Pilot-Wave Theories as Hidden Markov Models

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Abstract

The original version of the de Broglie-Bohm pilot-wave theory, also called Bohmian mechanics, attempted to treat the wave function or pilot wave as a part of the physical ontology of nature. More recent versions of the de Broglie-Bohm theory appearing in the last few decades have tried to regard the pilot wave instead as an aspect of the theory's nomology, or dynamical laws. This paper argues that neither of these views is correct, and that the de Broglie-Bohm pilot wave is best understood as a collection of latent variables in the sense of a hidden Markov model, a construct that was not available when de Broglie and Bohm originally formulated what became their pilot-wave theory. This paper also discusses several other challenges for the ontological view of the pilot wave. One such challenge is due to Foldy-Wouthuysen gauge transformations, which connect up with the Deotto-Ghirardi ambiguity in the de Broglie-Bohm theory. Another challenge arises from the freedom to carry out canonical transformations in the wave function's own notion of phase space, as defined by Strocchi and Heslot.

1 Introduction

1.1 The Interpretation of the Quantum State

The quantum state is often taken to be the starring protagonist of quantum theory. The earliest version of the quantum state, introduced by Erwin Schrödinger in 1926, took the form of a wave function defined in the abstract space of a system's kinematically possible configurations, and evolved in time according to a specific partial differential equation—the famous Schrödinger equation (Schrödinger 1926a–d). Later versions of the quantum state, introduced by Paul Dirac and John von Neumann, were represented either by state vectors in abstract Hilbert spaces, or by statistical operators or density operators on Hilbert spaces (Dirac 1930, von Neumann 1932). By the 1940s, quantum states had been abstracted further to positive, normalized linear functionals in the dual spaces to C*-algebras (Segal 1947a,b).

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Understandably, much of the discourse surrounding the interpretation of quantum theory over the past century has centered on the meaning and metaphysical status of the quantum state. Contemporary postures toward the quantum state typically lean toward one of the following views:

1. (Statistical) The quantum state is an instrumentalist statement about statistical distributions of possible measurement outcomes, as on the orthodox or 'textbook' interpretation of quantum theory. (See, for example, Shankar 1994; Sakurai, Napolitano 2010; Griffiths, Schroeter 2018.)

2. (Epistemic-over-measurements) The quantum state is a representation of the epistemology or knowledge of external observers about possible measurement results, as on the Copenhagen interpretation (Heisenberg 1955, 1958; Howard 2004) and on QBism (Fuchs 2010).

3. (Epistemic-over-ontology) The quantum state is a representation of the epistemology or knowledge of external observers about objective arrangements of physical or ontological entities, as on various "ψ-epistemic" accounts (Harrigan, Spekkens 2010).

4. (Ontological-monistic) The quantum state is the sole ontology of the universe, as on the Everett 'many worlds' interpretation (Everett 1956, 1957a; Everett, DeWitt, Graham 1973) and on certain interpretations of dynamical-collapse theories (Gisin 1984; Ghirardi, Rimini, Weber 1986).[1]

5. (Ontological-pluralistic) The quantum state is a part of the physical reality or ontology of the universe, as on some versions of the de Broglie-Bohm pilot-wave theory (de Broglie 1930; Bohm 1952a,b) and on more recent formulations of dynamical-collapse theories (Allori, Goldstein, Tumolka, Zanghì 2008; Goldstein, Tumulka, Zanghì 2012).

6. (Nomological) The quantum state is nomological, meaning a feature of a system's dynamical laws, as on other versions of the de Broglie-Bohm pilot-wave theory (Dürr, Goldstein, Zanghì 1996).

Arguably none of these views capture the full and proper meaning of the quantum state. All of them seem at least somewhat ill-fitting.

The first two views in the list above—the statistical and epistemic-over-measurement views—leave the measurement problem essentially unanswered (Maudlin 1995), and remain quiet over whether there is any ontology in the world beyond macroscopic measuring devices and external observers. The epistemic-over-ontology view does not seem to capture the functional relationship that quantum states have with the behavior of systems and the outcomes of measurements, and runs into difficulties in accommodating various no-go theorems, like the PBR theorem (Pusey, Barrett, Rudolph 2012). As for the fourth and fifth ontological views, quantum states are associated with abstract

[1]At least according to Bell's account of the original GRW theory (Bell 1987), the theory did not propose any ontology above and beyond the wave function. The "quantum jumps" of the GRW theory were, in Bell's words, "part of the wavefunction, not something else." In the same paper, Bell also wrote that "The GRW theory does not add variables." In later work, Bell wrote that "The GRW-type theories have nothing in their kinematics but the wavefunction" (Bell 1990).

spaces, like configuration spaces or Hilbert spaces, and have many other exotic properties that seem quite different from the sorts of entities traditionally assigned a physical or ontological meaning, as this paper will explain in detail, and as explored from a historical perspective in other work (Barandes 2026b). Finally, looking at the nomological view, quantum states have seemingly contingent, possibly complicated initial conditions of their own, and also typically feature highly complicated forms of time-dependence, as well as nontrivial behavior under time-reversal transformations, all of which make them awkward as dynamical laws.

1.2 Markovian Theories and Hidden Markov Models

With sufficient effort, one can mitigate most of the problems listed above to varying degrees, but the very need for that effort also motivates seeking out a more natural interpretation of quantum states. To that end, this paper will argue that quantum states are best understood through the lens of hidden Markov models, with quantum states playing the role of their latent variables—sometimes called 'hidden variables,' although that term should not be confused with the conventional notion of hidden variables in quantum theory.

In brief, for a physical theory to be called Markovian—or, perhaps more precisely, *dynamically* Markovian—is just to say that its dynamical laws feature a rather nice property: the dynamical laws, combined with the present-moment state of affairs, determine the state of affairs at the (possibly infinitesimally) next moment in time, with no dependence on the past except for whatever is mediated through the present-moment state of affairs. In somewhat more detail, given only the present-moment configuration or state of a system described by a Markovian theory, the dynamical laws fix either the unique configuration or state at the next moment in time, if the laws are deterministic, or fix a probability distribution over configurations or states at the next moment in time, if the laws are stochastic. A physical theory that fails to have this Markov property is said to be non-Markovian.

If a non-Markovian physical theory can be re-expressed as a Markovian theory by formally augmenting its configurations or states with a suitable collection of unobservable variables, then the resulting theory is called a hidden Markov model. The unobservable variables added to make the theory look Markovian are said to be latent or hidden variables, and they commonly have most of the following seven (and conceivably more) hallmark characteristics:

1. (Abstraction) They tend to be conceptually abstract.
2. (Non-uniqueness) They are highly non-unique, in the sense of admitting a diverse set of mathematical redefinitions.
3. (Unobservability) They are typically unobservable in principle.
4. (Non-spatiality) They have no notion of location in physical space.
5. (Absence of backreaction) They are not backreacted upon by the physical configurations or states of the system, meaning that they change in time on their own in a manner that is

completely insensitive to the goings-on of the physical configurations or states of the system.

6. (Multivariateness) They often encompass many degrees of freedom, depending on how difficult it is to turn the non-Markovian theory into a hidden Markov model.

7. (Contingency) They feature contingent patterns of time evolution depending on correspondingly contingent initial conditions.

Hidden Markov models first appeared in the research literature in the 1960s. They were originally called "probabilistic functions of Markov chains" (Baum, Petrie 1966; Baum, Eagon 1967; Baum, Petrie, Soules, Weiss 1970; Baum 1972). They only got their more modern name, "hidden Markov models," in 1970 in a set of unpublished lectures by Lee Neuwirth (Neuwirth 1970; Neuwirth, Cave 1980; Poritz 1988), and were quickly put to use in statistically modeling speech and language.[2] More modern examples include ϵ-machines, which have found widespread use in statistical physics and in the theory of complex systems (Crutchfield, Young 1989; Travers, Crutchfield 2011).

Given that these developments all took place long after quantum theory and its prominent interpretations had become firmly established in physics and philosophy, it is hardly surprising that no one thought to revisit the metaphysical status of quantum states from the standpoint of hidden Markov models until now.

1.3 Outline of this Paper

To narrow this paper's scope to a manageable degree, the primary focus will be on pilot-wave theories of fixed numbers of finitely many non-relativistic particles, like the original de Broglie-Bohm theory, also known as Bohmian mechanics, which will be reviewed in Section 2. On these theories, the quantum state generally takes the form of a complex-valued wave function $\Psi(q,t)$ evolving with the time t according to the Schrödinger equation in the system's configuration space, each of whose points q denotes a kinematically allowed 'classical' configuration of the system. This wave function, also called a pilot wave in the context of the de Broglie-Bohm theory, then guides or pilots the system's particles along their trajectories through three-dimensional physical space according to a precise guiding equation.

