3.7:

$$\Gamma^{\mathcal{D}}(t) = \Gamma^{\mathcal{D}}(t \leftarrow t')\Gamma^{\mathcal{D}}(t') \quad \text{for } t > t'. \tag{80}$$

Here, the measuring device's dynamics for times $t > t'$ is given by the relative unistochastic transition matrix

$$\Gamma^{\mathcal{D}}_{dd(\alpha')}(t \leftarrow t') \equiv |U^{\mathcal{D}}_{dd(\alpha')}(t \leftarrow t')|^2. \tag{81}$$

If the observable $\tilde{A}^{\mathcal{S}}$ is an emergeable, as opposed to one of the subject system's (diagonal) random variables, or beables, (19), then the subject system $\mathcal{S}$ does not experience a division event at t', in contrast with the measuring device $\mathcal{D}$. Instead, the subject system remains mired in indivisible time evolution at t', with some stochastically evolving underlying configuration. Moreover, if indeed $\tilde{A}^{\mathcal{S}}$ is an emergeable, then the measurement result obtained by the measuring device is an emergent effect of the interaction between the subject system and the measuring device rather than transparently revealing a physical aspect of the configuration of the subject system alone.

Despite t' not necessarily being a division event for the subject system $\mathcal{S}$, one can nevertheless compute the standalone probability $p^{\mathcal{S}}_i(t)$ for the subject system to be in its ith configuration for times $t > t'$ by marginalizing over the measuring device $\mathcal{D}$ and the environment $\mathcal{E}$. By another straightforward calculation, one finds

$$p^{\mathcal{S}}_i(t) = \sum_{\alpha'} \left[\sum_{i'_1, i'_2} \overline{U^{\mathcal{S}}_{ii'_1}(t \leftarrow t')} U^{\mathcal{S}}_{ii'_2}(t \leftarrow t') \tilde{e}^{\mathcal{S}}_{\alpha', i'_2} \overline{\tilde{e}^{\mathcal{S}}_{\alpha', i'_1}} \right] |\tilde{\Psi}^{\mathcal{S}}_{\alpha'}(t')|^2. \tag{82}$$

Recognizing $|\tilde{\Psi}^{\mathcal{S}}_{\alpha'}(t')|^2$ from (78) as the standalone probability $p^{\mathcal{D}}_{d(\alpha')}(t')$ for the measuring device $\mathcal{D}$ to end up in its configuration $d(\alpha')$ at the time t', and recalling both the configuration projectors $P^{\mathcal{S}}_i$ and the eigenprojectors $\tilde{P}^{\mathcal{S}}_{\alpha}$ appearing in the spectral decomposition (68) for $\tilde{A}^{\mathcal{S}}$, one can write $p^{\mathcal{S}}_i(t)$ more succinctly as

$$p^{\mathcal{S}}_i(t) = \mathrm{tr}(P^{\mathcal{S}}_i \rho^{\mathcal{S}}(t)). \tag{83}$$

Here, the subject system's density matrix $\rho^{\mathcal{S}}(t)$ for $t > t'$ is given by

$$\rho^{\mathcal{S}}(t) \equiv U^{\mathcal{S}}\big(t \leftarrow t'\big)\left[\sum_{\alpha'} p^{\mathcal{D}}_{d(\alpha')}\big(t'\big)\tilde{P}^{\mathcal{S}}_{\alpha'}\right]U^{\mathcal{S}\dagger}\big(t \leftarrow t'\big). \tag{84}$$

One can therefore recast the expectation value (79) for $\tilde{A}^{\mathcal{S}}$ as

$$\langle \tilde{A}^{\mathcal{S}}\big(t'\big)\rangle = \mathrm{tr}(\tilde{A}^{\mathcal{S}}\rho^{\mathcal{S}}\big(t'\big)), \tag{85}$$

which precisely mirrors the formula (18) for the expectation value of a (diagonal) random variable from Subsection 3.2.

