The ABL Rule and the Perils of Post-Selection

Jacob A. Barandes*

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Abstract

In 1964, Aharonov, Bergmann, and Lebowitz introduced their well-known 'ABL rule' with the intention of providing a time-symmetric formalism for computing novel kinds of conditional probabilities in quantum theory. Later papers attached additional significance to the ABL rule, including assertions that it supported violations of the uncertainty principle. The present work challenges these claims, as well as subsequent attempts to salvage the original interpretation of the ABL rule. Taking a broader view, this paper identifies a subtle category error at the heart of the ABL rule that consists of confusing observables that belong to a single system with emergent observables that arise only for physical ensembles. Along the way, this paper points out other problems and fallacious reasoning in the research literature surrounding the ABL rule, including the misuse of post-selection, a reliance on pattern matching to classical formulas, and a posture of 'measurementism' that takes experimental data as providing answers to interpretational questions.

1 Introduction

In 1964, Aharonov, Bergmann, and Lebowitz (ABL) published a highly influential paper titled "Time Symmetry in the Quantum Process of Measurement" in the journal *Physical Review* (Aharonov, Bergmann, Lebowitz, 1964). According to the *Physical Review* website, the ABL paper now has over 700 citations, which include papers from a variety of areas: quantum foundations (Griffiths 1984), quantum cosmology (Gell-Mann, Hartle 1994), closed timelike curves (Lloyd et al. 2011), and black holes (Horowitz, Maldacena; Lloyd 2006; Harlow 2016; Akers et al. 2024). It is worth noting that all these cited papers not only refer to the ABL paper as a whole, but explicitly refer to its claims at having provided a time-symmetric formulation of quantum theory.

Importantly, the ABL paper initiated the widespread use of post-selection in quantum theory. It also inspired the development of weak values in a 1988 paper by Aharonov, Albert, and Vaidman

*Departments of Philosophy and Physics, Harvard University, Cambridge, MA 02138; jacob_barandes@harvard.edu; ORCID: 0000-0002-3740-4418

that has since received over 2,200 citations (Aharonov, Albert, Vaidman 1988).[1]

The main result of the ABL paper was its derivation of the 'ABL rule,' a formula for a certain class of conditional probabilities that purportedly gives a time-symmetric formulation of quantum theory. The present work will clarify the precise meaning of the conditional probabilities that the ABL paper introduced, and argue that the ABL paper's formulation was not, in fact, time symmetric, but symmetric under a conceptually different class of transformations: chronological reverse-orderings of measurement sequences. By way of analogy, consider the distinction between, on the one hand, turning over a deck of face-down playing cards so that they are now all face-up, and, on the other hand, separating out all the playing cards and then re-stacking them in the opposite order while keeping them face-down the entire time.

The present work will also critique later papers that have attempted to salvage the original time-symmetric interpretation of the ABL rule, or that have tried to extend the ABL rule in ways that lead to supposed violations of the uncertainty principle. Other work will argue that weak values do not provide supporting evidence for the arguments made in the ABL paper, and run into fundamental interpretational difficulties of their own (Barandes 2026).

More broadly, this paper will argue that post-selection, whether used in the context of the ABL rule or for other purposes in quantum-physics research, is not an innocent or innocuous procedure, but routinely leads to statistical artifacts that are often mistakenly attributed to the exotic nature of quantum theory. A corollary is a call for research journals to insist that authors who rely on post-selection to obtain surprising results should explain why those results are features of quantum mechanics itself and not merely artifacts of post-selection.

With all that said, this paper will not argue that all research inspired by the ABL paper is wrong or does not work. To the contrary, the ABL paper has inspired new theoretical and experimental frameworks that stand on their own merits and do not depend on unsupported interpretational claims.

Section 2 will cover various preliminary topics that will be important for this paper's critical treatment of the ABL rule, including relevant foundational concepts in textbook quantum theory, fallacies related to ensembles and post-selection, and fallacies related to pattern matching and 'measurementism.' Section 3 will present a careful treatment of the ABL rule in detail, starting with a first look at the ABL rule itself, followed by a discussion of the role played by boundary conditions, a technical derivation of the ABL rule, an analysis of time symmetry in the ABL rule, a rigorous analysis of the original ABL paper, an assessment of historical attempts to justify key claims surrounding the ABL rule, and a novel perspective on arguments that the ABL rule provides a loophole in the uncertainty principle. Section 4 will conclude with a summary and a discussion of larger ramifications.

[1] A search using Google Scholar for {"quantum" AND ("postselection" OR "post-selection" OR "post-select" OR "postselect" OR "post-selected" OR "postselected")} yields no valid results from before 1964. For the span of years stretching from 1964 to 2025, it yields almost 19,000 results, the vast majority after the year 2000. A search for {"quantum" AND ("weak value" OR "weak values")} yields over 6,000 results.

2 Preliminaries

2.1 Textbook Quantum Theory

A sufficiently precise deconstruction of the claims made by the ABL paper and subsequent research will require engaging with the axiomatic foundations of orthodox or textbook quantum theory. To be self-contained and to put all the cards on the table, here is a retelling of the standard axioms for the textbook theory, as laid down by Dirac (1930) and von Neumann (1932) (DvN):

1. Quantum states: A quantum system is represented at any moment in time by a quantum state associated with a Hilbert space of some finite or infinite dimension. In general, a quantum state is a unit-trace, self-adjoint, positive semidefinite density matrix, or density operator, ρ. If the density matrix is rank-one, then one can instead use a unit vector defined up to arbitrary overall phase, called a state vector, or wave function, $|\Psi\rangle$. Sometimes one adds a mereological postulate that the Hilbert spaces of composite quantum systems are the tensor products of the respective Hilbert spaces of their constituent subsystems, where the quantum states of subsystems are related to the quantum states of their composite systems by the partial-trace operation.

2. Unitary time evolution: If a quantum system is a closed system, meaning that it is isolated from mutual interactions with any other systems, then the system's quantum state evolves according to a time-indexed family of unitary operators, which collectively define a time-dependent unitary operator known as the system's time-evolution operator. Under appropriate smoothness assumptions, one can express unitary time evolution as a differential equation that is first-order in time—the von Neumann equation for density matrices or the Schrödinger equation for state vectors.

3. Observables: A single quantum system has an associated set of observables. Each observable is represented by a self-adjoint operator $A = A^\dagger$ on the system's Hilbert space, where the possible numerical measurement outcomes that make up the observable's spectrum correspond to the eigenvalues of that self-adjoint operator. (More generally, one can work with positive-operator-valued measures, or POVMs.)

4. The Born rule: To compute the probability with which a measurement of an observable will yield an outcome belonging to some collection of eigenvalues of the associated self-adjoint operator, one projects the quantum state down to the subspace corresponding to that set of eigenvalues, and then one computes the trace of the resulting operator, or the norm-square of the resulting vector.

5. Collapse: Immediately following this measurement outcome, one projects or collapses the system's quantum state down to the appropriate subspace and then renormalizes it so that it has trace or norm equal to 1 again.

One should not take this paper's use of the DvN axioms as an endorsement. The DvN axioms famously do not define what precisely counts as a measurement, leading to ambiguities over when to apply the second axiom (unitary time evolution) and when to apply the fifth axiom (collapse). Unitary time evolution alone is not able to single out unique measurement outcomes, even with the invocation of open-system dynamics and decoherence. These difficulties make up the famous measurement problem. That said, most other axiomatic formulations and interpretations of quantum theory make essentially equivalent predictions for the kinds of experimental protocols that will be examined in this paper, so the DvN axioms will suffice.

2.2 Ensembles and Post-Selection

A central theme in this paper's critical analysis will be the distinction between single-system observables, which are empirically accessible features of a single system, and ensemble observables, which are irreducibly emergent features of a physical ensemble as a whole. As emphasized, for instance, by Hance, Rarity, and Ladyman (2023, Section VI), ensemble observables are categorically different from single-system observables—even if, in practice, it may sometimes be necessary to employ an ensemble to *study* the observables of a single system. That is, the mere use of an ensemble to probe a single-system observable does not make that observable an ensemble observable, because ensemble observables are a categorically distinct notion. For example, in quantum theory, the eigenvalue spectrum of an observable is a single-system concept, even if an experimentalist might use a physical ensemble to survey that spectrum fully.

A corollary is that it is a category error to try to identify single-system observables with ensemble observables, or to draw direct inferences about single-system observables from ensemble observables. This category error will be called the ensemble fallacy:

> The Ensemble Fallacy: The category error of attempting to identify a single-system observable with an ensemble observable, or to draw direct inferences about a single-system observable from an ensemble observable without independent rigorous justification. (1)

It is easy to confuse an emergent ensemble observable with a single-system observable, again because ensembles are sometimes used in practice to gain empirical access to single-system observables. A simple example might therefore be worthwhile here.