Section 3 will argue that this pilot-wave theory can be understood—and is perhaps best understood—as a new kind of hidden Markov model, with the pilot wave making up the model's latent or hidden variables. Indeed, as that section of the paper will show, the pilot wave exhibits all seven smoking-gun characteristics of latent variables listed above in Subsection 1.2.

Section 4 will present additional challenges both to the ontological view of the wave function, and also to the de Broglie-Bohm pilot-wave theory itself. These challenges will come from three

[2] Much of the research on hidden Markov models was classified at the time, and took place as part of the Institute for Defense Analyses, where Neuwirth was the deputy director of its Communications Research Division, despite Neuwirth's strongly anti-war political leanings (Neuwirth 2009). The fact that this research was classified may have further delayed its diffusion into other research communities, including those working in quantum foundations. As an interesting historical aside, one of Lee Neuwirth's children is the actress and singer Bebe Neuwirth, famous for, among other major roles, playing the character Lillith Sternin on the television sitcoms *Cheers* and *Frasier*.

primary directions: from thinking about interference effects, from a little-known class of gauge transformations first introduced by Foldy and Wouthuysen (1950), and from probing a notion of phase space for the wave function itself that was defined independently by Strocchi (1966) and Heslot (1985).

1.4 Some Terminological Disambiguation

Unfortunately, there are a significant number of terminological collisions between the theory of stochastic processes, the theory of causal modeling, and quantum theory. Before continuing on to the rest of this paper, it will be important to disambiguate some of this terminology.

In particular, the terms 'model,' 'Markov,' and 'latent variable' show up both in the theory of stochastic processes and in the theory of causal modeling. Both theories also use similar-looking graphical depictions, called directed graphs. Meanwhile, the terms 'Markov' and 'hidden variables' show up both in the theory of stochastic processes and in quantum theory.

For the theory of stochastic processes, a 'model' refers to a dynamical process in which one or more random variables and their probability distributions change with time. In that sense, a stochastic process generalizes a time series, which consists of a sequence of numerical values of a particular quantity indexed by time. By contrast, for the theory of causal modeling, a 'model' refers to a network of events or random variables connected to each other with directed causal links.

For the theory of stochastic processes, 'Markov' should be understood to mean 'dynamical Markovianity,' referring to the condition that future configurations or states of the system, or probability distributions over future configurations or states of the system, are always entirely fixed by just the (possibly infinitesimally) previous-moment configuration or state of the system (Milz, Modi 2021). By contrast, for the theory of causal modeling, 'Markov' usually refers to the 'causal Markov condition,' which is the condition that if the probability distribution of a random variable in a causal model is conditioned on specific values of all its immediate causal predecessors, or 'parents,' then that random variable has no correlations with any other random variables except for any of its causally descendant random variables. That is, for a causal model satisfying the causal Markov condition, the parents of a given random variable always screen off all the random variable's correlations except possibly for correlations with the random variable's causal descendants. (See, for example, Theorem 1.4.1 in Pearl 2009.)

It is possible for a stochastic process to be related to a causal model for which the stochastic process fails to be dynamically Markovian while the causal model satisfies the causal Markov condition, and vice versa. To see why, note that if the configuration of a stochastic process at some time depends on, say, its predecessors at two previous times, so that the stochastic process is not dynamically Markovian, then it may still be possible to construct a causal model in which those two previous moments in time, treated as parents, screen off all other correlations, so that the causal model satisfies the causal Markov condition. Going in the other direction, a dynamically Markovian process may feature correlations between variables at a single moment in time that, when written as a causal model, fail to feature the screening that is required by the causal Markov condition.

To make matters even more confusing, the term 'Markov' has a subtly different meaning in much of the contemporary research literature on quantum theory. Conventionally speaking, a quantum system is said to be Markovian if its present-moment quantum state fully determines its next quantum state (Milz, Modi 2021), as is the case for closed quantum systems evolving according to unitary dynamics or for open quantum systems evolving according to the Lindblad or GKLS equation (Gorin, Kossakowski, Sudarshan 1976; Lindblad 1976). Otherwise the quantum system is said to be non-Markovian.

The term 'latent' also merits careful untangling. For the theory of stochastic processes, a hidden Markov model is a stochastic process that formally satisfies dynamical Markovianity due to the augmentation of the system's configurations by additional, 'latent' variables, where those latent variables are often entirely unphysical and abstract. By contrast, for a causal model, 'latent structures' refer to unobservable components that are needed to ensure the validity of the causal Markov condition. A causal model that fails to satisfy the causal Markov condition, due to a missing latent structure, is usually considered to be an incomplete causal model. Indeed, one major purpose of causal modeling is to *find* latent structures, which are usually regarded as physical. (See, for example, Pearl 2009, Subsection 2.9.1, "On Minimality, Markov, and Stability.")

For a hidden Markov model, latent variables are sometimes called 'hidden variables,' which should be distinguished from the notion of hidden variables in quantum theory. In quantum theory, hidden variables usually refer to any variables other than the quantum state or wave function that some interpretations regard as representing necessary ontological components for providing a complete description of physical reality. For instance, the particles of a pilot-wave theory are paradigmatic examples of hidden variables according to the quantum-theoretic meaning of the term. One of the purposes of the present work will be to argue that the true 'hidden variables' of quantum theory, at least in the case of pilot-wave interpretations, are wave functions, in the sense of being best understood as latent variables of a hidden Markov model.

2 Pilot-Wave Theories

2.1 Relevant History

Before proceeding to the main arguments of this paper, it will be helpful to begin with some relevant history, which is covered in more detail in other work (Barandes 2026b). The purpose of this history will be to make clear the following four points about the idea of an ontological wave function or pilot wave defined in an abstract configuration space rather than in three-dimensional physical space:

- The idea was not a new contribution of David Bohm, Hugh Everett III, or John Bell, but appeared in Erwin Schrödinger's 1926 papers originally introducing his wave function.

- The idea was thoroughly studied and carefully considered in the late 1920s by the major figures responsible for wave-particle duality (Albert Einstein and Louis de Broglie), the wave function itself (Schrödinger), and the original pilot-wave theory (de Broglie).

- To a person, all of them resoundingly, repeatedly, and vociferously rejected the idea of treating the wave function as ontological, as did the people responsible for what later became known as the 'Copenhagen interpretation' (Niels Bohr, Werner Heisenberg, Max Born, Wolfgang Pauli, and Paul Dirac). Remarkably, denying the physical reality of the configuration-space wave function was one of the few things that all the prominent founders of quantum theory agreed on.

- More favorable attitudes toward the physical or ontological reality of the wave function showed up starting in the 1950s, and were largely due to Bohm and Everett, though Schrödinger himself apparently began warming to the idea around this time as well.

In a famous paper in 1905, Albert Einstein proposed that electromagnetic waves transported their energy in discrete quanta (Einstein 1905). Partly inspired by this idea, Louis de Broglie, writing in 1923, suggested that every material particle had an associated 'phase wave' that propagated in three-dimensional physical space, maintained phase harmony with any hypothetical periodic processes internal to the particle, and, in the geometrical-optics limit of short wavelength, guided or piloted the particle's trajectory along its rays (de Broglie 1923a–c).

In de Broglie's 1924 doctoral thesis, which would eventually win him the 1929 Nobel Prize in physics, he wrote down an early version of a guiding equation in the form

$$O_i = \frac{1}{h} J_i, \tag{1}$$

where, in de Broglie's notation, i was a Lorentz index. Here J_i was defined to be

$$J_i = m_0 c u_i + e\varphi_i, \tag{2}$$

where m_0 was the particle's proper or rest mass, c was the speed of light, e was its electric charge, φ_i were the electromagnetic gauge potentials (not to be confused with the notation for the wave's phase function, to be introduced momentarily), and $u_i = dx_i/ds$ was the particle's dimensionless four-velocity, with incremental parameter $ds = \sqrt{c^2dt^2 - dx^2 - dy^2 - dz^2}$. Meanwhile, O_i was given in terms of the phase function $\varphi(x, y, z, t)$ of the associated wave according to the differential identity

$$d\varphi = 2\pi \sum_i O_i dx^i, \tag{3}$$

so that O_i itself, from a more modern standpoint, can be identified as the wave's four-dimensional wave vector. The spatial parts $\mathbf{O} = (O_x, O_y, O_z)$ of O_i are then the gradient of the phase function $\varphi(x, y, z, t)$, up to a reciprocal factor of 2π:

$$\mathbf{O} = \frac{1}{2\pi} \nabla\varphi. \tag{4}$$

Hence, in the non-relativistic limit, for which $c\mathbf{u} \approx \mathbf{v}$, where $\mathbf{v}$ is the particle's ordinary velocity, and assuming vanishing electromagnetic gauge potentials, $\varphi_i = 0$, the primitive guiding equation

