The Stochastic-Quantum Theorem

Jacob A. Barandes*

February 5, 2026

Abstract

This paper introduces several new classes of mathematical structures that have close connections with physics and with the theory of dynamical systems. The most general of these structures, called indivisible stochastic processes, collectively encompass many important kinds of stochastic processes, including Markov chains and random dynamical systems. This paper then states and proves a new theorem that establishes a precise correspondence between any indivisible stochastic process and a unitarily evolving quantum system. This theorem therefore leads to a new formulation of quantum theory, alongside the Hilbert-space, path-integral, and quasi-probability formulations. The theorem also provides a first-principles explanation for why quantum systems are based on the complex numbers, Hilbert spaces, linear-unitary time evolution, and the Born rule. In addition, the theorem suggests that by selecting a suitable Hilbert space, together with an appropriate choice of unitary evolution, one can simulate any indivisible stochastic process on a quantum computer, thereby potentially opening up an extensive set of novel applications for quantum computing.

1 Introduction

In the development of physical theories, it sometimes turns out that existing definitions are too conceptually limiting, and that more flexible definitions are needed. Working with more flexible definitions at a higher level of abstraction or generality may make it easier to discover new connections or prove new theorems, which would then also apply down at the lower level of the original definitions.

This paper will argue that by appropriately generalizing standard definitions of dynamical systems to include various forms of non-Markovianity, one can obtain novel classes of mathematical structures that encompass an extensive array of physically important models. As with a traditionally defined dynamical system, each such mathematical structure describes a physical system

*Departments of Philosophy and Physics, Harvard University, Cambridge, MA 02138; jacob_barandes@harvard.edu; ORCID: 0000-0002-3740-4418

moving deterministically or stochastically along some trajectory in a configuration space, albeit with a more general set of laws than according to standard definitions.[1]

This paper also states and proves a new theorem showing that despite being based on trajectories in configuration spaces, the newly introduced class of indivisible stochastic processes actually includes all quantum systems with finite-dimensional Hilbert spaces.[2] As a consequence, this stochastic-quantum theorem potentially offers a more conceptually transparent way to understand quantum systems, with superpositions no longer regarded as literal blends of physical states. The theorem also provides a first-principles explanation for features of quantum theory that are usually taken to be axiomatic, including Hilbert spaces over the complex numbers, linear-unitary evolution, and the Born rule.

Seen from another point of view, this stochastic-quantum correspondence yields an alternative way to formulate quantum theory—a formulation that is phrased in the language of trajectories unfolding stochastically in configuration spaces according to ordinary notions of probability. This alternative formulation is distinct from the traditional Hilbert-space formulation (Dirac 1930, von Neumann 1932), the path-integral formulation (Dirac 1933; Feynman 1942, 1948), and the quasi-probability formulation (Wigner 1932, Moyal 1949).

From a more practical perspective, turning this stochastic-quantum correspondence around suggests that unitarily evolving quantum systems can be put to work simulating a very broad class of non-Markovian stochastic processes, thereby potentially opening up an extensive suite of new applications for quantum computers.

As this paper will explain, indivisible stochastic processes are inherently non-Markovian. The vast majority of research on non-Markovianity in quantum theory has focused on the phenomenological appearance of non-Markovian-like time evolution for density matrices of *open* quantum systems, due to interactions with their environments and associated feedback effects.

A remarkable and important exception was the work of Glick and Adami (2020), who constructed an iterated generalization of the 'Wigner's friend' thought experiment, whose original incarnation first showed up in Everett's unpublished 1956 dissertation and was developed further by Wigner several years later (Everett 1956, Wigner 1961). In their paper, Glick and Adami analyzed a hypothetical collection of small devices carrying out sequential measurements on a given quantum system, all inside a perfectly sealed container. In particular, Glick and Adami showed that it makes an *observable difference* whether those sequential measurements are treated as collapse events or not. If the individual measurements are treated as collapse events, then the overall system behaves effectively as a Markov chain, according to a definition of Markovianity that coincides with the notion of divisibility used in the present work. By contrast, if the individual measurements are treated as a form of unitary time evolution, so that the overall set-up is truly regarded as a *closed* system, then careful quantum-state tomography could reveal distinct empirical signatures of non-

[1]For pedagogical treatments of the standard theory of dynamical systems, see, for instance, Devaney (1989); Strogatz (1994); or Katok, Hasselblatt (1995).

[2]It is worth keeping in mind that "finite" can still mean *very large*, with a level of discreteness that can be well below any conceivable experimental resolution.

Markovianity, meaning that a fundamental form of non-Markovianity for closed systems has been lurking in quantum theory all along.

Section 2 begins by defining deterministic generalizations of dynamical systems, followed by the introduction of important distinctions between indivisible, Markovian, and Markovian-homogeneous dynamics. Section 3 provides a generalized definition of a system with stochastic laws, shows how to represent such a system in the formalism of linear algebra, describes connections between this work and the existing research literature, defines the relationship between a composite system and its subsystems, and introduces the crucial notion of a unistochastic process. Section 4 states the stochastic-quantum theorem, whose proof is this paper's primary goal, and then discusses some important corollaries and provides a simple example of the theorem in practice. Section 5 lays out the theorem's proof, which entails explicitly constructing the claimed correspondence between stochastic processes and quantum systems along the way. Section 6 concludes the paper with a brief discussion of future work.

2 Deterministic Systems

2.1 Indivisible Dynamical Systems

Dynamical systems are abstract mathematical structures that usefully model many deterministic physical processes. According to the standard definition (Devaney 1989; Strogatz 1994; Katok, Hasselblatt 1995), a dynamical system consists of a map representing some kind of evolution law that can be applied repeatedly to the elements of some set of states. A dynamical system is usually assumed to be divisible, in the sense that one can 'divide up' its evolution law over any time duration into well-defined evolution laws that describe intermediate time durations.[3] The more general case would be an indivisible dynamical system that might lack this feature.

The terms 'divisible' and 'indivisible' for dynamical laws are remarkably new. This terminology appears to be due to Wolf and Cirac, who introduced it in a 2008 paper on quantum channels (Wolf, Cirac 2008).

To accommodate the eventual possibility of indivisible evolution, this paper will define an indivisible dynamical system to mean a tuple of the form

$$(\mathcal{X}, \mathcal{T}, f) \tag{1}$$

that consists of the following data.

- The symbol $\mathcal{X}$ denotes a set that will be called the indivisible dynamical system's state space (or phase space), and whose individual elements $i \in \mathcal{X}$ denote the system's (allowed) states.

- Note that $\mathcal{X}$ may or may not be a finite set, and it may or may not involve additional

[3] Note that this terminology is unrelated to the much older concept of *infinite divisibility*, which refers to a probability distribution that can be expressed as the probability distribution of a sum of any integer number of independent and identically distributed random variables.

structure in its definition, such as a measure-theoretic structure or a vector-space structure. For the purposes of this paper, no such additional structure will be specified or assumed. More broadly, for reasons of brevity and simplicity, this paper will entirely set aside measure-theoretic considerations that arise for the case of uncountable sets.

- The symbol $\mathcal{T}$ denotes the system's set of target times $t \in \mathcal{T}$, where $\mathcal{T}$ may or may not be isomorphic to a subset of the real line $\mathbb{R}$ under addition.

- The symbol f denotes a map
$$f : \mathcal{X} \times \mathcal{T} \to \mathcal{X} \tag{2}$$
that will be called the system's dynamical map. This dynamical map f takes as inputs any state i and any target time t, and outputs a state $f(i, t) \in \mathcal{X}$:
$$i, t \mapsto f(i, t) \in \mathcal{X} \quad \text{[for all } i \in \mathcal{X},\, t \in \mathcal{T}\text{]}. \tag{3}$$

- Fixing the target time t turns f into a time-dependent dynamical map
$$f_t : \mathcal{X} \to \mathcal{X} \tag{4}$$
defined by
$$i \mapsto f_t(i) \equiv f(i, t) \quad \text{[for all } i \in \mathcal{X}\text{]}. \tag{5}$$

- Without any important loss of generality, the set of target times $\mathcal{T}$ will be assumed to include an element denoted by 0 and called the initial time. It will be further assumed that at the initial time 0, the time-dependent dynamical map f_t trivializes to the identity map $\mathrm{id}_{\mathcal{X}}$ on $\mathcal{X}$:
$$f_0 = \mathrm{id}_{\mathcal{X}}, \quad \text{or} \quad f_0(i) = i \quad \text{[for all } i \in \mathcal{X}\text{]}. \tag{6}$$

- One can regard the argument i appearing in the expression $f_t(i)$ as an initial state of the system at the initial time 0, with the time-dependent dynamical map f_t then describing the evolution of that state i from the initial time 0 to the target time t.

- Given a fixed state i, the set of states
$$\{f_t(i) \,|\, t \in \mathcal{T}\} \subset \mathcal{X} \tag{7}$$
describes the orbit, or trajectory, of the initial state i through the system's state space $\mathcal{X}$ according to the dynamical map f.

In many applications, one takes the set of target times $\mathcal{T}$ to be a semigroup, meaning that the definition of $\mathcal{T}$ includes an associative binary operation $\star$ (which is often denoted instead by $+$ in the commutative case):
$$t, t' \mapsto t \star t' \in \mathcal{T} \quad \big[\text{for all } t, t' \in \mathcal{T}\big], \tag{8}$$

$$(t \star t') \star t'' = t \star (t' \star t'') \quad \big[\text{for all } t, t', t'' \in \mathcal{T}\big]. \tag{9}$$

One usually also takes the initial time 0 to be the identity element under this binary operation,

$$0 \star t = t \star 0 = t \quad [\text{for all } t \in \mathcal{T}], \tag{10}$$

in which case $\mathcal{T}$ becomes a monoid, meaning a semigroup with an identity element. If, furthermore, every target time t has an inverse t' such that $t \star t' = t' \star t = 0$, then $\mathcal{T}$ becomes a group.

2.2 Markovian Dynamical Systems

In the most general case, an indivisible dynamical system will not provide a way to evolve a system from a *non-initial* target time $t' \neq 0$ to another target time t. An indivisible dynamical system will also generically lack any means of 'dividing up' the evolution from 0 to $t \neq 0$ into well-defined forms of evolution over intermediate time durations between 0 and t (even assuming that the set of times $\mathcal{T}$ has a notion of ordering). To make contact with the kinds of dynamical systems considered more widely in the research literature, it will therefore be necessary to introduce a somewhat less general class of mathematical structures.

This paper will define a Markovian (or divisible) dynamical system to be a tuple of the form

$$(\mathcal{X}, \mathcal{T}, g). \tag{11}$$

- Here $\mathcal{X}$ is a state space and $\mathcal{T}$ is a set of times (no longer called target times) forming a monoid, whose identity element 0 plays the role of an initial time, as usual.

- The symbol g denotes a map

  $$g : \mathcal{X} \times \mathcal{T}^2 \to \mathcal{X} \tag{12}$$

  that will be called the Markovian dynamical system's transition map. This transition map g takes as inputs any state i and any pair of times t, t', and outputs a state $g(i, (t, t')) \in \mathcal{X}$:

  $$i, t, t' \mapsto g\big(i, \big(t, t'\big)\big) \in \mathcal{X} \qquad \big[\text{for all } i \in \mathcal{X},\ t, t' \in \mathcal{T}\big]. \tag{13}$$

  The transition map g here should be understood as describing the evolution or transition of the state i at the time t' to the state $g(i, (t, t'))$ at the time t.

