Conceptual aspects of non-positive quantization

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Abstract

We draw a distinction between 'positive' and 'non-positive' quantization schemes. Although the former are somewhat known to philosophers (in both canonical and path integral versions), the latter are almost entirely unstudied in the philosophy literature; moreover, physics writings on non-positive quantization often proceed in a manner which is not maximally systematic. As such, this paper aims both to introduce to philosophers the powerful tools of non-positive quantization schemes and their applications and to provide a systematic lay-of-the-land for such approaches which should be of value to philosophers and physicists alike.

Contents

1 Introduction

For several decades, the topic of inter-theoretic relations has occupied and animated philosophers of science and of physics (Batterman 2002; Butterfield 2011; Nagel 1961; Oppenheim and Putnam 1958; Schaffner 1967). Often, these writings focus on recovering previous theories of physics in suitable limits of new

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theories of physics—consider e.g. work on the classical limit of quantum theory (Feintzeig 2023; Rosaler 2013), or on the non-relativistic limit of general relativity (Fletcher 2019; Malament 1986; March et al. 2024; Wolf et al. 2024). Such work is properly situated in what Reichenbach (1938) dubbed the 'context of justification', since the goal is indeed to *justify* new theories of physics via their ability to recover previous theories of physics (and, as such, the empirical successes of those previous theories of physics) in the appropriate regimes.[1] However, one might also be interested in inter-theoretic relations in what Reichenbach (1938) dubbed the 'context of discovery'—that is, moving beyond known theories of physics in the quest to *discover* new theories of physics, with the empirical successes of the latter ideally subsuming and transcending those of the former.[2]

As Linnemann (2022) notes, however, in contrast with the context of justification, discussion of inter-theoretic relations in the context of discovery remains lamentably scant. In this article, we hope to go some way towards redressing this issue, focussing in particular on the classical-to-quantum transition via the procedure of *quantization*.[3] Although there is already some good philosophical literature on this topic (e.g. Carosso (2022), Feintzeig (2023), Landsman (2017), Linnemann (2022), and Thébault (2016)), there still remains much work to be done. In particular, existing philosophical discussions of quantization mostly (and self-consciously) limit themselves to the simplest quantization schemes (e.g. canonical quantization—see Feintzeig (2023, p. 26, fn. 55) and Linnemann (2022, §2.1)) and do not discuss the kinds of approaches to quantization which most often occupy modern theoretical physicists. (One notable exception is Redhead (2002, §8), upon which our own work can be taken as building.) As such, there is a gap in the existing literature when it comes to exposing, evaluating, and propounding the philosophical upshots of these more modern quantization schemes. It is exactly that lacuna which we aim to fill with the present article.

As we just said, philosophical discussions of quantization focus almost exclusively on canonical Hilbert space quantization; even somewhat more sophisticated methods such as path integral quantization are rarely discussed (see again Feintzeig (2023, p. 26, fn. 55)). However, in quantum field theory and string theory, one encounters more general quantization prescriptions. These prescriptions are associated with authors such as Batalin and Vilkovisky (1981, 1983), Becchi et al. (1976), DeWitt (1967), Faddeev and Popov (1967), Fradkin and Vilkovisky (1975), and Tyutin (1975). They are of interest not only because they cover a wider range of theories, but also because they can offer practical and conceptual advantages in the treatment of theories that can be quantized

[1] Famously, for example, Einstein used considerations of the non-relativistic limit to justify general relativity: see e.g. Norton (1984).

[2] For more on the distinction between the context of justification and context of discovery, see Hoyningen-Huene (1987).

[3] This is not to assume that quantization is necessarily of interest only in the context of discovery. For example, semiclassical and perturbative analyses may be descriptively and explanatorily important in the context of justification (Batterman 2002; Bokulich 2008). However, philosophers typically discuss quantization as a matter of discovery, and in this paper we follow suit.

by simpler means (Weinberg 1995, §15.9). This paper is a philosopher's guide to these more general quantization prescriptions.

The paper therefore aims to survey a variety of quantization prescriptions and the reasons they are of philosophical interest. We accomplish the first part of this aim by proposing a certain organization of quantization prescriptions and understanding of the relationships among them. We accomplish the second by explaining the specific practical advantages of each and indicating salient interpretive questions.

With respect to the first part, we distinguish quantization prescriptions into two classes: what we call 'positive' and 'non-positive'. The former are marked by positivity properties like the positive definiteness of the Hilbert space inner product and reflection positivity of correlation functions. These are the prescriptions that have been received the most attention from philosophers up to this point. Non-positive quantization prescriptions—found in any quantum field theory textbook—lack these positivity properties. Positive and non-positive quantization prescriptions lead to qualitatively different quantum theories; for example, the former exhibit a spin–statistics connection that is absent in the latter.

The terminology and individuation of non-positive quantization prescriptions isn't completely uniform. For example, Weinberg (1995) distinguishes (i) De Witt–Faddeev–Popov, (ii) Becchi–Rouet–Stora–Tyutin (BRST), (iii) Batalin–Fradkin–Vilkovisky, and (iv) Batalin–Vilkovisky (BV), while Henneaux and Teitelboim (1992) call all of these 'BRST'. The question of which classification is the more apt depends in part on interpretive issues. Our approach will be to first present them as a unified class, *à la* Henneaux and Teitelboim (1992), and then explain how interpretive commitments can lead to a finer classification.

The physical interest of non-positive quantization is of two kinds. First, there are theories that admit no, or no well-behaved, positive quantization but which have appropriate analogues that admit a non-positive quantization. Examples are non-abelian Yang–Mills theory ('t Hooft 1971; 't Hooft and Veltman 1972; Becchi et al. 1976; Faddeev and Popov 1967; Henneaux and Teitelboim 1992) and (super)gravity (Fradkin and Vilkovisky 1973; Freedman and Van Proeyen 2012; Kallosh 1978). Second, even when a theory admits a positive quantization, the analysis of the quantum theory—in particular, its unitarity and renormalizability—can be eased by applying a more general quantization prescription (Zinn-Justin 1975; Weinberg 1995, Ch. 17).

Philosophically, the main question of interpretive interest concerns where to draw the line between the classical input to the quantization prescription and the by-products of the quantization procedure. Non-positive quantization methods were developed originally as more-or-less *ad hoc* modifications of canonical and path-integral quantization in the face of divergences, non-unitarity, and problems with renormalization. This suggests drawing the line so as to make the classical input minimal and to make the variety of quantization prescriptions large. From this perspective, each classical theory merits special treatment, with the quantization prescription varying from theory to theory. At the other extreme, one could draw the line such that there is 'one true (non-positive) quantization prescription', which takes as input a richer set of classical data and reduces to the other prescriptions in various special cases. Weinberg suggests this perspective when he says that "[w]e must give up the Faddeev–Popov–De Witt approach, and instead take the action from the beginning as the most general renormalizable function of the gauge, matter, ghost, and auxiliary fields that is invariant under BRST and the other symmetries of the theory" (Weinberg 1995, p. 91). One major difference between the two perspectives is that the latter recognizes many more distinctions among classical theories, as there are more ways for the input to differ.

The need for a specifically philosophical guide to non-positive quantization stems from the following gap. Calculation-oriented textbook treatments such Peskin and Schroeder (1995, §16.4) and Weinberg (1995, §15.9) introduce non-positive methods in passing and usually downplay them. They are presented as (often *ad hoc*) solutions to specific problems one encounters in doing calculations. The problem is that it is not totally clear where we end up, or what understanding of the conceptual lay of the land has been secured. More detailed treatments such as Henneaux and Teitelboim (1992) are thorough but have many details, so it's easy to lose the forest for the trees. The present paper is meant to serve as a guide to the forest.

As such, here is our plan for this article. In §2, we review and evaluate positive quantization schemes; as indicated already above, it is this section which is likely to be most familiar to philosophers. In §3, we introduce the relevant formalism of fermionic coordinates and Krein spaces, on which non-positive quantization is based. Then, in §4 and §5 (respectively), we discuss non-positive Hamiltonian and Lagrangian quantization, isolating along the way exactly where (conceptually distinct) devices such as ghosts and antifields arise. In §6, we discuss the interpretation of non-positive quantization schemes, distinguishing 'pluralist' and 'monist' approaches. We conclude in §7.

2 Positive quantization

As a reminder and to fix notation, we begin by recalling positive quantization in the simplest cases. Authors such as Linnemann (2022, §2.1) and Ruetsche (2011, ch. 2) discuss canonical quantization in detail, but even path integral quantization is rarely discussed in detail by philosophers.

At the most general level, quantization takes as input some data of the sort used to characterize a classical mechanical system, and outputs a description of a quantum system that is in some sense a quantum analogue of the input. The input to the canonical quantization prescription (discussed in §2.1) is a Hamiltonian description of a classical system, whereas path integral quantization (discussed in §2.2) takes as input a Lagrangian description. In the most familiar cases, the output of each prescription is a Hilbert space equipped with a collection of distinguished operators. We review these prescriptions and their relations to each other in the case of a harmonic oscillator (§2.3), and finally consider their extensions to field quantization (in §2.4).

2.1 Canonical quantization

The purpose of this subsection is to recall the canonical quantization prescription

$$\text{(Poisson bracket)} \mapsto \frac{1}{i\hbar}\text{(commutator)}, \tag{1}$$

with real functions mapping to self-adjoint operators, in a way that renders most straightforward our subsequent generalisations to field theory and to non-positive quantization.

Consider a particle moving in $\mathbb{R}$. The phase space of this system has global coordinates (q, p), and is equipped with a natural bracket

$$\{F, G\} = \frac{\partial F}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial G}{\partial q} \tag{2}$$

for all functions F and G on phase space. In particular, we have

$$\{q, q\} = \{p, p\} = 0, \qquad \{q, p\} = 1. \tag{3}$$

The dynamics are given by a real-valued function $H(q, p)$ of these coordinates in the sense that the equations of motion are

$$\dot{F} = \{F, H\} \tag{4}$$

for any function F on phase space.

A quantization of this system is a Hilbert space $\mathcal{H}$ and self-adjoint operators $q(t)$, $p(t)$, and H on $\mathcal{H}$ such that

$$\frac{1}{i\hbar}[q(t), q(t)] = \frac{1}{i\hbar}[p(t), p(t)] = 0, \qquad \frac{1}{i\hbar}[q(t), p(t)] = 1, \tag{5}$$

for all t and H has the functional dependence on q and p exhibited by their classical analogues. The result is a description of a quantum system whose states are given by density operators on $\mathcal{H}$ and whose dynamics are given by

$$\dot{F} = \frac{1}{i\hbar}[F, H] \tag{6}$$

for any operator F on $\mathcal{H}$.

For example, consider the Hamiltonian

$$H(q, p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2. \tag{7}$$

One quantization of this system is the Fock quantization; this is particularly useful for free fields or the interaction picture. Here the Hilbert space is $\ell^2(\mathbb{N})$, the space of square-summable sequences. Writing $|n\rangle$ for the unit-norm sequence supported only in the nth entry, there are natural operators

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \qquad a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,. \tag{8}$$

It follows that if we define

$$q_0 = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a), \qquad p_0 = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a),$$
$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right), \tag{9}$$

then defining

$$q(t) = e^{-Ht/i\hbar} q_0 e^{Ht/i\hbar}, \qquad p(t) = e^{-Ht/i\hbar} p_0 e^{Ht/i\hbar}, \tag{10}$$

gives a representation of the canonical commutation relations.