(1) reduces to

$$\mathbf{v} = \frac{1}{m_0}\left(\frac{h}{2\pi}\right)\nabla\varphi. \tag{5}$$

Today, the factor in parentheses would be called the reduced Planck constant $\hbar$ ('h-bar'), as originally introduced in 1928 by Paul Dirac (Dirac 1928):

$$\hbar = \frac{h}{2\pi}. \tag{6}$$

Partly inspired by de Broglie's phase-wave theory, and partly by Hamilton-Jacobi theory, Erwin Schrödinger introduced his theory of "undulatory mechanics" or "wave mechanics" in a series of four foundational papers in 1926 (Schrödinger 1926a–d). On this new theory, every quantum system as a whole had a complex-valued wave function $\psi(q, t)$ depending on points q in the system's abstract configuration space and on the time t, and satisfying what is now known as the Schrödinger equation, originally written as eq. $(4'')$ in Schrödinger fourth paper (Schrödinger 1926d):

$$\Delta\psi - \frac{8\pi^2}{h^2}V\psi \mp \frac{4\pi i}{h}\frac{\partial\psi}{\partial t} = 0. \tag{7}$$

Here Δ was a second-order differential operator acting on the system's configuration space that implicitly involved the masses of the various particles comprising the system, and V was a potential function defined on the system's configuration space. The ambiguous $\mp$ sign appearing in the equation reflected the freedom to work with either ψ or its complex-conjugate $\bar{\psi}$. Choosing the positive sign convention, rearranging, and using the definition of the reduced Planck constant $\hbar = h/2\pi$ from (6), the Schrödinger equation takes its more modern-looking form:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2}\Delta\psi + V\psi. \tag{8}$$

As Schrödinger pointed out in the first of these four foundational papers on undulatory mechanics (Schrödinger 1926a), his wave function $\psi(q, t)$ was defined not in three-dimensional physical space, like de Broglie's phase waves, but in the system's abstract, generically many-dimensional configuration space. As a consequence, Schrödinger spent a great deal of effort grappling with the physical meaning of the wave function in these four papers, in personal correspondence with colleagues, and in a series of four lectures that he delivered at the Royal Institution in London in 1928 (Schrödinger 1928).

By the late spring of 1926, Schrödinger had settled on the tentative idea that the wave function was a physical object of some kind, and that it manifested itself through its modulus-square $\psi\bar{\psi} = |\psi|^2$, which, in turn, indirectly determined the distribution of electric charge in three-dimensional physical space. Schrödinger described this view in the fourth of his foundational 1926 papers, where he also expressed the speculative idea that the wave function ψ represented the system being in all its kinematical configurations simultaneously, though in some configurations "more strongly" than in others, as a decades-early prefiguring of Hugh Everett's 'many worlds' interpretation (Schrödinger

1926d).[3]

During the period from 1926 to 1927, Hendrik Lorentz and Albert Einstein expressed their misgivings with the notion of physical waves propagating in many-dimensional configuration spaces. Einstein, in particular, wrote several letters during that period to Lorentz, Max Born, Paul Ehrenfest, and Arnold Sommerfeld criticizing the idea. Indeed, in Einstein's famous letter to Born on December 4, 1926, in which Einstein argued that "God does not play dice," Einstein also included a complaint about Schrödinger's waves in "$3n$-dimensional space."[4]

According to Born's statistical hypothesis, presented in a paper by Born in the summer of 1926, the modulus-square $|\psi(q,t)|^2$ of the wave function represented the probability density for a measurement of the system's configuration at the time t to yield the value q (Born 1926). The Born rule planted seeds of doubt in Schrödinger's mind about the physicality of his wave function. At the end of his four lectures on wave mechanics in 1928, Schrödinger said that he had abandoned his earlier view that the wave function was a physical object, although he moved back toward that view again later in his life (Schrödinger 1950, 1952a,b).

In 1927, de Broglie gave a presentation at the fifth Solvay Conference laying out a more detailed pilot-wave theory on which the waves associated with particles guided or piloted them. In a paper that year, he laid out a version of the theory, now known as his double-solution theory, that featured two conceptually distinct waves, both satisfying the same wave equation (de Broglie 1927). One wave was intended to feature a solitonic singularity that represented a material particle, and the other wave embodied Born's statistical probability for the particle's location. That 1927 paper was the first to feature the guiding equation in its more modern-looking form, as de Broglie's eq. (26′),

$$\overrightarrow{v_M} = \frac{1}{m_0}\overrightarrow{\text{grad}\varphi_1}, \tag{9}$$

where $\overrightarrow{v_M}$ was the particle's velocity, m_0 was its inertial mass, and φ_1 was the phase function of the particle's singularity-bearing wave.

Unable to work through the complicated mathematics of his double-solution theory, de Broglie began working on a second, conceptually distinct pilot-wave theory in 1928, eventually publishing a detailed description of this second pilot-wave theory in a 1930 book (de Broglie 1930). This alternative pilot-wave theory featured waves and particles as separate forms of ontology, with the waves again guiding or piloting the particles along their trajectories in physical space. De Broglie barely mentioned his double-solution theory in that 1930 book.

However, de Broglie ran into the same conceptual challenges as Schrödinger in trying to make sense of his theory in the case of multi-particle systems, for which the associated waves propagated not in three-dimensional physical space, but in the system's many-dimensional configuration space, which de Broglie called "abstract" and "fictitious." Toward the end of the book, he explained that he had decided to abandon this second pilot-wave theory as well.

[3] As Everett made clear in his unpublished long-form dissertation in 1956, Schrödinger's later views on the ontology of the wave function provided inspiration for Everett's own interpretation of quantum theory (Everett 1956).

[4] This complaint was mistranslated in the canonical English translations of the Born-Einstein letters (1971). The crucial "n" in "$3n$-dimensional" was missing (Howard 1990, Barandes 2026b).

David Bohm independently discovered de Broglie's second pilot-wave theory in 1951, eventually publishing his telling of the theory in a pair of 1952 papers submitted for publication simultaneously (Bohm 1952a,b). In the second of these two papers, Bohm used his newly formulated theory of decoherence, as laid out in the final chapters of his 1951 textbook (Bohm 1951, Chapter 22, "Quantum Theory of the Measurement Process"), to clarify how the pilot-wave theory provided a means of resolving the measurement problem.

In his pair of 1952 papers, Bohm referred to the wave function or pilot wave as a "ψ-field." He also described it in analogy with the electromagnetic field. However, he said little in these papers about the ψ-field's physical interpretation for multi-particle systems, although he did briefly acknowledge that the ψ-field would then have to propagate in a many-dimensional space.

In private letters to Wolfgang Pauli in 1951, Bohm explicitly acknowledged that he had rediscovered de Broglie's second pilot-wave interpretation, but argued that de Broglie had not taken his theory far enough, and that Bohm's approach to measurement had resolved several outstanding limitations of de Broglie's theory. In those letters to Pauli, Bohm also made an assertive case for regarding the pilot wave as a physical object, despite its "polydimensional" nature (Pauli 1996).

Einstein and Pauli suggested that Bohm should communicate with de Broglie. De Broglie immediately wrote a critical paper about the pilot-wave theory in 1951, emphasizing once again his problem with the idea of waves propagating in abstract configuration spaces (de Broglie 1951). However, Bohm's work rekindled de Broglie's interest in pilot-wave theories, and de Broglie eventually returned to working on his earlier double-solution theory from 1927.

Bohm's version of the pilot-wave theory, now known as the de Broglie-Bohm theory, or Bohmian mechanics, generated new debates over the physicality of the wave function. Just a few years later, in his unpublished 1956 long-form thesis, Hugh Everett III explicitly cited Bohm's interpretation, and argued that a system's wave function—regarded now as a vector in a Hilbert space, rather than as a function defined in a configuration space—was not only ontological, but was the sole form of ontology in the universe (Everett 1956, 1957a). In John Bell's retelling of the de Broglie-Bohm theory, Bell emphasized the physicality of the pilot wave as well (Bell 1980, 1982), even going so far as to write, in italics:

> *No one can understand this theory until he is willing to think of ψ as a real objective field rather than just a 'probability amplitude'. Even though it propagates not in 3-space but in 3N-space.* [Bell 1981, emphasis in the original]

The view that the wave function or pilot wave is physically real or ontological, despite being defined in a system's configuration space rather than in physical space, has come to be called 'wave-function realism,' and continues to generate controversy to this day (Albert 1996; Lewis 2004; Ney, Albert 2013; Myrvold 2015; Chen 2019; Wallace 2020; Ney 2021; Ney 2023).

Pushing back on wave-function realism, Dürr, Goldstein, and Zanghì have argued that the pilot wave should be understood not as a physical or ontological object, but as a form of nomology, meaning a feature of a quantum system's dynamical laws. (See, for example, Dürr, Goldstein, Zanghì 1996.) More will be said about this nomological view later.