- Fixing two times t, t' turns g into a time-dependent transition map

  $$g_{t \leftarrow t'} : \mathcal{X} \to \mathcal{X} \tag{14}$$

defined by

$$i \mapsto g_{t \leftarrow t'}(i) \equiv g\big(i, (t, t')\big) \qquad (15)$$
$$[\text{for all } i \in \mathcal{X}].$$

This time-dependent transition map will be required to trivialize to the identity map $\mathrm{id}_{\mathcal{X}}$ on $\mathcal{X}$ when $t' = t$,

$$g_{t \leftarrow t} = \mathrm{id}_{\mathcal{X}} \quad [\text{for all } t \in \mathcal{T}], \qquad (16)$$
$$\text{or} \quad g_{t \leftarrow t}(i) = i \quad [\text{for all } i \in \mathcal{X},\, t \in \mathcal{T}],$$

as well as satisfy the 'divisibility condition'

$$g_{t \leftarrow t''} = g_{t \leftarrow t'} \circ g_{t' \leftarrow t''} \qquad (17)$$
$$\big[\text{for all } t, t', t'' \in \mathcal{T}\big],$$

where $\circ$ denotes function composition. The divisibility condition means that the time-dependent transition map $g_{t \leftarrow t''}$ factorizes or 'divides up' into a part $g_{t' \leftarrow t''}$ that carries out the evolution from t'' to t', followed by a part $g_{t \leftarrow t'}$ that carries out the evolution from t' to t.

- Choosing the time t' in the time-dependent transition map $g_{t \leftarrow t'}$ to be the initial time 0 naturally defines a time-dependent dynamical map (4),

$$f_t \equiv g_{t \leftarrow 0} \quad [\text{for all } t \in \mathcal{T}], \qquad (18)$$

  which then also defines an overall dynamical map $f : \mathcal{X} \times \mathcal{T} \to \mathcal{X}$ according to $f(i, t) \equiv f_t(i)$. The trivialization condition $g_{0 \leftarrow 0} = \mathrm{id}_{\mathcal{X}}$ from (16) ensures that f_t satisfies the corresponding trivialization condition $f_0 = \mathrm{id}_{\mathcal{X}}$ in (6). It follows that every Markovian dynamical system is, in particular, a special case of an indivisible dynamical system, one that includes the additional structure that corresponds to having a transition map g.

- Meanwhile, setting $t'' = 0$ in the divisibility condition (17) on the transition map g yields the subsidiary divisibility condition

$$f_t = g_{t \leftarrow t'} \circ f_{t'} \quad \big[\text{for all } t, t' \in \mathcal{T}\big], \qquad (19)$$

  which means that the time-dependent dynamical map f_t 'divides up' into a part $f_{t'}$ that carries out the evolution from the initial time 0 to t', followed by a part $g_{t \leftarrow t'}$ that carries out the evolution from t' to t.

- It also follows from the subsidiary divisibility condition (19), together with the trivialization condition $f_0 = \mathrm{id}_{\mathcal{X}}$ in (6), that the time-dependent dynamical map f_t has a well-defined inverse f_t^{-1} given by

$$f_t^{-1} = g_{0\leftarrow t} \quad [\text{for all } t \in \mathcal{T}]. \tag{20}$$

That is, a Markovian dynamical system's time-dependent dynamical maps f_t are invertible.[4]

2.3 Markovian-Homogeneous Dynamical Systems

A Markovian dynamical system will be called *Markovian-homogeneous* (or *time-homogeneous*) if it has the special property

$$g_{t\star t'\leftarrow t'} = g_{t\leftarrow 0} \quad \left[\text{for all } t, t' \in \mathcal{T}\right]. \tag{21}$$

Equivalently, using $f_t \equiv g_{t\leftarrow 0}$ from (18), one can write this special property as

$$g_{t\star t'\leftarrow t'} = f_t \quad \left[\text{for all } t, t' \in \mathcal{T}\right]. \tag{22}$$

It then follows immediately from the subsidiary divisibility condition (19) that the dynamical map $f : \mathcal{X} \times \mathcal{T} \to \mathcal{X}$ gives a semigroup action of the set of times $\mathcal{T}$ on the state space $\mathcal{X}$, in the sense that the semigroup operation $\star$ on $\mathcal{T}$ is mapped to function composition:

$$f_{t\star t'} = f_t \circ f_{t'} \quad \left[\text{for all } t, t' \in \mathcal{T}\right]. \tag{23}$$

This equation is an example of homogeneity in time, which implies that time evolution depends only on duration and not on absolute times.

Notice that the homogeneity property (23) is phrased directly in terms of a dynamical map f, so it can be imposed on an indivisible dynamical system $(\mathcal{X}, \mathcal{T}, f)$, without any need to invoke a transition map g. In that case, it also follows that there is no longer a meaningful distinction between an indivisible dynamical system $(\mathcal{X}, \mathcal{T}, f)$ and a Markovian dynamical system $(\mathcal{X}, \mathcal{T}, g)$. As such, if the homogeneity property (23) is imposed, then the resulting mathematical structure will simply be called a Markovian-homogeneous dynamical system.

What most references call a dynamical system corresponds to what this paper would call a Markovian-homogeneous dynamical system. That is, a dynamical system, without any further qualifiers, is a tuple $(\mathcal{X}, \mathcal{T}, f)$ for which $\mathcal{X}$ is a state space, $\mathcal{T}$ is a set of times forming a monoid, and $f : \mathcal{X} \times \mathcal{T} \to \mathcal{X}$ is a dynamical map satisfying the homogeneity property (23). It follows that an indivisible dynamical system (1) is a generalization of the kinds of dynamical systems that are usually considered in textbooks, in the research literature, and in many applications.

In the most general cases, however, indivisible dynamical systems will not satisfy the homogeneity property (23). Nor will it necessarily be possible to obtain a given indivisible dynamical system $(\mathcal{X}, \mathcal{T}, f)$ by starting with a Markovian dynamical system $(\mathcal{X}, \mathcal{T}, g)$ and then defining a dynamical map f according to $f_t \equiv g_{t\leftarrow 0}$ for all times t, as in (18). Indeed, in the case in which there do not

[4]If the set of times $\mathcal{T}$ has a well-defined ordering relation, and if one modifies the definition of a Markovian dynamical system $(\mathcal{X}, \mathcal{T}, g)$ by restricting the transition map (12) so that $g_{t\leftarrow t'}$ is only defined when t' comes before t according to that ordering relation, then the arguments leading to (20) will break down. In that case, the time-dependent dynamical maps f_t will not necessarily have inverses.

exist inverse time-dependent dynamical maps f_t^{-1}, then, in light of the inversion formula (20), there would be an immediate obstruction to deriving a dynamical map f from a transition map g.

An indivisible dynamical system $(\mathcal{X}, \mathcal{T}, f)$ that cannot be derived from a Markovian dynamical system $(\mathcal{X}, \mathcal{T}, g)$ will lack the structure needed to divide up its time evolution into intermediate durations.

3 Stochastic Processes

3.1 Indivisible Stochastic Processes

This paper will be concerned primarily with a mathematical structure that replaces the deterministic behavior of an indivisible dynamical system (1) with probabilistic, or stochastic, behavior. An indivisible stochastic process will be defined to mean a tuple of the form

$$(\mathcal{C}, \mathcal{T}, \mathcal{T}_0, \Gamma, p, \mathcal{A}) \tag{24}$$

that consists of the following data.

- The symbol $\mathcal{C}$ denotes a set called the system's configuration space, and the elements of $\mathcal{C}$ are called the system's (allowed) configurations. Configurations and configuration spaces will play an analogous role for indivisible stochastic processes that states and state spaces play for indivisible dynamical systems.

- The reason for switching the terminology from 'states' to 'configurations' is conceptual. In applications of the theory of dynamical systems to physical situations, such as in classical Hamiltonian mechanics, the notion of a 'state' is often taken to include rates of change or momenta in addition to configurations, because defining a 'state' in that way can make it possible to obtain deterministic laws in the form of first-order differential equations. For an indivisible stochastic process, by contrast, the probabilistic laws may mean that rates of change and momenta are not well-defined in the absence of specifying a trajectory. In that case, the only available notion of a 'state' is limited to the more rudimentary notion of a 'configuration,' which is more like a static arrangement of things.

- For the purposes of this paper, $\mathcal{C}$ will be assumed to be a finite set, with some (possibly very large) number N of elements denoted by $1, 2, 3, \ldots, N$, and indexed by Latin letters $i, j, \ldots$. For brevity, and by an abuse of notation, the symbol $\mathcal{C}$ will sometimes be used to refer to the indivisible stochastic process as a whole.

- The symbol $\mathcal{T}$ denotes the system's set of target times, including a time 0 that will be called the system's initial time.

- The symbol $\mathcal{T}_0$ denotes the system's set of conditioning times, and is taken to be a subset $\mathcal{T}_0 \subset \mathcal{T}$ of the set of target times. Without any real loss of generality, $\mathcal{T}_0$ will be assumed to include the initial time 0. In practical cases, $\mathcal{T}_0$ will often be assumed to be a 'sparse' subset of $\mathcal{T}$, in the sense that $\mathcal{T}$ will contain many times not in $\mathcal{T}_0$.

- The symbol Γ denotes a map

$$\Gamma : \mathcal{C}^2 \times \mathcal{T} \times \mathcal{T}_0 \to [0,1] \subset \mathbb{R} \tag{25}$$

that will be called the transition map of the indivisible stochastic process, where $[0,1]$ denotes the closed unit interval $0 \leq x \leq 1$. Each value of this transition map will be labeled as

$$\Gamma_{ij}(t \leftarrow t_0) \equiv \Gamma((i,j),(t,t_0)) \tag{26}$$
$$[\text{for all } i,j \in \mathcal{C},\ t \in \mathcal{T},\ t_0 \in \mathcal{T}_0],$$

and will be called the conditional transition probability $p(i,t|j,t_0)$ for the system to be in its ith configuration at the target time t, given that the system is in its jth configuration at the conditioning time t_0:

$$\Gamma_{ij}(t \leftarrow t_0) \equiv p(i,t|j,t_0) \in [0,1] \tag{27}$$
$$[\text{for all } i,j \in \mathcal{C},\ t \in \mathcal{T},\ t_0 \in \mathcal{T}_0].$$

- The transition map Γ is required to satisfy the standard normalization condition

$$\sum_{i=1}^{N} \Gamma_{ij}(t \leftarrow t_0) = \sum_{i=1}^{N} p(i,t|j,t_0) = 1 \tag{28}$$
$$[\text{for all } j \in \mathcal{C},\ t \in \mathcal{T},\ t_0 \in \mathcal{T}_0],$$

as well as the trivialization condition

$$\Gamma_{ij}(t_0 \leftarrow t_0) \equiv \delta_{ij} \equiv \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j \end{cases} \tag{29}$$
$$[\text{for all } t_0 \in \mathcal{T}_0],$$

where δ_{ij} is the usual Kronecker delta.

- The symbol p denotes a map

$$p : \mathcal{C} \times \mathcal{T} \to [0,1] \subset \mathbb{R} \tag{30}$$

that will be called the system's standalone probability distribution. Each value of this map will be labeled as

$$p_i(t) \equiv p(i,t) \in [0,1] \tag{31}$$
$$[\text{for all } i \in \mathcal{C},\ t \in \mathcal{T}],$$

and will be called the standalone probability for the system to be in its ith configuration at the target time t.