2.2 Path integral quantization

The goal of this subsection is to recall the path integral quantization prescription in a generalizable way.

Consider again a particle moving in $\mathbb{R}$. A possible history for this particle is given by a function $q(t)$ and its time derivatives; its dynamics are described by a real-valued function $L(q, \dot{q})$ in the sense that the equations of motion are the Euler–Lagrange expressions

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0. \tag{11}$$

The path integral quantization of this system is most simply formulated in the Hilbert space $L^2(\mathbb{R})$ of square-integrable functions on the real line. Elements of this space are states of a system at a time, and the dynamics are given by

$$\psi(x', t') = \int dx\, K(x', t'; x, t)\, \psi(x, t), \tag{12}$$

$$K(x', t'; x, t) = \int_{q(t)=x}^{q(t')=x'} \mathcal{D}q\, e^{\frac{i}{\hbar}S(q)}, \tag{13}$$

$$S(q) = \int_t^{t'} ds\, L(q, \dot{q}), \tag{14}$$

where the integral defining K is taken over all curves $q : [t, t'] \to \mathbb{R}$ such that $q(t) = x$ and $q(t') = x'$. The position and momentum operators at $t = 0$ are given by

$$(q_0\psi)(x) = x\,\psi(x), \qquad (p_0\psi)(x) = -i\hbar\,\psi'(x), \tag{15}$$

with operators at other times determined as before.

The path integral formulation is not particularly convenient for computing dynamics explicitly. For example, in the case of a simple harmonic oscillator

$$L_{\mathrm{SHO}}(q, \dot{q}) = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2 \tag{16}$$

the kernel $K(x', t'; x, t)$ is

$$\sqrt{\frac{m\omega}{2\pi i\hbar \sin\omega t}} \exp\left[-\frac{m\omega}{2i\hbar\sin\omega t}\left((x^2+x'^2)\cos\omega t - 2xx'\right)\right]. \tag{17}$$

However, this formulation does give a convenient expression for time-ordered expectation values. Writing $q(t)\,|x,t\rangle = x\,|x,t\rangle$ for the eigenstates of the position operator, we have

$$\langle x', t'|T\left\{q(t_1)\cdots q(t_n)\right\}|x,t\rangle = \int_{q(t)=x}^{q(t')=x'} \mathcal{D}q\, e^{\frac{i}{\hbar}S(q)} q(t_1)\cdots q(t_n), \tag{18}$$

where $T\{-\}$ re-orders its contents so that the times of the operators increase from right to left. Time-ordered expectation values of this sort are useful in scattering problems, and axiomatic approaches to field theory usually take these quantities as basic.

Time-ordered expectation values are also central to perturbation theory. For example, consider the anharmonic oscillator

$$L_{\mathrm{AHO}}(q,\dot q) = L_{\mathrm{SHO}}(q,\dot q) - \frac{\lambda}{4}q^4 \qquad (\lambda > 0). \tag{19}$$

Expanding the integration measure as a series in λ, we have

$$e^{\frac{i}{\hbar}S_{\mathrm{AHO}}(q)} = \sum_{n\geq 0}\frac{1}{n!}\frac{(-i\lambda)^n}{4^n\hbar^n}\int dt_1\cdots dt_n\, e^{\frac{i}{\hbar}S_{\mathrm{SHO}}(q)} q(s_1)^4\cdots q(s_n)^4. \tag{20}$$

So any path integral in the anharmonic oscillator theory perturbatively reduces to a series of path integrals in the simple harmonic oscillator theory involving insertions of q^4.

This approach to perturbation theory works whenever the action of the theory can be decomposed as $L = L_{\mathrm{free}} + L_{\mathrm{pert}}$ where L_{free} is Gaussian. In a theory with coordinates q^i for $i = 1, \ldots, n$, a sufficient condition for this decomposition to be possible is that the Lagrangian be *non-degenerate*,

$$\det\left(\frac{\partial L}{\partial \dot q^i\,\partial \dot q^j}\right) \neq 0, \tag{21}$$

when evaluated at the stationary point about which the perturbation occurs.

2.3 Correspondence of the prescriptions

In good cases, the two quantization prescriptions just reviewed are compatible in the sense that the quantum theory produced from a Lagrangian using path integration coincides with the theory produced by canonically quantizing the Hamiltonian induced by the Lagrangian. In particular, this holds when the Lagrangian is non-degenerate.

A non-degenerate Lagrangian gives rise to a Hamiltonian. Any Lagrangian $L(q^i, \dot{q}^i)$ defines a canonical momentum and a Hamiltonian through the Legendre transformation

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \qquad H(q^i, p_i) = \sum_{i=1}^{n} p_i \dot{q}^i - L(q^i, \dot{q}^i). \tag{22}$$

For the second equation to define H as a function of canonical coordinates and momenta, it must be possible to write $\dot{q}^i$ as an algebraic function of these by solving the first equation. A sufficient condition for this is that the Lagrangian be non-degenerate.

A non-degenerate Lagrangian also induces a Poisson bracket on the phase space coordinatized by (q^i, p_i). Varying the Lagrangian density $L(q^i, \dot{q}^i)$ gives

$$\delta(L\,dt) = \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}\right) dt\,\delta q^i + d\theta, \qquad \theta = \frac{\partial L}{\partial \dot{q}^i}\,\delta q^i, \tag{23}$$

and varying θ in turn gives

$$\delta\theta = \frac{\partial L}{\partial q^i\,\partial \dot{q}^j}\,\delta q^i \wedge \delta q^j + \frac{\partial L}{\partial \dot{q}^i\,\partial \dot{q}^j}\,\delta \dot{q}^i \wedge \delta q^j, \tag{24}$$

which—because L is non-degenerate—pulls back to phase space to give a symplectic form ω. We therefore obtain a bracket

$$\{F, G\} = \omega(X_F, X_G), \tag{25}$$

where F and G are smooth functions on phase space and X_F is the Hamiltonian vector field associated with F under ω: that is, the unique vector field satisfying $dF = -\iota_{X_F}\omega$. If the Lagrangian is degenerate, then ω might only be a presymplectic form, in which case the Poisson bracket can only be defined on those smooth functions F for which there exists some X_F such that $dF = -\iota_{X_F}\omega$.

For example, in the case of the simple harmonic oscillator we have

$$p = m\dot{q}, \qquad \omega = dp \wedge dq, \tag{26}$$

thereby recovering the Hamiltonian theory of §2.1. The map

$$|n\rangle \mapsto \psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) \tag{27}$$

with

$$H_n(z) = (-)^n e^{z^2}\frac{d^n}{dz^n}e^{-z^2} \tag{28}$$

identifies $\ell^2(\mathbb{N})$ with $L^2(\mathbb{R})$, and under this identification H acts as

$$(H\psi)(x) = -\frac{\hbar^2}{2m}\psi''(x) + \frac{1}{2}m\omega^2 x^2\,\psi(x). \tag{29}$$

These dynamics are compatible with those induced by the path integral in the sense that

$$(e^{Ht/i\hbar}\psi)(x) = \int dx_0\,K(x, t; x_0, 0)\,\psi(x_0). \tag{30}$$

2.4 Field quantization

As a final preliminary, recall the extension of these methods to the field-theoretic case. In the context of field theories, we adopt natural units and set $c = \hbar = 1$, flagging factors of $\hbar$ when they are interpretively relevant.[4] The main items to introduce here are Fock space quantization for free theories and the interaction picture.

In the simplest field-theoretic cases, everything proceeds analogously to what has already been introduced above. For example, consider a complex scalar field φ with Lagrangian[5]

$$\mathcal{L} = (\partial^\mu \varphi^*)(\partial_\mu \varphi) - m^2 \varphi^* \varphi. \tag{31}$$

This gives a Hamiltonian system

$$\pi = \partial_0 \varphi^*, \qquad \mathcal{H} = \pi^* \pi - (\partial^i \varphi^*)(\partial_i \varphi) + m^2 \varphi^* \varphi, \tag{32}$$

and a Poisson bracket such that

$$\{\varphi(\vec{x}, t), \pi(\vec{y}, t)\} = \{\varphi^*(\vec{x}, t), \pi^*(\vec{y}, t)\} = \delta^3(\vec{x} - \vec{y}), \tag{33}$$

and such that all other brackets involving φ, φ^*, π, and π^* vanish.

Canonical quantization requests that we replace the Poisson bracket with the commutator:

$$[\varphi(x, t), \pi(y, t)] = [\varphi^*(x, t), \pi^*(y, t)] = i\delta^3(x - y). \tag{34}$$

This is typically done by mimicking the Fock space construction of §2.1. The underlying vector space $\mathcal{H}$ is spanned by elements of the form

$$a^\dagger_{p_1} \cdots a^\dagger_{p_m} b^\dagger_{p_1} \cdots b^\dagger_{p_n} |0\rangle, \qquad [a^\dagger_p, a^\dagger_q] = [b^\dagger_p, b^\dagger_q] = 0, \tag{35}$$

for a distinguished vector $|0\rangle$, where the $a^\dagger_p$ and $b^\dagger_p$ are indexed by 3-momenta. Further operators a_p and b_p are introduced, defined by

$$\begin{aligned} a_p |0\rangle &= b_p |0\rangle = 0, \\ [a_p, a_q] &= [b_p, b_q] = 0, \\ [a_p, b_q] &= [a_p, b^\dagger_q] = [a^\dagger_p, b_q] = [a^\dagger_p, b^\dagger_q] = 0, \\ [a_p, a^\dagger_q] &= [b_p, b^\dagger_q] = \delta^3(p - q). \end{aligned} \tag{36}$$

Setting $\langle 0|0\rangle = 1$ gives a positive-definite Hermitian form on $\mathcal{H}$, making it a Hilbert space. Writing our fields as families of oscillators indexed by momentum

$$\begin{aligned} \varphi(x) &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2k^0}} \left(a_k e^{-ik\cdot x} + b^\dagger_k e^{ik\cdot x} \right), \\ \pi(x) &= (\partial_0 \varphi^*)(x), \qquad \varphi^*(x) = \varphi(x)^*, \qquad \pi^*(x) = \pi(x)^*, \end{aligned} \tag{37}$$

[4] We remain at the level of rigour of, say, Peskin and Schroeder (1995).

[5] We consider a *complex* scalar field here to set up for the fermionic fields which we consider in later sections.

with k on shell, we obtain the required commutation relations, and the Hamiltonian is

$$H = \int \frac{d^3k}{(2\pi)^{3/2}} k^0 \left(a_k^\dagger a_k + b_k^\dagger b_k \right). \tag{38}$$

On the other hand, path integral quantization produces expressions of the form

$$\langle \varphi(x_1) \cdots \varphi(x_n) \rangle = \frac{1}{Z} \int \mathcal{D}\varphi \, e^{i \int d^4x \, \mathcal{L}} \varphi(x_1) \cdots \varphi(x_n). \tag{39}$$

As before, this formula is particularly useful in perturbation theory.