2.2 General Formulation

It will be useful to lay out the general structure of the de Broglie-Bohm pilot-wave theory, again also known today as Bohmian mechanics. Let $\Psi(q,t)$ denote the complex-valued configuration-space wave function or pilot wave for a quantum system with $n = 3N$ degrees of freedom and whose configurations are labeled by $q = (q_1, \ldots, q_n)$, with t denoting the time. Suppose that the configuration-space wave function satisfies the following schematic version of the Schrödinger equation (8):

$$i\hbar\,\partial_t\Psi = -\frac{\hbar^2}{2}\Delta\Psi + V\Psi. \tag{10}$$

Here V is a potential function defined on the system's configuration space, $\partial_t = \partial/\partial t$ is the partial derivative with respect to the time t, and Δ is a second-order differential operator acting on the system's configuration space and having the general form

$$\Delta = \sum_{i,j=1}^{n} \partial_i(\mu_{ij}\partial_j), \tag{11}$$

where $\mu_{ij} = \mu_{ji}$ is a symmetric array of real-valued functions of the coordinates (assumed for simplicity to have unit determinant), $\partial_i = \partial/\partial q_i$ and $\partial_j = \partial/\partial q_j$ are the partial derivatives with respect to the respective coordinates q_i and q_j, and Δ is understood to act on test functions $f(q,t)$ as $\sum_{i,j}\partial_i(\mu_{ij}\partial_j f)$. Writing the wave function in polar form in terms of a real-valued radial function $R(q,t)$ and a real-valued phase function $S(q,t)/\hbar$,

$$\Psi(q,t) = R(q,t)e^{iS(q,t)/\hbar}, \tag{12}$$

the Schrödinger equation (10) breaks up into a pair of coupled, real-valued equations involving R and S. One then imposes a guiding equation on the velocities $\dot{Q}_i(t)$ based on the values of the gradients $\partial_j S$ of the phase function evaluated on the system's actual trajectory $Q(t) = (Q_1(t), \ldots, Q_n(t))$,

$$\dot{Q}_i(t) = \sum_{j=1}^{n} \mu_{ij}\partial_j S\Big|_{Q(t)}, \tag{13}$$

where dots denote time derivatives. It follows that the modulus-square of the wave function, as given by

$$\rho = \Psi\overline{\Psi} = |\Psi|^2 = R^2, \tag{14}$$

satisfies the following continuity equation,

$$\partial_t\rho = -\sum_{i=1}^{n}\partial_i J_i, \tag{15}$$

where the probability current densities $J_i(q,t)$ are given by

$$J_i = \sum_{j=1}^{n} \hbar\mu_{ij} \text{Im}\, \bar{\Psi}\partial_j \Psi = \rho \sum_{j=1}^{n} \mu_{ij}\partial_j S, \tag{16}$$

as one can show with a short calculation.[5] One can then write the guiding equation (13) in the equivalent form

$$\dot{Q}_i(t) = \left.\frac{J_i}{\rho}\right|_{Q(t)} = \frac{J_i(Q(t),t)}{\rho(Q(t),t)}. \tag{17}$$

These equations guarantee a crucial property called 'equivariance,' which ensures that if the system's initial probability density is $\rho(q, t_0) = |\Psi(q, t_0)|^2$ at an initial time t_0, a condition called the 'quantum equilibrium hypothesis,' then the system's probability density at all later times t will continue to coincide with $\rho(q, t) = |\Psi(q, t)|^2$, in keeping with the Born rule.

Notice that the final version (17) of the guiding equation does not lead to divide-by-zero errors, because, by construction, the system has zero probability of ever having an actual configuration $Q(t)$ at any time t such that $\rho(Q(t), t) = 0$. If the system approaches regions of its configuration space for which the probability density $\rho(q, t)$ gets very small, then the system's velocities $\dot{Q}_i(t)$ become highly unstable, typically driving the system away from those regions of its configuration space.

3 Hidden Markov Models

3.1 The Shoemaker Universe

It will be useful now to explain what hidden Markov models are, before establishing their relationship with the de Broglie-Bohm pilot-wave theory. A good starting place will be a simple but profound thought experiment.

In a 1969 paper, Sydney Shoemaker presented an argument intended to establish the conceivability of durations of time without physical change (Shoemaker 1969). Following Shoemaker, one imagines a hypothetical universe that is entirely static and empty except for three civilizations A, B, and C that live in three separate solar systems far apart in space, but still close enough together

[5] Here is that calculation:

$$\begin{aligned}\partial_t\rho &= \partial_t\big(\Psi\bar{\Psi}\big) = (\partial_t\Psi)\bar{\Psi} + \Psi\big(\partial_t\bar{\Psi}\big)\\ &= \left(-\frac{\hbar}{2i}\Delta\Psi + \cancel{\frac{1}{i\hbar}V\Psi}\right)\bar{\Psi} + \Psi\left(\frac{\hbar}{2i}\Delta\bar{\Psi} - \cancel{\frac{1}{i\hbar}V\bar{\Psi}}\right) = -\sum_{i=1}^{n}\partial_i J_i,\end{aligned}$$

where

$$\begin{aligned}J_i &= \sum_{j=1}^{n}\frac{\hbar}{2i}\mu_{ij}\big(\bar{\Psi}\partial_j\Psi - \Psi\partial_j\bar{\Psi}\big) = \sum_{j=1}^{n}\hbar\mu_{ij}\text{Im}\,\bar{\Psi}\partial_j\Psi\\ &= \sum_{j=1}^{n}\hbar\mu_{ij}\text{Im}\, Re^{-iS/\hbar}\partial_j\Big(Re^{iS/\hbar}\Big) = \rho\sum_{j=1}^{n}\mu_{ij}\partial_j S.\end{aligned}$$

that sentient beings in each solar system can see the other civilizations through telescopes. Each civilization uses the same notions of days and years, with 365 days making up one year. Furthermore, the Shoemaker universe features some rather unusual behavior:

- Every three years, civilizations B and C see civilization A suddenly pause completely for one full year, after which civilization A unpauses and then proceeds normally, with the inhabitants of civilization A having no awareness of the paused year.

- Every four years, civilizations A and C similarly see civilization B pause for one full year before unpausing again.

- Every five years, civilizations A and B see civilization C pause for one full year before unpausing again.

The inhabitants of the Shoemaker universe would seem to be quite reasonable if they claimed that every $3 \times 4 \times 5 = 60$ years, their entire universe paused for a full year, even though no physical changes would have occurred during that paused year, and the inhabitants would not be able to obtain any direct empirical evidence that such a year-long pause had happened.

Accepting this view for the sake of argument, what could the dynamical laws of the Shoemaker universe be like? The laws would seem to be manifestly non-Markovian, because the state of the Shoemaker universe at the end of Day 1 of the 60th year looks just like the state of the Shoemaker universe at the end of Day 365 of the 60th year, and yet the state at the beginning of Day 2 is still paused, whereas the state at the beginning of Day 366 looks like a physical change has taken place. It is as though, according to the dynamical laws of the Shoemaker universe, the state of the universe at the beginning of Day 366 has somehow been determined by the state of the universe a year in the past. Indeed, Shoemaker said that this universe exhibited "action at a temporal distance."

If one is uncomfortable with non-Markovian laws of nature like these, then there is a simple solution that is always available. One can turn the Shoemaker universe into a hidden Markov model by positing the existence of a latent variable $Q(t)$ that grows steadily with time, perhaps according to the simple differential equation

$$\frac{dQ(t)}{dt} = 1. \tag{18}$$

Then the dynamical laws of the Shoemaker universe can check on the value of $Q(t)$ along with the state of the universe at the present-time t in order to determine the state of the universe at the infinitesimally next moment in time $t + dt$. If the latent variable $Q(t)$ has reached a value corresponding to the end of Day 365 of a year that is a multiple of 60, then the laws should prescribe that the next moment should exhibit ordinary physical change in the three civilizations.

The latent variable $Q(t)$ features six of the seven telltale characteristics listed earlier, in Subsection 1.2

1. It is a conceptually abstract variable.

2. It is non-unique and can be deformed by various mathematical transformations and redefinitions. Indeed, even its differential equation (18) could easily be altered.

3. It is unobservable in principle.
4. It has no location in physical space.
5. It is not backreacted upon by the physical material in the Shoemaker universe.
6. It is very simple, *in violation* of the generically complicated nature of latent variables.
7. It has contingent time evolution depending on correspondingly contingent initial conditions, in the sense that its 60-year 'internal clock' conceivably could have started with various initial values.

Suppose that one is committed to interpreting $Q(t)$ as more than just a formal latent variable. Then one could, in principle, either take $Q(t)$ to be an exotic part of the ontology of the Shoemaker universe, or one could choose to regard $Q(t)$ as part of the nomology or dynamical laws themselves. However, there does not seem to be any knock-down argument for favoring the ontological view over the nomological view, or vice versa.