A star has various observable features, like mass, volume, electromagnetic spectrum, chemical composition, age, location, and so forth. One can certainly study these single-star observables by using ensembles—say, by looking at lots of stars of similar mass at different moments during their life cycle. For millennia, it has also been the case that human societies have selected ensembles of stars in the sky that form various mythologically inspired shapes that we know as constellations. A constellation is an ensemble observable that depends on arbitrary choices made by people, and is not a single-star observable. The stars that make up a constellation are typically far apart in three-dimensional space, and have essentially no mutual interactions. The fact that the constellation

Orion refers to a hunter tells us exceedingly little about the observables of any of its individual stars. It certainly does not suggest that each star contains a small degree of 'hunter-ness' among its observables.[2]

One can also define ensembles by post-selection, which is a key component of the ABL rule. Post-selection refers to culling—or, perhaps less politely, cherry-picking—members of an ensemble after the fact, meaning after the experimental procedure, with the effect of changing the ensemble's statistical properties. Post-selection is a delicate issue in statistics, and can easily lead one to draw erroneous conclusions about a set of data, an effect that is a form of selection bias (Hernán, Hernández-Díaz, Robins 2004). Indeed, one goal of many statisticians is to *avoid* or *correct* for post-selection, as a way to escape or mitigate its potential consequences, whereas for the ABL rule, one is supposed to *implement* post-selection quite deliberately. Also, by construction, post-selection produces ensemble observables, not single-system observables, because the post-selection criteria are entirely up to whoever does the culling or cherry-picking and are inherently statements about the ensemble as a whole. It will be helpful to give this general class of errors a name:

> The Post-Selection Fallacy: Drawing erroneous conclusions about a system or ensemble due to the implicit or explicit use of post-selection. (2)

One can easily come up with examples that illustrate both the ensemble fallacy and the post-selection fallacy:

- Consider a classical ensemble of experimental trials consisting of identical coins with the unusual observable feature that each time a coin lands on tails, it becomes 1% darker in appearance. In each trial, a coin is tossed 100 times in a row, for the purpose of estimating whether the coins are fair. If the ensemble of trials is post-selected on the coins being, say, at least 70% dark at the end, then the post-selected ensemble will under-sample coins that land frequently on heads. Any skewing of the final frequency distribution due to this post-selection decision is obviously not an intrinsic feature of the individual coins, but an ensemble-level statistical artifact of the arbitrary choice of post-selection. In particular, if the final frequency distribution of heads and tails fails to be 50/50, then one would be incorrect to conclude that the coins were not fair.

- Consider a health symptom S that arises if and only if a patient has a condition A, a condition B, or both. Suppose that the conditions A and B are statistically independent in the general population, and that each condition occurs with a probability of 10%. In terms of the probabilities p_A for A, p_B for B, and p_{AB} for the logical conjunction of A *and* B, one therefore has

$$p_A = \frac{1}{10}, \quad p_B = \frac{1}{10}, \quad p_{AB} = p_A p_B = \frac{1}{100}.$$

[2] This particular example does not feature a cooperative form of emergence, because the stars essentially do not interact with each other. By contrast, in the case of superradiance, an ensemble of systems interact with each other to produce an irreducibly emergent pattern of radiation. However, superradiance, like a constellation, is not a single-system observable, and to confuse it with one would be to commit the ensemble fallacy.

It follows that if one imagines an ensemble of, say, $N = 10{,}000$ people in the general population, then one would expect that $N_A \approx p_A N = 1{,}000$ have the condition A, that $N_B \approx p_B N = 1{,}000$ have the condition B, and that $N_{AB} \approx p_{AB} N = 100$ have both conditions. However, suppose that one post-selects on patients who have the health symptom S. By assumption, the number of such patients is $N_S = N_A + N_B - N_{AB} \approx 1{,}900$, where the subtraction of N_{AB} avoids double-counting patients who have both the conditions A and B. From among this post-selected ensemble, one sees that $N_A/N_S \approx 10/19$ have A, that $N_B/N_S \approx 10/19$ have B, and that $N_{AB}/N_S \approx 1/19$ have both A and B. However, $1/19 \neq 10/19 \times 10/19$, so the post-selected sample shows a (negative) statistical correlation between A and B. This correlation, however, is an ensemble-level statistical artifact of the choice of post-selection, and does not reflect a true interdependence between A and B. This example is an instance of Berkson's paradox (Berkson 1946), or collider bias (Cole et al., 2009).

- Consider an ensemble of trials involving stones and holes. In each trial, one tosses a small stone in a slightly random direction toward a collection of holes. Suppose, further, that in each trial, precisely one hole, picked at random, is completely covered up by a metal cover or shutter. If the ensemble is post-selected to include only trials in which a metallic clanging noise is recorded, then one will find that every stone in the post-selected ensemble of trials has been blocked from entering a hole. It would obviously be incorrect to conclude from this statistical analysis that the shutter itself possesses the mysterious capacity to cover all the holes at once. The hole-blocking is an emergent property of the post-selected ensemble, and not a single-system property of the shutter, so making inferences about the shutter from the post-selected ensemble would be to commit the ensemble fallacy (1). The choice of this specific example was not accidental—it is closely related to the subject of several papers on the use of the ABL rule for quantum systems (Aharanov, Vaidman 2003; Kastner 2004).

- Consider an ensemble of trials in which an object can be placed in any of three boxes. In each trial, there is only one object. Suppose, moreover, that with 50/50 odds, a detection machine will check either the first box for the object, or the second box, but not both. The machine will show a bright green light if it succeeds in finding the object, and the machine never checks the third box in any of the trials. If the trials are post-selected on the logically conjunctive proposition that the machine has checked the first box and ends up displaying its green light, then, among the members of the resulting ensemble, the odds of the object being in the first box are 100%. If the trials are instead post-selected on the logical conjunction of the machine checking the second box and showing its green light, then, among the members of *that* ensemble, the odds of the object being in the second box are likewise 100%. However, despite finding 100% odds in both cases, one would be incorrect to conclude from this experiment that the object was somehow in *both* the first and second boxes. Again, this example was chosen for a reason—a very similar example became a significant focus of interest in the research literature on the ABL rule (Aharonov, Albert, D'Amato 1985; Aharonov, Vaidman 1991; Cohen 1995; Vaidman 1996a; Vaidman 1998; Kastner 1999).

This paper will show that the ABL rule similarly refers to emergent observables of physical ensembles that cannot be identified as single-system observables and cannot be used to draw direct inferences about single-system observables. To make such identifications or inferences would, again, be precisely to commit the ensemble fallacy (1). This paper will also argue that some claimed implications of the ABL rule run aground on the post-selection fallacy (2). Other work will argue that weak values suffer from similar problems (Barandes 2026).

Again, none of this criticism should be taken to mean that all the spin-offs of the ABL rule should be discarded. Ensemble observables are still observables, after all, and have their uses. Indeed, constellations have helped people navigate the world for many generations (Huth 2013). Those successful applications of constellations, however, have not revealed very much about the intrinsic nature of individual stars.

2.3 Pattern Matching and Measurementism

Beyond the ensemble and post-selection fallacies introduced in Subsection 2.2, the present work will identify two other, larger problems with the ABL paper and related research literature. The first is an overuse of pattern matching to classical formulas or interpretations in quantum-physics research. The second is 'measurementism,' which will refer to the philosophical posture that if an experiment ends up obtaining a specific quantity, then that experiment alone confirms a previously favored interpretation of that quantity. ("We have measured it many times in the lab, so how could it be wrong?" or "The fact that our theory leads to an experiment that can be performed is enough to justify the theory.")

In this paper, pattern matching will refer to the common practice of trying to impose, by decree, relationships between classical notions and quantum notions. This practice is especially common when the relevant notions play similar or analogous *functional* roles in classical physics and in quantum physics, despite consisting of fundamentally different *mathematical* structures.

One specific form of pattern matching consists of looking at classical formulas that have reasonably transparent conceptual or physical meanings, and then trying to guess corresponding formulas in the quantum case by formal analogy—say, by replacing random variables with self-adjoint operators, replacing Poisson brackets with commutators, replacing classical probability distributions with density matrices, replacing marginalization with partial traces, or replacing stochastic processes with quantum channels. Dirac's canonical quantization is explicitly a form of pattern matching of this kind, and a great deal of ongoing research in quantum causal modeling involves explicit pattern matching to the ingredients of classical causal models.