3 Fermions and Krein spaces

So much for the familiar, 'positive' approaches to quantization. The goal of this section (in §3.1) is to introduce anticommutation relations and odd coordinates (which we take to be more-or-less familiar) and (in §3.2) to motivate the use of Krein spaces (which we do not) using the fact that nondegenerate Lagrangians for odd coordinates typically lead to negative-norm modes.

3.1 Fermions and odd coordinates

The Fock space quantization of §2.4 produced a theory of bosonic quanta. If we want a theory of fermions, then because fermionic states must be antisymmetric, we must have

$$\begin{aligned} [a_k, a_{k'}]_+ &= [a_k, b_{k'}]_+ = [b_k, b_{k'}]_+ = 0, \\ [a_k, a_{k'}^\dagger]_+ &= [b_k, b_{k'}^\dagger]_+ = \delta^3(k - k'), \end{aligned} \tag{40}$$

where $[-,-]_+$ is the anticommutator. This is inconsistent with the canonical commutation prescription, because the Poisson bracket and the anticommutator have different algebraic properties.

The familiar solution to this problem is to generalize the class of classical systems under consideration to include *graded* Poisson algebras. In the simplest case, we suppose that every function on classical phase space uniquely decomposes as the sum of an even part and an odd part. A function that is purely even or purely odd is called 'homogeneous', and for every homogeneous F we write ϵ_F for its parity, with $\epsilon_F = 0$ if F is even and $\epsilon_F = 1$ if it is odd. Multiplication and the Poisson bracket respect the grading:

$$\epsilon_{FG} = \epsilon_{\{F,G\}} = \epsilon_F + \epsilon_G \tag{41}$$

—the former is graded commutative while the latter is even and graded symmetric:

$$FG = (-)^{\epsilon_F \epsilon_G} GF, \qquad \{F, G\} = -(-)^{\epsilon_F \epsilon_G} \{G, F\}. \tag{42}$$

The Poisson bracket is a derivation and satisfies the Jacobi identity:

$$\begin{aligned} \{F, GH\} &= \{F, G\}H + (-)^{\epsilon_F \epsilon_G} G\{F, H\}, \\ \{F, \{G, H\}\} &= \{\{F, G\}, H\} + (-)^{\epsilon_F \epsilon_G} \{G, \{F, H\}\}. \end{aligned} \tag{43}$$

The canonical formalism generalizes to graded Poisson algebras as follows. Consider a system with even coordinates q^i ($i = 1, \ldots, n_+$) and odd coordinates θ^α ($\alpha = 1, \ldots, n_-$), and suppose given a Lagrangian $L(q^i, \dot{q}^i, \theta^\alpha, \dot{\theta}^\alpha)$. The canonical momenta are defined by

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \qquad \pi_\alpha = \frac{\partial L}{\partial \dot{\theta}^\alpha}, \tag{44}$$

and the Hamiltonian is

$$H = \dot{q}^i p_i + \dot{\theta}^\alpha \pi_\alpha - L(q^i, \dot{q}^i, \theta^\alpha, \dot{\theta}^\alpha). \tag{45}$$

The generalization of the Poisson bracket of Eq. (2) to the graded case is

$$\{F, G\} = \left[\frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i}\right] + (-)^{\epsilon_F}\left[\frac{\partial F}{\partial \theta^\alpha}\frac{\partial G}{\partial \pi_\alpha} + \frac{\partial F}{\partial \pi_\alpha}\frac{\partial G}{\partial \theta^\alpha}\right]. \tag{46}$$

In particular we have $\{\theta^\alpha, \pi_\beta\} = -\delta^\alpha{}_\beta$, and so canonical quantization requires

$$\frac{1}{i\hbar}[\theta^\alpha, \pi_\beta]_+ = -\delta^\alpha{}_\beta. \tag{47}$$

which has the correct positivity properties for fermions.

Though the canonical formalism generalizes cleanly, fermionic theories are qualitatively different to bosonic theories. *Inter alia*, non-degenerate fermionic Lagrangians generically lead to negative-norm states. For example, consider a theory with an odd real coordinate θ and an odd imaginary coordinate $\bar{\theta}$ and Lagrangian

$$L = \dot{\bar{\theta}}\dot{\theta}. \tag{48}$$

This is the natural generalization to odd coordinates of the Gaussian free theory and is non-degenerate. The canonical momenta and Poisson brackets are

$$\begin{gathered}\bar{\pi} = \frac{\partial L}{\partial \dot{\theta}} = -\dot{\bar{\theta}}, \qquad \pi = \frac{\partial L}{\partial \dot{\bar{\theta}}} = \dot{\theta}, \\ \{\theta, \bar{\pi}\} = \{\bar{\theta}, \pi\} = -1,\end{gathered} \tag{49}$$

and so upon quantization we must have

$$\theta\bar{\pi} + \bar{\pi}\theta = \bar{\theta}\pi + \pi\bar{\theta} = -i\hbar. \tag{50}$$

Because θ and $i\bar{\pi}$ are real, their quantum counterparts must be self-adjoint. The operator $A = \hbar^{-1/2}(\theta - i\bar{\pi})$ is therefore self-adjoint and satisfies $A^2 = -1$. This is a contradiction, because it implies that for any state ψ we have

$$\langle A\psi, A\psi\rangle = \langle \psi, A^2\psi\rangle = -\langle \psi, \psi\rangle \tag{51}$$

so that either ψ or $A\psi$ has negative norm—a contradiction.

There are two reactions one might have to this. One might search for graded Poisson brackets which don't lead to negative-norm states (e.g., the Dirac equation). Alternatively, one might try to learn to live with negative-norm states by generalizing also the quantum side of the canonical quantization prescription. Some motivation for this is how much non-degeneracy is assumed in the above.

3.2 Krein spaces

The goal of this subsection is to introduce Krein spaces and to explain how one might extract a Hilbert space as a subquotient thereof. The obvious problem to keep in view is that the probability interpretation of the norm requires non-negative values.[6]

3.2.1 Introducing Krein spaces

Put more carefully, the argument at the end of the previous subsection shows that the canonical quantization prescription cannot be satisfied, because the condition on the brackets implies $A^2 = -1$, contradicting the condition on hermiticity. It can't show that there are negative-norm states, since the concept is contradictory. So if we want to include them, then we need a generalization of Hilbert spaces that allows for vectors v such that $\langle v, v\rangle < 0$.

Krein spaces generalize Hilbert spaces by weakening the positivity requirements on the scalar product. Whereas a Hilbert space carries a positive definite Hermitian form, a Krein space is a vector space $\mathcal{K}$ equipped with a merely *non-degenerate* Hermitian form $\langle -, -\rangle$.[7] This is a strictly more general structure, so that any Hilbert space is a Krein space, but there are Krein spaces that are not Hilbert spaces.

For a simple example, consider the odd charged scalar field, by analogy with the discussion in §2.4 and §3.1.[8] The Lagrangian is

$$\mathcal{L} = (\partial^\mu \bar{C})(\partial_\mu C), \tag{52}$$

where C is a real odd field and $\bar{C}$ an imaginary odd field. The canonical momenta are

$$\bar{\mathcal{P}} = \frac{\delta \mathcal{L}}{\delta(\partial_0 C)} = -\partial_0 \bar{C}, \qquad \mathcal{P} = \frac{\delta \mathcal{L}}{\delta(\partial_0 \bar{C})} = \partial_0 C, \tag{53}$$

and so we have

$$\{C(\vec{x},t), \bar{\mathcal{P}}(\vec{y},t)\} = \{\bar{C}(\vec{x},t), \mathcal{P}(\vec{y},t)\} = -\delta^3(\vec{x}-\vec{y}), \tag{54}$$

with other brackets zero. Canonical quantization then requires

$$[C(\vec{x},t), \bar{\mathcal{P}}(\vec{y},t)]_+ = [\bar{C}(\vec{x},t), \mathcal{P}(\vec{y},t)]_+ = -i\delta^3(\vec{x}-\vec{y}). \tag{55}$$

[6] For further background on indefinite inner product spaces, see Bognár (1974).

[7] As usual, the infinite-dimensional case additionally involves an appropriate topological vector space structure. In this case, the Krein space must be equipped with the norm topology of a positive definite hermitian form with respect to which the Krein space inner product induces a bijection $\mathcal{K} \to \mathcal{K}'$ via the assignment $u \mapsto \langle u, -\rangle$, with $\mathcal{K}'$ the continuous dual of $\mathcal{K}$. In what follows, we will only consider the special case of Krein spaces obtained via the Fock space construction, where the topology takes care of itself.

[8] See Bogolubov et al. (1990, p. 423) for how this looks in a more Wightman-style formalism.

As before, we can achieve this by expanding in oscillators

$$
\begin{aligned}
c_p |0\rangle &= \bar{c}_p |0\rangle = 0, \\
[c_p, c_q]_+ &= [\bar{c}_p, \bar{c}_q]_+ = [c_p^\dagger, c_q^\dagger]_+ = [\bar{c}_p^\dagger, \bar{c}_q^\dagger]_+ = 0, \\
[c_p, c_q^\dagger]_+ &= [c_p, \bar{c}_q]_+ = [\bar{c}_p, c_q]_+ = [\bar{c}_p, \bar{c}_q^\dagger]_+ = 0, \\
[c_p, \bar{c}_q^\dagger]_+ &= [\bar{c}_p, c_q^\dagger]_+ = \delta^3(p-q).
\end{aligned}
\tag{56}
$$

Requiring that $\langle 0|0\rangle = 1$ and that $\dagger$ denote the adjoint, we get a Hermitian form on $\mathcal{K}$. Taking

$$C(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2k^0}} \left(c_k e^{-ik\cdot x} + c_k^\dagger e^{ik\cdot x} \right), \tag{57}$$

$$\bar{C}(x) = -\int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2k^0}} \left(\bar{c}_k e^{-ik\cdot x} - \bar{c}_k^\dagger e^{ik\cdot x} \right), \tag{58}$$

$$\bar{\mathcal{P}} = -\partial_0 \bar{C}, \qquad \mathcal{P} = \partial_0 C. \tag{59}$$

then gives the canonical anticommutation relations.

Observe that the Hermitian form on $\mathcal{K}$ is not positive definite. For example, the state $c_p^\dagger |0\rangle$ is nonzero but has norm 0, and the state $c_p^\dagger \bar{c}_q^\dagger |0\rangle$ has negative norm. Note in turn further that the spin–statistics connection fails here, because the spin–statistics theorem assumes positivity of the norm; see Bain (2016) for further discussion.[9]

3.2.2 Probability and interpretation

The obvious problem with negative-norm states is that they obstruct the interpretation of inner products in terms of probability. To recover such an interpretation from a non-positive quantization, one must extract a Hilbert space whose self-adjoint operators can serve as observables for the quantum system. Negative-norm states are eliminated by restricting to a subspace, and zero-norm states by a quotient. The extra flexibility of non-positive quantization comes from the fact that this observable Hilbert space need not be a subspace of the Krein space, but will in general be a subquotient. The choice of this subquotient is an extra choice that must be input to the Krein quantization machine.

