DEFINABILITY IN PHYSICS

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Abstract The concept of definability of physical fields in a set-theoretical foundation is introduced. We propose an axiomatic set theory and show that the Schrödinger equation and, more generally, a nonlinear sigma model can be derived from a null postulate and that quantization of fields is equivalent to definability. We find that space-time is relational in this theory. Some examples of the relevance to physics are suggested.

A set U of finite integers is definable in the set theory if and only if there exists a formula $\Phi_U(n)$ from which we can unequivocally determine whether a given finite integer n is a member of U or not.[1] We can extend this concept to physical fields by asserting that a physical field in a finite region of space is definable in a set-theoretical foundation if and only if the set of distributions of the fields energy among eigenstates can be mirrored in that theory by a definable set of finite integers. This concept of definability is appropriate because, were there a field whose set of energy distributions among eigenstates corresponded to an undefinable set of finite integers, that field would have at least one energy distribution whose presence or absence is impossible to determine, so the field could not be verifiable or falsifiable. Therefore, our task is to find a foundation in which it is possible to specify the definable sets of finite integers and which must also contain the mathematics necessary for the physical fields corresponding to the sets.

Extensionality	Two sets with just the same members are equal.
Pairs	For every two sets, there is a set that contains just them.
Union	For every set of sets, there is a set with just all their members.
Infinity	There is at least one set ω^* with members determined in infinite succession.
Power Set	For every set, there is a set containing just all its subsets.
Regularity	Every non-empty set has a minimal member (i.e. "weak" regularity).
Replacement	Replacing members of a set one-for-one creates a set.

These axioms are the well-known theory of Zermelo-Frankel (ZF) with the axiom schema of subsets deleted. As a result of that deletion, all theorems must hold for every ω^*. The minimal ω^*, usually called ω, cannot be obtained, so that both finite and infinite integers exist in ω^*. This implies that all sets of finite integers are finite and hence definable.

We can now adjoin to this theory another axiom asserting that all subsets of ω^* are constructible. By constructible sets we mean sets that are generated sequentially by some process, one after the other, so that the process well-orders the sets. Gödel has shown that an axiom asserting that all sets are constructible can be consistently added to ZF, giving a theory called ZFC$^+$.[2] It has also been shown that no more that countably many subsets of ω^* can be proven to exist in ZFC$^+$. Both these results will, of course, hold for the sub-theory ZFC$^+$ minus the axiom schema of subsets. Therefore we can adjoin a new axiom asserting that the subsets of ω^* are constructible and there are countably many such subsets. We shall call these eight axioms Theory T.

We first show that T contains a real line. Recall the definition of "rational numbers" as the set of ratios, usually called Q, of any two members of the set ω. In T, we can likewise, using the axiom of unions, establish for ω^* the set of ratios of any two of its integers, finite or infinite. This will be an "enlargement" of the rational numbers and we shall call this enlargement Q*. Two members of Q* are called "identical" if their ratio is 1. We now employ the symbol "$\equiv$" for "is identical to". An "infinitesimal" is a member of Q* "equal" to 0, i.e., letting y signify the member and employing the symbol "=" to signify equality, $y = 0 \leftrightarrow \forall k[y < l/k]$, where k is a finite integer. The reciprocal of an infinitesimal is "infinite". A member of Q* that is not an infinitesimal and not infinite is "finite". The constructibility axiom well-orders the power set of ω^*, creating a metric space composed of the subsets of ω^*. These subsets represent the binimals making up a real line R^*.

An *equality-preserving* bijective map $\Phi(x,u)$ between intervals X and U of R^* in which $x \in X$ and $u \in U$ such that $\forall x_1, x_2, u_1, u_2[\phi(x_1,u_2) \wedge \phi(x_2,u_2) \rightarrow (x_1, -x_2 = 0 \leftrightarrow u_1 - u_2 = 0)]$ creates pieces that are biunique and homeomorphic. It is clear that U vanishes if and only if X vanishes.