Furthermore, the formula (82) for $p^{\mathcal{S}}_i(t)$ above yields a *linear* relationship between the standalone probabilities $p^{\mathcal{D}}_{d(\alpha')}(t')$ for the measuring device $\mathcal{D}$ at t' and the standalone probabilities $p^{\mathcal{S}}_i(t)$ for the subject system $\mathcal{S}$ at $t > t'$:

$$p^{\mathcal{S}}_i(t) = \sum_{\alpha'} \Gamma^{\mathcal{SD}}_{i,d(\alpha')}\big(t \leftarrow t'\big)p^{\mathcal{D}}_{d(\alpha')}\big(t'\big). \tag{86}$$

The entries $\Gamma^{\mathcal{SD}}_{i,d(\alpha')}(t \leftarrow t')$ of the *hybrid* relative transition matrix appearing here are given explicitly by

$$\Gamma^{\mathcal{SD}}_{i,d(\alpha')}\big(t \leftarrow t'\big) \equiv \sum_{i'_1,i'_2} \overline{U^{\mathcal{S}}_{ii'_1}(t \leftarrow t')}U^{\mathcal{S}}_{ii'_2}\big(t \leftarrow t'\big)\tilde{e}^{\mathcal{S}}_{\alpha',i'_2}\overline{\tilde{e}^{\mathcal{S}}_{\alpha',i'_1}}. \tag{87}$$

Because these matrix entries do not depend on the measuring device's standalone probabilities $p^{\mathcal{D}}_{d(\alpha')}(t')$, they naturally serve as *conditional probabilities* for the subject system $\mathcal{S}$ to be in its ith configuration at the time $t > t'$, given that the measuring device $\mathcal{D}$ is in its configuration $d(\alpha')$ at t':

$$p^{\mathcal{SD}}\big(i,t|d(\alpha'),t'\big) \equiv \Gamma^{\mathcal{SD}}_{i,d(\alpha')}\big(t \leftarrow t'\big). \tag{88}$$

4.3 Wave-function collapse

Importantly, notice that one can write the hybrid transition matrix (87) from the previous subsection in a form that resembles the dictionary (15):

$$\Gamma^{\mathcal{SD}}_{i,d(\alpha')}\big(t \leftarrow t'\big) = \mathrm{tr}(U^{\mathcal{S}\dagger}\big(t \leftarrow t'\big)P^{\mathcal{S}}_i U^{\mathcal{S}}\big(t \leftarrow t'\big)\tilde{P}^{\mathcal{S}}_{\alpha}). \tag{89}$$

Rearranging the right-hand side gives the equation

$$\Gamma^{\mathcal{SD}}_{i,d(\alpha')}\big(t \leftarrow t'\big) = \mathrm{tr}(P^{\mathcal{S}}_i\rho^{\mathcal{S}|\alpha',t'}(t)), \tag{90}$$

with a *conditional* density matrix $\rho^{\mathcal{S}|\alpha',t'}(t)$ for the subject system $\mathcal{S}$ at the time $t > t'$ naturally defined by time-evolving the eigenprojector $\tilde{P}^{\mathcal{S}}_{\alpha'}$ from t' to t:

$$\rho^{\mathcal{S}|\alpha',t'}(t) \equiv U^{\mathcal{S}}\big(t \leftarrow t'\big)\tilde{P}^{\mathcal{S}}_{\alpha'}U^{\mathcal{S}\dagger}\big(t \leftarrow t'\big). \tag{91}$$

Thus, the calculation (82) of the standalone probabilities $p_i^{\mathcal{S}}(t)$ for the subject system at $t > t'$ in the previous subsection reduces to the statement that they are given by

$$p_i^{\mathcal{S}}(t) = \mathrm{tr}(P_i^{\mathcal{S}} \rho^{\mathcal{S}}(t)), \tag{92}$$

where the subject system's density matrix $\rho^{\mathcal{S}}(t)$, which was originally defined in (84) in the previous subsection, can equivalently be expressed as a probabilistic mixture of the conditional density matrices $\rho^{\mathcal{S}|\alpha',t'}(t)$ defined in (91), statistically weighted by the measurement probabilities $p^{\mathcal{D}}_{d(\alpha')}(t')$:

$$\rho^{\mathcal{S}}(t) \equiv \sum_{\alpha'} \rho^{\mathcal{S}|\alpha',t'}(t) p^{\mathcal{D}}_{d(\alpha')}\big(t'\big). \tag{93}$$

Taking stock of these results, one sees that to make future predictions for $t > t'$ about the subject system $\mathcal{S}$, conditioned on the measuring device's result $d(\alpha')$ at t', one uses the conditional probabilities $\Gamma^{\mathcal{SD}}_{i,d(\alpha')}(t \leftarrow t') = \mathrm{tr}(P_i^{\mathcal{S}} \rho^{\mathcal{S}|\alpha',t'}(t))$ from (90), in which the subject system's density matrix has effectively been replaced by the conditional density matrix $\rho^{\mathcal{S}|\alpha',t'}(t)$. This conditional density matrix corresponds to a *collapsed* state vector or wave function defined as

$$\Psi^{\mathcal{S}|\alpha',t'}(t) \equiv U\big(t \leftarrow t'\big)\tilde{e}^{\mathcal{S}}_{\alpha}. \tag{94}$$

The phenomenon of *wave-function collapse* therefore reduces to a prosaic example of conditioning.