If a Krein space has negative-norm states, then it has states that cannot be normalized.[10] If there are negative-norm states then there are zero-norm states, which can't be normalized and so can't be given the usual probabilistic interpretation in terms of the Born rule.

If the Krein space is positive definite, then it is a Hilbert space, and everything can proceed as usual.

If the Krein space is positive semidefinite, then there are vectors with zero norm but no vectors of negative norm. In this case, one can extract a Hilbert

[9] For an insightful review of Bain's book, see Swanson (2018).

[10] The point is that the inner product doesn't induce a norm, or even a seminorm. So the phrase 'negative norm' isn't really apt, although it is common.

space by quotienting out the zero-norm vectors. That is, if $\mathcal{K}$ is a positive semidefinite Krein space and $\mathcal{K}_0 \subseteq \mathcal{K}$ is the subspace of vectors with norm zero, then the Hermitian form on $\mathcal{K}$ induces a positive-definite Hermitian form on the quotient $\mathcal{K}/\mathcal{K}_0$. Completing this gives a Hilbert space $\mathcal{H} = \overline{\mathcal{K}/\mathcal{K}_0}$.

In the most general case, we can recover a Hilbert space by first restricting to a positive semidefinite subspace. That is, suppose that $\mathcal{K}_+ \subseteq \mathcal{K}$ is a subspace on which the Hermitian form is positive semidefinite. Proceeding as in the previous paragraph, we can define $\mathcal{H} = \overline{\mathcal{K}_+/\mathcal{K}_0}$, where $\mathcal{K}_0 \subseteq \mathcal{K}_+$ is the subspace of zero-norm vectors in $\mathcal{K}_+$.

In general, then, non-positive quantization requires a further ingredient: namely, a choice $\mathcal{K}_+ \subseteq \mathcal{K}$ of positive semidefinite subspace which induces a Hilbert space $\mathcal{H}$ on which the observables act as self-adjoint operators. The physical significance of this choice will (with any luck) be illuminated further by the cases discussed in subsequent sections. But the thing to keep in mind for now is that the observables of the quantized theory are operators on the Hilbert space $\mathcal{H}$.

This extra ingredient allows for a more flexible relationship between the classical theory and unitarity. Canonical quantization will in general produce a dynamics that does not restrict to a positive-definite subspace. That is, such a dynamics will generally carry a vector with positive norm into a superposition of states with positive norm and states with zero norm. However, passing to the quotient kills zero norm states, and so such a dynamics can still be unitary on the observable Hilbert space.

The upshot is that non-positive quantization takes as input not only a graded Poisson algebra of functions on phase space but also a specification of the positive semidefinite subspace defining the observable Hilbert space $\mathcal{H}$. The dynamics will be unitary in $\mathcal{H}$ if it is pseudo-unitary in the Krein space and preserves the distinguished positive semidefinite subspace.

4 Non-positive Hamiltonian quantization

Non-positive canonical quantization uses the same 'Poisson bracket goes to commutator' prescription as the positive case, adding a specification of the observable Hilbert space in terms of a symmetry of the classical theory. This section describes the general prescription (§4.1) and then walks through the simplest non-trivial example (§4.2).

4.1 Symmetry and the observable Hilbert space

The upshot of §3.2.2 is that non-positive quantization requires an additional input as compared to positive quantization. Positive canonical quantization takes (i) a graded Poisson algebra and (ii) a Hamiltonian as input, and as output it prescribes (a) an algebra of Hilbert space operators and (b) a quantum Hamiltonian satisfying certain conditions (c): the Poisson bracket is sent to

$(i\hbar)^{-1}$(commutator), real elements of the Poisson algebra are taken to self-adjoint operators, and the Hamiltonian gives a unitary dynamics. Non-positive quantization prescribes in addition the specification of a positive-semidefinite subspace to which the dynamics restricts.

In the cases of interest in this paper, the positive-semidefinite subspace is characterized by a symmetry of the classical theory. It works like this. Suppose that Ω is an odd, real function on the classical phase space such that

$$\{\Omega, \Omega\} = 0, \qquad \{H, \Omega\} = 0, \tag{60}$$

where H is the Hamiltonian. The first of these conditions says that Ω is nilpotent, and it is nontrivial because Ω is odd. The second says that Ω generates a symmetry of the dynamics.

If these conditions persist into a quantum theory on a Krein space $\mathcal{K}$, then Ω becomes a self-adjoint operator satisfying

$$\Omega^2 = \frac{1}{2}[\Omega, \Omega]_+ = 0, \qquad [H, \Omega] = 0. \tag{61}$$

The second of these conditions implies that the dynamics induced by H restricts to $\ker\Omega$, because

$$\Omega H\,|v\rangle = H\Omega\,|v\rangle - [H, \Omega]\,|v\rangle = 0 \tag{62}$$

for all $|v\rangle$ in $\ker\Omega$, where the first term vanishes by hypothesis on $|v\rangle$ and the second by hypothesis on H. The nilpotency of Ω then implies that

$$\langle \Omega v, \Omega v\rangle = \langle \Omega^2 v, v\rangle = 0 \tag{63}$$

for all $|v\rangle$ in $\ker\Omega$, and so $\operatorname{im}\Omega$ is a subset of $\ker\Omega$ consisting of zero norm vectors. Supposing that $\mathcal{K}_+ = \ker\Omega$ is a positive-semidefinite subspace and $\mathcal{K}_0 = \operatorname{im}\Omega$ is the subspace of zero-norm vectors, the observable Hilbert space is given by the subquotient

$$H(\Omega) = \frac{\ker\Omega}{\operatorname{im}\Omega}. \tag{64}$$

In other words, it is given by the cohomology of Ω.

More generally, if the classical algebra is equipped with an appropriate grading, then one can take the observable Hilbert space to be the cohomology of Ω in specified degree. That is, suppose the classical algebra carries a $\mathbb{Z}$-grading with respect to which the Poisson bracket is a bilinear form of degree zero and Ω is a homogeneous element of degree $+1$. This grading is usually called the *ghost number*, and it is independent of the Grassmann parity: one can have even elements of odd ghost number and odd elements of even ghost number. Suppose furthermore that the ghost number operator has a canonical generator $\mathcal{G}$,

$$\{A, \mathcal{G}\} = i(\operatorname{gh} A)\,A. \tag{65}$$

Here the factor of i is a convention that gives $\mathcal{G}$ integer eigenvalues upon quantization. Indeed, supposing that the dynamics has ghost number zero,

$$\{H, \mathcal{G}\} = 0, \tag{66}$$

and that all of this persists into the quantum theory, the reasoning of the previous paragraph leads to the cohomology

$$H^0(\Omega) = \frac{\{v \in \ker \Omega : \mathcal{G}v = 0\}}{\{v \in \operatorname{im} \Omega : \mathcal{G}v = 0\}} \tag{67}$$

and if the Krein space Hermitian form is positive definite on this subquotient, then we can take this to be the observable Hilbert space.

The upshot is that there is a non-positive quantization prescription that takes as inputs (i) a graded Poisson algebra, (ii) a Hamiltonian H, (iii) a symmetry Ω, and (iv) a ghost number operator $\mathcal{G}$, and that prescribes as outputs (a) an algebra of Krein space operators and (b) quantum realizations of H, Ω, and $\mathcal{G}$, such that (c): the Poisson bracket is sent to $(i\hbar)^{-1}$(commutator), real elements of the Poisson algebra are taken to self-adjoint operators, the Hamiltonian gives a unitary dynamics, and $H^0(\Omega)$ is a Hilbert space.[11]

4.2 Example

In the simplest non-trivial case, Ω comes from a group (or, more generally, Lie algebroid) action on the classical space of fields under which the action is invariant. In this case, one can think of Ω as being given prior to the classical action. This is analogous to the fact that when the action is in 'canonical form', the induced Poisson bracket can be thought of as prior to the dynamics.

For example, consider the Lagrangian

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + (\partial^\mu \bar{C}_a)(D_\mu C^a) - \frac{1}{2}(B_a)^2 - (\partial^\mu B_a)A^a_\mu. \tag{68}$$

Here A^a_μ is a real, even, ghost number zero $\mathfrak{su}(n)$-valued 1-form, the field B_a is a real, even scalar of ghost number 0, the field C^a is a real, odd scalar of ghost number +1, the field $\bar{C}_a$ is an imaginary, odd scalar of ghost number -1, and

$$D_\mu C^a = \partial_\mu C^a + g f^a{}_{bc} A^b_\mu C^c, \tag{69}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^a{}_{bc} A^b_\mu A^c_\nu. \tag{70}$$

This Lagrangian is invariant under the BRST transformation $\varphi \mapsto \epsilon\, \delta\varphi$ (cf. §6.2.1) with ϵ odd and imaginary and the differential given by

$$\begin{aligned} \delta C^a &= -\frac{1}{2} g f^a{}_{bc} C^b C^c, & \delta B_a &= 0, \\ \delta A^a_\mu &= D_\mu C^a, & \delta \bar{C}_a &= B_a, \end{aligned} \tag{71}$$

and so gives the conserved current

$$\begin{aligned} J^\mu &= \sum_n \delta\varphi_n \frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi_n)} \\ &= (B_a \eta^{\mu\nu} - F^{\mu\nu}_a) D_\nu C^a + \frac{1}{2} g f^a{}_{bc} (\partial^\mu \bar{C}_a) C^b C^c. \end{aligned} \tag{72}$$

[11] Some readers might already identify Ω with the BRST charge. We hesitate to do so at this point, since we have not yet introduced the notion of BRST invariance (on which see the next subsection); what we have presented here is more general.

Defining the canonical momenta

$$\begin{aligned} \pi_a^0 &= B_a, & \bar{\mathcal{P}}_a &= -\partial_0 \bar{C}_a, \\ \pi_a^i &= F_a^{i0}, & \mathcal{P}^a &= D_0 C^a, \end{aligned} \tag{73}$$

the induced graded Poisson bracket is

$$\left\{A_\mu^a(\vec{x},t), \pi_b^\nu(\vec{y},t)\right\} = \delta^a{}_b \eta_\mu{}^\nu \delta^3(\vec{x}-\vec{y}), \tag{74}$$

$$\left\{C^a(\vec{x},t), \bar{\mathcal{P}}_b(\vec{y},t)\right\} = \left\{\bar{C}_a(\vec{x},t), \mathcal{P}^b(\vec{y},t)\right\} = -\delta_{ab}\delta^3(\vec{x}-\vec{y}), \tag{75}$$

and the Noether charge—'BRST charge'—associated with the BRST invariance is

$$\Omega = \int d^3x \left(\pi_a^0 \mathcal{P}^a + \pi_a^i \, D_i C^a - \frac{1}{2} g f^a{}_{bc} \bar{\mathcal{P}}_a C^b C^c \right), \tag{76}$$

which indeed generates the transformations

$$\begin{aligned} \left\{A_\mu^a, \epsilon\Omega\right\} &= \epsilon\, \delta A_\mu^a, \\ \left\{C^a, \epsilon\Omega\right\} &= \epsilon\, \delta C^a, \\ \left\{\bar{C}_a, \epsilon\Omega\right\} &= \epsilon\, \delta \bar{C}_a. \end{aligned} \tag{77}$$

The ghost number is generated by

$$\mathcal{G} = \int d^3x \left(i C^a \bar{\mathcal{P}}_a - i \bar{C}_a \mathcal{P}^a \right), \tag{78}$$

so that

$$\{C^a, \mathcal{G}\} = i C^a, \qquad \{\bar{C}_a, \mathcal{G}\} = -i \bar{C}_a. \tag{79}$$

4.2.1 Fock space quantization

In perturbation theory, we proceed as in the Fock space construction above, taking $g = 0$.