- Only the standalone probabilities $p_1(0), \ldots, p_N(0)$ at the initial time 0 will be taken to be freely adjustable, subject to the standard normalization condition

$$\sum_{j=1}^{N} p_j(0) = 1. \tag{32}$$

The standalone probabilities at every other target time t will be assumed to be defined by the following law of total probability or marginalization condition:

$$\begin{aligned} p_i(t) &\equiv \sum_{j=1}^{N} \Gamma_{ij}(t \leftarrow 0) p_j(0) \\ &= \sum_{j=1}^{N} p(i, t|j, 0) p_j(0) \\ &\quad [\text{for all } i \in \mathcal{C},\ t \in \mathcal{T}]. \end{aligned} \tag{33}$$

The normalization condition (28) on the transition map Γ and the normalization condition (32) on the standalone probabilities at the initial time 0 then together ensure that the probability distribution p satisfies the standard normalization condition more generally:

$$\sum_{i=1}^{N} p_i(t) = 1 \quad [\text{for all } t \in \mathcal{T}]. \tag{34}$$

- Note that the definition of the transition map Γ is independent of the choice of standalone probabilities $p_1(0), \ldots, p_N(0)$ at the initial time 0. That is, Γ can be freely adjusted independently of those initial standalone probabilities. Importantly, the law of total probability (33) therefore defines a *linear relationship* between the standalone probabilities $p_1(0), \ldots, p_N(0)$ at the initial time 0 and the standalone probabilities $p_1(t), \ldots, p_N(t)$ at any target time t. In the work ahead, it will be argued that this linear relationship is ultimately responsible for the linearity of time evolution in quantum theory.

- The transition map Γ will be assumed to satisfy the following divisibility condition for any target time $t \in \mathcal{T}$ and any pair of conditioning times $t_0, t' \in \mathcal{T}_0 \subset \mathcal{T}$:

$$\begin{aligned} \Gamma_{ij}(t \leftarrow t_0) &= \sum_{k=1}^{N} \Gamma_{ik}\big(t \leftarrow t'\big) \Gamma_{kj}\big(t' \leftarrow t_0\big) \\ &\quad \big[\text{for all } i, j \in \mathcal{C},\ t \in \mathcal{T},\ t_0, t' \in \mathcal{T}_0\big]. \end{aligned} \tag{35}$$

As such, conditioning times like t' will alternatively be called division events.

- If t' is a target time but *not* a conditioning time, then the values $\Gamma_{ik}(t \leftarrow t')$ appearing in the divisibility condition (35) will not be well-defined, and the divisibility condition will not hold.[5] The process described here is therefore indivisible for generic target times t', in a sense that parallels the notion of indivisibility for an indivisible dynamical system, as introduced earlier in this paper.

- By combining the marginalization condition (33) with the divisibility condition (35), it follows that the standalone probabilities $p_1(t), \dots, p_N(t)$ at any target time $t \in \mathcal{T}$ and the standalone probabilities $p_1(t'), \dots, p_N(t')$ at any conditioning time t' are related by the following marginalization condition of their own:

$$p_i(t) = \sum_{j=1}^{N} \Gamma_{ij}\big(t \leftarrow t'\big) p_j\big(t'\big) \tag{36}$$

$$= \sum_{j=1}^{N} p\big(i, t|j, t'\big) p_j\big(t'\big) \tag{37}$$

$$\big[\text{for all } i \in \mathcal{C},\ t \in \mathcal{T},\ t' \in \mathcal{T}_0\big]. \tag{38}$$

- The transition map Γ will *not* be assumed to satisfy anything like a homogeneity property. More precisely, no assumption will be made that given any pair of target times t, t', there will exist a target time t'' such that the following homogeneity property holds:

$$\Gamma_{ij}(t \leftarrow 0) = \sum_{k=1}^{N} \Gamma_{ik}\big(t'' \leftarrow 0\big) \Gamma_{kj}\big(t' \leftarrow 0\big) \tag{39}$$

$$[\text{for all } i, j \in \mathcal{C}].$$

That is, an indivisible stochastic process will generically be non-Markovian.

- These notions of indivisibility and non-Markovianity represent a distinct way to generalize the Markovian case, as compared with most forms of non-Markovianity described in textbooks and in the research literature. According to those more traditional forms of non-Markovianity, as emphasized by Gillespie (1998, 2000), one assumes that the set of target times $\mathcal{T}$ has an ordering relation, and one further assumes the existence of higher-order conditional probabilities $p(i, t|j_1, t_1; j_2, t_2; \dots)$ that are conditioned on arbitrarily many conditioning times $t_1, t_2, \dots$. From that more traditional standpoint, the system is Markovian if and only if the latest conditioning time t_1 always screens off all the earlier conditioning times $t_2, \dots$, so that $p(i, t|j_1, t_1; j_2, t_2; \dots) = p(i, t|j_1, t_1)$. The definition of an indivisible stochastic process presented in this paper does not assume fixed values of such higher-order conditional prob-

[5] Note that if the quantities $\Gamma_{kj}(t' \leftarrow t_0)$ are regarded as forming an $N \times N$ stochastic matrix, a term to be defined shortly, then the inverse matrix, if it exists, will always have negative entries unless both matrices are permutation matrices, as follows from an elementary theorem of linear algebra. It follows that one cannot safely define the non-negative quantities $\Gamma_{ik}(t \leftarrow t')$ by invoking inverse matrices.

abilities in the first place, and therefore, in a sense, represents a whole equivalence class of non-Markovian models, each of which is called a non-Markovian realizer. In particular, an indivisible stochastic process is defined by fixing less information than any of its non-Markovian realizers.

- The symbol $\mathcal{A}$ denotes a commutative algebra of maps of the form

$$A : \mathcal{C} \times \mathcal{T} \to \mathbb{R} \tag{40}$$

under the usual rules of function arithmetic, and will be called the system's algebra of random variables. The individual values of each random variable $A \in \mathcal{A}$ will be labeled as

$$a_i(t) \equiv A(i,t) \in \mathbb{R} \quad [\text{for all } i \in \mathcal{C},\, t \in \mathcal{T}]. \tag{41}$$

Each such value $a_i(t)$ will be called the magnitude or value of the random variable A when the system is in its ith configuration at the target time t. No assumption will be made here that these magnitudes are all distinct, even at any fixed target time t.

- For the purposes of this paper, the algebra of random variables $\mathcal{A}$ will always be taken to be maximal, in the sense of containing every well-defined map of the form (40).

- Fixing the target time t, the expectation value $\langle A(t) \rangle$ of a random variable A will denote its statistical average according to the standalone probability distribution p at the target time t:

$$\langle A(t) \rangle \equiv \sum_{i=1}^{N} a_i(t) p_i(t). \tag{42}$$

3.2 Ingredients from Linear Algebra

Given an indivisible stochastic process $(\mathcal{C}, \mathcal{T}, \mathcal{T}_0, \Gamma, p, \mathcal{A})$, with $\mathcal{C}$ a configuration space of finite integer size N, it will be convenient to introduce some formalism from linear algebra.

- Fixing a target time t, let $p(t)$, called the system's (time-dependent) probability vector, denote the $N \times 1$ column vector whose ith entry is the ith standalone probability $p_i(t)$:

$$p(t) \equiv \begin{pmatrix} p_1(t) \\ \vdots \\ p_N(t) \end{pmatrix}. \tag{43}$$

- Fixing a target time t and a conditioning time t_0, let $\Gamma(t \leftarrow t_0)$, called the system's (time-dependent) transition matrix, denote the $N \times N$ matrix for which the entry in the ith row,

jth column is the conditional transition probability $\Gamma_{ij}(t \leftarrow t_0) \equiv p(i,t|j,t_0)$:

$$\Gamma(t \leftarrow t_0) \equiv \begin{pmatrix} \Gamma_{11}(t \leftarrow t_0) & \Gamma_{12}(t \leftarrow t_0) & \\ \Gamma_{21}(t \leftarrow t_0) & \ddots & \\ & & \Gamma_{NN}(t \leftarrow t_0) \end{pmatrix}$$
$$= \begin{pmatrix} p(1,t|1,t_0) & p(1,t|2,t_0) & \\ p(2,t|1,t_0) & \ddots & \\ & & p(N,t|N,t_0) \end{pmatrix}. \tag{44}$$

It follows that for any target time t and any conditioning time t_0, the transition matrix $\Gamma(t \leftarrow t_0)$ is a (column) stochastic matrix in the mathematical sense, meaning that its entries are all non-negative real numbers,

$$\Gamma_{ij}(t \leftarrow t_0) \geq 0 \quad [\text{for all } i,j \in \mathcal{C},\ t \in \mathcal{T},\ t_0 \in \mathcal{T}_0], \tag{45}$$

and that its columns each sum to 1, as required in (28). The trivialization condition (29) on the transition map Γ then becomes the statement that the transition matrix $\Gamma(t_0 \leftarrow t_0)$ for any conditioning time t_0 is just the $N \times N$ identity matrix $\mathbb{1}$:

$$\Gamma(t_0 \leftarrow t_0) = \mathbb{1} \equiv \begin{pmatrix} 1 & 0 & \\ 0 & \ddots & \\ & & 1 \end{pmatrix}. \tag{46}$$

Observe that the law of total probability (36) for any target time t and any conditioning time t' naturally takes the form of matrix multiplication:

$$p(t) = \Gamma\big(t \leftarrow t'\big)p\big(t'\big) \quad \big[\text{for all } t \in \mathcal{T},\ t' \in \mathcal{T}_0\big]. \tag{47}$$

3.3 Connections with Other Constructions

There is a definite sense in which an indivisible stochastic process $(\mathcal{C}, \mathcal{T}, \mathcal{T}_0, \Gamma, p, \mathcal{A})$, as introduced in (24), provides a probabilistic extension of an indivisible dynamical system $(\mathcal{X}, \mathcal{T}, f)$, as introduced in (1). Indeed, if the set of conditioning times $\mathcal{T}_0$ contains only the initial time 0, and the transition map $\Gamma : \mathcal{C}^2 \times \mathcal{T} \times \mathcal{T}_0 \to [0,1]$ from (25) outputs only the trivial probabilities 1 and 0, then Γ is effectively deterministic. In that case, one can naturally set $\mathcal{X} \equiv \mathcal{C}$ and define a dynamical map $f : \mathcal{X} \times \mathcal{T} \to \mathcal{X}$ according to

$$f(j,t) = i \quad \text{if and only if} \quad \Gamma((i,j),(t,0)) = 1 \tag{48}$$
$$[\text{for all } i,j \in \mathcal{X},\ t \in \mathcal{T}].$$

An indivisible stochastic process is distinct from a mathematical structure already in the research literature, called a stochastic dynamical system or a random dynamical system (Honerkamp 1996, Arnold 1998).

- The definition of a random dynamical system starts with what this paper would call an indivisible dynamical system $(\mathcal{X}, \mathcal{T}, f)$, with $\mathcal{T}$ a monoid, and replaces the single dynamical map f with a probabilistic family or ensemble of dynamical maps $\{f_\omega \,|\, \omega \in \Omega\}$ that are indexed by a point ω in some sample space Ω. That is, a randomly sampled point ω in Ω picks out an entire dynamical map f_ω as a whole.