More to the point, there does not appear to be a good argument that confronting this ontological-nomological fork is obligatory. Indeed, there is a perfectly reasonable alternative way to understood $Q(t)$: treat it as a latent variable in a hidden Markov model for a fundamentally non-Markovian universe. This alternative view only becomes available when one is aware of the concept of a hidden Markov model, which Shoemaker almost certainly could not have known about in 1969.

3.2 Continuous Stochastic Processes

As a more complicated but also much more relevant example, one can consider a continuous stochastic process of a very general form. To be precise, suppose that a given system with configurations labeled smoothly by $q = (q_1, \ldots, q_n)$ has a time-dependent probability density $\rho(q,t)$ that is a smooth function of the coordinates $q_1, \ldots, q_n$ and the time t, where the total number n of degrees of freedom is assumed to be finite. This stochastic process will generically be non-Markovian.

Introduce a 'field' $r(q,t)$ on the system's configuration space according to the differential equation

$$\frac{\partial\bigl(r^2(q,t)\bigr)}{\partial t} = \frac{\partial \rho(q,t)}{\partial t}, \tag{19}$$

and define a corresponding set of probability current densities $J_i(q,t)$ according to[6]

$$J_i(q_1, \ldots, q_i, \ldots, q_n, t) = -c_i \frac{\partial}{\partial t} \int_{a_i}^{q_i} dq_i' \, r^2\bigl(q_1, \ldots, q_i', \ldots, q_n, t\bigr), \tag{20}$$

where a_i are arbitrary constants and where c_i are constants satisfying the summation identity

$$\sum_{i=1}^{n} c_i = 1. \tag{21}$$

[6]The time derivative here needs to be a partial derivative because the integration yields a function of $n+1$ variables.

It follows that if the following guiding equation is imposed on the velocities $\dot{Q}_i(t)$ of the system's actual trajectory $Q(t) = (Q_1(t), \ldots, Q_n(t))$,

$$\dot{Q}_i(t) = \left.\frac{J_i}{r^2}\right|_{Q(t)} = \frac{J_i(Q(t), t)}{r^2(Q(t), t)}, \tag{22}$$

then the system will obey equivariance, in the sense that the identification

$$\rho(q, t) = r^2(q, t) \tag{23}$$

will consistently hold at future times if it holds at any initial time, in accordance with the continuity equation

$$\frac{\partial \rho}{\partial t} = -\sum_{i=1}^{n} \frac{\partial J_i}{\partial q_i}, \tag{24}$$

which is ensured by the formula (20) for the probability current densities.

The result is then a *deterministic* hidden Markov model for the continuous stochastic process that is also, furthermore, a pilot-wave theory. The values of the 'field' $r(q, t)$ at all points q in the system's configuration space play the role of both the latent or hidden variables for the hidden Markov model, and $r(q, t)$ as a whole is also a pilot wave.

Notice that $r(q, t)$, again regarded as an infinite distribution of variables at all points q in the system's configuration space, exhibits all seven of the smoking-gun characteristics of latent variables listed in Subsection 1.2:

1. It is an abstract function defined over the system's abstract configuration space.

2. It is non-unique in a variety of ways. For example, one can multiply it by an arbitrary constant without affecting the guiding equation (22) or the equivariance property (23). One can also modify the definition of the current densities (20) and therefore the guiding equation by changing the constants c_i or by adding terms with vanishing divergence. (See, for example, Deotto, Ghirardi 1998, which will come up again later.)

3. It is unobservable in principle, because the only given configurations are those of the original system.

4. It has no notion of location in physical space.

5. It is not backreacted upon by the system's configuration.

6. It encompasses an infinite number of its own degrees of freedom, one at each point q in the system's configuration space, even if the original system has only a finite number n of degrees of freedom.

7. It has its own initial conditions, which should satisfy $r^2(q, t_0) = \rho(q, t_0)$ at the initial time t_0.

Notice that the infinite collection of variables that make up $r(q,t)$ therefore constitute an even better example of latent variables than the latent variable $Q(t)$ for the Shoemaker universe from Subsection 3.1 because $Q(t)$, by contrast, did not properly exhibit the sixth listed characteristic (multivariateness).

Looking back at the de Broglie-Bohm pilot-wave theory from Subsection 2.2, one sees a remarkable and telling resemblance. Indeed, it is evident that the de Broglie-Bohm pilot-wave $\Psi(q,t)$ exhibits all seven of these hallmark characteristics of latent variables as well. The lack of backreaction, or "back action," in particular, is a widely acknowledged property of the de Broglie-Bohm pilot wave (Dürr, Goldstein, Zanghì 1996).[7]

One should therefore take seriously the idea that the de Broglie-Bohm pilot-wave theory is nothing more or less than a deterministic hidden Markov model of the same general kind as the one constructed here, with the pilot wave comprising just another class of latent variables. Indeed, the very facts that the construction laid out in this subsection was so simple, so general, so non-unique, and so inaccessible to empirical falsification, suggest that it is essentially just a parlor trick—an example of a collapse into triviality—and not a fundamental statement about physics.

This new deterministic hidden Markov model even gets the measurement process to work out correctly, and in a manner very similar to how the de Broglie-Bohm pilot-wave theory handles measurements (Bohm 1952b). Suppose that the system's overall configuration encompasses a measuring device containing a 'macroscopically large' number of its own degrees of freedom, and suppose that the measuring device has, say, two distinct possible outcomes, each of which corresponds to a macroscopically distinct rearrangement of its pre-measurement degrees of freedom. It follows that if the stochastic process is capable of accommodating measurements in the first place, and if the overall system goes through a measurement process, then the final probability density should take the form $\rho(q,t) = \rho_1(q,t) + \rho_2(q,t)$, where the two terms correspond to the two macroscopically distinct measurement outcomes and therefore have non-overlapping support on the overall system's configuration space. The system's actual configuration $Q(t)$ at the time t will belong to just one of these two regions of support, and so $\rho(Q(t),t)$ will reduce to just one of the two terms $\rho_i(Q(t),t)$, meaning that only that term $\rho_i(Q(t),t)$ will show up in the guiding equation (22). By continuity, only that term $\rho_i(Q(t),t)$ will make a difference to the system's actual trajectory $Q(t)$ for at least some period of future time. (Beyond that time, there can, in principle, be 'interference effects,' but the same is also possible for the de Broglie-Bohm theory in the case of re-coherence.)

As was the case for the latent variable $Q(t)$ for the Shoemaker universe, one is free to claim that the 'field' $r(q,t)$ is ontological, or even nomological (meaning a time-dependent part of the system's dynamical laws), provided that one is willing to grapple with making sense of ontological objects in a many-dimensional configuration space, or nomological ingredients with arbitrarily complicated time dependence and contingent initial conditions. However, given the arguably more fitting option of understanding $r(q,t)$ as a collection of latent variables that are part of a hidden Markov model, one is no longer obligated—nor perhaps even well-advised—to insist on interpreting $r(q,t)$ as ontological

[7] One can, of course, introduce backreactions on the de Broglie-Bohm pilot wave if one wishes, but the point is that putting in backreactions by hand is unnecessary.

or nomological.

Crucially, the same option is arguably available for the wave function or pilot wave $\Psi(q,t)$ for the de Broglie-Bohm theory. Why not interpret the wave function as a collection of latent variables in a hidden Markov model, given its close resemblance to $r(q,t)$ and the fact that it features all seven smoking-gun characteristics of a latent variable? If it walks like a duck, and quacks like a duck, and satisfies *five additional characteristics* of a duck, then what should one conclude?

4 Further Challenges to the Ontological View

4.1 Interference Effects

Beyond the discussion presented so far, there are still other challenges to the ontological view of the wave function or pilot wave. One place to start is with questions surrounding the famous interference effects of quantum theory.

Actually, far from being seen as *undermining* the thesis that the wave function is ontological, the appearance of interference effects in various experiments is often held up as evidence *in favor* of the ontological thesis. It is therefore worth taking a moment to explain how interference shows up in these sorts of experiments, and why they are not definitive evidence for an ontological interpretation of the wave function. A key example is the well-known double-slit experiment. (See, for example, Feynman, Leighton, Sands 1965, Volume 3, Chapter 1; or Heisenberg 1958, Chapter III. Note that the actual experiment was not performed with electrons until the work of Jönsson in 1961.)

In the double-slit experiment, particles are sent, *one at a time*, toward a wall with two small slits or holes in it. The slits are close together, but are still significantly farther apart than their individual widths. Far away, on the other side of the wall, is a detection screen that identifies the precise landing site of each particle that arrives there. The idea is to collect detailed statistics on the individual landing sites so that one can construct an overall histogram.

Crucially, on each successful run of the experiment, there is always only a single landing site on the detection screen. Obtaining any statistical patterns among the landing sites therefore requires running the experiment many times.