The ghost sector is the same as in §3.2.1. That is, the nonzero anticommutators are

$$[c_p^a, \bar{c}_q^{b\dagger}]_+ = [\bar{c}_p^a, c_q^{b\dagger}]_+ = \delta^{ab}\delta^3(p-q), \tag{80}$$

with the fields expanded in oscillators as

$$\begin{aligned} C^a(x) &= -\int \frac{d^3k}{2(2\pi k^0)^{3/2}} \left(c_k^a \, e^{-ik\cdot x} + c_k^{a\dagger} e^{ik\cdot x} \right), \\ \bar{C}_a(x) &= \int \frac{d^3k}{(2\pi)^{3/2}} (k^0)^{1/2} \left(\bar{c}_k^a e^{-ik\cdot x} - \bar{c}_k^{a\dagger} e^{ik\cdot x} \right), \\ \bar{\mathcal{P}}_a(x) &= i \int \frac{d^3k}{(2\pi)^{3/2}} (k^0)^{3/2} \left(\bar{c}_k^a \, e^{-ik\cdot x} + \bar{c}_k^{a\dagger} e^{ik\cdot x} \right), \\ \mathcal{P}^a(x) &= i \int \frac{d^3k}{2(2\pi)^{3/2} (k^0)^{1/2}} \left(c_k^a \, e^{-ik\cdot x} - c_k^{a\dagger} e^{ik\cdot x} \right), \end{aligned} \tag{81}$$

so that

$$\begin{aligned}\left[C^a(\vec{x},t),\bar{\mathcal{P}}_b(\vec{y},t)\right]_+ &= -i\delta^a{}_b\delta^3(\vec{x}-\vec{y}),\\ \left[\bar{C}_a(\vec{x},t),\mathcal{P}^b(\vec{y},t)\right]_+ &= -i\delta_a{}^b\delta^3(\vec{x}-\vec{y}).\end{aligned} \tag{82}$$

In the Yang–Mills sector, we introduce annihilation and creation operators $a^a_{\sigma k}$, where a is the Lie algebra index and $\sigma = 0, 1, 2, 3$. The nonzero commutators are

$$\begin{gathered}[a^a_{0p}, a^{b\dagger}_{0q}] = -\delta^{ab}\delta^3(\vec{p}-\vec{q}),\\ [a^a_{1p}, a^{b\dagger}_{1q}] = [a^a_{2p}, a^{b\dagger}_{2q}] = [a^a_{3p}, a^{b\dagger}_{3q}] = \delta^{ab}\delta^3(\vec{p}-\vec{q}),\end{gathered} \tag{83}$$

and the fields are

$$\begin{aligned}A^a_\mu(x) &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2k^0}}\left(a^a_{\sigma k}\epsilon^\sigma_\mu(k)\,e^{-ik\cdot x} + a^{a\dagger}_{\sigma k}\epsilon^\sigma_\mu(k)\,e^{ik\cdot x}\right),\\ \pi^0_a &= i\int \frac{d^3k}{(2\pi)^{3/2}}\sqrt{k^0}\left(b^a_k e^{-ik\cdot x} - b^{a\dagger}_k e^{ik\cdot x}\right),\\ \pi^{ai} &= i\int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2}}\sqrt{k^0}\\ &\quad\times\Big(\left(a^a_{\sigma k}\epsilon^{\sigma i}(k) - a^a_{0k}\epsilon^{3i}(k)\right)e^{-ik\cdot x}\\ &\qquad - \left(a^{a\dagger}_{\sigma k}\epsilon^{\sigma i}(k) - a^{a\dagger}_{0k}\epsilon^{3i}(k)\right)e^{ik\cdot x}\Big),\end{aligned} \tag{84}$$

where the polarization vectors $\epsilon^\sigma_\mu(\vec{k})$ are given by

$$\epsilon^{\sigma\mu}(\hat{z}) = \delta^{\sigma\mu}, \qquad \epsilon^{\sigma\mu}(\vec{k}) = R(\vec{k})^\mu{}_\nu\epsilon^{\sigma\nu}(\hat{z}), \tag{85}$$

for $\hat{z}$ the unit vector in the third direction and $R(\vec{k})$ the pure rotation taking $\hat{z}$ to the unit vector in the $\vec{k}$ direction. We define

$$\begin{gathered}b^a_k = \frac{1}{\sqrt{2}}(a^a_{0k} - a^a_{3k}), \qquad \bar{b}^a_k = -\frac{1}{\sqrt{2}}(a^a_{0k} + a^a_{3k}),\\ [\bar{b}^a_p, b^{b\dagger}_q] = [b^a_p, \bar{b}^{b\dagger}_q] = \delta^{ab}\delta^3(\vec{p}-\vec{q}).\end{gathered} \tag{86}$$

These give the commutation relations

$$\left[A^a_\mu(\vec{x},t), \pi^\nu_b(\vec{y},t)\right] = i\delta^a{}_b\eta_\mu{}^\nu\delta^3(\vec{x}-\vec{y}). \tag{87}$$

The Noether charge becomes

$$\Omega = \int d^3k\left(b^{a\dagger}_k c^a_k + c^{a\dagger}_k b^a_k\right), \tag{88}$$

and satisfies

$$\begin{aligned}\left[A^a_\mu, \Omega\right] &= i\partial_\mu C^a, & [C^a, \Omega]_+ &= 0,\\ [\Omega, \Omega]_+ &= 0, & \left[\bar{C}_a, \Omega\right]_+ &= i\pi^0_a,\end{aligned} \tag{89}$$

while the ghost number operator becomes (after normal ordering)

$$\mathcal{G} = \int d^3k \, \left(c_k^{a\dagger} \bar{c}_k^a - \bar{c}_k^{a\dagger} c_k^a \right) \tag{90}$$

and satisfies

$$\begin{aligned} [\mathcal{G}, C^a] &= C^a, & [\mathcal{G}, \bar{\mathcal{P}}_a] &= -\bar{\mathcal{P}}_a, \\ [\mathcal{G}, \bar{C}_a] &= -\bar{C}_a, & [\mathcal{G}, \mathcal{P}^a] &= \mathcal{P}^a, \end{aligned} \tag{91}$$

along with $[\mathcal{G}, A_\mu^a] = [\mathcal{G}, B_a] = 0$.

4.2.2 The observable Hilbert space

The observable Hilbert space in this case admits a nice description in terms of particle content. The negative norm quanta created by $c^{a\dagger}$ and $\bar{c}^{a\dagger}$ assemble into a 'quartet' with the quanta $b^{a\dagger}$ and $\bar{b}^{a\dagger}$, which ensures that none of these quanta appear in the observable Hilbert space. However, we can't merely think of the observable Hilbert space as the subspace of Krein space states without b and c quanta, because the dynamics take us out of this subspace. We really are interested in the subquotient.

The quartet mechanism amounts to the fact that

$$\begin{aligned} \Omega \bar{b}_p^{a\dagger} |0\rangle &= c_p^{a\dagger} |0\rangle \,, \\ \Omega \bar{c}_p^{a\dagger} |0\rangle &= b_p^{a\dagger} |0\rangle \,. \end{aligned} \tag{92}$$

The $\bar{b}^{a\dagger}$ and $\bar{c}^{a\dagger}$ states on the left are therefore not in the kernel of Ω, so they do not appear in the observable Hilbert space, while the $b^{a\dagger}$ and $c^{a\dagger}$ states on the right belong to the image of Ω, so they are quotiented out in passing to the observable Hilbert space.

More generally, passing to the observable Hilbert space sets states with b and c (anti)quanta to zero. Consider the operators

$$\begin{aligned} N &= \int d^3k \, \left(\bar{b}_k^{a\dagger} b_k^a + b_k^{a\dagger} \bar{b}_k^a + \bar{c}_k^{a\dagger} c_k^a + c_k^{a\dagger} \bar{c}_k^a \right), \\ K &= \int d^3k \, \left(\bar{c}_k^{a\dagger} \bar{b}_k^a + \bar{b}_k^{a\dagger} \bar{c}_k \right), \end{aligned} \tag{93}$$

where the first counts b and c (anti)quanta and the latter exhibits N as a coboundary:

$$[N, \Omega] = 0, \qquad [K, \Omega]_+ = N. \tag{94}$$

If $|v\rangle$ is in the kernel of Ω and is an eigenstate of N with nonzero eigenvalue n, then

$$|v\rangle = \frac{1}{n} N |v\rangle = \frac{1}{n} K\Omega |v\rangle + \frac{1}{n} \Omega K |v\rangle = \Omega \left(\frac{1}{n} K |v\rangle \right), \tag{95}$$

and so $|v\rangle$ is in the image of Ω, meaning that it gets killed by the quotient.

It is worth emphasizing that the observable Hilbert space really is a subquotient of the Krein space in this case, rather than a subspace. Because the Fock basis is an eigenbasis of N, the argument just given allows one to identify the observable Hilbert space with $\ker N$, where the quotient map $\mathcal{K}_+ \to \ker N$ sets all of the Fock basis vectors containing a b or c to zero. Since $\ker N$ is naturally a subspace of $\mathcal{K}$, one might be tempted to think of $\mathcal{K}$ as a direct sum of the observable Hilbert space and a space of unobservable states. But such a decomposition is generally incompatible with the dynamics, which mixes states in $\ker N$ with states outside of $\ker N$. For example, when the theory discussed in this section is extended to include matter quanta q charged under the A^a_μ field, it gives nonzero amplitudes for $q\bar{q} \to b\bar{b}$ and $q\bar{q} \to c\bar{c}$ scattering. The dynamics on the observable Hilbert space therefore cannot be given by restricting the dynamics on $\mathcal{K}$ to the subspace $\ker N$, as this restriction is not well defined. Rather, we restrict the dynamics to $\mathcal{K}_+$ and then project to $\ker N$ using the Fock basis—that is, we treat the observable Hilbert space as a subquotient of $\mathcal{K}$, rather than a subspace. The resulting dynamics is unitary because the amplitudes for $q\bar{q} \to b\bar{b}$ and $q\bar{b} \to c\bar{c}$ are equal in magnitude: the negative norm of the c states gives these amplitudes opposite signs, so setting the state $\Omega c^\dagger \bar{b}^\dagger |0\rangle = b^\dagger \bar{b}^\dagger |0\rangle + c^\dagger \bar{c}^\dagger |0\rangle$ to zero in the quotient does not lose any amplitude.