Notice that in each pair of mathematical objects in the previous paragraph—for instance, random variables and self-adjoint operators—the functional role played by the latter member in quantum physics is similar or analogous to the functional role played by the former member in classical physics. However, the two members of each pair are based on fundamentally different mathematical structures.[3]

[3]By contrast, if two physical theories each contain a feature that, while perhaps playing different functional roles in the two theories, turn out to consist of sufficiently similar mathematical structures in both theories, then one

A different form of pattern matching consists of taking a formula already obtained in the quantum case and then assigning it an interpretation or meaning through an appeal to a similar-looking classical formula. An example with particular relevance to the ABL paper would be assuming that the initial and final boundary conditions used in Lagrangian mechanics are analogous to pre-selections and post-selections in quantum theory, as will be discussed in Subsection 3.2.

These forms of pattern matching, however, are never justified on their own. While pattern matching can be a useful heuristic or provisional first step, one must ultimately be able to derive any claimed formula in quantum theory either from the DvN axioms, as reviewed in Subsection 2.1, or from some rigorously self-consistent modification or replacement of the DvN axioms. Anything else would lie strictly outside of an axiomatic framework, and so would either need to be based on a direct appeal to scientific induction or abduction, or would essentially require engaging in a form of hand-waving.

It will be convenient to give these illicit forms of pattern matching a name:

$$\left.\begin{array}{l}\text{The Pattern-Matching Fallacy: Declaring the validity of a new mathematical con-}\\ \text{struct in quantum theory by analogy with a known classical mathematical construct,}\\ \text{or assigning an interpretation to an existing mathematical construct in quantum the-}\\ \text{ory by analogy with the interpretation of a known classical mathematical construct.}\end{array}\right\} \quad (3)$$

When even pattern matching is insufficient to vindicate or justify a claimed interpretation of some physical quantity, another tempting option might be 'measurementism,' a pervasive philosophical attitude that this paper will define as the following fallacy:

$$\left.\begin{array}{l}\text{The Measurementist Fallacy (or Measurementism): If a quantity of ambiguous in-}\\ \text{terpretation can be measured experimentally, then experiments alone can provide}\\ \text{confirmatory support for a favored interpretation of that quantity, or a justification}\\ \text{for theoretical work that led to the consideration of that quantity.}\end{array}\right\} \quad (4)$$

As an immediate application, the mere fact that a specific frequency ratio appearing in the ABL rule is experimentally measurable—and, indeed, has been measured in the laboratory in specific cases—does not vindicate many of the strong interpretational claims that have been made about that frequency ratio or about the ABL rule.

may be justified in learning about the feature of the first theory from the corresponding feature of the other theory. Importantly, this case arises in the use of 'analogue models,' for which one indirectly studies a feature of an empirically inaccessible lower-level theory, like quantum theory or general relativity, by examining or running measurements that probe mathematically similar features of an empirically accessible higher-level theory.

3 The ABL Rule

3.1 A First Look at the ABL Rule

At its core, the ABL paper was concerned with the application of quantum theory to a specific kind of experimental protocol.

Consider a large ensemble of $N \gg 1$ identical quantum systems with negligible internal time evolution. Suppose that for each system in the ensemble, an external agent or observer measures $n+2$ possibly distinct observables $A, C_1, \ldots, C_n, B$ in succession, while sequentially recording the associated measurement outcomes. For the purposes of this experimental protocol, assume that each observable is complete, in the sense that its measurement outcome, together with the DvN collapse axiom reviewed in Subsection 2.1, yields a single, specific state vector. The observer counts up the number N_{ab} of members of the ensemble that share a specific pre-selected measurement outcome a for the first observable A and a specific post-selected measurement outcome b for the last observable B. Then, from that subensemble labeled by the pair a and b, the observer counts up the number $N_{ac_1\cdots c_n b} \leq N_{ab}$ of members that share a specific sequence $c_1, \ldots, c_n$ of measurement results for the n respective intermediate observables $C_1, \ldots, C_n$. The numerical fraction

$$0 \leq \frac{N_{ac_1\cdots c_n b}}{N_{ab}} \leq 1 \tag{5}$$

defines a specific, experimentally accessible probability—namely, the statistical probability that if one member of the subensemble labeled by the pair a and b is randomly selected according to a uniform probability measure, then the selected member will exhibit the specific sequence $c_1, \ldots, c_n$ of intermediate measurement outcomes.

As a concrete example, consider an ensemble of $N = 10{,}000$ experimental trials involving a single spin-1/2 degree of freedom that is free from any nontrivial internal dynamics for the duration of the procedure. In each trial, the system is prepared in the spin-z eigenstate $|z+\rangle$, and then is subjected to a spin-x measurement, followed by a spin-y measurement, and then finally a spin-z measurement. If one focuses on trials in which the final spin-z measurement yields $|z-\rangle$, then it follows from the usual algebraic relationships between the spin-x eigenbasis, the spin-y eigenbasis, and the spin-z eigenbasis, together with simple arithmetic manipulations, that the number of trials reduces to approximately $N_{z+,z-} \approx 5{,}000$. It also follows that each of the four possible configurations of intermediate spin-x and spin-y measurements $(x+, y+)$, $(x+, y-)$, $(x-, y+)$, and $(x-, y-)$ shows up in approximately 1,250 of those 5,000 trials. One can therefore compute the ratio (5) explicitly for, say, the intermediate outcome pair $(x+, y-)$, thereby yielding the answer

$$\frac{N_{z+,x+,y-,z-}}{N_{z+,z-}} \approx \frac{1{,}250}{5{,}000} = \frac{1}{4} = 25\%. \tag{6}$$

Textbook quantum theory, based on the DvN axioms reviewed in Subsection 2.1, gives a general theoretical expression for predicting the empirical probability (5), called the 'ABL rule.' Its derivation originally appeared in the ABL paper, and will be derived in a different way in Subsection 3.3.

3.2 Boundary Conditions

Newtonian mechanics provides a description of classical objects in physical space, with trajectories that satisfy dynamical laws consisting of differential equations that are second-order in time. Because these dynamical laws are second-order in time, obtaining a definite trajectory requires specifying both the initial positions of all objects in the system and also their initial velocities. Once one obtains a definite trajectory satisfying the dynamical laws, one can ask and answer a great many questions about the properties of the system at arbitrary times along that trajectory.

Again because Newton's dynamical laws are second-order in time, one can instead obtain trajectories by augmenting initial positions with *final* positions, rather than with initial *velocities*, and then making use of appropriate variational methods. These variational methods include the principle of least action of Lagrangian mechanics.

In many cases, one obtains a unique trajectory from such a variational method, when combined with boundary conditions on the past and the future arrangements of the system. In such a situation, if one knows both the initial and final arrangements of a system of objects—that is, the past and future boundary conditions—then one is entitled to make precise inferences about the system at intermediate times, including inferences about the positions of the system's constituent objects, their velocities, their kinetic energies, and so forth.

As explained in Subsection 3.1, the experimental protocol for the ABL rule involves a pre-selection at an initial time as well as a post-selection at a final time. These two operations might seem analogous to boundary conditions that specify the initial and final arrangements of objects in Lagrangian mechanics, but to impose that interpretation would be precisely to engage in a form of unjustified pattern matching between classical and quantum concepts—that is, to impose that interpretation would mean committing the pattern-matching fallacy (3).

It is true that pre-selecting a quantum state is very much like specifying the initial arrangement of a Newtonian system. However, the rules for time evolution given by the DvN axioms in Subsection 2.1 are first-order in time. If one knows the quantum state of a given system at an initial time, then unitary time evolution as well as the Born rule and the collapse axioms provide the system's later quantum states. These time-evolution rules are not second-order in time, and cannot accommodate the specification of final quantum states as additional boundary conditions, unless one is willing to violate those time-evolution rules.

When post-selection is imposed on an ensemble of quantum systems in the manner in which it is used for the ABL rule, it is due to an external agent, and represents an abrupt change to the time evolution of each system that the system has no means of anticipating. The post-selection is not a boundary condition internal to any particular quantum system in question, or a boundary condition internal to the ensemble of quantum systems. In particular, the post-selection is not a boundary condition in the sense of classical Lagrangian mechanics as merely indicating where a system's trajectory, on its own, ended up taking the system. As a consequence, although post-selection might well *reveal* facts about the past of a particular system or an ensemble of systems, nothing about each system's quantum state at times before the post-selection can actually *physically depend* on the later choice of post-selected quantum state, because any such physical dependence would require some form of clairvoyance on the part of the system. That is, post-selection cannot itself be responsible for any past features of the system, and to assume otherwise would be to commit the post-selection fallacy (2).[4]

3.3 A Derivation of the ABL Rule

Deriving the ABL rule is a straightforward exercise in the application of textbook quantum theory. The derivation presented below also turns out to be closely related to the derivation of weak values, to be addressed in other work (Barandes 2026).