If the particles are classical objects, like small stones, then, after many runs of the experiment, the histogram consists of a blend of two normal or Gaussian distributions of dots, where each Gaussian distribution is lined up with one of the two slits.

By contrast, if the experiment is carried out with electrons, then, again after many runs of the experiment, the histogram appears to show peaks and valleys of dots, in a manner that looks just like the pattern of constructive and destructive interference of waves propagating through the two slits. To be clear, no wavelike interference pattern is seen on any one run of the experiment—it is only after many runs of the experiment that the histogram of landing sites shows such a pattern of dots.

The textbook explanation for this interference-like pattern of dots is that each electron is—or is guided by—a wave propagating in three-dimensional physical space, just as de Broglie originally imagined in his 1923 papers. However, the moment one considers doing the experiment with multi-electron systems on each run, even if electrical repulsive effects can be ignored, this intuitive picture breaks down, because then it becomes salient that the wave function or pilot wave propagates not in three-dimensional physical space, but in the system's many-dimensional configuration space. That is, one encounters precisely the sort of confusing picture that led all the founders of quantum theory to abandon the ontological view of the wave function, as outlined in Subsection 2.1 and described more extensively in other work (Barandes 2026b). Rather than evidence in favor of the ontological view of the wave function, interference, at least when involving multiple particles at a time, undermines that view.

Ultimately the wave function is not observable, and is not visualized in these experiments. What is visualized is the set of landing sites of the electrons, and whatever one's interpretation of the wave function, all empirically adequate formulations of quantum theory agree on those landing sites. On traditional versions of the de Broglie-Bohm pilot-wave theory, those landing sites are explained by an ontological wave function interfering with itself and physically guiding the particles preferentially to regions of the detection screen where the wave function exhibits constructive interference. On nomological versions of the pilot-wave theory, the particles are simply obeying complicated dynamical laws, where those laws have contingent initial conditions and intricate time dependence of their own.

Alternatively, on the view that the wave function is merely a latent variable of a hidden Markov model, as argued in this paper, the underlying physical story is non-Markovian, and the landing sites are the indirect result of those non-Markovian laws.

An analogy due to Hugh Everett III may be apt here. In a private letter to Bryce DeWitt, dated May 31, 1957, Everett wrote:

> A crucial point in deciding on a theory is that one does *not* accept or reject the theory on the basis of whether the basic world picture it presents is compatible with everyday experience. Rather, one accepts or rejects on the basis of whether or not the *experience which is predicted by the theory* is in accord with actual experience.
>
> Let me clarify this point. One of the basic criticisms leveled against the Copernican theory was that the "mobility of the earth as a real physical fact is incompatible with the common sense interpretation of nature." In other words, as any fool can plainly see[,] the earth doesn't *really* move[,] because we don't experience any motion. However, a theory which involves the motion of the earth is not difficult to swallow if it is a complete enough theory that one can also deduce that no motion will be felt by the earth's inhabitants (as was possible with Newtonian physics). Thus, in order to decide whether or not a theory contradicts our experience, it is necessary to see what the theory itself predicts our experience will be. [Everett 1957b, emphasis and parenthetical in the original]

Everett included similar comments in the shorter, published version of his doctoral thesis (Everett 1957a). Those comments appeared in a footnote added during the proofing stage. In that footnote, he criticized arguments against his own theory, writing that such arguments

> [...] are like the criticism of the Copernican theory that the mobility of the earth as a real physical fact is incompatible with the common sense interpretation of nature because we feel no such motion. In both cases the argument fails when it is shown that the theory itself predicts that our experience will be what it in fact is. (In the Copernican case the addition of Newtonian physics was required to be able to show that the earth's inhabitants would be unaware of any motion of the earth.) [Ibid., parenthetical in the original][8]

Everett's words here echo a famous statement made by Einstein, as quoted in Heisenberg's memoir *Physics and Beyond* (Heisenberg 1971): "It is the theory which decides what we can observe."

The planets in the night sky might provide an even better analogy.

Over many nights, some of the planets appear to follow trajectories that take them on retrograde-prograde 'loops.' From a modern perspective, one accounts for this strange apparent motion by appealing to Newton's complicated theory of mechanics, which features planets *qua* orbs, inertial masses, forces, accelerations, equations of motion that are slightly non-Markovian because they feature second-order time derivatives, and relative perspectives between the orbs.

However, the resulting trajectories in the sky end up being nearly periodic over sufficiently large stretches of time, so Fourier's theorem guarantees that each such planet's observed trajectory in the sky can be expressed in terms of a discrete Fourier series. One can organize the modes in that Fourier series into a collection of unobservable 'epicycles upon epicycles upon epicycles,' in much the same sense that Ptolemy originally imagined his epicycles, where each epicycle is a perfect circle of fixed radius rotating at a constant rate. These Fourier epicycles are just formal visualizations of the modes in a discrete Fourier series, of course, so they are unobservable in principle. Indeed, they satisfy most of the characteristics of latent variables from Subsection 1.2, except that they have spatial locations and are unique, again by Fourier's theorem.

One can also add, of course, that the Newtonian model is slightly non-Markovian, because its equations of motion are second-order in time derivatives. The Fourier-epicycle model is therefore, in a literal sense, a hidden Markov model for the Newtonian model, although admittedly not a unique way to embed Newtonian mechanics into a Markovian framework. Indeed, the Hamiltonian phase-space framework provides a conceptually distinct such approach (Barandes 2026a).

In simple cases, the Fourier epicycles give a much more intuitive and direct explanation of the retrograde-prograde loops that some of the planets make in the night sky. With Fourier's theorem in hand, it follows that all of observational planetary astronomy could, in principle, be reduced to the marvelously parsimonious axiom that planetary motion in the sky is nearly periodic, so that the only task of scientific astronomy would be to calculate the Fourier amplitudes to any desired precision based on empirical observation. Planets on perfect circles that rotate at constant rates—what could be conceptually simpler than that? Why prefer Newton's much more complicated theory that

[8] This Copernican analogy has become a repeated anecdote in the literature on the Everett interpretation of quantum theory, where it is usually retold in terms of an encounter between Elizabeth Anscombe and Ludwig Wittgenstein that Anscombe reported in an introduction to the *Tractatus* (Anscombe 1959). See, for instance, Coleman (1994), Wallace (2012), and Carroll (2019).

removes the unobservable Fourier epicycles? Why not just reify the Fourier epicycles and treat them as ontological? Why treat the retrograde-prograde loops in the sky as red herrings, rather than as clear signals that Fourier epicycles should be taken seriously as part of the ontology of nature?

One good reason to be suspicious of regarding the Fourier epicycles as ontological is that they have too much in common with latent variables in a hidden Markov model, which should make them immediately suspect as aspects of the ontology of nature. Moreover, beyond the simplest cases, the Fourier epicycles become so ornate and complicated that they do not give an intuitive or easily visualizable explanation of the retrograde-prograde loops anymore. Finally, the motion of the planets is not exactly periodic, and the solar system features many other objects, like the moons of the various planets, and comets on non-periodic hyperbolic orbits, that cannot be accommodated into a discrete system of Fourier epicycles. This discussion, of course, puts aside additional complications from relativity.

One should note the resemblances of these problems to some of the outstanding challenges of the de Broglie-Bohm theory, which features a pilot wave satisfying all the characteristics of a latent variable in a hidden Markov model, and has run into well-known difficulties in being generalized beyond the case of systems consisting of fixed numbers of finitely many non-relativistic particles. One might then be within one's rights to arrive at the following conclusion: wave functions are the epicycles of the modern age.

4.2 Foldy-Wouthuysen Gauge Transformations

One of the seven characteristics of a latent variable in a hidden Markov model, as listed in Subsection 1.2, is non-uniqueness. As this next part of the present work will show, the degree of non-uniqueness of the quantum state—whether treated as a configuration-space wave function or as a state vector in a Hilbert space—turns out to be even more substantial than for the 'field' $r(q,t)$ of the generic hidden Markov model constructed in Subsection 3.2. That is, a quantum state is even more like a set of latent variables than $r(q,t)$, despite the fact that $r(q,t)$ was intended by construction to comprise a set of latent variables.

The starting place is a remarkable feature of general quantum systems that is not widely known. Given a quantum system described in terms of a Hilbert-space formulation with state vector $|\Psi(t)\rangle$, observables $A(t)$, and a Hamiltonian $H(t)$, there exists a highly general family of gauge transformations that leave all empirical quantities invariant. These gauge transformations were first identified in 1950 by Foldy and Wouthuysen in the specific context of relativistic spin-half particles (Foldy, Wouthuysen 1950). Brown later wrote them in a more general form in a paper concerned with questions of objectivity for quantum systems (Brown 1999).