5 Non-positive Lagrangian quantization

Path integral quantization assigns quantum expectation values to observables on the quantum space of states. As in canonical quantization, in the non-positive case these observables are cohomology classes of operators on the Krein space. However, in path integral quantization we lack the Poisson bracket of the Hamiltonian formulation, and so the cohomology classes of interest cannot be formulated in terms of distinguished quantities like $\mathcal{G}$ and Ω. Instead, non-positive path integral quantization takes the differential δ of §4.2 and the ghost number grading to be additional pieces of classical data, and it assigns quantum expectation values to the cohomology of δ.

Because the objects of interest are cohomology classes of a differential, non-positive path integral quantization imposes further conditions and allows for more flexibility than the positive case. On the one hand, the path integral must assign the same quantum expectation value to every representative of a given cohomology class. This requires the integration measure to be closed, which can be expressed in terms of a new differential Δ as

$$\Delta e^{\frac{i}{\hbar} S} = 0. \tag{96}$$

On the other hand, because a closed integration measure assigns 0 to exact terms, the integration cycle can be varied with impunity. Non-positive path integral quantization therefore takes as classical input also a choice of Lagrangian submanifold in the form of the graph of a derivative $d\Psi$. The dependence on this graph is encoded by a further set of fields, called the *antifields*.

5.1 Antifields

In general, the formalism is as follows. The classical theory contains a set of fields Φ^A $(A = 1, \ldots, N)$, each of which has a Grassmann parity and ghost number, as well as a set of 'antifields' $\Phi^\ddagger_A$ $(A = 1, \ldots, N)$ of opposite Grassmann parity and with ghost number satisfying $\mathrm{gh}\,\Phi^\ddagger_A = -\,\mathrm{gh}\,\Phi^A - 1$. This algebra of fields and antifields is equipped with an odd bracket of ghost number 1,

$$\epsilon_{(F,G)} = \epsilon_F + \epsilon_G + 1, \qquad \mathrm{gh}(F,G) = \mathrm{gh}\,F + \mathrm{gh}\,G + 1, \tag{97}$$

that is a shifted Poisson bracket:[12]

$$\begin{aligned} (F,G) &= -(-)^{(\epsilon_F+1)(\epsilon_G+1)}(G,F), \\ (F,GH) &= (F,G)H + (-)^{(\epsilon_F+1)\epsilon_G} G(F,H), \\ (F,(G,H)) &= ((F,G),H) + (-)^{(\epsilon_F+1)(\epsilon_G+1)}(G,(F,H)), \end{aligned} \tag{98}$$

and a differential Δ that generates the bracket

$$\begin{gathered} \Delta^2 = 0, \qquad \Delta(1) = 0, \\ \Delta(FG) = (\Delta F)G + (-)^{\epsilon_F} F\,\Delta G + (-)^{\epsilon_F}(F,G). \end{gathered} \tag{99}$$

For the path integral to consistently assign expectation values to cohomology classes, it must satisfy the *quantum master equation*,[13]

$$\Delta e^{\frac{i}{\hbar}S} = 0 \qquad \Longleftrightarrow \qquad \frac{1}{2}(S,S) - i\hbar\,\Delta S = 0, \tag{100}$$

where $S(\Phi, \Phi^\ddagger)$ is an even function of ghost number zero. If $S = \sum_{n\geq 0} S^{(n)}(-i\hbar)^n$ is a formal power series in $\hbar$, then this is equivalent to the system of equations

$$\begin{aligned} 0 &= (S^{(0)}, S^{(0)}), \\ 0 &= \Delta S^{(0)} + (S^{(0)}, S^{(1)}), \\ 0 &= \Delta S^{(n-1)} + \sum_{k=0}^{n} \frac{1}{2}(S^{(k)}, S^{(n-k)}), \end{aligned} \tag{101}$$

where $n > 1$. So in the small-$\hbar$ limit the action must satisfy the *classical master equation* $(S,S) = 0$.

In order for a function $\mathcal{O}$ of the fields to define an element in cohomology, it must satisfy

$$0 = \delta_S \mathcal{O} = (S, \mathcal{O}) - i\hbar\Delta\mathcal{O}. \tag{102}$$

In this case, the expectation value of $\mathcal{O}$ is given by

$$\langle \mathcal{O} \rangle = \frac{1}{Z_\Psi} \int \mathcal{D}\Phi^A \, \exp\left[\frac{i}{\hbar} S\left(\Phi, \frac{\delta\Psi}{\delta\Phi}\right)\right] \mathcal{O}, \tag{103}$$

[12] It is worth keeping in mind that Grassmann grading and ghost grading are conceptually distinct; for simplicity here, we follow much of the literature in restricting to the case that all fields with even ghost number are Grassmann even (respectively odd).

[13] See e.g. Mnev (2019, §4.5).

where Ψ is an odd function of ghost number -1 and Z_Ψ is the normalization factor determined by the action $S(\Phi, \delta\Psi/\delta\Phi)$. That is, the path integral is taken over all values of the fields Φ^A, with the antifields fixed by

$$\Phi^\ddagger_A = \frac{\delta \Psi}{\delta \Phi^A} \tag{104}$$

For this path integral to give a good perturbation theory, the action $S(\Phi, \delta\Psi/\delta\Phi)$ must be nondegenerate. This constrains the possible choices of Ψ.

5.2 Example

The path integral quantization of the theory of §4.2 begins with the Lagrangian

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + A^{\ddagger\mu}_a D_\mu C^a + \frac{1}{2} g f^a{}_{bc} C^\ddagger_a C^b C^c - \bar{C}^{\ddagger a} B_a. \tag{105}$$

Here, as before, A^a_μ is a real, even, ghost number zero $\mathfrak{su}(n)$-valued 1-form, the field B_a is a real, even scalar of ghost number 0, the field C^a is a real, odd scalar of ghost number $+1$, and the field $\bar{C}_a$ is an imaginary, odd scalar of ghost number -1.

The differential and the antibracket it generates are

$$\begin{aligned} \Delta &= (-)^{\epsilon_A} \frac{\delta}{\delta \Phi^A} \frac{\delta}{\delta \Phi^\ddagger_A}, \\ (F, G) &= F \left(\frac{\overleftarrow{\delta}}{\delta \Phi^A} \frac{\delta}{\delta \Phi^\ddagger_A} - \frac{\overleftarrow{\delta}}{\delta \Phi^\ddagger_A} \frac{\delta}{\delta \Phi^A} \right) G, \end{aligned} \tag{106}$$

where Φ^A runs over A^a_μ, C^a, $\bar{C}_a$, and B_a where ϵ_A is the parity of Φ^A, and where

$$F \frac{\overleftarrow{\delta}}{\delta \Phi^A} = (-)^{(\epsilon_F + 1)\epsilon_A} \frac{\delta F}{\delta \Phi^A}. \tag{107}$$

It follows that

$$\begin{aligned} (S, C^a) &= -\frac{1}{2} g f^a{}_{bc} C^b C^c, & (S, B_a) &= 0, \\ (S, A^a_\mu) &= D_\mu C^a, & (S, \bar{C}_a) &= B_a, \\ (S, \bar{C}^{\ddagger a}) &= 0, & (S, B^{\ddagger a}) &= -\bar{C}^{\ddagger a}, \\ (S, A^{\ddagger\mu}_a) &= D_\nu F^{\nu\mu}_a - g f^i{}_{ja} A^{\ddagger\mu}_i C^j, & (S, C^\ddagger_a) &= -D_\mu A^{\ddagger\mu}_a + g f^i{}_{ja} C^\ddagger_i C^j, \end{aligned} \tag{108}$$

where

$$\begin{aligned} D_\mu A^{\ddagger\nu}_a &= \partial_\mu A^{\ddagger\nu}_a + g f^c{}_{ab} A^b_\mu A^{\ddagger\nu}_c, \\ D_\lambda F^{\mu\nu}_a &= \partial_\lambda F^{\mu\nu}_a + g f^c{}_{ab} A^b_\lambda F^{\mu\nu}_c, \end{aligned} \tag{109}$$

and so the classical master equation

$$(S, S) = 0 \tag{110}$$

is satisfied.

For the integration surface, we can take for example

$$\Psi = \int d^4x\, \bar{C}_a \left(\frac{1}{2}B_a - \partial^\mu A^a_\mu\right), \tag{111}$$

so that

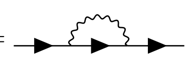
$$A^{\ddagger\mu}_a = \partial^\mu \bar{C}_a, \qquad C^\ddagger_a = 0, \qquad \bar{C}^{\ddagger a} = \frac{1}{2}B_a - \partial^\mu A^a_\mu, \tag{112}$$

recovering the Lagrangian of §4.2.

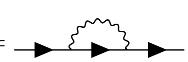
5.3 Renormalization

The antifield formalism is particularly useful in the analysis of renormalization. The quantum master equation (100) implies relations among the fields and couplings that can lead to cancellations among divergences in the theory. Such cancellations are desirable if renormalizability is taken as a guide to theory choice, but they are also of interest in analyzing the renormalization behavior of effective theories.[14]

As a simple illustration of the cancellations imlied by the quantum master equation, consider QED. In perturbative calculations of quantum expectation values, one typically encounters divergent integrals. For example, at one loop the self-energy of the electron goes as

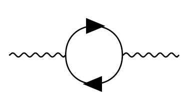
$i\Sigma_2 =$  $\sim \int d^4k\, \frac{1}{k^2}\frac{1}{\not{k}} \sim \Lambda$ (113)

with Λ a UV cutoff. This diverges in the large-Λ limit. To deal with this divergence in renormalized perturbation theory, one observes that differentiating the integrand lowers the degree of divergence by one, so that after Taylor expansion in the momentum p of the external electron the diagram goes as

$i\Sigma_2 =$  $\sim A\not{p} + B + \text{finite}$ (114)

with A and B functions of Λ that diverge in the large-Λ limit. These divergences are absorbed by renormalizing the parameters of the Lagrangian so as to absorb the infinite parts of A and B.

Symmetries can reduce the superficial degree of divergence of diagrams like this. For example, applying the same reasoning to the vacuum polarization would suggest that its infinite part goes as

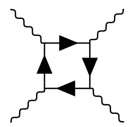
$i\Pi_2^{\mu\nu} =$  $\sim Ap^\mu p^\nu + Bp^2\eta^{\mu\nu} + \text{finite}$ (115)

[14] On which see Burgess (2020).

so that again there are two divergences to be absorbed. However, the symmetry of QED implies

$$p_\mu \Pi_2^{\mu\nu} = 0 \qquad \Longrightarrow \qquad A + B = 0 \tag{116}$$

and so the divergences of A and B must cancel, reducing the divergence of the vacuum polarization from quadratic to merely logarithmic. This remaining logarithmic divergence is then absorbed by renormalizing the parameters of the Lagrangian. Similarly, the superficial degree of divergence of light-by-light scattering at one loop is logarithmic



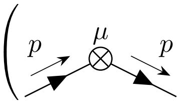
$$\sim \int d^4k \frac{1}{\not{k}\not{k}\not{k}\not{k}} \sim \Lambda^0 \tag{117}$$

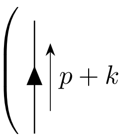
but the Ward identity implies that the diagram is actually convergent. These cancellations subvert naïve power counting arguments, and their source is in the symmetries of QED.