By contrast, for an observer who does not know the specific measurement result $d(\alpha')$, the correct density matrix $\rho^{\mathcal{S}}(t)$ to use is the one defined in (93) in the previous subsection. Again, this density matrix consists of an appropriate probabilistic mixture of conditional or collapsed density matrices that are statistically weighted over the measurement results.

4.4 The measurement problem

According to the foregoing treatment of the measurement process, a measuring device is an ordinary physical system that can carry out a measurement of an observable and then ends up in a final configuration that reflects a definite measurement outcome. The probabilities for a measuring device's various possible measurement outcomes are given by the textbook Born rule (78), and conditioning on the specific measurement outcome leads to the textbook formula (94) for wave-function collapse. Hence, this picture arguably has the resources to solve the measurement problem (Myrvold, 2022).

The stochastic-quantum correspondence is also helpful for understanding the measurement process in another important way. Textbook treatments typically regard measuring devices as axiomatic primitives, without providing clear principles for deciding which kinds of systems merit being called measuring devices. The approach taken toward the measurement process in this paper not only gives a candidate resolution of the measurement problem but also yields a natural set of criteria for defining what counts as a good measuring device in the first place, without the need to regard measuring devices as special among all other systems in any truly fundamental way. Based on this

approach, one sees that a good measuring device should be a physical system with at least as many configurations as possible outcomes for the observable to be measured (at least up to the desired level of experimental resolution), it should admit an overall form of dynamics that results in the correct final correlations, and it should be in sufficiently strong contact with a noisy environment to generate a robust division event at the conclusion of the measurement interaction.[22]

4.5 The uncertainty principle

Again, the preceding treatment of the measurement process leads to the textbook Born rule (78) and the textbook formula (94) for wave-function collapse. As a consequence, any pair of observables $\tilde{A}, \tilde{B}$ and their respective standard deviations $\Delta\tilde{A}, \Delta\tilde{B}$ will satisfy the *Heisenberg-Robertson uncertainty principle* (Heisenberg 1927, Robertson 1929),

$$\Delta\tilde{A}\,\Delta\tilde{B} \geq \frac{1}{2}|\mathrm{tr}(i[\tilde{A}\tilde{B} - \tilde{B}\tilde{A}]\rho)|, \tag{95}$$

as follows from any of the standard proofs.

The stochastic-quantum correspondence goes beyond replicating the uncertainty principle by painting a clearer picture of what the uncertainty principle physically means. Consider for simplicity the case in which $\tilde{A} = A$ is a random variable, or beable, and $\tilde{B}$ is an emergeable, in the language of Subsection 4.1. Then A has a direct interpretation solely in terms of the subject system's configuration space, whereas $\tilde{B}$ encodes an emergent pattern in the subject system's dynamics that can nonetheless show up in the measurement outcomes of a measuring device.

Suppose that A has a definite value or magnitude at some initial time 0. Then, assuming that A has no degeneracies in its spectrum, the subject system must be in a specific configuration with probability 1 at the initial time 0. The overall stochastic dynamics will then lead to uncertainty in the outcome of any measurement of $\tilde{B}$.

Suppose that one goes ahead and measures $\tilde{B}$, so that a definite measurement outcome emergently shows up in the configuration of a measuring device at some time $t' > 0$. The analysis in Subsection 4.3 then implies that there is an inevitable disturbance in the subject system that leads its density matrix to end up effectively as a non-diagonal matrix equal to an eigenprojector of $\tilde{B}$. A non-diagonal density matrix signifies that the system is in the midst of an indivisible stochastic process, as explained in Subsection 3.8. In the present circumstances, that indivisible stochastic process is precisely one that would ensure that if $\tilde{B}$ were measured again shortly after t', then the measuring device would obtain the same outcome for $\tilde{B}$ as before. However, being in the midst of an indivisible stochastic process also implies uncertainty in the subject system's underlying configuration, thereby rendering the value of A uncertain.

[22]The first two of these three criteria would be standard requirements for a measuring device even without worrying about indivisible stochastic dynamics or quantum theory. Note that without the third criterion—strong contact with an environment—one obtains a "latent measurement" (Dicke 1989; Glick, Adami 2020).