- For any randomly sampled point ω and any fixed target time t, one can define a map

$$f_{\omega,t} : \mathcal{X} \to \mathcal{X} \tag{49}$$

by

$$i \mapsto f_{\omega,t}(i) \equiv f_\omega(i,t) \quad [\text{for all } i \in \mathcal{X}]. \tag{50}$$

A random dynamical system is then assumed to satisfy an initial condition that generalizes (6),

$$f_{\omega,0} = \mathrm{id}_{\mathcal{X}} \quad [\text{for all } \omega \in \Omega], \tag{51}$$

as well as a homogeneity property that generalizes (23),

$$f_{\omega',t\star t'} = f_{\theta(\omega',t'),t} \circ f_{\omega',t'} \\ \big[\text{for all } \omega' \in \Omega,\, t, t' \in \mathcal{T}\big]. \tag{52}$$

Here $\theta : \Omega \times \mathcal{T} \to \Omega$ is a map that is part of the definition of the random dynamical system, and specifies a deterministic rule for updating the sample point ω' and the dynamical map $f_{\omega'}$ to accommodate replacing the original initial time 0 with the effectively new initial time t'. That is, θ is necessary to account for any deterministic evolution in the underlying source of randomness itself.

- It follows from the foregoing definitions that the conditional probability $p(i,t|j,0)$ for the system to be in its ith state at the target time t, given that the system is in its jth state at the initial time 0, is obtained by adding up the probabilities for all the points ω in the sample space Ω whose corresponding dynamical maps f_ω take the state j at the initial time 0 and yield the state i at the final time t:

$$p(i,t|j,0) = \text{probability}(\{\omega \in \Omega \,|\, f_{\omega,t}(j) = i\}) \\ [\text{for all } i, j \in \mathcal{X},\, t \in \mathcal{T}]. \tag{53}$$

In a sense, the absolutely *most general* kind of stochastic mathematical structure is simply called a stochastic process, and essentially requires only the specification of a state space $\mathcal{X}$, a set of target times $\mathcal{T}$, an initial probability distribution p, and a set of one or more time-dependent random variables $\mathcal{A}$.[6] Importantly, the definition of a stochastic process lacks the specification of a dynamical law. By requiring a dynamical law in the form of a transition map Γ, an indivisible stochastic process $(\mathcal{C}, \mathcal{T}, \mathcal{T}_0, \Gamma, p, \mathcal{A})$ is not quite as general as a stochastic process, but will still be general enough to encompass a large class of physical and mathematical models.

In particular, one can regard any Markov chain as a special case of an indivisible stochastic process. The starting place is to assume that the set of target times $\mathcal{T}$ and the set of conditioning times $\mathcal{T}_0$ are identical and are both isomorphic to the integers, $\mathcal{T} = \mathcal{T}_0 \cong \mathbb{Z}$, with each time $t = n\,\delta t$ an integer number $n \in \mathbb{Z}$ of steps of some fixed, elementary time scale δt. One then further assumes that for each integer n, the time-dependent transition matrix $\Gamma(n\,\delta t \leftarrow 0)$ originally defined in (44) can be expressed as the nth power of the transition matrix $\Gamma(\delta t) \equiv \Gamma(\delta t \leftarrow 0)$ that implements the evolution for just the first time step δt:

$$\Gamma(n\,\delta t \leftarrow 0) = [\Gamma(\delta t)]^n \quad [\text{for all } n \in \mathbb{Z}]. \tag{54}$$

More broadly, an indivisible stochastic process can therefore be understood as a kind of non-Markovian generalization of a Markov chain.

As this paper will also show, the class of indivisible stochastic processes essentially includes all quantum systems as well (at least those that have or can be approximated as having finite-dimensional Hilbert spaces).

3.4 Composite Systems and Subsystems

Introducing a notation of primes and tildes now to distinguish between different indivisible stochastic processes, an indivisible stochastic process $(\tilde{\mathcal{C}}, \tilde{\mathcal{T}}, \tilde{\mathcal{T}}_0, \tilde{\Gamma}, \tilde{p}, \tilde{\mathcal{A}})$ will be called a composite system if its configuration space $\tilde{\mathcal{C}}$ is naturally expressible as a nontrivial Cartesian product of two sets $\mathcal{C}$ and $\mathcal{C}'$:

$$\tilde{\mathcal{C}} = \mathcal{C} \times \mathcal{C}'. \tag{55}$$

A composite system has the following salient features.

- Letting N denote the size of $\mathcal{C}$, and letting N' denote the size of $\mathcal{C}'$, the composite system's configuration space $\tilde{\mathcal{C}}$ has size $\tilde{N} \equiv NN'$.

- Labeling the elements of $\mathcal{C}$ by unprimed Latin letters $i, j, \dots$, and labeling the elements of $\mathcal{C}'$ by primed Latin letters $i', j', \dots$, the composite system's transition map $\tilde{\Gamma} : \tilde{\mathcal{C}}^2 \times \tilde{\mathcal{T}} \times \tilde{\mathcal{T}}_0 \to [0,1]$ has individual values that will be denoted by

$$\begin{aligned} \tilde{\Gamma}_{ii',jj'}(t \leftarrow t_0) &\equiv \tilde{\Gamma}\big(\big((i,i'),(j,j')\big),(t,t_0)\big) \\ &\equiv \tilde{p}\big((i,i'),t|(j,j'),t_0\big) \\ &\quad [\text{for all } i,j \in \mathcal{C},\ i',j' \in \mathcal{C}',\ t \in \tilde{\mathcal{T}},\ t_0 \in \tilde{\mathcal{T}}_0]. \end{aligned} \tag{56}$$

[6] See Rosenblatt (1962), Parzen (1962), Doob (1990), or Ross (1995) for textbook treatments.

The normalization condition (28) on the transition map $\tilde{\Gamma}$ takes the form

$$\sum_{i=1}^{N}\sum_{i'=1}^{N'}\tilde{\Gamma}_{ii',jj'}(t \leftarrow t_0) = 1 \tag{57}$$
$$[\text{for all } j \in \mathcal{C},\, j' \in \mathcal{C}',\, t \in \tilde{\mathcal{T}},\, t_0 \in \tilde{\mathcal{T}}_0].$$

- The composite system's probability distribution $\tilde{p} : \tilde{\mathcal{C}} \times \tilde{\mathcal{T}} \to [0,1]$ has individual values at any fixed target time t that will be denoted by

$$\tilde{p}_{ii'}(t) \equiv p\big((i,i'),t\big) \tag{58}$$
$$[\text{for all } i \in \mathcal{C},\, i' \in \mathcal{C}',\, t \in \tilde{\mathcal{T}}].$$

The normalization condition (34) on the probability distribution $\tilde{p}$ takes the form

$$\sum_{i=1}^{N}\sum_{i'=1}^{N'}\tilde{p}_{ii'}(t) = 1 \quad [\text{for all } t \in \tilde{\mathcal{T}}], \tag{59}$$

and the law of total probability (36) becomes

$$\tilde{p}_{ii'}(t) \equiv \sum_{j=1}^{N}\sum_{j'=1}^{N'}\tilde{\Gamma}_{ii',jj'}\big(t \leftarrow t'\big)\tilde{p}_{jj'}\big(t'\big) \tag{60}$$
$$[\text{for all } i \in \mathcal{C},\, i' \in \mathcal{C}',\, t \in \tilde{\mathcal{T}},\, t' \in \tilde{\mathcal{T}}_0].$$

Given a composite system $(\tilde{\mathcal{C}}, \tilde{\mathcal{T}}, \tilde{\mathcal{T}}_0, \tilde{\Gamma}, \tilde{p}, \tilde{\mathcal{A}})$ with configuration space $\tilde{\mathcal{C}} = \mathcal{C} \times \mathcal{C}'$, as in (55), the composite system's set of target times $\tilde{\mathcal{T}}$ trivially defines two sets of target times $\mathcal{T}$ and $\mathcal{T}'$ according to

$$\mathcal{T} \equiv \mathcal{T}' \equiv \tilde{\mathcal{T}}, \tag{61}$$

and the composite system's set of conditioning times $\tilde{\mathcal{T}}_0$ trivially defines two sets of conditioning times $\mathcal{T}_0$ and $\mathcal{T}'_0$ according to

$$\mathcal{T}_0 \equiv \mathcal{T}'_0 \equiv \tilde{\mathcal{T}}_0. \tag{62}$$

Meanwhile, the composite system's probability distribution $\tilde{p}$ defines two probability distributions $p : \mathcal{C} \times \mathcal{T} \to [0,1]$ and $p' : \mathcal{C}' \times \mathcal{T}' \to [0,1]$ according to the respective marginalization formulas

$$p_i(t) \equiv \sum_{i'=1}^{N'}\tilde{p}_{ii'}(t) \quad [\text{for all } i \in \mathcal{C},\, t \in \mathcal{T}], \tag{63}$$

$$p'_{i'}(t) \equiv \sum_{i=1}^{N}\tilde{p}_{ii'}(t) \quad [\text{for all } i' \in \mathcal{C}',\, t \in \mathcal{T}']. \tag{64}$$

Note that these two probability distributions p and p' do not necessarily turn $\mathcal{C}$ and $\mathcal{C}'$ into indivisible stochastic processes of their own, in the sense of (24), due to the generic lack of well-defined transition maps $\Gamma : \mathcal{C}^2 \times \mathcal{T} \times \mathcal{T}_0 \to [0,1]$ and $\Gamma' : \mathcal{C}'^2 \times \mathcal{T}' \times \mathcal{T}'_0 \to [0,1]$. Indeed, in place of the law of total probability (36), one instead has the relations

$$p_i(t) \equiv \sum_{i'=1}^{N'} \sum_{j=1}^{N} \sum_{j'=1}^{N'} \tilde{\Gamma}_{ii',jj'}(t \leftarrow t') \tilde{p}_{jj'}(t') \tag{65}$$

$$[\text{for all } i \in \mathcal{C},\ t \in \mathcal{T},\ t' \in \mathcal{T}_0],$$

$$p'_{i'}(t) \equiv \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{j'=1}^{N'} \tilde{\Gamma}_{ii',jj'}(t \leftarrow t') \tilde{p}_{jj'}(t') \tag{66}$$

$$[\text{for all } i' \in \mathcal{C}',\ t \in \mathcal{T}',\ t' \in \mathcal{T}'_0].$$

The two sets $\mathcal{C}$ and $\mathcal{C}'$ will be called the configuration spaces of subsystems of the composite system $(\tilde{\mathcal{C}}, \tilde{\mathcal{T}}, \tilde{\mathcal{T}}_0, \tilde{\Gamma}, \tilde{p}, \tilde{\mathcal{A}})$.

3.5 Unistochastic Processes

Recall that an $N \times N$ matrix U with complex-valued entries is called unitary if it satisfies

$$U^\dagger U = U U^\dagger = \mathbb{1}. \tag{67}$$

Here $\dagger$ denotes the adjoint operation, meaning complex conjugation combined with the transpose operation.