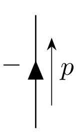
More generally, the symmetries of QED imply *Ward–Takahashi identities.* Letting $j^\mu(x) = \bar{\psi}(x)\,\gamma^\mu\psi(x)$ be the QED current, these identities are of the form

$$\frac{\partial}{\partial z^\mu}\langle j^\mu(z)\,\Phi^{A_1}(x_1)\cdots\Phi^{A_n}(x_n)\rangle = \sum_{i=1}^{n} Q_i\delta(z-x_i)\langle\Phi^{A_1}(x_1)\cdots\Phi^{A_n}(x_n)\rangle \tag{118}$$

for Q_i the charge of Φ^{A_i}. This implies for example that

$k_\mu \cdot \Bigg($ 

$\Bigg) = Q \Bigg($ 

$-$ 

$\Bigg),$ (119)

making the divergences in the electric charge and the photon field strength proportional.

For more general theories, define the generating function

$$Z_\Psi(J,K) = \int \mathcal{D}\Phi^A \, \exp\left[iS(\Phi, K + \tfrac{\delta\Psi}{\delta\Phi}) + i\int d^4x\, J_A(x)\,\Phi^A(x)\right]. \tag{120}$$

Under the change of integration variables

$$\Phi^A \mapsto \Phi^A + \left.\frac{\delta S}{\delta \Phi_A^\ddagger}\right|_{\Phi, K+\frac{\delta\Psi}{\delta\Phi}} \epsilon \tag{121}$$

with ϵ odd and real to give the right parity and reality properties, we have

$$\begin{aligned} \mathcal{D}\Phi^A &\mapsto \mathcal{D}\Phi^A\,(1 + (\Delta S)\epsilon)\,, \\ S &\mapsto S - i\frac{1}{2}(S,S)\epsilon, \end{aligned} \tag{122}$$

where the functions on the right are evaluated at $(\Phi, K + \frac{\delta\Psi}{\delta\Phi})$. These cancel by the quantum master equation, giving the identity

$$0 = \int d^4x\, J_A(x)\, \frac{\delta \log Z_\Psi}{\delta K_A(x)}. \tag{123}$$

This can be re-expressed in terms of the 1PI generating functional

$$\Gamma[\Phi^A] = -i \log Z_\Psi[J^{\Phi,K}, K] - \int d^4x\, J_A^{\Phi,K}(x)\, \Phi^A(x), \tag{124}$$

where $J^{\Phi,K}$ is defined implicitly by

$$-i\, \frac{\delta}{\delta J_A(x)} \log Z_\Psi[J,K]\bigg|_{J=J^{\Phi,K}} = \Phi^A(x). \tag{125}$$

In terms of these quantities, the identity above becomes

$$(\Gamma, \Gamma) = 0. \tag{126}$$

This is an equation relating 1PI diagrams, which is most convenient for BPHZ renormalization.[15]

6 Interpretation of non-positive quantization

Up to this point, we have seen the essential aspects of non-positive Hamiltonian quantization (§4) and non-positive Lagrangian quantization (§5). We turn in this section to interpreting what is going on in non-positive quantization.

Roughly, there are two perspectives on non-positive quantization. We call these the *pluralist perspective* and the *monist perspective*. The extent to which one thinks that non-positive quantization has conceptual merits will depend upon which of these perspectives one adopts.

From the pluralist perspective, different classical theories merit different quantization prescriptions. This perspective recognizes a wide variety of prescriptions, with the correct choice in any given case determined by features of the classical theory at hand. From this perspective, quantization prescriptions which make different use of ghosts and antifields are *per se* distinct.

From the monist perspective, there is essentially one quantization prescription that applies to every classical theory in the same way. This perspective recognizes a wider variety of classical theories than does the pluralist perspective, because the monist perspective sees the classical theory as involving more data—at the very least, it includes a specification of the observable Hilbert space, usually through the data of a charge Ω or a differential Δ. From this perspective, ghosts and antifields reflect features of the classical theory.

This section compares these two perspectives. For the sake of simplicity, we restrict attention to Lagrangian quantization methods; the Hamiltonian case is parallel.

[15] On which see e.g. Abers and Lee (1973).

6.1 Antifields

§5 explained how to quantize an action containing antifields. On both pluralist and monist perspectives, this antifield action comes from an action that does not contain antifields. The two perspectives differ in how they understand the relation of the two actions.

From both perspectives, the starting point is a classical action $S_{\mathrm{cl}}(\varphi)$ containing some fields φ^i $(i = 1, \ldots, n)$. The action $S_{\mathrm{cl}}(\varphi)$ is quantized in two steps: first one constructs an action $S(\Phi, \Phi^\ddagger)$ that depends on a family of fields Φ^A $(A = 1, \ldots, n)$ and antifield partners $\Phi^\ddagger_A$, then one quantizes $S(\Phi, \Phi^\ddagger)$ as in §5. The result is then taken to be the antifield quantization of the original action $S_{\mathrm{cl}}(\varphi)$.

For the sake of simplicity, consider the case in which $S_{\mathrm{cl}}(\varphi)$ is non-degenerate. We return to the degenerate case in §6.3.1.

From the pluralist perspective, the antifield action $S(\Phi, \Phi^\ddagger)$ is constrained by a set of postulates involving the classical action $S_{\mathrm{cl}}(\varphi)$. In our non-degenerate case, the fields Φ^A coincide with the fields φ, so the antifields $\Phi^\ddagger_A$ are all odd and have ghost number -1. The antifield action $S(\Phi, \Phi^\ddagger)$ must satisfy

$$S^{(0)}(\Phi, 0) = S_{\mathrm{cl}}(\varphi), \qquad \frac{1}{2}(S, S) - i\hbar\Delta S = 0, \tag{127}$$

where $S = \sum_{n\geq 0} S^{(n)}(-i\hbar)^n$. These postulates do not determine the antifield action $S(\Phi, \Phi^\ddagger)$ in general; for this, one needs further principles such as unitarity. However, in the non-degenerate case it is always possible to take $S(\Phi, \Phi^\ddagger) = S_{\mathrm{cl}}(\varphi)$ and $\Psi = 0$, which reproduces the ordinary path integral quantization of §2.2.

From the monist perspective, antifields are coordinates on a resolution of the space of solutions of $S_{\mathrm{cl}}(\varphi)$. The function $S(\varphi, \varphi^\ddagger) = S_{\mathrm{cl}}(\varphi)$ defines an operator $(S, -)$ on the algebra generated by φ^i and $\varphi^\ddagger_i$ that satisfies

$$(S, \varphi^i) = 0, \qquad (S, \varphi^\ddagger_i) = \frac{\delta S}{\delta \varphi^i}. \tag{128}$$

The first of these equations means that the φ^i define elements of the cohomology of $(S, -)$. The second means that the equation of motion for φ^i is a coboundary, hence vanishes in cohomology. As a result, the algebra of functions generated by φ^i and $\varphi^\ddagger_i$ and equipped with the differential is the homotopically correct algebra of functions on the critical locus $\{\delta S/\delta\varphi^i = 0\}$ (and exists even when the ordinary critical locus does not). It follows that the differential

$$\delta_S = (S, -) - i\hbar\Delta \tag{129}$$

is a perturbation of $(S, -)$, and so the usual techniques of deformation theory apply, giving expressions for the cohomology of δ_S in terms of the cohomology of $(S, -)$.

From the pluralist perspective, antifield quantization is an alternative quantization prescription to the path integral quantization of §2.2. In the nondegenerate case there is little use for it, since it effectively just reproduces ordinary path integral quantization.

From the monist perspective, antifields are a feature of the classical theory, and specifically its space of solutions. In many cases it is convenient to take the underived truncation, dropping the $\varphi_i^\ddagger$ (which are not cocycles) and setting $\delta S/\delta\varphi^i$ to zero (since it is a coboundary). But this eliminates potentially useful information. As noted in §5.3, antifields can be useful in the analysis of renormalization, and this is true even in the simple case of the Klein–Gordon field (Costello 2011; Costello and Gwilliam 2017; Gwilliam 2012).

6.2 Ghosts

As we have seen, non-positive quantization prescriptions are distinguished by their use of fields that violate the spin–statistics connection by exhibiting negative-norm modes. In both the Hamiltonian and Lagrangian prescriptions of §4 and §5 (respectively), these fields are graded by a ghost number, with antifields appearing in negative degree. The theories presented in those sections also exhibited fields in positive degree. The pluralist and monist perspectives differ over the relation of these fields to the classical theory.

From the pluralist perspective, ghosts are conceptually similar to antifields: one begins with a classical action $S_{\text{cl}}(\varphi)$ that contains no ghosts (or antifields) and constructs from this an action $S(\varphi, C)$ containing a set of ghosts C^a and determined by postulates involving $S_{\text{cl}}(\varphi)$. In fact, there are various sets of postulates for constructing ghostly actions, and the pluralist takes these to be different quantization prescriptions, each well adapted to different families of classical actions.

From the monist perspective, ghosts are coordinate functions on the classical configuration space. This configuration space is in general a smooth (higher) groupoid, and in a coordinate chart the functions in positive ghost degree are those that depend on the (higher) arrows in this groupoid.

6.2.1 Pluralism

From the pluralist perspective, there are various sets of postulates one can use to construct a ghostly action $S(\varphi, C)$ from a classical one. Here, we review two.

DeWitt–Faddeev–Popov (DFP) quantization arose in the study of unitarity in Einstein gravity and Yang–Mills theory. In the latter case, the classical action is given by the Lagrangian

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4}(F^a_{\mu\nu})^2, \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^a{}_{bc} A^b_\mu A^c_\nu, \tag{130}$$

where A^a_μ ($a = 1, \ldots, n^2-1$) is an $\mathfrak{su}(n)$-valued 1-form and $f^a{}_{bc}$ are the structure constants of $\mathfrak{su}(n)$. In the free theory limit $g \to 0$, this is essentially a theory of $n^2 - 1$ photons, and so the propagator for each of these fields should be the same as that of the photon. It is usually assumed that the photon is described equally well by the Fermi Lagrangian

$$\mathcal{L}_{\text{Fermi}} = -\frac{1}{2}(\partial^\nu A^\mu)(\partial_\nu A_\mu) \tag{131}$$

and so it should be possible to quantize Yang–Mills theory using the analogous Lagrangian for each of the $n^2 - 1$ fields. As we now review, this is roughly the reasoning that Feynman followed in the first paper to introduce ghosts in nonabelian Yang–Mills theory (Feynman 1963).

In his treatment, Feynman observed that quantizing the Fermi Yang–Mills Lagrangian gave a non-unitary theory. He observed furthermore that unitarity could be restored by introducing a family of charged odd scalar fields C^a that do not appear as external legs. The effect of such fields is to add loops with a factor of -1 (due to the odd parity), which cancel the violations of unitarity.