We can now define "functions of a real variable in T". $u(x)$ is a function of a real variable in T only if it is a constant or a sequence in x of continuously connected biunique pieces such that the derivative of u with respect to x is also a function of a real variable in T. These functions are thus of bounded variation. If some derivative is a constant, they are polynomials. If no derivative is a constant, these functions do not exist in T. They can, however, be approached arbitrarily closely by some linear combination of polynomials of very high degree given by many iterations of the following variational form for the Sturm-Liouville problem, so that these polynomials are effectively eigenfunctions:

$\int_a^b [p\left(\frac{du}{dx}\right)^2 - qu^2]dx \equiv \lambda \int_a^b ru^2 dx : \lambda$ is locally minimum for $\int_a^b ru^2 dx$ constant;
where a $\neq b, u\left(\frac{du}{dx}\right) \equiv 0$ at a and b; p, q, r are functions of the real variable x

Let u_1 and u_2 be functions of the real variables x_1 and x_2 respectively. Let us consider a one-dimensional string, where x_1 is space and x_2 is time. We can write Hamilton's Principle using a null identity, since:

$$\int \left[\left(\frac{\partial u_1 u_2}{\partial x_1}\right)^2 - \left(\frac{\partial u_1 u_2}{\partial x_2}\right)^2 \right] dx_1 dx_2 \equiv 0 \rightarrow \delta \int \left[\left(\frac{\partial u_1 u_2}{\partial x_1}\right)^2 - \left(\frac{\partial u_1 u_2}{\partial x_2}\right)^2 \right] dx_1 dx_2 = 0$$

A generalization to finitely many space-like and time-like dimensions can now be obtained. Let $u_{\ell mi}(x_i)$ and $u_{\ell mj}(x_j)$ be eigenfunctions with non-negative eigenvalues $\lambda_{\ell mi}$ and $\lambda_{\ell mj}$ respectively. We assert a "field" is a sum of eigenstates: $\underline{\Psi}_m = \sum_\ell \Psi_{\ell m} \underline{i_\ell}, \Psi_{\ell m} = \prod_i u_{\ell mi} \prod_j u_{\ell mj}$, subject to the postulate that for every eigenstates m the value of the integral of the Lagrange density over all space-time is *identically* null. Let $dsd\tau = \prod_i r_i dx_i \prod_j r_j dx_j$:

$$\sum_\ell \int \left\{ \sum_i \frac{1}{r_i} \left[P_{\ell mi} \left(\frac{\partial \Psi_{\ell m}}{\partial x_i}\right)^2 - Q_{\ell mi} \Psi_{\ell m}^2 \right] - \sum_j \frac{1}{r_j} \left[P_{\ell mj} \left(\frac{\partial \Psi_{\ell m}}{\partial x_j}\right)^2 - Q_{\ell mj} \Psi_{\ell m}^2 \right] \right\} dsd\tau \equiv 0 for\ all\ m.$$

In this integral expression the P and Q can be functions of any of the x_i and x_j, thus of any $\Psi_{\ell m}$ as well. This is a *nonlinear sigma model.* The $\Psi_{\ell m}$ can be determined by an algorithm with coordinated iterations that are constrained by the indicial expression $\sum\limits_{\ell i} \lambda_{\ell mi} - \sum\limits_{\ell j} \lambda_{\ell mj} \equiv 0$ *for all m.*

The following is a proof of quantization using the mathematics of T plus the null postulate. Let
$\sum\limits_{\ell mi} \int \left\{ \frac{1}{r_i} \left[P_{\ell mi} \left(\frac{\partial \Psi_{\ell m}}{\partial x_i} \right)^2 - Q_{\ell mi} \Psi_{\ell m}^2 \right] \right\} ds d\tau$
and $\sum\limits_{\ell mj} \int \left\{ \frac{1}{r_j} \left[P_{\ell mj} \left(\frac{\partial \Psi_{\ell m}}{\partial x_j} \right)^2 - Q_{\ell mj} \Psi_{\ell m}^2 \right] \right\} ds d\tau$ both be represented by α, since they are identical:

I. α is positive and must be closed to addition and to the absolute value of subtraction; In T we must have that α is an integer times a constant which is infinitesimal or finite.