A stochastic matrix X is called unistochastic if each of its entries X_{ij} is the modulus-square $|U_{ij}|^2$ of the corresponding entry U_{ij} of a unitary matrix U:

$$X = \begin{pmatrix} X_{11} & X_{12} & \\ X_{21} & \ddots & \\ & & X_{NN} \end{pmatrix} = \begin{pmatrix} |U_{11}|^2 & |U_{12}|^2 & \\ |U_{21}|^2 & \ddots & \\ & & |U_{NN}|^2 \end{pmatrix}. \tag{68}$$

The study of unistochastic matrices was initiated in a 1954 paper (Horn 1954), which originally called them 'ortho-stochastic matrices.' Unistochastic matrices were given their modern name in 1989 (Thompson 1989; Nylen, Tam, Hulig 1993; Bengtsson 2004), and today orthostochastic matrices refer to the special case in which U can be taken to be a real-orthogonal matrix, meaning a unitary matrix with entries that are all real-valued.

Every unistochastic matrix is doubly stochastic or bistochastic, meaning that its columns *and* its rows each sum to 1.[7] All 2×2 doubly stochastic matrices are unistochastic,[8] but this equivalence

[7] Proof: $\sum_i |V_{ij}|^2 = \sum_i \overline{V_{ij}} V_{ij} = [V^\dagger V]_{jj} = 1$ and $\sum_j |V_{ij}|^2 = \sum_j V_{ij} \overline{V_{ij}} = [V V^\dagger]_{ii} = 1$, where overlines denote complex conjugation. QED

[8] Proof: Every 2×2 doubly stochastic matrix is of the form $\left(\begin{smallmatrix} x & 1-x \\ 1-x & x \end{smallmatrix}\right)$, where $0 \le x \le 1$, and the matrix $\left(\begin{smallmatrix} \sqrt{x} & -\sqrt{1-x} \\ \sqrt{1-x} & \sqrt{x} \end{smallmatrix}\right)$ is easily seen to be unitary. QED

does not extend beyond the 2×2 case.[9] Moreover, beyond the 2×2 case, not every unistochastic matrix is orthostochastic.[10]

An indivisible stochastic process $(\mathcal{C}, \mathcal{T}, \mathcal{T}_0, \Gamma, p, \mathcal{A})$ whose transition map Γ defines a unistochastic matrix $\Gamma(t \leftarrow t_0)$ at every target time t and conditioning time t_0 will be called a unistochastic process. One of the main goals of this paper will be to establish that the study of indivisible stochastic processes essentially reduces to the study of unistochastic processes, which will also turn out to correspond to unitarily evolving quantum systems.

4 The Stochastic-Quantum Theorem

4.1 Statement of the Theorem

Focusing on the case of indivisible stochastic processes with finite configuration spaces (leaving the more general case to future work), this paper's next major goal will be to present a self-contained proof of the following theorem, which is a new result that implicitly also appeared in other work (Barandes 2025).

The Stochastic-Quantum Theorem

Every indivisible stochastic process can be regarded as a subsystem of a unistochastic process. (69)

Remarkably, to study the class of indivisible stochastic process, this theorem implies that it suffices to restrict one's attention to the subclass of unistochastic processes.

4.2 Corollaries of the Theorem

The proof of the stochastic-quantum theorem (69) will involve the construction of a representation of the given indivisible stochastic process in the formalism of Hilbert spaces, and will show that every indivisible stochastic process corresponds to a unitarily evolving quantum system in a Hilbert space. One thereby turns some of the puzzling axiomatic ingredients of quantum theory—the complex numbers,[11] Hilbert spaces, linear-unitary time evolution, and the Born rule in particular—into the output of a theorem.

It follows from the stochastic-quantum theorem that all indivisible stochastic processes (at least

[9] For example, the 3×3 doubly stochastic matrix $\frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ is provably not unistochastic.

[10] For example, the 3×3 matrix $\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & z & z^2 \\ 1 & z^2 & z \end{pmatrix}$ is unitary for $z = \exp(2\pi i/3)$, so the 3×3 matrix $\frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ is unistochastic. But this unistochastic matrix is not orthostochastic, because there does not exist a set of three mutually orthogonal 3×1 column vectors that feature only 1s and -1s as entries.

[11] Time-reversal transformations for quantum systems are carried out by anti-unitary operators, which take the general form VK, where V is unitary and K is an abstract operator that carries out complex-conjugation. By definition, $K^2 = 1$ and $Ki = -iK$, where $i \equiv \sqrt{-1}$ is the imaginary unit. One can therefore show that i, K, and iK generate a Clifford algebra called the pseudo-quaternions (Stueckelberg 1960). As a consequence, there is a sense in which quantum systems are actually defined over the pseudo-quaternions, rather than merely over the complex numbers, although only the complex numbers are typically used in the construction of observables.

those based on finite configuration spaces) can be modeled in terms of unitarily evolving quantum systems. From this perspective, unitarily evolving quantum systems actually represent the most general way to model a system with stochastic dynamical laws.

As shown in other work (Barandes 2025), one can go in the other logical direction and show that any comprehensive quantum system that includes measuring devices and observers as part of the system can be modeled as an indivisible stochastic process. Altogether, one thereby arrives at an important new correspondence between indivisible stochastic processes and quantum systems, called the stochastic-quantum correspondence, that provides a new way to think about the physical meaning of quantum theory.

4.3 A Simple Example

The notion of embedding an indivisible stochastic process into a unistochastic process does not always require an elaborate construction. As a simple example, consider a discrete Markovian-homogeneous dynamical system $(\mathcal{X}, \mathcal{T}, f)$ whose state space $\mathcal{X}$ has some finite size N, whose set of target times $\mathcal{T}$ is isomorphic to the integers $\mathbb{Z}$ under addition, and whose dynamical map $f : \mathcal{X} \times \mathcal{T} \to \mathcal{X}$ satisfies the appropriate version of the homogeneity property (23):

$$f_{t+t'} = f_t \circ f_{t'} \quad \big[\text{for all } t, t' \in \mathcal{T} \cong \mathbb{Z}\big]. \tag{70}$$

Based on these definitions and assumptions, it follows from the homogeneity property and the trivialization condition $f_0 = \mathrm{id}_{\mathcal{X}}$ from (6) that the time-dependent dynamical map f_t is invertible for any time $t \in \mathcal{T}$, with inverse given by

$$f_t^{-1} = f_{-t} \quad [\text{for all } t \in \mathcal{T} \cong \mathbb{Z}]. \tag{71}$$

Markovian-homogeneous dynamical systems of this kind provide a discretization of many of the kinds of deterministic, logically time-reversible systems that show up in classical physics.

Letting δt be the elementary duration of a single time step, and letting $n \in \mathbb{Z}$ denote an integer number of time steps, the system's specific state i at the time $t = n\,\delta t$ can be identified with an $N \times 1$ column vector with a 1 in its ith entry and 0s in all its other entries. Moreover, the time-dependent dynamical map $f_{n\,\delta t}$ can be expressed as the nth power of a fixed permutation matrix Σ, meaning a matrix consisting of only 1s and 0s, with a single 1 in each row and in each column.

Every permutation matrix is unitary, and is also unistochastic, because computing the modulus-squares of 1s and 0s gives back 1s and 0s. Hence, a discrete Markovian-homogeneous dynamical system defined in this way is already a unistochastic process, and therefore trivially satisfies the stochastic-quantum theorem (69).

Interestingly, there exists an analytic interpolation of this *discrete-time* unistochastic process that yields a corresponding *continuous-time* unistochastic process, at the cost of introducing non-trivial stochasticity into the intervals between the discrete time steps. This new unistochastic process is based on a time-dependent unistochastic matrix $\Gamma(t \leftarrow 0)$ whose entries are the modulus-squares of the corresponding entries of the $N \times N$ unitary time-evolution operator defined by $U(t \leftarrow 0) \equiv \Sigma^{t/\delta t}$, and therefore satisfies $\Gamma(n\,\delta t \leftarrow 0) = \Sigma^n$ for any integer n. One can even go on to define an $N \times N$ self-adjoint Hamiltonian H as the infinitesimal generator of $U(t \leftarrow 0)$, so there is ultimately a Schrödinger equation underlying this system.[12]

5 Proof of the Theorem

5.1 The Time-Evolution Operator

To commence the proof of the stochastic-quantum theorem (69), one starts with a given indivisible stochastic process $(\mathcal{C}, \mathcal{T}, \mathcal{T}_0, \Gamma, p, \mathcal{A})$ with a finite configuration space $\mathcal{C}$ of size N. Singling out one conditioning time t_0, which will be taken to be the initial time 0 for convenience, the non-negativity (45) of the system's conditional transition probabilities, $\Gamma_{ij}(t \leftarrow 0) \geq 0$, means that each transition probability can be written as the modulus-square of a non-unique complex number $\Theta_{ij}(t \leftarrow 0)$:

$$\Gamma_{ij}(t \leftarrow 0) = |\Theta_{ij}(t \leftarrow 0)|^2 \quad [\text{for all } i, j \in \mathcal{C},\, t \in \mathcal{T}]. \tag{72}$$

It is worth emphasizing that this formula is an identity, not a postulate. Any non-negative real number can be written non-uniquely as the modulus-square of a complex number.

For each fixed target time t, the complex numbers $\Theta_{ij}(t \leftarrow 0)$ collectively form their own $N \times N$ matrix $\Theta(t \leftarrow 0)$, which one can think of as a non-unique 'potential matrix' for $\Gamma_{ij}(t \leftarrow 0)$, and will be called the system's time-evolution operator:

$$\Theta(t \leftarrow 0) \equiv \begin{pmatrix} \Theta_{11}(t \leftarrow 0) & \Theta_{12}(t \leftarrow 0) & \\ \Theta_{21}(t \leftarrow 0) & \ddots & \\ & & \Theta_{NN}(t \leftarrow 0) \end{pmatrix}. \tag{73}$$

(As a concession to terminological conventions, the terms 'matrix' and 'operator' will be used more-or-less interchangeably in what follows.)

The normalization condition (28) on the system's transition matrix $\Gamma(t \leftarrow 0)$ then becomes the summation condition

$$\sum_{i=1}^{N} |\Theta_{ij}(t \leftarrow 0)|^2 = 1 \quad [\text{for all } j \in \mathcal{C},\, t \in \mathcal{T}], \tag{74}$$

which can roughly be regarded as a generalization of a unitarity constraint. In keeping with the trivialization condition $\Gamma(0 \leftarrow 0) = \mathbb{1}$ from (46), the time-evolution operator $\Theta(0 \leftarrow 0)$ for $t = 0$

[12] To define the unitary matrix $\Sigma^{t/\delta t}$, one starts by using the fact that every permutation matrix is also unitary to write the permutation matrix Σ as $V^\dagger D V$ for some unitary matrix V and some diagonal matrix D, where the entries of D are the eigenvalues of Σ. Because Σ is unitary, its eigenvalues are all phase factors $\exp(i\theta_m)$, where $m = 1, \ldots, N$, and where each phase θ_m is a real number. Setting all the eigenvalues to the power $t/\delta t$, so that they each take the new form $\exp(i\theta_m t/\delta t)$, one ends up with the matrix $\Sigma^{t/\delta t} \equiv V^\dagger D^{t/\delta t} V$, which is still unitary and now depends analytically on the time parameter t. The quantum-theoretic Hamiltonian matrix H for this system then has energy eigenvalues defined by $E_m \equiv -\hbar\theta_m/\delta t$ for $m = 1, \ldots, N$, and is diagonalized by the same unitary matrix V that diagonalizes Σ.

will be taken to be the $N \times N$ identity matrix $\mathbb{1}$:

$$\Theta(0 \leftarrow 0) \equiv \mathbb{1} \equiv \begin{pmatrix} 1 & 0 & \\ 0 & \ddots & \\ & & 1 \end{pmatrix}. \tag{75}$$

5.2 The Hilbert Space

The time-evolution operator $\Theta(t \leftarrow 0)$ acts on an N-dimensional Hilbert space $\mathcal{H}$ defined as the space $\mathbb{C}^N$ of $N \times 1$ column vectors with complex-valued entries,

$$\mathcal{H} \equiv \mathbb{C}^N, \tag{76}$$

together with the standard inner product $v^\dagger w$ for all $v, w \in \mathbb{C}^N$. The standard orthonormal basis $e_1, \ldots, e_N$ is defined by

$$e_1 \equiv \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \ldots, \quad e_N \equiv \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{77}$$

These vectors are labeled as e_i, where each value of $i = 1, \ldots, N$ denotes a configuration in the system's configuration space $\mathcal{C}$, so this basis will be called the system's configuration basis.