Consider three quantum systems: a subject system to be studied, a measuring device, and an external agent or observer. The appropriate Hilbert space for the total system has the tensor-product form

$$\mathcal{H}_{\text{tot}} = \mathcal{H} \otimes \mathcal{H}_{\text{dev}} \otimes \mathcal{H}_{\text{obs}}, \tag{7}$$

where $\mathcal{H}$, $\mathcal{H}_{\text{dev}}$, and $\mathcal{H}_{\text{obs}}$ are the respective Hilbert spaces for the subject system, the measuring device, and the observer.

Next, suppose that the initial quantum state $|\Psi_{\text{tot}}\rangle$ of the total system assigns a generic state vector $|\Psi\rangle$ to the subject system, a 'ready' state vector $|\text{dev}(\emptyset)\rangle$ to the measuring device, and a 'ready' state vector $|\text{obs}(\emptyset)\rangle$ to the observer:

$$|\Psi_{\text{tot}}\rangle = |\Psi\rangle \otimes |\text{dev}(\emptyset)\rangle \otimes |\text{obs}(\emptyset)\rangle. \tag{8}$$

Let $A, C_1, \ldots, C_n, B$ be a collection of observables belonging to the subject system, with respective eigenvalues denoted by the variables $a, c_1, \ldots, c_n, b$. Assume that each of these observables is complete, in the sense that a measurement of any of them fixes the quantum state of the subject system completely.

Suppose, moreover, that the measuring device and observer are arranged in advance so that the *observer* measures A, then the *measuring device* measures the chronological sequence $C_1, \ldots C_n$, and then the *observer* measures B. Applying unitary time evolution to the total system, one sees that the first measurement takes the form

$$\left.\begin{aligned} |\Psi_{\text{tot}}\rangle &= |\Psi\rangle \otimes |\text{dev}(\emptyset)\rangle \otimes |\text{obs}(\emptyset)\rangle \\ &= \sum_a \langle a|\Psi\rangle |a\rangle \otimes |\text{dev}(\emptyset)\rangle \otimes |\text{obs}(\emptyset)\rangle \\ [\text{observer measurement of } A] &\mapsto \sum_a \langle a|\Psi\rangle |a\rangle \otimes |\text{dev}(\emptyset)\rangle \otimes |\text{obs}(a)\rangle, \end{aligned}\right\} \tag{9}$$

[4] In principle, the sort of clairvoyance described here could be permissible on certain retrocausal interpretations of quantum theory. Alternatively, on the Everett interpretation (Everett 1956, 1957), one could argue that post-selection places the observer on a specific branch of the universal wave function, complete with its own particular branch-history. Hence, in the case of the Everett interpretation, the post-selection does not alter the past, but merely singles out one past from among many others that also exist. Either of these alternative views, however, would clearly be interpretation-dependent.

where $\text{obs}(a)$ indicates that the observer has obtained the measurement outcome a for the observable A. The second measurement takes the form

$$\left.\begin{aligned} &\sum_a \langle a|\Psi\rangle |a\rangle \otimes |\text{dev}(\emptyset)\rangle \otimes |\text{obs}(a)\rangle \\ &= \sum_a \sum_{c_1} \langle a|\Psi\rangle \langle c_1|a\rangle |c_1\rangle \otimes |\text{dev}(\emptyset)\rangle \otimes |\text{obs}(a)\rangle \\ [\text{device measurement of } C_1] \mapsto &\sum_a \sum_{c_1} \langle a|\Psi\rangle \langle c_1|a\rangle |c_1\rangle \otimes |\text{dev}(c_1)\rangle \otimes |\text{obs}(a)\rangle, \end{aligned}\right\} \tag{10}$$

where $\text{dev}(c_1)$ indicates that the measuring device has obtained the measurement outcome c_1 for the observable C_1. Similarly, after the measuring device carries out its next measurement, one has

$$\sum_a \sum_{c_1, c_2} \langle a|\Psi\rangle \langle c_1|a\rangle \langle c_2|c_1\rangle |c_2\rangle \otimes |\text{dev}(c_1, c_2)\rangle \otimes |\text{obs}(a)\rangle, \tag{11}$$

where $\text{dev}(c_1, c_2)$ indicates that the measuring device has obtained the measurement outcome c_1 for the observable C_1 and then the measurement outcome c_2 for the observable C_2, in that chronological order. Continuing, one ends up finding that the final state vector $|\Psi'_{\text{tot}}\rangle$ for the total system is given by

$$|\Psi'_{\text{tot}}\rangle = \sum_{a,b} |b\rangle \otimes \left[\sum_{c_1, \dots, c_n} \langle b|c_n\rangle \langle c_n|c_{n-1}\rangle \cdots \langle c_1|a\rangle \langle a|\Psi\rangle |\text{dev}(c_1, \dots, c_n)\rangle \right] \otimes |\text{obs}(a, b)\rangle. \tag{12}$$

Here $\text{obs}(a, b)$ indicates that the observer has obtained the measurement outcome a for the observable A and b for the observable B, whereas $\text{dev}(c_1, \dots, c_n)$ indicates that the measuring device has obtained the chronologically ordered sequence of measurement outcomes $c_1, \dots, c_n$ for the respective observables $C_1, \dots, C_n$.

It will be convenient to introduce projection operators

$$P_a \equiv |a\rangle\langle a|, \quad P_b \equiv |b\rangle\langle b|, \quad P_{c_i} \equiv |c_i\rangle\langle c_i|. \tag{13}$$

Each of these three sets of projection operators gives a complete and mutually exclusive set, so they each make up a projection-valued-measure (PVM), in the sense that

$$\left.\begin{aligned} P_a P_{a'} &= \delta_{aa'} P_a, \quad \sum_a P_a = \mathbb{1}, \\ P_b P_{b'} &= \delta_{bb'} P_b, \quad \sum_b P_b = \mathbb{1}, \\ P_{c_i} P_{c'_i} &= \delta_{c_i c'_i} P_{c_i}, \quad \sum_{c_i} P_{c_i} = \mathbb{1}, \end{aligned}\right\} \tag{14}$$

where $\mathbb{1}$ is the identity operator on the subject system's Hilbert space. (Note that for the third PVM here, the sum is *not* over the values of i that distinguish the different observables $C_1, \dots, C_n$, but over the full spectrum c_i of eigenvalues of just the *single* observable C_i.) One can then write (12) somewhat more compactly as

$$|\Psi'_{\text{tot}}\rangle = \sum_{a,b} \sum_{c_1,\ldots,c_n} [P_b P_{c_n} \cdots P_{c_1} P_a |\Psi\rangle] \otimes |\text{dev}(c_1,\ldots,c_n)\rangle \otimes |\text{obs}(a,b)\rangle. \tag{15}$$

Introducing another projection operator

$$P_{\text{obs}(a,b)} \equiv |\text{obs}(a,b)\rangle\langle\text{obs}(a,b)|, \tag{16}$$

the overall probability $p[\text{obs}(a,b)|\Psi]$ for the observer to obtain the specific pair of measurement outcomes (a,b), conditioned on the initial state vector $|\Psi\rangle$ of the subject system, then follows from the Born rule:

$$p[\text{obs}(a,b)|\Psi,(A,C_1,\ldots,C_n,B)] = \text{tr}\big([\mathbb{1} \otimes \mathbb{1}_{\text{dev}} \otimes P_{\text{obs}(a,b)}]\,[|\Psi'_{\text{tot}}\rangle\langle\Psi'_{\text{tot}}|]\big). \tag{17}$$

Here $\mathbb{1}_{\text{dev}}$ is the identity operator on the measuring device's Hilbert space. The additional conditioning on $(A,C_1,\ldots,C_n,B)$, as an ordered list, is a reminder that the evolution of the total system's state vector from $|\Psi_{\text{tot}}\rangle$ to $|\Psi'_{\text{tot}}\rangle$ involved measurements of that specific ordered sequence of observables. By a straightforward calculation, one finds

$$p[\text{obs}(a,b)|\Psi,(A,C_1,\ldots,C_n,B)] = \sum_{c_1,\ldots,c_n} \text{tr}[P_b P_{c_n} \cdots P_{c_1} P_a P_\Psi P_a P_{c_1} \cdots P_{c_n}], \tag{18}$$

where P_Ψ is the rank-one density matrix defined by

$$P_\Psi \equiv |\Psi\rangle\langle\Psi|. \tag{19}$$

Defining the projection operator

$$P_{\text{dev}(c_1,\ldots,c_n)} \equiv |\text{dev}(c_1,\ldots,c_n)\rangle\langle\text{dev}(c_1,\ldots,c_n)|, \tag{20}$$

the joint probability for the observer to obtain (a,b) and for the measuring device to obtain the ordered sequence $(c_1,\ldots,c_n)$ likewise follows from the Born rule:

$$p[\text{obs}(a,b),\text{dev}(c_1,\ldots,c_n)|\Psi,(A,C_1,\ldots,C_n,B)] = \text{tr}\big([\mathbb{1} \otimes P_{\text{dev}(c_1,\ldots,c_n)} \otimes P_{\text{obs}(a,b)}]\,[|\Psi'_{\text{tot}}\rangle\langle\Psi'_{\text{tot}}|]\big). \tag{21}$$