II. There is either no field, in which case $\alpha \equiv 0$, or otherwise in T the field is finite in which case $\alpha \neq 0$); thus $\alpha = 0 \leftrightarrow \alpha \equiv 0$.

III. Therefore $\alpha \equiv nI$, where n is an integer and I is a finite constant such that $\alpha = 0 \leftrightarrow n \equiv 0$

Quantization is thus obtained within this theory utilizing both the relational nature of space-time in T and a physical null postulate. The Schrödinger equation can now be derived for finitely many space variables and one time variable. Thence we can get the eigenstates and the energy levels associated with each state. When there are finitely many space-like dimensions and one time-like dimension, we can obtain the Schrödinger equation from the nonlinear sigma model: Let ℓ = 1,2 and suppress m. We can now introduce $\Psi = A \prod\limits_i u_i(x_i)[u_1(\tau) + \iota u_2(\tau)]$, where $\iota = \sqrt{-1}$, normalizing $[u_1^2(\tau) + u_2^2(\tau)] \int \prod\limits_i u_i^2(x_i) ds \equiv 1$. Then we see that $\frac{du_1}{d\tau} = -u_2$ and $\frac{du_2}{d\tau} = u_1$ or $\frac{du_1}{d\tau} = u_2$ and $\frac{du_2}{d\tau} = -u_1$. In either case, for each and every *irreducible* bin-unique time-eigenfunction piece the least non-zero value of α is the finite constant I. Thus $A^2 \int \int_0^{\pi/2} \left[\left(\frac{du_1}{d\tau} \right)^2 + \left(\frac{du_2}{d\tau} \right)^2 \right] \prod\limits_i u_i^2(x_i) ds d\tau \equiv A^2 \pi/2 \equiv I$. Substituting the Planck constant h for $4I$ and letting τ be ωt, the integrand becomes the well-known time-part of the Lagrange density for the Schrödinger equation, $(h/4\pi\iota) \left[\Psi^* \left(\frac{\partial \Psi}{\partial t} \right) - \left(\frac{\partial \Psi^*}{\partial t} \right) \Psi \right]$. It immediately follows that the energy in the m^{th} eigenstate, obtained from this time-part of the Lagrange density, exists only in quanta of $h\omega_m/2\pi$. The sum of energies in all of the eigenstates E_t is thus $\Sigma n_m h\omega_m/2\pi$ where n_m is the number of quanta in the m^{th} eigenstate. For any finite energy, every ordered set of finite integers n_m corresponding to the number of quanta residing in each of the eigenstates, using the fundamental theorem of arithmetic, maps bi-uniquely with $\prod_m P_m^{n_m}$, where P_m is the m^{th} prime starting with 2. The set of these finite integers must always be definable in T. Furthermore, conversely, for classical physics a definable field does not exist in T. Given finite E_t and ω_m, then, if h were infinitesimal, some n_m would have to be infinite and thus the set of all distributions of energy among the eigenstates could not be mirrored by any set (in T) of finite integers. Accordingly, **definability in T is equivalent to quantization.**

In summary,

☐ Quantization is derivable from a null postulate in the set-theoretical foundation T.

☐ Schrödinger equation and, more generally, a nonlinear sigma model, are obtained.

☐ There are inherently no singularities in these physical fields.

☐ Space-time is relational, giving a possible foundation for quantum gravity.

In addition, though we do not have the opportunity to discuss these points in any detail, it should be more or less obvious that

☐ By similar reasoning, definability in T can be shown equivalent to compactification of all the spatial dimensions effectively.

☐ Dyson's problem regarding divergence of perturbation series in QED goes away.

☐ Wigner's metaphysical question regarding the apparent unreasonable effectiveness of mathematics in physics is directly answered.

References

[1] Tarski, A., Mostowski, A., Robinson, R., *Undecidable Theories.* North Holland, Amsterdam, 1953.

[2] Gödel, K., The consistency of the axiom of choice and of the generalized continuum hypothesis. *Annals of Math. Studies*, 1940.