There exists an associated set of rank-one projectors $P_1, \ldots, P_N$ defined by

$$P_i \equiv e_i e_i^\dagger = \mathrm{diag}(0, \ldots, 0, \underset{\substack{\uparrow \\ i\text{th entry}}}{1}, 0, \ldots, 0) \quad [\text{for all } i \in \mathcal{C}], \tag{78}$$

and called configuration projectors. These configuration projectors $P_1, \ldots, P_N$ form a projection-valued measure (PVM) (Mackey 1952, 1957), meaning that they satisfy the conditions of mutual exclusivity,

$$P_i P_j = \delta_{ij} P_i \quad [\text{for all } i, j \in \mathcal{C}], \tag{79}$$

with δ_{ij} again the usual Kronecker delta, and completeness,

$$\sum_{i=1}^{N} P_i = \mathbb{1}, \tag{80}$$

with $\mathbb{1}$ again the $N \times N$ identity matrix.

5.3 The Dictionary

It follows from a short calculation that the relationship $\Gamma_{ij}(t \leftarrow 0) = |\Theta_{ij}(t \leftarrow 0)|^2$ in the identity (72) can be expressed in terms of a matrix trace as

$$\boxed{\Gamma_{ij}(t \leftarrow 0) = \mathrm{tr}\big(\Theta^\dagger(t \leftarrow 0) P_i \Theta(t \leftarrow 0) P_j\big)} \tag{81}$$

$$[\text{for all } i, j \in \mathcal{C},\ t \in \mathcal{T}].$$

This 'dictionary' essentially translates between the theory of indivisible stochastic processes, as expressed by the left-hand side, and a corresponding Hilbert-space representation, as expressed by the right-hand side. This equation turns out to provide the foundation for a stochastic-quantum correspondence (Barandes 2025) and will play a crucial role in the proof of the stochastic-quantum theorem (69).

5.4 The Density Matrix

Fixing the conditioning time t_0 to be the initial time 0 and inserting the dictionary (81) into the law of total probability (33) yields the following equation:

$$p_i(t) = \mathrm{tr}(P_i \rho(t)) \quad [\text{for all } i \in \mathcal{C},\ t \in \mathcal{T}]. \tag{82}$$

Here $\rho(t)$, which will be called the system's density matrix, is an $N \times N$ time-dependent, self-adjoint, unit-trace, generically non-diagonal matrix defined for any target time t by

$$\rho(t) \equiv \Theta(t \leftarrow 0)\rho(0)\Theta^\dagger(t \leftarrow 0), \tag{83}$$

where its value at the initial time 0 is the following $N \times N$ diagonal matrix:

$$\rho(0) \equiv \sum_{j=1}^{N} p_j(0) P_j = \mathrm{diag}(p_1(0), \dots, p_N(0)) = \begin{pmatrix} p_1(0) & 0 & \\ 0 & \ddots & \\ & & p_N(0) \end{pmatrix}. \tag{84}$$

Importantly, one sees from this analysis that the linearity of the law of total probability (33) underlies the linearity of the relationship (83) between the system's density matrix $\rho(0)$ at the initial time 0 and its density matrix $\rho(t)$ at target times t.

For any target time t, one can similarly express the expectation value (42) of a random variable A as

$$\langle A(t) \rangle = \mathrm{tr}(A(t)\rho(t)). \tag{85}$$

Here $A(t)$ denotes the $N \times N$ diagonal matrix

$$A(t) \equiv \sum_{i=1}^{N} a_i(t) P_i = \mathrm{diag}(a_1(t), \dots, a_N(t)) = \begin{pmatrix} a_1(t) & 0 & \\ 0 & \ddots & \\ & & a_N(t) \end{pmatrix}. \tag{86}$$

Observe, in particular, that the magnitudes $a_1(t), \dots, a_N(t)$ of the random variable A become the eigenvalues of the $N \times N$ matrix $A(t)$.

Notice also that the formula for the standalone probability $p_i(t)$ in (82) and the formula for the expectation value $\langle A(t) \rangle$ in (85) are both special cases of the Born rule.

5.5 An Aside on 'Classical Wave Functions'

Pausing for a moment, recall the smooth unistochastic interpolation of a discrete Markovian-homogeneous dynamical system described in Subsection 4.3. For that system, the unitary time-evolution operator $U(t \leftarrow 0) \equiv \Sigma^{t/\delta t}$ trivializes to a permutation matrix Σ^n at every integer time step $n\,\delta t$. It follows that the system's density matrix $\rho(t)$, as defined in terms of its initial value $\rho(0)$ from (83), reduces to a diagonal matrix $\rho(n\,\delta t)$ at each integer time step. Taking the square root of each of its diagonal entries $p_1(n\,\delta t), \dots, p_N(n\,\delta t)$, and allowing for arbitrary phase factors, one obtains a 'classical wave function' with components $\Psi_1(n\,\delta t), \dots, \Psi_N(n\,\delta t)$ satisfying $|\Psi_i(n\,\delta t)|^2 = p_i(n\,\delta t)$ and capturing precisely the same information as the diagonal density matrix $\rho(n\,\delta t)$.

This classical wave function is the starting place for a representation of classical deterministic physics known popularly as the 'Koopman-von Neumann' formulation, due to its superficial resemblance to work by Koopman (1931) and von Neumann (1932a, 1932b) in the 1930s. However, as pointed out explicitly by Jordan and Sudarshan (1961), and in other work (Barandes 2026), Koopman and von Neumann were actually using Hilbert spaces to represent *observables*, rather than to represent *probability distributions*. The formulation of classical physics in terms of 'classical wave functions' is more properly due to Schönberg (1953), Loinger (1962), Della Riccia and Wiener (1966), and Sudarshan (1976).

5.6 The Kraus Decomposition

Returning to the proof, fixing the target time t, and letting β denote an integer from 1 to N that will play a conceptually different role from a configuration index, let $K_\beta(t \leftarrow 0)$ denote the $N \times N$ matrix whose βth column agrees with the βth column of the time-evolution operator $\Theta(t \leftarrow 0)$, with 0s in all its other entries:

$$K_\beta(t \leftarrow 0) = \begin{pmatrix} 0 & \cdots & 0 & \Theta_{1\beta}(t \leftarrow 0) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \Theta_{2\beta}(t \leftarrow 0) & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \Theta_{N\beta}(t \leftarrow 0) & 0 & \cdots & 0 \end{pmatrix}. \tag{87}$$

That is, the entry in the ith row, jth column of $K_\beta(t \leftarrow 0)$ is given by

$$K_{\beta,ij}(t) \equiv (\Theta(t \leftarrow 0)P_\beta)_{ij} \equiv \delta_{\beta j}\Theta_{ij}(t) \quad [\text{for all } \beta, i, j \in \mathcal{C},\, t \in \mathcal{T}]. \tag{88}$$

It follows from the summation condition (74) on $\Theta(t \leftarrow 0)$ that these new matrices satisfy the Kraus condition:

$$\sum_{\beta=1}^{N} K_\beta^\dagger(t \leftarrow 0)K_\beta(t \leftarrow 0) = \mathbb{1} \quad [\text{for all } t \in \mathcal{T}]. \tag{89}$$

Moreover, one can write the dictionary (81) as

$$\Gamma_{ij}(t \leftarrow 0) = \sum_{\beta=1}^{N} \operatorname{tr}\big(K_\beta^\dagger(t \leftarrow 0)P_i K_\beta(t \leftarrow 0)P_j\big) \tag{90}$$
$$[\text{for all } i, j \in \mathcal{C},\, t \in \mathcal{T}],$$

and one can express the time-evolution rule (83) for the system's density matrix as

$$\rho(t) = \sum_{\beta=1}^{N} K_\beta(t \leftarrow 0)\rho(0)K_\beta^\dagger(t \leftarrow 0). \tag{91}$$

The matrices $K_1(t \leftarrow 0), \ldots, K_N(t \leftarrow 0)$ are therefore Kraus operators (Kraus 1971), and (91) gives a Kraus decomposition of $\rho(t)$.

Abstracting these results, one obtains a completely positive trace-preserving (CPTP) map, or quantum channel,

$$\mathcal{E}_{t \leftarrow 0} : \mathbb{C}^{N \times N} \to \mathbb{C}^{N \times N}, \tag{92}$$

defined on the algebra $\mathbb{C}^{N \times N}$ of $N \times N$ matrices over the complex numbers according to

$$X \mapsto \mathcal{E}_{t \leftarrow 0}(X) \equiv \sum_{\beta=1}^{N} K_\beta(t \leftarrow 0)XK_\beta^\dagger(t \leftarrow 0). \tag{93}$$

5.7 Dilation

By the Stinespring dilation theorem (Stinespring 1955, Keyl 2002), the quantum channel (92) can be purified, meaning made into a form of unitary time evolution. Specifically, for some integer $\tilde{N}$ in the *bounded* interval

$$N \leq \tilde{N} \leq N^3, \tag{94}$$

where $\tilde{N}$ is an integer multiple of N, there exists an $\tilde{N} \times \tilde{N}$ unitary matrix $\tilde{U}(t \leftarrow 0)$ on a potentially enlarged or dilated Hilbert space

$$\tilde{\mathcal{H}} \equiv \mathbb{C}^{\tilde{N}} \tag{95}$$

such that the dictionary (81) can be written as[13]

$$\Gamma_{ij}(t \leftarrow 0) = \mathrm{tr}\Big(\mathrm{tr}'\Big(\tilde{U}^\dagger(t \leftarrow 0)[P_i \otimes \mathbb{1}']\tilde{U}(t \leftarrow 0)[P_j \otimes P'_{j'}]\Big)\Big) \tag{96}$$
$$[\text{for all } i, j \in \mathcal{C},\ t \in \mathcal{T}].$$

The formula (96) involves a number of ingredients.

- The dilated Hilbert space $\tilde{\mathcal{H}}$ is defined as the tensor product

$$\tilde{\mathcal{H}} \equiv \mathcal{H} \otimes \mathcal{H}', \tag{97}$$

where $\mathcal{H} \equiv \mathbb{C}^N$ is the system's original Hilbert space (76), and where

$$\mathcal{H}' \equiv \mathbb{C}^{N'} \tag{98}$$

is an ancillary Hilbert space whose dimension N' satisfies $\tilde{N} = NN' \leq N^3$. That is, N' is an integer lying in the bounded interval

$$1 \leq N' \leq N^2. \tag{99}$$

- The first trace $\mathrm{tr}(\cdots)$ in (96) denotes the partial trace over just the original Hilbert space $\mathcal{H}$, and the second trace $\mathrm{tr}'(\cdots)$ similarly denotes the partial trace over the ancillary Hilbert space $\mathcal{H}'$.