The result is

$$p[\text{obs}(a,b),\text{dev}(c_1,\ldots,c_n)|\Psi,(A,C_1,\ldots,C_n,B)] = \text{tr}[P_b P_{c_n} \cdots P_{c_1} P_a P_\Psi P_a P_{c_1} \cdots P_{c_n}]. \tag{22}$$

Notice, as expected, that these overall and joint probabilities are manifestly related by marginalization over the full set of ordered sequences $(c_1, \ldots, c_n)$:

$$p[\mathrm{obs}(a,b)|\Psi,(A,C_1,\ldots,C_n,B)] = \sum_{c_1,\ldots,c_n} p[\mathrm{obs}(a,b),\mathrm{dev}(c_1,\ldots,c_n)|\Psi,(A,C_1,\ldots,C_n,B)]. \tag{23}$$

If one applies the DvN collapse axiom to single out the term in the superposition $|\Psi'_{\mathrm{tot}}\rangle$ corresponding to the observer's measurement results (a,b), then the reduced density matrix for the measuring device alone is obtained from the partial trace over the Hilbert spaces of the subject system and the observer according to

$$\rho_{\mathrm{dev}|\mathrm{obs}(a,b),\Psi} = \mathrm{tr}_{\mathcal{H},\mathcal{H}_{\mathrm{obs}}}\left(\frac{\big[\mathbb{1}\otimes\mathbb{1}_{\mathrm{dev}}\otimes P_{\mathrm{obs}(a,b)}\big]|\Psi'_{\mathrm{tot}}\rangle\langle\Psi'_{\mathrm{tot}}|\big[\mathbb{1}\otimes\mathbb{1}_{\mathrm{dev}}\otimes P_{\mathrm{obs}(a,b)}\big]}{\mathrm{tr}\big(\big[\mathbb{1}\otimes\mathbb{1}_{\mathrm{dev}}\otimes P_{\mathrm{obs}(a,b)}\big]|\Psi'_{\mathrm{tot}}\rangle\langle\Psi'_{\mathrm{tot}}|\big)}\right). \tag{24}$$

The result is a rank-one density matrix that can be expressed in terms of a state vector given by

$$|\Psi_{\mathrm{dev}|\mathrm{obs}(a,b),\Psi}\rangle = \frac{\sum_{c_1,\ldots,c_n}\langle b|P_{c_n}\cdots P_{c_1}P_a|\Psi\rangle|\mathrm{dev}(c_1,\ldots,c_n)\rangle}{\sqrt{\sum_{c'_1,\ldots,c'_n}\mathrm{tr}\Big(P_bP_{c'_n}\cdots P_{c'_1}P_aP_\Psi P_aP_{c'_1}\cdots P_{c'_n}\Big)}}. \tag{25}$$

If one were to measure the pointer variables of the measuring device itself, then the Born rule would imply that the corresponding probability would be

$$p[\mathrm{dev}(c_1,\ldots c_n)|\Psi,\mathrm{obs}(a,b),(A,C_1,\ldots,C_n,B)] = \frac{\mathrm{tr}(P_bP_{c_n}\cdots P_{c_1}P_aP_\Psi P_aP_{c_1}\cdots P_{c_n})}{\sum_{c'_1,\ldots,c'_n}\mathrm{tr}\Big(P_bP_{c'_n}\cdots P_{c'_1}P_aP_\Psi P_aP_{c'_1}\cdots P_{c'_n}\Big)}. \tag{26}$$

Notice that this conditional probability is related to the joint probability in (22) and the overall probability in (18) according to

$$p[\mathrm{dev}(c_1,\ldots c_n)|\Psi,\mathrm{obs}(a,b),(A,C_1,\ldots,C_n,B)] = \frac{p[\mathrm{obs}(a,b),\mathrm{dev}(c_1,\ldots,c_n)|\Psi,(A,C_1,\ldots,C_n,B)]}{p[\mathrm{obs}(a,b)|\Psi,(A,C_1,\ldots,C_n,B)]}, \tag{27}$$

which is just a form of Bayes' theorem:

$$p(x|y,z) = \frac{p(x,y|z)}{p(y|z)}. \tag{28}$$

One can simplify the conditional probability (26) by using the identity

$$P_aP_\Psi P_a = |a\rangle\langle a|\Psi\rangle\langle\Psi|a\rangle\langle a| = |\langle a|\Psi\rangle|^2 P_a \tag{29}$$

in both the numerator and the denominator, thereby leading to a cancellation of a common factor of $|\langle a|\Psi\rangle|^2$, assuming that this factor is nonzero. The conditional probability (26) therefore reduces to the formula

$$p[\mathrm{dev}(c_1, \ldots c_n)|\mathrm{obs}(a,b), (A, C_1, \ldots, C_n, B)] = \frac{\mathrm{tr}(P_b P_{c_n} \cdots P_{c_1} P_a P_{c_1} \cdots P_{c_n})}{\sum_{c'_1,\ldots,c'_n} \mathrm{tr}\Big(P_b P_{c'_n} \cdots P_{c'_1} P_a P_{c'_1} \cdots P_{c'_n}\Big)}, \quad (30)$$

which no longer depends on $|\Psi\rangle$. The formula (30) is known as the ABL rule.

3.4 Time Symmetry

Due to the cyclic property of the trace, the ABL rule (30) satisfies the following reverse-ordering symmetry:

$$p[\mathrm{dev}(c_1, \ldots c_n)|\mathrm{obs}(a,b), (A, C_1, \ldots, C_n, B)] = p[\mathrm{dev}(c_n, \ldots c_1)|\mathrm{obs}(b,a), (B, C_n, \ldots, C_1, A)]. \quad (31)$$

That is, the conditional probability has the same numerical value if the measurement sequence $A, C_1, \ldots, C_n, B$ is carried out in the opposite chronological order. In the notation of Subsection 3.1, the reverse-ordering symmetry (31) of the ABL rule means that the probability $N_{ac_1\cdots c_n b}/N_{ab}$ appearing in (5) is equal to the probability $N_{bc_n\cdots c_1 a}/N_{ba}$, with the sequence of the $n+2$ measurements carried out in the opposite chronological order.

The ABL paper called this property "time symmetry." Again, the ABL paper's *title* was "Time Symmetry in the Quantum Process of Measurement."

This notion of time symmetry was incorrect. Carrying out the $n + 2$ measurements in the opposite chronological order simply fails to be the true time-reverse of the experimental protocol. A measurement process is intrinsically time-directed, as noted by Shimony (1996) and acknowledged by Vaidman (1996b, 1998). It logically follows that the true time-reverse of the experimental protocol would entail not only reversing the order of $n + 2$ measurements, but also internally reversing each individual measurement process itself.

An analogy might be helpful here. If one buys bread from the store, and then later one buys fruit from the store, then the time-reverse of the overall process would not consist of buying fruit first and then buying bread second. The time-reverse would instead mean something like *selling* fruit and then later *selling* bread, for the simple reason that the time-reverse of exchanging money for an item of food would be exchanging an item of food for money. (Presumably, the time-reverse would also involve walking backward, speaking backward, thinking backward, and so forth.) By the same reasoning, the time-reverse of one measurement followed by a different measurement would not consist of the *same* pair of measurements merely occurring in the opposite chronological order, but would instead mean the corresponding *reverse-time* measurements occurring in the opposite chronological order.

Hence, just as a truly time-symmetric shopping plan would involve buying food during the first trip and selling food during the second trip, a truly time-symmetric experimental protocol for a quantum system would involve carrying out a forward-time measurement at the beginning and a reverse-time measurement at the end.