To begin, consider an arbitrary quantum system in its Hilbert-space formulation. Letting $V(t)$ be an arbitrary time-dependent unitary operator (not to be confused with the notation for a potential function), a Foldy-Wouthuysen gauge transformation is defined by

$$\left.\begin{array}{lll} |\Psi(t)\rangle & \mapsto & |\Psi'(t)\rangle = V(t)|\Psi(t)\rangle, \\ A(t) & \mapsto & A'(t) = V(t)A(t)V^{\dagger}(t), \\ H(t) & \mapsto & H'(t) = V(t)H(t)V^{\dagger}(t) - i\hbar V(t)\partial_t V^{\dagger}(t). \end{array}\right\} \tag{25}$$

The time-dependence of the unitary operator $V(t)$ here is significant. The gauge transformations defined above are not the ordinary kinds of *constant* unitary transformations that are familiar from most textbooks on quantum theory.

Foldy-Wouthuysen gauge transformations preserve all expectation values $\langle A(t)\rangle = \langle\Psi(t)|A(t)|\Psi(t)\rangle$, so they leave all the empirical predictions of quantum theory exactly unchanged. Note that Foldy-Wouthuysen gauge transformations do not produce a *new* quantum system or theory, but merely yield a new 'gauge' for the *same* quantum system or theory.

Notice also that the Hamiltonian $H(t)$ does not transform like an observable.[9] Instead, the transformation law for the Hamiltonian $H(t)$ has precisely the same form as the transformation law for a non-Abelian gauge connection. (For a pedagogical treatment of non-Abelian gauge theories, see, for example, Peskin, Schroeder 1999; Weinberg 1996). Indeed, if one regards a time-evolving quantum system as a bundle of identical Hilbert spaces fibered over a one-dimensional manifold representing the time t, then a Foldy-Wouthuysen gauge transformation can be understood as an independent unitary rotation by $V(t)$ of the Hilbert-space fiber at each time t, with the Hamiltonian $H(t)$ serving as the gauge connection. Indeed, one can re-express the Hilbert-space Schrödinger equation

$$i\hbar\frac{\partial|\Psi(t)\rangle}{\partial t} = H(t)|\Psi(t)\rangle \tag{26}$$

in the manifestly gauge-covariant form

$$\mathcal{D}_t|\Psi(t)\rangle = 0, \tag{27}$$

where the Foldy-Wouthuysen gauge-covariant derivative $\mathcal{D}_t$ is defined to be

$$\mathcal{D}_t \equiv \frac{\partial}{\partial t} + \frac{i}{\hbar}H(t), \tag{28}$$

in much the same way as for a non-Abelian gauge theory.

The existence of Foldy-Wouthuysen gauge transformations should raise doubts about the viability of any 'ontological-monistic' interpretation of quantum theory (in the terminology of Subsection 1.1) that asserts that the ontology of nature is exhausted entirely by abstract state vectors in Hilbert spaces, which are manifestly not gauge invariant. These doubts ought to be of particular

[9] It may come as a surprise to learn that the Hamiltonian is not an observable. Is it not the case that for a single non-relativistic particle of mass m, position q, momentum p, and potential $V(q)$, the Hamiltonian H is given by $p^2/2m + V(q)$? The reply is that although the arithmetic combination of observables $p^2/2m + V(q)$ is indeed gauge-invariant, the Hamiltonian H will not remain equal to that specific combination of observables under Foldy-Wouthuysen gauge transformations.

concern for some versions of the Everett interpretation, including Everett's own version of his interpretation, which Everett explicitly took to consist only of the wave function of the universe as its fundamental ontology (Everett 1956, 1957a; Barandes 2026b).

The de Broglie-Bohm pilot-wave theory does not fall into the category of such interpretations, as the theory explicitly posits additional ontology beyond the wave function, in the form of particles. However, the existence of Foldy-Wouthuysen gauge transformations also has potential implications for the de Broglie-Bohm theory, and these implications will turn out to put serious pressure not only on the ontological view of the pilot wave, but on the view that the de Broglie-Bohm theory is well-defined in the first place.

Working in configuration space, consider a quantum system consisting of N non-relativistic particles with configurations q labeled by $n = 3N$ degrees of freedom $q_1, \dots, q_n$ (usually assumed to be position coordinates) and Hamiltonian

$$H(q,t) = -\frac{\hbar^2}{2}\Delta + V(q,t). \tag{29}$$

Here the potential function $V(q,t)$ should not be confused with the Foldy-Wouthuysen unitary operator $V(t)$, and Δ is the second-order differential operator on the system's configuration space originally defined in (11):

$$\Delta = \sum_{i,j=1}^{n} \partial_i(\mu_{ij}\partial_j).$$

Now consider the subclass of Foldy-Wouthuysen gauge transformations (25) that commute with all the coordinate operators of the system. In that case, the time-dependent unitary operator $V(t)$, when expressed in the q-representation, reduces to a simple phase factor,

$$e^{i\lambda(q,t)/\hbar}, \tag{30}$$

where $\lambda(q,t)$ is a smooth but otherwise arbitrary function of the configuration $q = (q_1, \dots, q_n)$ and the time t. The wave function $\Psi(q,t)$ and the Hamiltonian $H(q,t)$ then transform respectively as

$$\left.\begin{array}{lcl} \Psi & \mapsto & \Psi' = e^{i\lambda/\hbar}\Psi, \\ H & \mapsto & H' = e^{i\lambda/\hbar}He^{-i\lambda/\hbar} - \partial_t\lambda, \end{array}\right\} \tag{31}$$

where the first term in H' should be understood as acting on arbitrary test functions $f(q,t)$ as $e^{i\lambda/\hbar}H\big(e^{-i\lambda/\hbar}f\big)$.

Under this subclass of Foldy-Wouthuysen gauge transformations, the Born-rule probability density $\rho(q,t) = |\Psi(q,t)|^2$ in (14) is manifestly gauge invariant:

$$\rho = |\Psi|^2 \mapsto \rho' = \rho. \tag{32}$$

Meanwhile, the radial function $R(q,t)$ and the phase function $S(q,t)$ appearing in the polar decomposition (12) of the wave function transform respectively as

$$R \mapsto R' = R, \tag{33}$$

$$S \mapsto S' = S + \lambda. \tag{34}$$

Because the phase function $S(q,t)$ is not gauge invariant, the probability current densities $J_i(q,t)$ defined in (16) are also not gauge invariant. Instead, they transform nontrivially under Foldy-Wouthuysen gauge transformations according to

$$J_i = \rho \sum_{j=1}^{n} \mu_{ij} \partial_j S \mapsto J_i' = J_i + \rho \sum_{j=1}^{n} \mu_{ij} \partial_j \lambda. \tag{35}$$

Unsurprisingly, with the additional structure that turns textbook quantum mechanics into the de Broglie-Bohm pilot-wave theory, not all of the Foldy-Wouthuysen gauge invariance of textbook quantum mechanics is preserved.

Importantly, however, if one restricts further to the smaller subclass of Foldy-Wouthuysen gauge transformations for which the inhomogeneous additive term appearing in the transformation formula (35) for J_i has vanishing generalized divergence, in the precise sense that

$$\sum_{i=1}^{n} \partial_i \left(\rho \sum_{j=1}^{n} \mu_{ij} \partial_j \lambda \right) = 0, \tag{36}$$

then the continuity equation (15) still holds, as before:

$$\partial_t \rho = - \sum_{i=1}^{n} \partial_i J_i. \tag{37}$$

That is, although some of the Foldy-Wouthuysen gauge invariance is broken by moving from textbook quantum mechanics to the de Broglie-Bohm theory, there can still be some lingering Foldy-Wouthuysen gauge invariance that leaves the de Broglie-Bohm theory's self-consistency intact.

To be clear, these Foldy-Wouthuysen gauge transformations do not change the de Broglie-Bohm pilot-wave theory into some other theory. They are a family of gauge transformations inherent to the de Broglie-Bohm theory itself, akin to the electromagnetic gauge transformations of the Maxwell theory. These Foldy-Wouthuysen gauge transformations also do not replace the de Broglie-Bohm particles with some other qualitatively distinct form of ontology.

Nevertheless, the fact that the current densities (35) are not gauge invariant means that the velocities $\dot{Q}_i(t)$ of the de Broglie-Bohm particles, as given by the guiding equation in either the forms (13) or (17), likewise fail to be gauge invariant, but transform nontrivially according to

$$\dot{Q}(t) \mapsto \dot{Q}'(t) = \dot{Q}(t) + \sum_{j=1}^{n} \mu_{ij} \partial_j \lambda. \tag{38}$$

It follows that the resulting trajectories of the de Broglie-Bohm particles fail to be gauge invariant as well. Nor is there any preferred or canonical gauge choice of velocities and trajectories from among all the gauge choices related by (38). These distinct gauge choices are all indistinguishable at the level of the empirical output of the de Broglie-Bohm theory.