- Importantly, for each fixed time t, $\tilde{U}(t \leftarrow 0)$ is an $\tilde{N} \times \tilde{N}$ unitary matrix that reduces to the $\tilde{N} \times \tilde{N}$ identity matrix $\tilde{U}(0 \leftarrow 0) = \tilde{\mathbb{1}}$ at the initial time 0.

- The symbol $\mathbb{1}'$ denotes the $N' \times N'$ identity matrix on the ancillary Hilbert space $\mathcal{H}'$.

- Letting $e'_1, \ldots, e'_{N'}$ denote the standard orthonormal basis for the ancillary Hilbert space $\mathcal{H}'$, in analogy with the configuration basis (77) for the system's original Hilbert space $\mathcal{H}$, and letting the primed Latin letters $i', j', \ldots$ each denote an element of an ancillary configuration

[13] From the starting assumptions presented here, one can sketch the following proof: Given $N \times N$ Kraus matrices $K_\beta(t \leftarrow 0)$, with $\beta = 1, \ldots, N$, define an $N^3 \times N^2$ matrix $\tilde{V}(t)$ according to $\tilde{V}_{(i\beta m)(jl)}(t \leftarrow 0) \equiv K_{\beta,ij}(t \leftarrow 0)\delta_{lm}$, treating $(i\beta m)$ as the first index of $\tilde{V}(t \leftarrow 0)$ and treating (jl) as its second index. One can show that this matrix satisfies $\tilde{V}^\dagger(t \leftarrow 0)\tilde{V}(t \leftarrow 0) = \mathbb{1}_{N^2 \times N^2}$, so it defines a partial isometry, which can always be extended to a unitary $N^3 \times N^3$ matrix $\tilde{U}_{(i\beta m)(ja)}(t \leftarrow 0)$ by adding $N^3 - N^2$ additional columns that are mutually orthogonal with each other and with the previous N^2 columns already in $\tilde{V}(t \leftarrow 0)$, where the new index a runs through N^2 possible values. These additional columns can always be chosen so that at the initial time 0, where $V(0 \leftarrow 0) = \mathbb{1}$ is the $N \times N$ identity matrix, they make $\tilde{U}(0 \leftarrow 0)$ coincide with the $N^3 \times N^3$ identity matrix. The last step is to show that $\tilde{U}(t \leftarrow 0)$ satisfies (96), whose right-hand side reduces to $\sum_{\beta,m} |\tilde{U}_{(i\beta m)(jj')}(t \leftarrow 0)|^2 = \sum_\beta |K_{\beta,ij}(t \leftarrow 0)|^2$. QED

space $\mathcal{C}'$ consisting of the integers from 1 to N', the symbol $P'_{i'}$ denotes the rank-one projector

$$P'_{i'} \equiv e'_{i'} e'^{\dagger}_{i'} = \mathrm{diag}(0, \dots, 0, \underset{\substack{\uparrow \\ i'\text{th entry}}}{1}, 0, \dots, 0) \qquad \left[\text{for all } i' \in \mathcal{C}'\right], \tag{100}$$

which is an $N' \times N'$ diagonal matrix with a 1 in its i'th diagonal entry and 0s in all its other entries. These projectors form a projection-valued measure (PVM) on $\mathcal{H}'$ satisfying the conditions of mutual exclusivity,

$$P'_{i'} P'_{j'} = \delta_{i'j'} P'_{i'} \quad \left[\text{for all } i', j' \in \mathcal{C}'\right], \tag{101}$$

and completeness,

$$\sum_{i'=1}^{N'} P'_{i'} = \mathbb{1}'. \tag{102}$$

- Note that the left-hand side of (96) is insensitive to the specific choice of $j' \in \mathcal{C}'$ on the right-hand side.

Extending the foregoing construction, and fixing the target time t, one obtains an $\tilde{N} \times \tilde{N}$ transition matrix given by the new dictionary

$$\tilde{\Gamma}_{ii',jj'}(t \leftarrow 0) \equiv \tilde{\mathrm{tr}}\big(\tilde{U}^{\dagger}(t \leftarrow 0)\tilde{P}_{ii'}\tilde{U}(t \leftarrow 0)\tilde{P}_{jj'}\big) \qquad \left[\text{for all } i, j \in \mathcal{C},\, i', j' \in \mathcal{C}',\, t \in \mathcal{T}\right]. \tag{103}$$

Here the trace is now over the dilated Hilbert space $\tilde{\mathcal{H}}$ defined in (95), and

$$\tilde{P}_{ii'} \equiv P_i \otimes \tilde{P}_{i'} \quad \left[\text{for all } i \in \mathcal{C},\, i' \in \mathcal{C}'\right] \tag{104}$$

defines a rank-one projector on $\tilde{\mathcal{H}}$.

Unfolding the notation, the dilated dictionary (103) reduces to the statement that

$$\tilde{\Gamma}_{ii',jj'}(t \leftarrow 0) = |\tilde{U}_{ii',jj'}(t \leftarrow 0)|^2 \qquad \left[\text{for all } i, j \in \mathcal{C},\, i', j' \in \mathcal{C}',\, t \in \mathcal{T}\right]. \tag{105}$$

That is, each entry $\tilde{\Gamma}_{ii',jj'}(t \leftarrow 0)$ of the $\tilde{N} \times \tilde{N}$ transition matrix $\tilde{\Gamma}(t)$ is the modulus-square of the corresponding entry $\tilde{U}_{ii',jj'}(t \leftarrow 0)$ of an $\tilde{N} \times \tilde{N}$ unitary matrix $\tilde{U}(t \leftarrow 0)$. Again, a stochastic matrix with this special feature is called a unistochastic matrix.

5.8 The Dilated Indivisible Stochastic Process

One can now define a dilated indivisible stochastic process $(\tilde{\mathcal{C}}, \tilde{\mathcal{T}}, \tilde{\mathcal{T}}_0, \tilde{\Gamma}, \tilde{p}, \tilde{\mathcal{A}})$ in the following way.

- Let the dilated system's configuration space $\tilde{\mathcal{C}}$ be defined as the Cartesian product

$$\tilde{\mathcal{C}} \equiv \mathcal{C} \times \mathcal{C}', \tag{106}$$

where $\mathcal{C}$ is the original system's configuration space, with N elements labeled by unprimed Latin letters $i, j, \dots$, and where $\mathcal{C}'$ is the configuration space of an ancillary subsystem, with N' elements labeled by primed Latin letters $i', j', \dots$.

- Let the dilated system's set of target times $\tilde{\mathcal{T}}$ be the original system's set of target times $\mathcal{T}$:

$$\tilde{\mathcal{T}} \equiv \mathcal{T}. \tag{107}$$

- Let the dilated system's set of conditioning times $\tilde{\mathcal{T}}_0$ be the singleton set $\{0\}$.

- Let $\tilde{\Gamma} : \tilde{\mathcal{C}}^2 \times \tilde{\mathcal{T}} \times \{0\} \to [0,1]$ be the transition map defined according to the dilated dictionary (105). Then for each target time t, the $\tilde{N} \times \tilde{N}$ matrix $\tilde{\Gamma}(t \leftarrow 0)$ is unistochastic. Moreover, by construction, $\tilde{\Gamma}(t \leftarrow 0)$ satisfies the marginalization condition

$$\sum_{i'=1}^{N'} \tilde{\Gamma}_{ii',jj'}(t \leftarrow 0) = \Gamma_{ij}(t \leftarrow 0) \tag{108}$$
$$\big[\text{for all } i, j \in \mathcal{C},\ j' \in \mathcal{C}',\ t \in \mathcal{T}\big],$$

where, as in (96), the value of $j' \in \mathcal{C}'$ is irrelevant.

- Let the map $\tilde{p} : \tilde{\mathcal{C}} \to [0,1]$ be the probability distribution defined by

$$\tilde{p}_{ii'}(t) \equiv \sum_{j=1}^{N} \tilde{\Gamma}_{ii',jj'}(t \leftarrow 0) p_j(0) \tag{109}$$
$$\big[\text{for all } i \in \mathcal{C},\ i', j' \in \mathcal{C}',\ t \in \mathcal{T}\big].$$

It follows from the marginalization condition (108), together with the law of total probability (33), that

$$\sum_{i'=1}^{N'} \tilde{p}_{ii'}(t) = p_i(t) \quad [\text{for all } i \in \mathcal{C},\ t \in \mathcal{T}], \tag{110}$$

as in (63).

- Let the algebra $\tilde{\mathcal{A}}$ of random variables be the set of all maps of the form $\tilde{A} : \tilde{\mathcal{C}} \times \tilde{\mathcal{T}} \to \mathbb{R}$.

These results establish that $(\tilde{\mathcal{C}}, \tilde{\mathcal{T}}, \tilde{\mathcal{T}}_0, \tilde{\Gamma}, \tilde{p}, \tilde{\mathcal{A}})$ is a composite unistochastic process, and that the original indivisible stochastic process $(\mathcal{C}, \mathcal{T}, \mathcal{T}_0, \Gamma, p, \mathcal{A})$ can be regarded as one of its subsystems. This conclusion completes the proof of the stochastic-quantum theorem (69). QED

5.9 The Corresponding Quantum System

Corresponding to the dilated unistochastic process $(\tilde{\mathcal{C}}, \tilde{\mathcal{T}}, \tilde{\mathcal{T}}_0, \tilde{\Gamma}, \tilde{p}, \tilde{\mathcal{A}})$ is a quantum system based on a Hilbert space $\tilde{\mathcal{H}}$ of dimension $\tilde{N} \leq N^3$, with linear-unitary time evolution encoded in the unitary time-evolution operator $\tilde{U}(t \leftarrow 0)$.

Unistochastic matrices are not generally orthostochastic, meaning that they are not guaranteed to be expressible in terms of real-orthogonal matrices. As a consequence, if one is given an indivisible stochastic process whose $N \times N$ transition matrix $\Gamma(t \leftarrow 0)$ is *already* unistochastic, then there is no guarantee that the corresponding $N \times N$ unitary time-evolution operator $U(t \leftarrow 0)$ can be assumed to be a real-orthogonal matrix. The stochastic-quantum correspondence therefore implies that in order to provide Hilbert-space representations for the most general kinds of indivisible stochastic processes, the complex numbers $\mathbb{C}$ will be an important feature of quantum theory.

Whether in that case or more generally, of course, one is always free to start with an $N \times N$ time-evolution operator $\Theta(t \leftarrow 0)$ in (72) whose individual entries are all real. With that choice, the $\tilde{N} \times \tilde{N}$ unitary time-evolution operator $\tilde{U}(t \leftarrow 0)$ obtained from the Stinespring dilation theorem will likewise be real, and will therefore be an orthogonal matrix.