After the ABL paper's publication in 1964, subsequent papers continued to advocate for the ABL paper's incorrect notion of time symmetry. For example, a 1990 paper by Aharonov and Vaidman made that claim, even going as far as calling the pre-selection and post-selection "boundary conditions," overlooking the sorts of pattern-matching problems laid out in Subsection 3.2:

> However, if our task is a description of a quantum system between two successive measurements, then we know the boundary conditions in the future as well as in the past. (We assume that both measurements are complete.) Therefore for the intermediate time interval we have a complete symmetry under time reversal. The contribution to the description of the quantum system from the result of the initial measurement is the usual wave function evolving from the past toward the future, from the initial measurement to the final measurement. Because of the symmetry under time reversal, the contribution of the final measurement should be similar: the wave function evolving backwards in time from the final measurement to the initial measurement. [Aharonov, Vaidman 1990, p. 12]

A 1991 paper by Aharonov and Vaidman contained the following statements in its concluding section: "What we have presented here is a novel approach to standard quantum theory. ... It has an advantage that it is symmetrical under time reversal." (Aharonov, Vaidman 1991, p. 2327) Papers by other authors have made similar statements, such as a 2011 paper by Lloyd et al., which included the following assertion: "it is a time-symmetrical formulation of quantum mechanics in which not only the initial state, but also the final state is specified." (Lloyd et al. 2011, p. 025007-5)

3.5 The ABL Paper

The ABL paper's abstract began with a strong claim:

> We examine the assertion that the "reduction of the wave packet," implicit in the quantum theory of measurement[,] introduces into the foundations of quantum physics a time-asymmetric element, which in turn leads to irreversibility. We argue that this time asymmetry is actually related to the manner in which statistical ensembles are constructed. If we construct an ensemble time symmetrically by using both initial and final states of the system to delimit the sample, then the resulting probability distribution turns out to be time symmetric as well. [Aharonov, Bergmann, Lebowitz 1964; p. B1410.]

On their face, these claims seem doubtful. It is difficult to believe that one can obtain a time-symmetric formulation of quantum theory merely by constructing ensembles differently, for reasons already explained in Subsection 3.4. The authors of the ABL paper did not suggest that they were working outside of the framework of the DvN axioms, as reviewed in Subsection 2.1, and for which the reduction or collapse of the quantum state arises from measurement processes. Again, a generic measurement process is structured, has a nonzero duration, and, as emphasized by Shimony (1996), is time-directed. That is, a measurement process has an intrinsic temporal direction from set-up, to initiation, to detection, and then to recording. It follows that whenever one invokes a measurement process, there will be some practical source of time-asymmetry in the overall system, regardless of how one tries to set up an ensemble.

Even if one were to attempt to treat measurements as axiomatically instantaneous events, one would still have to contend with the fact that a quantum state's time evolution, according to the DvN axioms, is *discontinuous* to the immediate *past* of a measurement, but is *continuous* to the immediate *future* of a measurement, and leads to measurement-outcome probabilities only in the future direction. A measurement-induced form of time-asymmetry in the system is, once again, unavoidable.

Given the thermodynamic-level difficulty of implementing a realistic measurement process involving macroscopic measuring devices running in reverse, it is hard to imagine how one could institute a reduction or collapse of the quantum state at both temporal ends of a duration of time that could lead to a time-symmetric formulation relevant to any practical experimental protocol. One would need to construct a physical ensemble by imposing a forward-time measurement at the beginning and an infeasible reverse-time measurement at the end, with no clear way to combine two such opposite-time measurements into a single probability formula. These basic facts alone present a fundamental obstruction to the sort of time-symmetric theory that the ABL paper attempted to formulate. A secondary consequence is that 'pre-selection' and 'post-selection' are inherently different from each other in a very physical sense, at a level beyond merely the fact that one precedes the other in time, as acknowledged by Vaidman (1996b, 1998).

It follows from this reasoning that the source of the time asymmetry examined by the ABL paper would not appear to lie in the construction of ensembles. A corollary is that one should not expect that one could eliminate that time asymmetry merely by changing how one sets up ensembles.

A couple of paragraphs later in the ABL paper, one finds these statements:

> In this paper we propose to examine the nature of the time symmetry in the quantum theory of measurement. Rather than delve into the measurement process itself, which involves a specialized interaction between the atomic system and a macroscopic device, we shall simply accept the standard expressions for probabilities of values furnished by the conventional theory. [Aharonov, Bergmann, Lebowitz 1964, p. B1411]

It is understandable that one might not wish to get mired in the fine-grained details of measurement processes, especially given the measurement problem. However, one must still take into account the fact that a measurement is not an instantaneous, irreducible, structureless event—or, if one were to choose to treat a measurement as if it *were* instantaneous, then one should be mindful that the measurement separates discontinuous evolution to the past of a system's quantum state from continuous evolution to the future of the system's quantum state. Either way, a measurement is a time-directed process.

The rest of that paragraph went on to say:

> Whereas the conventional theory deals with ensembles of quantum systems that have been "preselected" on the basis of some initial observation, we shall deduce from it probability expressions that refer to ensembles that have been selected from combinations of data favoring neither past nor future. A theory that concerns itself exclusively with such symmetrically selected ensembles (the "time-symmetric theory") will contain only time-symmetric expressions for the probabilities of observations. Logically this time-symmetric theory is contained in the conventional theory but lacks one of the latter's postulates. [Ibid., B1411]

Notice that the ABL paper here regarded ensembles selected from "data favoring neither past nor future" as "symmetrically selected," and suggested that an alternative formulation of quantum theory limited to such ensembles would "contain only time-symmetric expressions for the probabilities of observations." However, one cannot get around the need for measurements in pre-selections and post-selections as long as one is relying on the DvN axioms, or on any other axiomatic framework for quantum theory that relies on measurements in an essential way. Again, the ABL paper's interpretation elided the time-directed nature of the measurements inherent to any such pre-selections or post-selections. This elision led the ABL paper to employ a notion of "time symmetry" that referred only to changing the sequential ordering of measurements, rather than truly time-reversing the whole experimental protocol, including each individual, time-directed measurement itself.

The ABL paper assumed that the initial state vector $|\Psi\rangle$ of the subject system was an exact eigenvector $|a\rangle$ of a given pre-selected observable. The ABL paper's notation used the labels $d_j, \ldots, d_n$ for the intermediate outcomes instead of $c_1, \ldots, c_n$, used A in place of P_a, used B in place of P_b, used D_i in place of P_{c_i}, and used a diagonal line / rather than a vertical line | to denote the 'given' delimiter. Adjusting the ABL paper's notation to align it with the notation of the present work, the ABL paper's version of the ABL rule (30) took the form

$$\left.\begin{aligned} p(c_1, \ldots, c_n|a, b) &= \frac{p(c_1, \ldots, c_n, b|a)}{p(b|a)} \\ &= \frac{1}{H(a, b)} \mathrm{tr}(P_a P_{c_1} \cdots P_{c_n} P_b P_{c_n} \cdots P_{c_1}) \qquad \text{[ABL's eq. (2.4)]}, \end{aligned}\right\} \tag{32}$$

with $H(a, b)$ playing the role of $p(b|a)$ and determined by overall normalization to be

$$H(a, b) = \sum_{c'_1, \ldots, c'_n} \mathrm{tr}\Big(P_a P_{c'_1} \cdots P_{c'_n} P_b P_{c'_n} \cdots P_{c'_1}\Big) \qquad \text{[ABL's eq. (2.5)]}. \tag{33}$$

Immediately below these formulas, the ABL paper included these statements:

> This expression is manifestly time symmetric. If we change the sequence of measurements to $[B, C_n, \ldots, C_1, A]$, Eqs. (2.4), (2.5) remain unchanged. [Ibid., p. B1412]

The first line of (32) contained the ambiguous-looking notation

$$p(c_1, \ldots, c_n|a, b) = \frac{p(c_1, \ldots, c_n, b|a)}{p(b|a)}. \tag{34}$$

This formula made the ABL paper's conditional probability, as it appears on the left-hand side of (34), look like it referred to a logical conjunction of separate propositions $c_1, \ldots, c_n$, to be read as "c_1 and c_2 and ... and c_n," conditioned on a logical conjunction of separate propositions a, b, to be read as "a and b." However, to assign that meaning or interpretation to the ABL paper's conditional probability would be to commit the pattern-matching fallacy (3). In particular, one cannot sum on, say, c_2 to marginalize down to a shorter measurement sequence that skips the measurement carried out on the intermediate observable C_2. The ABL paper's notation also suppressed the important role played by time-directed measurements in the overall physical process.

One should compare the ABL paper's formula (34) with the more precise (if admittedly more cumbersome) notation (27) from the present work, adapted to the case in which $|\Psi\rangle = |a\rangle$:

$$p[\text{dev}(c_1, \ldots c_n)|a, \text{obs}(a,b), (A, C_1, \ldots, C_n, B)] = \frac{p[\text{obs}(a,b), \text{dev}(c_1, \ldots, c_n)|a, (A, C_1, \ldots, C_n, B)]}{p[\text{obs}(a,b)|a, (A, C_1, \ldots, C_n, B)]}. \tag{35}$$

This latter version makes clear that $\text{dev}(c_1, \ldots c_n)$ is an atomic proposition, and not the logical conjunction of separate propositions $c_1, \ldots, c_n$. Similarly, $\text{obs}(a, b)$ is an atomic proposition, and not the logical conjunction of separate propositions a and b. This latter version also makes manifest the time-directed nature of the measurements in the experimental protocol.