5 Discussion and Future Work

5.1 Indivisible quantum theory

This paper has shown that one can reconstruct the mathematical formalism and all the empirical predictions of quantum theory using simpler, more physically transparent axioms than the standard Dirac-von Neumann axioms. Rather than postulating Hilbert spaces and their ingredients from the beginning, one instead posits a physical model, called an *indivisible stochastic process*, based on trajectories in configuration spaces following generically indivisible stochastic dynamics. The *stochastic-quantum correspondence* then shows that every quantum system can be viewed as the Hilbert-space representation of an underlying indivisible stochastic processes.

This new axiomatic approach naturally suggests a new interpretation of quantum theory grounded in the theory of stochastic processes. According to this highly adaptable interpretation, which one could naturally call the *indivisible interpretation* of quantum theory, or just *indivisible quantum theory*, systems have underlying physical configurations in configuration spaces at all times.

This perspective deflates some of the most mysterious features of quantum theory. In particular, one sees that density matrices, wave functions, and all the other ingredients of Hilbert spaces, while highly useful, are merely mathematical appurtenances. These appurtenances should therefore not be assigned direct physical meanings or treated as though they directly represent physical objects, any more than Lagrangians or Hamilton's principal functions directly represent physical objects. Superposition is then not a literal smearing of physical objects but is merely a mathematical artifact of catching a system in the middle of an indivisible stochastic process, as represented using a Hilbert-space formulation and wave functions.

Moreover, from this standpoint, *canonical quantization* need not be regarded as the promotion of classical observables to noncommutative operators by fiat but can be implemented (when mathematically feasible) simply by generalizing a classical system's dynamics from being deterministic to being stochastic, with all the exotic features of quantum theory then emerging automatically. As a consequence, this formulation of canonical quantization potentially offers more straightforward techniques for coupling classical systems to quantum systems in real-world applications.

5.2 The category problem

In an important sense, the stochastic-quantum correspondence and the indivisible interpretation also legitimize many standard practices followed in physics and in other scientific areas like astronomy, chemistry, biology, and paleontology. To see why, notice that according to the thoroughly instrumentalist and operationalist Dirac-von Neumann axioms, the *only* predictions provided by textbook quantum theory are predictions about a rather narrow category of things: measurement outcomes, probabilities of measurement outcomes, and expectation values that are averages of measurement outcomes statistically weighted by measurement-outcome probabilities (Griffiths 2018; Townsend 2012; Shankar 1994; Sakurai, Napolitano 2010; Schumacher, Westmoreland 2010). Meanwhile, scientists in all areas of research talk about a much broader category of phenomena—from the mixing

of gases in the primordial universe to the spontaneous appearance of genetic mutations—that presumably just *happen* in some way, according to *happening* probabilities, in the past, present, or future. Strictly speaking, however, the *happening* of phenomena, as a category, lies outside the axiomatic ambit of textbook quantum theory, which refers only to connecting the measurement settings of chemical detectors and telescopes to the probabilities of their measurement outcomes. The inability of textbook quantum theory to account for the happening of phenomena represents what one might call the *category problem*. The category problem either means that scientists are not speaking honestly or coherently about their research, or that textbook quantum theory is inadequate as a physical theory.

Decoherence alone cannot bridge the categorical gap between measurement-outcome probabilities and happening probabilities, because decoherence can only temporarily change whatever orthonormal basis momentarily diagonalizes a system's density matrix (and, after all, every density matrix is always diagonal in *some* orthonormal basis). After a system undergoes decoherence, textbook quantum theory then still requires one to make a direct appeal to the measurement axioms to translate the final density matrix into a statement about probabilities, which will then axiomatically end up being measurement-outcome probabilities rather than happening probabilities.

Nor can appealing to some sort of *thermodynamic limit* resolve the discrepancy either. In order for a limit in a physical context to make sense, there should be clearly physical ingredients or constituents involved. Furthermore, the end result of the limit should gradually emerge as a better and better physical approximation at *finite* stages of the limiting process, simply because a rigorous limit consists of inequalities between finite (if arbitrarily large or small) parameters. For example, in the *hydrodynamic limit* of a system of classical interacting particles, the particles are the physical ingredients, and one sees fluid-like behavior gradually emerge as a better and better physical approximation as the number of particles progressively increases. In the case of textbook quantum theory, by contrast, every finite stage of any purported thermodynamic limit features only measurement outcomes and measurement-outcome probabilities, so there are no clearly physical ingredients or constituents, and the categorical gap between measurement outcomes and the happening of phenomena never closes.

The stochastic-quantum correspondence and the indivisible interpretation yield a much richer version of quantum theory in which physical phenomena really happen, with probabilities that are really happening probabilities, and therefore not only resolves the category problem but also vindicates the ways that scientists talk about the world. Measurement-outcome probabilities are then merely a special case, arising when what is actually happening is a change to the configuration of a measuring device.