The ambiguity (38) in the velocities and trajectories of the de Broglie-Bohm particles first appeared in a paper by Deotto and Ghirardi (1998), who showed that the guiding equation—and thus the trajectories of the de Broglie-Bohm particles—were empirically underdetermined in a very explicit sense: one could add an arbitrary divergence-free term to the current densities without changing the empirical predictions of the de Broglie-Bohm theory. A new result of the present work is thus to connect this Deotto-Ghirardi ambiguity with Foldy-Wouthuysen gauge transformations. More precisely, the present work shows that the Deotto-Ghirardi ambiguity does not refer to an *ad hoc* modification of the de Broglie-Bohm theory, but is an expression of the fact that the velocities and trajectories of the de Broglie-Bohm theory, along with the pilot wave itself and the probability current densities, all fail to be gauge-invariant notions under Foldy-Wouthuysen gauge transformations.

In other physical theories, structures that fail to be invariant under a relevant notion of gauge transformations, such as the gauge potentials of Maxwellian electromagnetism or the metric tensor of general relativity, tend not to be assigned a physical or ontological meaning. Physical or ontological meaning is instead usually reserved for gauge-invariant structures. The de Broglie-Bohm theory has a continuously infinite collection of suitably restricted Foldy-Wouthuysen gauge transformations that leave the theory internally consistent (in the sense of preserving the continuity equation) while not leaving invariant the pilot wave, the probability current densities, the particle velocities, or the particle trajectories. This fact is potentially a significant problem for trying to take the pilot wave and the particle trajectories of the de Broglie-Bohm theory seriously as part of the ontology of nature.

4.3 Strocchi-Heslot Phase Spaces

To get a better handle on the meaning of the Foldy-Wouthuysen gauge transformations described in Subsection 4.2, it may be illuminating to examine the mathematical structures underlying the pilot wave or wave function in more detail. In particular, it will be worthwhile to spend a moment thinking about phase spaces. The key point is that by involving both particles and a wave function $\Psi(q,t)$, the de Broglie-Bohm theory features two distinct notions of phase space.

On the one hand, if the de Broglie-Bohm particles are distinguishable and are finite in number, then they have a finite-dimensional phase space in essentially the familiar classical sense. Specifically, the ith particle has canonical coordinates $(q_{ix}, q_{iy}, q_{iz}) = (x_i, y_i, z_i)$ given by the particle's position, and canonical momenta $(p_{ix}, p_{iy}, p_{iz}) = (m_i\dot{x}_i, m_i\dot{y}_i, m_i\dot{z}_i)$ given by the components of the particle's physical momentum, where m_i is the mass of the ith particle and where dots denote time derivatives. (These particular formulas for the canonical momenta can be modified if there are electromagnetic fields present.)

On the other hand, the wave function $\Psi(q,t)$ itself, as a complex-valued function defined in the space of configurations q of the particles and depending on the time t, consists of a pair of real-valued functions $X(q,t)$ and $Y(q,t)$ that form their own, infinite-dimensional notion of phase space, and that satisfy a linear system of first-order, classical-looking Hamilton's equations of motion that descend from the linear Schrödinger equation. This additional notion of phase space was first identified by Strocchi (1966) and then independently by Heslot (1985), and so will be called the Strocchi-Heslot phase space in the present work.[10]

One can then understand general Foldy-Wouthuysen gauge transformations (25) as linear, time-dependent canonical transformations on the Strocchi-Heslot phase space of the wave function, rather than as canonical transformations of any kind on the phase space of the de Broglie-Bohm particles.

It is important to keep in mind that these are canonical transformations on the Strocchi-Heslot phase space of the wave function. In particular, they should not be confused with canonical transformations on the phase space of the de Broglie-Bohm particles, which were examined by Stone (1994). In particular, Stone's paper explored how the de Broglie-Bohm pilot wave could be referred to different choices of canonical coordinates for the particles—even by replacing, say, the positions of the particles with their momenta. The present work, by contrast, leaves the canonical coordinates for the particles equal to their positions, so that the pilot wave always refers to those positions.

A classical phase space is a manifold coordinatized by pairs of variables called canonical coordinates q_i and canonical momenta p_i, together with a Poisson bracket $\{\ ,\ \}$ defined for arbitrary functions $f(q,p,t)$ and $g(q,p,t)$ on phase space by

$$\{f,g\} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} \right). \tag{39}$$

Once one has specified a Hamiltonian $H(q,p,t)$, there is then an associated flow expressible as a system of differential equations, called Hamilton's equations of motion, that are first-order in time derivatives:

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}, \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}, \end{aligned} \tag{40}$$

A change of phase-space variables

$$\begin{aligned} q'_i &= q'_i(q,p,t), \\ p'_i &= p'_i(q,p,t), \end{aligned} \tag{41}$$

is said to be a canonical transformation if all the various Poisson brackets, when expressed in terms of the new phase-space variables, maintain their structure, and if the new phase-space variables satisfy Hamilton's equations for a potentially modified Hamiltonian $H'(q',p',t)$.

Each canonical transformation therefore defines an alternative choice of phase-space variables,

[10] Technically speaking, Strocchi and Heslot identified this additional notion of phase space only for finite-dimensional quantum systems. Ashtekar and Schilling later generalized Strocchi-Heslot phase spaces to quantum systems with infinite-dimensional Hilbert spaces (Ashtekar, Schilling 1999). For more on Strocchi-Heslot phase spaces, see Barandes (2026a).

and each such choice is called a canonical frame. An obvious question, then, is whether some canonical frames are more physical, or more transparently revealing of the system's ontology, than other canonical frames.

For example, if one is given only that a specific system has a Hamiltonian formulation with one canonical coordinate q, one canonical momentum p, and Hamiltonian $H = (1/2m)p^2 + (1/2)kq^2$ for positive constants $m, k > 0$, then the system could be a simple harmonic oscillator with position q, momentum p, mass m, and spring constant k, or the system could instead be a simple harmonic oscillator with position p, momentum $-q$, mass $1/k$, and spring constant $1/m$, as follows from a straightforward calculation of the corresponding canonical transformation. (The minus sign turns out to be needed for self-consistency.)

It is easy to come up with much more exotic examples in which different choices of canonical frame can lead to much more radical differences in a system's apparent ontology. A pure Hamiltonian formulation of a classical system turns out to contain very little definite ontological structure on its own (Curiel 2014).

In practice, how does one resolve this ambiguity over the correct canonical frame? The standard approach is to identify the system's observable features, along with their physical meanings. Recalling the simple harmonic oscillator described earlier, if just observing the system reveals that q identifies its location in physical space, and p identifies its momentum in the form $m\dot{q}$, then those observations single out a canonical frame, at least up to less ontological questions like one's choice of coordinate system for physical space or one's choice of measurement units.

As explained above, the wave function has its own notion of phase space, as first identified by Strocchi and Heslot. However, a wave function, unlike a particle, or even the local value of an electric field, is not observable. As a consequence, there are no observables definable from within a Strocchi-Heslot phase space to single out an ontological canonical frame for the wave function itself. The subclass of Foldy-Wouthuysen gauge transformations defined in (31) and satisfying the subsidiary condition (36) represent just one kind of canonical transformation on the Strocchi-Heslot phase space. There are infinitely many other canonical transformations, leading to infinitely many other canonical frames, each of which describes a different-looking ontology for the wave function, without any principled way to single out any one of them. It is not enough to try to identify the pilot wave with 'all these canonical frames' in some broad collective or equivalence-class sense, because that strategy would not even succeed for the elementary case of the simple harmonic oscillator described above.

5 Conclusion

This paper has argued that the de Broglie-Bohm pilot-wave theory is best understood as a hidden Markov model, and that the configuration-space wave function, which serves as the pilot wave for the theory, is optimally interpreted as consisting of a set of latent variables for that hidden Markov model. On this view, the latent-variable reading is a better conceptual fit for the wave function than ontological, nomological, or epistemological readings.

This paper has argued, moreover, that the de Broglie-Bohm theory suffers from a potentially catastrophic ill-definiteness under a class of gauge transformations that do not preserve the pilot wave or the trajectories of the de Broglie-Bohm particles. For reasons why the invocation of 'weak measurements' and 'weak values' do not provide true empirical evidence of these trajectories, see Barandes (2026c,d). As was also mentioned in the present work, similar problems of gauge-invariance may present a problem for some versions of the Everett interpretation as well.

Hidden Markov models are ultimately just formal ways of representing processes whose dynamics are non-Markovian. One might then naturally wonder whether quantum theory more broadly should be interpreted as describing non-Markovian physical systems, beyond the simple cases of non-relativistic systems of particles or the de Broglie-Bohm theory. In other words, perhaps when one takes physically fundamental non-Markovian processes and tries to shoehorn them into a Markovian paradigm, the result is quantum theory, with all its seemingly exotic features. For arguments along these lines, see Barandes (2025, 2026a).

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