3.6 Vaidman's Interpretation

A 1996 paper by Vaidman expressly acknowledged the fundamentally different roles played by pre-selection and post-selection in the ABL rule:

> Note the asymmetry between the [pre-selection] measurement at t_1 and the [post-selection] measurement at t_2. Given an ensemble of quantum systems, it is always possible to prepare all of them in a particular state $|\Psi_1\rangle$, but we cannot ensure finding the system in a particular state $|\Psi_2\rangle$. Indeed, if the pre-selection measurement yielded a result different from projection on $|\Psi_1\rangle$ we can always change the state to $|\Psi_1\rangle$, but if the measurement at t_2 did not show $|\Psi_2\rangle$, our only choice is to discard such a system from the ensemble. Note also the asymmetry of the measurement procedures. The measurement device has to be prepared before the measurement interaction in the "ready" state and we cannot ensure finding the "ready" state after the interaction. [Vaidman 1996b, p. 3]

However, shortly thereafter, the paper set aside this concern by asserting that the only relevant notion of time-symmetry should refer to the intermediate measurements alone:

> These asymmetries, however, are not relevant to the problem we consider here. We study the symmetry relative to the measurements at [the intermediate] time t for a given pre- and post-selected system, and we do not investigate the time-symmetry of obtaining such a system. [Ibid., p. 4]

The paper restated this assertion, in a section titled "Time asymmetry prejudice":

> In my approach the pre- and post-selected states are given. Only intermediate measurements are to be discussed. So the frequently posed question about the probability of the result of the post-selection measurement is irrelevant. It seems to me that the critics of the time-symmetrized quantum theory use in their arguments the preconception of an asymmetry. [Ibid., p. 12]

These statements represent a departure from the time-symmetric interpretation obtained from a plain reading of the ABL paper, and an implicit admission that the time-symmetric interpretation cannot be sustained.

3.7 The AAD Paper and the Uncertainty Principle

Recalling the spin-1/2 example presented in Subsection 3.1, consider again an ensemble of $N = 10{,}000$ experimental trials involving a single spin-1/2 degree of freedom, where, in each trial, the system is prepared in the spin-z eigenstate $|z+\rangle$. Suppose, however, that the system is then subjected to a spin-x measurement, followed by a *second* spin-x measurement, and no further measurements. If one collects only trials in which the final spin-x measurement yields $|x+\rangle$, then the number of trials reduces to approximately $N_{z+,x+} \approx 5{,}000$. In all these trials, the DvN collapse axiom trivially implies that the second spin-x measurement has to give the same result as the first spin-x measurement. Thus, for this ensemble, the ABL ratio (30) trivially yields

$$\frac{N_{z+,x+,x+}}{N_{z+,x+}} \approx \frac{5{,}000}{5{,}000} = 1 = 100\% \qquad [p(x_+|z_+, x_+) \text{ in the original ABL notation}]. \tag{36}$$

Notice here the the *middle* measurement value appearing in this protocol, showing up as the *second* subscript of the numerator $N_{z+,x+,x+}$, corresponds to a spin-x measurement, in between the pre-selected spin-z value and the post-selected spin-x value.

If one instead considers an ensemble of $N = 10{,}000$ trials in which the spin-1/2 degree of freedom is prepared in the spin-z eigenstate $|z+\rangle$, then is subjected to a *second* spin-z measurement, followed by a spin-x measurement, where one keeps only trials with the final result $|x+\rangle$, then the number of trials is again approximately $N_{z+,x+} \approx 5{,}000$, for which the DvN collapse axiom ensures that every second spin-z measurement yields $|z+\rangle$. Hence, for this ensemble, the ABL ratio (30) trivially gives

$$\frac{N_{z+,z+,x+}}{N_{z+,x+}} \approx \frac{5{,}000}{5{,}000} = 1 = 100\% \qquad [p(z_+|z_+, x_+) \text{ in the original ABL notation}]. \tag{37}$$

Here the middle measurement value, showing up as the second subscript of the numerator $N_{z+,z+,x+}$, now corresponds to a spin-z measurement, in between the pre-selected spin-z value and the post-selected spin-x value.

Is one justified in concluding from (36) and (37) that the spin-x and spin-z observables measured in the middle step of each of these protocols, despite being represented by noncommuting self-adjoint operators, both occur with probability 100%, in contravention of the uncertainty principle? The answer would appear to be negative, because (36) and (37) are ensemble properties that refer to fundamentally different ensembles, so to conclude instead in the affirmative would precisely be to commit the ensemble fallacy (1).

Nor can one justify claiming that (36) and (37) imply a violation of the uncertainty principle merely because both ratios are amenable to experimental measurement in the laboratory. To assert the opposite would be to commit the measurementist fallacy (4).

Moreover, notice the crucial role played by the post-selection of $|x+\rangle$. Without that post-selection, both the frequency ratios (36) and (37) would instead have been 5,000/10,000 = 50%, perfectly in keeping with the uncertainty principle. The preceding example makes abundantly clear the perils of trying to make statistical inferences when post-selection is involved—and, indeed, when post-selection is invoked on purpose. This illicit invocation of post-selection is an example of the post-selection fallacy (2), and gives yet another reason to doubt that the preceding example represents a true exception to the uncertainty principle.

However, a 1985 paper by Albert, Aharonov, and D'Amato (AAD), titled "Curious New Statistical Prediction of Quantum Mechanics" and appearing in the journal *Physical Review Letters*, (Albert, Aharonov, D'Amato 1985), argued that the ABL rule provided a way to violate the uncertainty principle in just this manner:[5]

> Suppose that [a given quantum] system is measured at time t_i to be in the state $|A = a\rangle$ (where A represents some complete set of commuting observables of the system, and a represents some particular set of eigenvalues of those observables), and is measured at time t_f ($t_f > t_i$) to be in the state $|B = b\rangle$. What do these results imply about the results of other experiments that might have been carried out within the interval ($t_i < t < t_f$) between them? It turns out that the probability (which was first written down by Aharonov, Bergmann, and Lebowitz) that a measurement of some complete set of observables C within that interval, *if* it were carried out, would find that $C = c_j$ is
>
> $$P(c_j) = \frac{|\langle A = a|C = c_j\rangle|^2 |\langle C = c_j|B = b\rangle|^2}{\sum_i |\langle A = a|C = c_i\rangle|^2 |\langle C = c_i|B = b\rangle|^2} \qquad \text{[AAD's eq. (1)];} \qquad (38)$$
>
> and that formula entails, among other things, that $P(a) = P(b) = 1$. Consequently, these authors maintain that such a system, within such an interval, must have definite, dispersion-free values of both A and B, whether or not A and B may happen to commute. In their view, the proper quantum mechanical descriptions of the past and the future are essentially different: Our knowledge of the past is not restricted, in the same way as our ability to predict the future, by the uncertainty relations; indeed, so far as the past is concerned, the quantal formalism itself *requires* that those relations be violated. [Ibid., pp. 5–6, emphasis in the original]

[5]The quoted passage attributes these claims to the authors of the original ABL paper. However, As Sharp and Shanks later pointed out, there does not appear to be evidence that these views were expressed in the original 1964 ABL paper: "Actually, Albert et al. attribute these conclusions to Aharonov et al. (1964) but we can find no evidence of either conclusion in that work." (Sharp, Shanks 1993, p. 494, footnote 2) The AAD paper may have been referring to a different paper, published the previous year (Aharonov, Albert 1984).

The AAD paper acknowledged that claims of dispersion-free values of non-commuting observables, and violations of the uncertainty principle, might sound surprising, in light of various no-go theorems. The AAD paper nevertheless doubled down:

> Is it somehow mistaken, then, or somehow misleading, to suppose that (1) attributes definite values to A and B? Is it that (1) itself produces some contradiction? How? Where?
>
> No. It turns out (and this is the subject of the present note) that there is a remarkable and heretofore unknown property of the quantal statistics whereby quantum mechanical systems, within the interval between two measurements, *fail* to satisfy that assumption (the assumption about the projection operators), and so evade its consequences. [Ibid., p. 6, emphasis in the original]

Once again, this erroneous conclusion may have been precipitated by an overly minimalist choice of notation. In the more expansive and precise notation of (27), the probability (38) in the AAD paper would have been written instead as

$$p[\mathrm{dev}(c_j)|a, \mathrm{obs}(a,b), (A,C,B)] = \frac{p[\mathrm{obs}(a,b), \mathrm{dev}(c_j)|a, (A,C,B)]}{p[\mathrm{obs}(a,b)|a, (A,C,B)]}. \tag{39}$$

The operational meaning of the probability (39), as outlined in Subsection 3.1, concerns the fraction of the appropriately constructed ensemble whose members show the intermediate measurement result c_j for the complete set of observables represented by C. The choice of C for the intermediate measurement is central to defining the physical ensemble in question.