However, it is important to keep in mind that from the point of view of the stochastic-quantum correspondence, the Hilbert-space representations of a given indivisible stochastic process are convenient fictions, and so one is entirely free to assume that they involve the complex numbers anyway. Assuming that a given choice of Hilbert-space representation is defined over the complex numbers, rather than merely over the real numbers, allows one to take advantage of the spectral theorem, eigenvectors of the time-evolution operator $\tilde{U}(t \leftarrow 0)$, and anti-unitary operators. Assuming appropriate smoothness conditions in time, one can further make use of a self-adjoint Hamiltonian $\tilde{H}(t)$ with real-valued energy eigenvalues, as well as the Schrödinger equation. To the extent that these mathematical constructs are often taken by textbooks to be indisputable features of quantum systems, the complex numbers become an unavoidable part of quantum theory.[14]

6 Discussion and Future Work

The unitarily evolving quantum system that lies on the other side of the stochastic-quantum correspondence is not limited to a *commutative* algebra of observables represented by operators that are diagonal in the configuration basis.

[14] Even if one assumes that a given choice of Hilbert-space representation is defined over the complex numbers, one can always double the dimension of the Hilbert space from N to $2N$ and represent the imaginary unit $i \equiv \sqrt{-1}$ by the 2×2 real matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, in which case all $N \times N$ unitary matrices become $2N \times 2N$ real orthogonal matrices (Myrheim 1999). Interestingly, one can then also represent the complex-conjugation operation K needed for anti-unitary operators as a 2×2 real matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, which happens to coincide with the first Pauli sigma matrix σ_x (Stueckelberg 1960). However, this approach still ultimately preserves the algebraic structure of the complex numbers in the Hilbert space, in the sense that they will correspond to a subalgebra of matrices on the overall Hilbert space that commute with the Hamiltonian and all the system's observables. Hence, this approach does not truly eliminate the complex numbers from the Hilbert-space formalism of quantum theory. Moreover, using this 'real' representation means that the Hilbert spaces of composite systems will not factorize as neatly into Hilbert spaces for their constituent subsystems.

Indeed, as explained in other work (Barandes 2025), one can model the quantum measurement process of an observable represented by an *arbitrary* self-adjoint operator in terms of a unistochastic process that contains a subject system, a measuring device, and an environment, all as explicitly defined subsystems. By taking the measuring device's allowed configurations to correspond to definite readings of outcomes, and by taking the transition matrix for the overall unistochastic process to be based on precisely the type of unitary time-evolution operator employed in standard textbook treatments of the measurement process, one inevitably finds that the measuring device ends up in one of its measurement-reading configurations with a stochastic probability given by the general form of the Born rule. Hence, in principle, one has access to a quantum system's entire *noncommutative* algebra of observables.

In essence, the stochastic-quantum correspondence therefore suggests that every quantum system may ultimately be an indivisible stochastic process in disguise. From this standpoint, the usual Hilbert-space formulation is then just a convenient mathematical layer on top of a system evolving stochastically along some trajectory through a configuration space in a highly non-Markovian way.

With the stochastic-quantum correspondence in hand, one can refer other exotic features of quantum systems back to their associated indivisible stochastic processes to give those features a more physically transparent interpretation. For example, interference and entanglement can be understood as artifacts of the generic indivisibility of the stochastic dynamics. Moreover, from the perspective of this stochastic-quantum correspondence, what gives quantum computers their possible advantage over classical computers is their access to dynamical laws that are indivisible, and therefore extremely non-Markovian.

This overall approach to quantum foundations therefore sheds new light on some of the strangest features of quantum theory, in addition to suggesting novel applications of quantum computers. This approach might even provide a helpful stepping stone for the development of self-consistent generalizations of quantum theory itself.

Acknowledgments

The author would like to thank Scott Aaronson, Emily Adlam, David Baker, Craig Callender, Iris Cong, Jenann Ismael, David Kagan, Alex Meehan, Logan McCarty, Wayne Myrvold, Jill North, and Noel Swanson for helpful conversations. This research was supported by the American Institute of Mathematics.

References

[1] R. Devaney. "An Introduction to Chaotic Dynamical Systems". 1989.

[2] S. H. Strogatz. *Nonlinear Dynamics and Chaos*. Westview Press, 1994.

[3] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, 1995.

[4] P. A. M. Dirac. *The Principles of Quantum Mechanics*. Oxford University Press, 1st edition, 1930.

[5] J. von Neumann. *Mathematische Grundlagen der Quantenmechanik*. Berlin: Springer, 1932.

[6] P. A. M. Dirac. "The Lagrangian in Quantum Mechanics". *Physikalische Zeitschrift der Sowjetunion*, 3(1):64–72, 1933.

[7] R. P. Feynman. *The Principle of Least Action in Quantum Mechanics*. PhD thesis, Princeton University, Princeton, NJ, 1942.

[8] R. P. Feynman. "Space-Time Approach to Non-Relativistic Quantum Mechanics". *Reviews of Modern Physics*, 20(2):367, April 1948. `doi:10.1103/RevModPhys.20.367`.

[9] E. P. Wigner. "On the Quantum Correction for Thermodynamic Equilibrium". *Physical Review*, 40(5):749–759, June 1932. `doi:10.1103/PhysRev.40.749`.

[10] J. E. Moyal. "Quantum Mechanics as a Statistical Theory". *Mathematical Proceedings of the Cambridge Philosophical Society*, 45(1):99–124, January 1949. `doi:10.1017/S0305004100000487`.

[11] J. R. Glick and C. Adami. "Markovian and Non-Markovian Quantum Measurements". *Foundations of Physics*, July 2020. `arXiv:1701.05636`, `doi:10.1007/s10701-020-00362-4`.

[12] H. Everett III. "The Theory of the Universal Wave Function". Unpublished draft Ph.D. thesis (137 pp.), Princeton University, 1956.

[13] E. P. Wigner. "Remarks on the Mind-Body Question". In I. J. Good, editor, *The Scientist Speculates*, pages 284—302. Heinemann, 1961.

[14] M. M. Wolf and J. I. Cirac. "Dividing Quantum Channels". *Communications in Mathematical Physics*, 279:147–168, 2008. `arXiv:math-ph/0611057`, `doi:10.1007/s00220-008-0411-y`.

[15] D. T. Gillespie. "Describing the State of a Stochastic Process". *American Journal of Physics*, 66(6):533–536, June 1998. `doi:10.1119/1.18895`.

[16] D. T. Gillespie. "Non-Markovian Stochastic Processes". In *AIP Conference Proceedings*, Volume 511, pages 49–56. American Institute of Physics, July 2000. `doi:10.1063/1.60002`.

[17] J. Honerkamp. *Stochastic Dynamical Systems: Concepts, Numerical Methods, Data Analysis*. Wiley-VCH, 1996.

[18] L. Arnold. *Random Dynamical Systems*. Springer, 1998.

[19] M. Rosenblatt. *Random Processes*. Oxford University Press, 1962.

[20] E. Parzen. *Stochastic Processes*. Holden-Day, 1962.

[21] J. L. Doob. *Stochastic Processes*. Wiley-Interscience, 1990.

[22] S. Ross. *Stochastic Processes*. John Wiley and Sons, 2nd edition, 1995.

[23] A. Horn. "Doubly Stochastic Matrices and the Diagonal of a Rotation Matrix". *American Journal of Mathematics*, 76(3):620–630, 1954. doi:10.2307/2372705.

[24] R. C. Thompson. "Lecture Notes from a Johns Hopkins University Lecture Series". Unpublished lecture notes, 1989.

[25] P. Nylen, T.-Y. Tam, and F. Uhlig. "On the Eigenvalues of Principal Submatrices of Normal, Hermitian and Symmetric Matrices". *Linear and Multilinear Algebra*, 36(1):69–78, 1993. doi:10.1080/03081089308818276.

[26] I. Bengtsson. "The Importance of Being Unistochastic", March 2004. URL: https://arxiv.org/abs/quant-ph/0403088, arXiv:quant-ph/0403088v1.

[27] J. A. Barandes. "The Stochastic-Quantum Correspondence". *Philosophy of Physics*, 3(1):8, June 2025. arXiv:2302.10778, doi:10.31389/pop.186.

[28] E. C. G. Stueckelberg. "Quantum Theory in Real Hilbert Space". *Helvetica Physica Acta*, 33(4):727–752, 1960. URL: https://www.e-periodica.ch/digbib/view?pid=hpa-001:1960:33::715#735.

[29] G. W. Mackey. "Induced Representations of Locally Compact Groups I". *Annals of Mathematics*, 55(1):101–139, 1952. URL: https://www.jstor.org/stable/1969423, doi:10.2307/1969423.

[30] G. W. Mackey. "Quantum Mechanics and Hilbert Space". *American Mathematical Monthly*, 64(8):45–57, 1957. URL: https://www.jstor.org/stable/2308516, doi:10.2307/2308516.

[31] B. O. Koopman. "Hamiltonian Systems and Transformations in Hilbert Space". *Proceedings of the National Academy of Sciences*, 17(5):315–318, 1931. doi:10.1073/pnas.17.5.315.

[32] J. von Neumann. "Zur Operatorenmethode In Der Klassischen Mechanik". *Annals of Mathematics*, 33(3):587–642, 1932. doi:10.2307/1968537.

[33] J. von Neumann. "Zusatze Zur Arbeit 'Zur Operatorenmethode...' ". *Annals of Mathematics*, 33(4):789–791, 1932. doi:10.2307/1968225.

[34] T. F. Jordan and E. C. G. Sudarshan. "Dynamical Mappings of Density Operators in Quantum Mechanics". *Journal of Mathematical Physics*, 2(6):772–775, 1961. URL: https://scitation.aip.org/content/aip/journal/jmp/2/6/10.1063/1.1724221, doi:10.1063/1.1724221.

[35] J. A. Barandes. "The History of Hilbert-Space Formulations of Classical Physics". *The European Physical Journal H*, 51(1):1, January 2026. doi:10.1140/epjh/s13129-025-00113-x.

[36] M. Schönberg. "Application of Second Quantization Methods to the Classical Statistical Mechanics (II)". *Il Nuovo Cimento*, 10(4):419–472, April 1953. doi:10.1007/BF02781980.

[37] A. Loinger. "Galilei Group and Liouville Equation". *Annals of Physics*, 20(1):132–144, 1962. doi:10.1016/0003-4916(62)90119-7.

[38] G. Della Riccia and N. Wiener. "Wave Mechanics in Classical Phase Space, Brownian Motion, and Quantum Theory". *Journal of Mathematical Physics*, 7(8):1372–1383, August 1966. doi:10.1063/1.1705047.

[39] E. C. G. Sudarshan. "Interaction between Classical and Quantum Systems and the Measurement of Quantum Observables". *Pramana*, 6(3):117–126, March 1976. doi:10.1007/BF02847120.

[40] K. Kraus. "General State Changes in Quantum Theory". *Annals of Physics*, 64(2):311–335, June 1971. URL: https://www.sciencedirect.com/science/article/pii/0003491671901084, doi:10.1016/0003-4916(71)90108-4.

[41] W. F. Stinespring. "Positive Functions on C*-algebras". *Proceedings of the American Mathematical Society*, 6(2):211–216, April 1955. doi:10.2307/2032342.

[42] M. Keyl. "Fundamentals of Quantum Information Theory". *Physics Reports*, 369(5):431–548, October 2002. arXiv:0202122, doi:10.1016/S0370-1573(02)00266-1.

[43] J. Myrheim. "Quantum Mechanics on a Real Hilbert Space". 1999. URL: https://arxiv.org/abs/quant-ph/9905037, arXiv:quant-ph/9905037.