5.3 Interpretational issues

A formulation or interpretation of quantum theory that posits physical configurations separate from—or, in the present case, instead of—the standard ingredients of Hilbert spaces can be thought of as a kind of *hidden-variables theory*. In keeping with the Bell-Kochen-Specker theorem (Bell

1966; Kochen, Specker 1967), indivisible quantum theory is a manifestly contextual theory, with a given quantum system's beables belonging to a specific measurement context, and various classes of emergeables belonging to other measurement contexts, as detailed in Subsection 4.2.

Indivisible quantum theory is based on non-Markovian stochastic dynamics, so it lies outside the *ontological models* framework of Harrigan and Spekkens (2010). In particular, the wave function is neither ontic nor entirely epistemic but has a law-like or nomic character, as is clear from its definition (20) as part of the time-evolution operator.[23] As such, the theorem of Pusey, Barrett, and Rudolph (2012) does not apply.

Because indivisible quantum theory invokes hidden variables in the form of underlying physical configurations, this framework for quantum theory shares some aspects with the *de Broglie-Bohm formulation*, or *Bohmian mechanics* (de Broglie 1930; Bohm 1952a, 1952b). However, in contrast to indivisible quantum theory, Bohmian mechanics employs deterministic dynamics and features a fundamental guiding equation that explicitly breaks Lorentz invariance by singling out a preferred foliation of spacetime into spacelike hypersurfaces. The indivisible interpretation instead takes seriously what experiments strongly suggest—that the dynamics of quantum theory is indeterministic, that there is no fundamentally preferred foliation of spacetime, and that quantum systems can exhibit genuinely non-Markovian behavior (Glick, Adami 2020). Indivisible quantum theory is also more flexible and model-independent than Bohmian mechanics and works for all kinds of quantum systems, beyond the case of systems of fixed numbers of finitely many non-relativistic particles.

In contrast with the *Everett interpretation* (Everett 1957, 1973; Wallace 2012), also known as the *'many worlds' interpretation*, the indivisible interpretation assumes that quantum systems, like classical systems, have definite configurations in configuration spaces and does not attempt to derive probability from non-probabilistic assumptions or grapple with fundamental aspects of personal identity in a universe continuously branching into large (and somewhat undefined) numbers of parallel worlds. Simply put, there are no fundamental wave functions in the indivisible interpretation, meaning that there is nothing in the ontology that branches into a multitude of worlds, so the approach taken in this paper is more metaphysically modest than the Everett interpretation.

Unlike *stochastic-collapse theories* (Ghirardi, Remini, Weber 1986; Bassi, Ghirardi 2003), indivisible quantum theory does not invoke any fundamental violations of unitarity and does not require introducing any new constants of nature to specify dynamical-collapse rates. That said, there are common threads between indivisible quantum theory and some approaches to stochastic-collapse theories that demote the wave function from having a physical status (Bedingham 2018).[24]

The indivisible interpretation shares some features with the *modal interpretations* (Krips 1969; Van Fraassen 1972; Vermaas, Dieks 1995; Bacciagaluppi, Hemmo 1996; Lombardi, Dieks 2021), including an insistence that systems always have definite configurations of some kind at every moment in time, while assigning a law-like, objective role to at least some forms of probability. One difference between the indivisible interpretation and most of the modal interpretations, however,

[23] Indeed, in Section 6 of their 2010 paper, Harrigan and Spekkens specifically note that models based on "nomic" wave functions and stochastic dynamical laws lie outside their framework.

[24] The author would like to thank an anonymous reviewer for suggesting this point.

is the indivisible interpretation's insistence that the definite configuration of a given system is an element of a classical-looking configuration space rather than corresponding more abstractly to features of a Hilbert space. The indivisible interpretation also avoids some of the ontological instabilities that are a serious challenge for most of the modal interpretations (Vermaas 1999).

5.4 Future directions

Future work will address questions of locality and causation. Looking forward, it would also be interesting to see what implications the stochastic-quantum correspondence could have for both phenomenological stochastic processes, like those in biology or finance, as well as for future work in fundamental physics, like quantum gravity.

More broadly, by recasting the Hilbert-space formulation of quantum theory as merely a convenient way to represent a large class of stochastic processes, one opens the door to searching for totally different representations that might look nothing at all like Hilbert spaces and that could allow for the construction of more general kinds of stochastic processes. Perhaps one could even find a way to generalize the theory beyond stochastic processes altogether.

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Declarations

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