To ask instead for the probability that an intermediate measurement of A should yield a would mean to construct a *different* experimental protocol producing a *different* physical ensemble, for which the ratio yields 1:

$$p[\mathrm{dev}(a)|a, \mathrm{obs}(a,b), (A,A,B)] = \frac{p[\mathrm{obs}(a,b), \mathrm{dev}(a)|a, (A,A,B)]}{p[\mathrm{obs}(a,b)|a, (A,A,B)]} = 1. \tag{40}$$

Similarly, to ask for the probability that an intermediate measurement of B should yield b would mean to construct yet *another* experimental protocol producing a distinct physical ensemble of its own, for which the ratio again yields 1:

$$p[\mathrm{dev}(b)|a, \mathrm{obs}(a,b), (A,B,B)] = \frac{p[\mathrm{obs}(a,b), \mathrm{dev}(b)|a, (A,B,B)]}{p[\mathrm{obs}(a,b)|a, (A,B,B)]} = 1. \tag{41}$$

The three probabilities (39), (40), and (41) here refer to three separate physical ensembles, and so it would be a mistake to try to draw inferential conclusions about any one of them from either or both of the other two. As explained earlier, it is also dangerous to jump to conclusions about statistical inferences when one's set-up involves post-selection, on pain of committing the post-selection fallacy (2).

To make completely clear why the AAD authors did not find a loophole in the uncertainty principle, it will be useful to present an operational argument for how one can experimentally verify the uncertainty principle. One imagines setting up *one* physical ensemble of identical quantum systems whose definition consists solely of a preparation or 'pre-selection' of each member of the ensemble in the same initial quantum state ρ. One then carries out a measurement of some observable C for half of the members of the ensemble, and a measurement of some other observable D for the other half of the members of the ensemble, where C and D, as self-adjoint operators on the system's Hilbert space, may fail to commute, in the sense that $CD - DC \neq 0$. That is, one carries out a *controlled experiment* by *fixing* ρ as the definition of the entire physical ensemble and then *independently varying* just the choice of C or D within that fixed ensemble, *without* any post-selection. One finds that the respective spreads or standard deviations ΔC and ΔD of measurement-outcome distributions for C and D have a product $\Delta C\,\Delta D$ that satisfies the inequality

$$\Delta C\,\Delta D \geq \frac{1}{2}|\mathrm{tr}[(CD - DC)\rho]|, \tag{42}$$

which is just the uncertainty principle. In some cases, such as for a particle's position $C = x$ and corresponding momentum $D = p_x$, the right-hand side of (42) reduces to $\hbar/2$, and it is impossible for C and D both to have vanishing dispersion, regardless of the initial quantum state ρ.

For the AAD argument, by contrast, one not only explicitly carries out a form of post-selection, but one must also have *two* physical ensembles produced according to *different* experimental protocols: a first ensemble produced by measuring the chronological sequence A, C, B and keeping only members of the ensemble for which $A = a$ and $B = b$, and a second ensemble produced by measuring the chronological sequence A, D, B and similarly keeping only members for which $A = a$ and $B = b$. Crucially, notice that one cannot include B in the experimental protocols that produce these two ensembles without also including either C or D, in contrast with the single ensemble constructed for the uncertainty principle in the previous paragraph. For each of these two ensembles, the physical post-selection process for B implicitly depends on the previous measurement of C (for the first ensemble) or on the previous measurement of D (for the second ensemble). Thus, one cannot imagine replacing C with D without *also* changing the physical post-selection process for B as well. It is true that replacing C with D does not alter the pre-selection on A, but to make the same assumption about the post-selection on B would be precisely to assume the erroneous form of time-symmetry for quantum measurements that the present paper rigorously argued against in Subsection 3.4.

As a consequence, for the AAD argument, it is not possible to carry out a controlled experiment by *fixing* the pre-selection on A and the post-selection on B to produce a single physical ensemble, and then *independently varying* the choice of C or D to obtain a violation of the uncertainty principle. This lack of independence between the choice of intermediate observable C or D and the post-selection on B is concealed by the notation $P(c_j)$ used in the AAD version of the ABL rule, (38), as well as in the notation $p(c_1, \ldots, c_n|a, b)$ used in the original ABL rule (32), but is manifest in the more precise notation used in this paper's formula for the ABL rule in (30).

The present discussion highlights another important feature of textbook quantum theory that is clear from the review of the DvN axioms in Subsection 2.1: at least in some cases, the textbook theory can make concrete, reliable predictions for a *single* quantum system, without the need to invoke an ensemble. This feature is due, in part, to the fact that the DvN axioms prescribe that observables are assigned at the single-system level, not at an ensemble level. It therefore makes sense to talk about measuring observables for a single system, to regard measurement outcomes as statements about that single system, and to regard measurement probabilities and expectation values as *referring* to observables of that single system. (This key feature of textbook quantum theory also turns out to have important implications for weak values, to be discussed in other work.)

As an example that is highly relevant to the uncertainty principle, consider a single system with two noncommuting observables C and D. If one repeatedly measures C, without giving the system significant time for internal evolution between the measurements, then the DvN collapse axiom will ensure that one reliably obtains the same result each time, within reasonably small error bars. However, an intervening measurement of the observable D will generically lead to a significant change in subsequent measurements of C, to a degree controlled in part by the commutator $CD - DC$. This *experimental* noncommutativity between *algebraically* noncommuting observables C and D is perhaps the most universal feature of all quantum systems, and provides a concrete, operational meaning to the uncertainty principle at the single-system level. In particular, a core part of the uncertainty principle is precisely that it has this implication for a single system.

The AAD argument, by contrast, always requires (multiple) ensembles, so it refers exclusively to patterns of behavior in ensembles. As a consequence, the AAD argument has no meaning *in principle* for a single system, and therefore inevitably misses a core part of the uncertainty principle. Thus, *a fortiori*, one cannot interpret the AAD argument as giving any real insight into the uncertainty principle for a single system, let alone providing a loophole, without committing the ensemble fallacy (1).

Over the years since the AAD paper was published, several other papers have pointed out, in more narrow ways, the failure of the AAD paper to show that the ABL rule supports counterfactual reasoning. For instance, in a 1986 paper, Bub and Brown wrote:

> That [the AAD] argument is fallacious can be seen by noting that the subensemble of the preselected [a] ensemble that is post-selected for [b] via an intervening [C] measurement differs from the subensemble that is post-selected for [b] via an intervening [D] measurement. [Bub, Brown 1986, p. 2338]

Sharp and Shanks made a similar case in a 1993 paper, and came to a conclusion not very different from the one reached in the present work:

> Interpreted correctly, the ABL-Rule makes predictions only as to the outcomes of actual measurements conducted upon systems subject to both and pre- and post-selection. So interpreted, the rule is not in conflict with orthodox quantum mechanics, but neither does it yield fresh insights about the fundamental interpretive issues in quantum mechanics. [Sharp, Shanks 1993, p. 499]

One finds similar criticisms over the use of the ABL rule for counterfactuals in papers concerned with the consistent-histories interpretation of quantum theory (Griffiths 1984, Cohen 1995, Kastner 1999).

4 Conclusion

The present work has reviewed the ABL rule, and raised several challenges to some of the ways that it has been interpreted in the research literature. In particular, this paper has argued that the ABL rule does not provide a time-symmetric formulation of quantum theory and does not lead to true violations of the uncertainty principle. Along the way, this paper has highlighted several relevant fallacies that are relevant to this critical analysis, including the ensemble fallacy (1), the post-selection fallacy (2), the pattern-matching fallacy (3), and the measurementist fallacy (4).

Are these challenges worth discussing? Does it really matter if the ABL rule is not really time-symmetric, or if post-selection can lead to erroneous statistical inferences? The answer to these questions is yes, not only for reasons of philosophical rigor, but also because physics research since 2000 has made substantial use of these interpretational claims about the ABL rule and post-selection, and that research merits scrutiny. In particular, other work will investigate the connections between the ABL rule and weak values (Barandes 2026).

At a broader level, this paper should be taken as an argument against the cavalier use of post-selection to generate publication-worthy results, and as a call for research journals to insist on more careful explanations by authors that when they use post-selection, their results are actually produced by quantum theory itself, and are not merely manifestations of selection bias. It is heartening that this important form of self-corrective introspection is already happening in the research literature (see, for example, Wharton, Price 2025).

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