The earliest systematic treatments of ghosts treated them as a result of modifying the path integral measure. The Yang–Mills Lagrangian is invariant under the transformation

$$A^a_\mu \mapsto (\alpha \rhd A)^a_\mu := A^a_\mu + \partial_\mu \alpha^a + g f^a{}_{bc} A^b_\mu \alpha^c \tag{132}$$

for α^c any smooth $\mathfrak{su}(n)$-valued function on spacetime. This invariance leads to a divergent propagator. The idea of the DeWitt–Faddeev–Popov approach is to adjust the integrand for each value of A^a_μ so as to replace it with one that coincides with that of (say) the Fermi Lagrangian. This is effected by also integrating over $\mathfrak{g}$-valued functions with a modified path integral measure

$$\mathcal{D}A^a_\mu \mapsto \int \mathcal{D}\alpha\, \mathcal{D}A^a_\mu\, \delta(G(\alpha \rhd A)) \tag{133}$$

where G is some function like $G(A) = \partial^\mu A^a_\mu$. Things then go as in e.g. Dougherty (2021, §2.3).

Becchi–Rouet–Stora–Tyutin (BRST) quantization modifies the action directly, without modifying the path integral measure. We begin with a classical action $S_{\text{cl}}(\varphi)$ as usual. Here, we are to think of BRST symmetry as being different from gauge symmetry. We proceed as in DFP quantization with the observation that the action is invariant under the transformation

$$A^a_\mu \mapsto A^a_\mu + \partial_\mu \alpha^a + g f^a{}_{bc} A^b_\mu \alpha^c \tag{134}$$

for an $\mathfrak{su}(n)$-valued function α. On the basis of this we introduce a set of ghosts C^a and antighosts $\bar{C}_a$ with the same indices as the gauge transformation parameters.

The next step is to impose BRST symmetry on the action $S(\varphi, C, \bar{C})$. The idea is that there is a transformation $\varphi^i \mapsto \epsilon\, \delta\varphi^i$ with ϵ odd and imaginary given by replacing the parameter α^a in the transformation above with ϵC^a, and the transformation acts on the ghosts as $C^a \mapsto -\frac{1}{2} g f^a{}_{bc} C^b C^c$. The idea is to then look for an action of the form

$$S(\varphi, C, \bar{C}) = S_{\mathrm{cl}}(\varphi) + S_{\mathrm{gauge}}(\varphi) + S_{\mathrm{ghost}}(\varphi, C, \bar{C}), \tag{135}$$

where $S_{\mathrm{gauge}}(\varphi)$ is a term making the sum of the first two terms non-degenerate. The term $S_{\mathrm{ghost}}(\varphi, C, \bar{C})$ is then supposed to be determined by BRST symmetry.

Since $S(\varphi, C, \bar{C})$ is non-degenerate, we can proceed with ordinary path integral quantization. Proponents of this approach then restrict attention to BRST-invariant observables—although we note from a conceptual point of view this restriction is often not justified.

This is a generalization of DFP quantization in the sense that the latter systematically produces an action with BRST symmetry. The piece $S_{\mathrm{gauge}}(\varphi)$ is $G(\varphi)$, and the BRST transformation of $\bar{C}_a$ is such that it cancels the BRST transformation of $G(\varphi)$.

There are two benefits to this. First, as discussed in §5.3, symmetries of the classical action lead to cancellations among divergences in the renormalization of the theory. This means that a theory with BRST symmetry is more likely to be 'good', in the sense of leading to said cancellations. Put somewhat differently: this means that BRST symmetry constrains the terms in the effective Lagrangian.

The second benefit of BRST quantization is its generality. First, DFP quantization can only give terms bilinear in C^a and $\bar{C}_a$, since its additions to the action come from functional determinants. This means we can only use a $G(A)$ if it is linear in A and its derivatives. But in BRST quantization we can make $S_{\mathrm{gauge}}(A)$ anything we like, as long as we can find a $S_{\mathrm{ghost}}(A, C, \bar{C})$ that gives BRST invariance. Moreover, we can apply BRST quantization to higher gauge theories.[16]

6.2.2 Monism

From the monist perspective, (higher) ghosts are a feature of a theory whose classical configuration space is a (higher) groupoid.

For example, the configuration space of Yang–Mills theory is the stack of principal SU(n)-bundles with connection. In perturbative quantization, we choose some solution—the flat connection on the trivial $SU(n)$-bundle, say—and take the infinitesimal neighborhood of that solution, which will be a Lie algebroid. Expressing this Lie algebroid in terms of its Chevalley–Eilenberg complex,[17] we find that it is generated by coordinates A^a_μ and C^a in degrees zero and one, respectively, with differential

$$\delta A^a_\mu = D_\mu C^a, \qquad \delta C^a = -\frac{1}{2} g f^a{}_{bc} C^b C^c. \tag{136}$$

[16] See e.g. Borsten et al. (2025).
[17] See e.g. nLab authors (2026).

Any function S on the full configuration space induces a function that we'll also call S on the infinitesimal neighborhood of any point, which in this case can be represented by a function $S(A, C)$ such that $\delta S = 0$.

From this perspective, BRST quantization involves a choice of a different representative of the cohomology class of S, one that is non-degenerate at the solution about which we're perturbing. Another representative will be of the form $S + \delta\Psi$ where Ψ is an element of degree -1. Since the complex under consideration is concentrated in non-negative degree, we first move to a quasi-isomorphic complex by adding generators $\bar{C}_a$ and B_a in degrees -1 and 0, respectively, such that

$$\delta\bar{C}_a = B_a, \qquad \delta B_a = 0. \tag{137}$$

Then we can choose, for example, the Ψ at the end of §5.2, and we're off to the races.

6.3 Combining antifields and ghosts

From the pluralist point of view, at least, antifields are unnecessary in the non-degenerate case. Their real use comes in the degenerate case, where further postulates relate the classical action $S_{\rm cl}(\varphi)$ to the antifield action $S(\Phi, \Phi^\ddagger)$, which is now nontrivially different.

For the sake of simplicity we consider the case where both the fields in $S_{\rm cl}$ and its Noether identities have even parity.

6.3.1 Pluralism

Batalin–Vilkovisky (BV) quantization applies to a classical action $S_{\rm cl}(\varphi)$ and to a choice of stationary point φ_0 of $S_{\rm cl}$ about which one is perturbing. These must satisfy the following postulate: there is a neighborhood of the stationary point φ_0 on which $S_{\rm cl}$ satisfies Noether identities

$$\begin{aligned} &\frac{\delta S_{\rm cl}}{\delta\varphi^i} R^i{}_\alpha = 0, && \alpha = 1, \ldots, m, \\ &\operatorname{rank} R^i{}_\alpha\big|_{\varphi=\varphi_0} = m, && \operatorname{rank} \left.\frac{\delta^2 S_{\rm cl}}{\delta\varphi^i\,\delta\varphi^j}\right|_{\varphi=\varphi_0} = n - m, \end{aligned} \tag{138}$$

where $R^i{}_\alpha$ is an even function of the fields φ.

BV quantization involves a set of fields Φ^A ($A = 1, \ldots, N$), an action $S(\Phi, \Phi^\ddagger)$, a stationary point $(\Phi_0, \Phi_0^\ddagger)$ about which one perturbatively expands, and a choice of values for the antifields in the path integral. These are constrained by five postulates:

1. The action S must satisfy the quantum master equation

$$\frac{1}{2}(S, S) - i\hbar\Delta S = 0. \tag{139}$$

Writing S as a series in $\hbar$, this is equivalent to the set of equations

$$\begin{aligned}
S(\Phi, \Phi^{\ddagger}) &= \sum_{k\geq 0} (-i\hbar)^k S^{(k)}(\Phi, \Phi^{\ddagger}), \\
(S^{(0)}, S^{(0)}) &= 0, \\
(S^{(1)}, S^{(0)}) &= \Delta S^{(0)}, \\
(S^{(k)}, S^{(0)}) &= \Delta S^{(k-1)} - \frac{1}{2}\sum_{p=1}^{k-1} (S^{(p)}, S^{(k-p)}) \qquad k > 1.
\end{aligned} \tag{140}$$

2. The fields φ^i are among the Φ^A and are assigned ghost number zero, and the classical (that is, $\hbar^0$) part $S^{(0)}(\Phi, \Phi^{\ddagger})$ of the action must reduce to $S_{\mathrm{cl}}(\varphi)$ on the surface $\Phi^{\ddagger} = 0$:

$$S^{(0)}(\Phi, 0) = S_{\mathrm{cl}}(\varphi). \tag{141}$$

3. The fields Φ^A include a set of odd fields C^α $(\alpha = 1, \ldots, m)$ of ghost number 1, and

$$\left. \frac{\delta S^{(0)}}{\delta \varphi^{\ddagger}_i} \frac{\overleftarrow{\delta}}{\delta C^\alpha} \right|_{\Phi^{\ddagger}=0} = R^i{}_\alpha. \tag{142}$$

4. Write $\Phi_{\min} = \{\varphi, C\}$ for the minimal set of fields required by the previous two postulates. The non-minimal fields in Φ come in pairs (Λ^A, Π^A) of opposite parity and ghost numbers $\mathrm{gh}\,\Pi^A = \mathrm{gh}\,\Lambda^A + 1$, and the action depends on these non-minimal fields as

$$S(\Phi, \Phi) = S_{\min}(\Phi_{\min}, \Phi^{\ddagger}_{\min}) + \Lambda^{\ddagger}_A \Pi^A. \tag{143}$$

It follows that S satisfies the quantum master equation if and only if $S_{\min}$ satisfies the quantum master equation in the minimal sector.

5. After choosing values for the antifields in the path integral, the resulting action must be non-degenerate. This requires first that the Hessian of $S^{(0)}(\Phi, \Phi^{\ddagger})$ at the stationary point $(\Phi_0, \Phi^{\ddagger}_0)$ have rank N. The constraint on the value of the antifields in the path integral is most simply expressed by adding to Φ sets of odd fields $\bar{C}_\alpha$ of ghost number -1 and even fields B_α of ghost number 0 $(\alpha = 1, \ldots, m)$ and choosing a function Ψ satisfying the condition on the left

$$\det\left(\frac{\delta \Psi}{\delta \bar{C}_\alpha} \frac{\overleftarrow{\delta}}{\delta \varphi^i} R^i{}_\beta \right) \neq 0, \qquad\qquad \Phi^{\ddagger}_A = \frac{\delta \Psi}{\delta \Phi^A}, \tag{144}$$

and then setting $\Phi^{\ddagger}_A$ to the value on the right. Some such Ψ always exists in light of rank assumptions on $R^i{}_\alpha$ and S_{cl}.

These five postulates amount to a system of equations constraining the action $S(\Phi, \Phi^\ddagger)$ and the integration surface Ψ. The first postulate inductively constrains $S^{(k)}$ for $k > 0$ and imposes the classical master equation on $S^{(0)}$. In the $S^{(0)}$ base case, consider the expansion

$$S^{(0)}(\Phi, \Phi^\ddagger) = \sum_{p \geq 0} \frac{1}{p!} \Phi^\ddagger_{A_1} \cdots \Phi^\ddagger_{A_p} F^{A_1 \cdots A_p}(\Phi) \tag{145}$$

in antifield degree, where the $F^{A_1 \cdots A_p}$ are (anti)symmetric in the appropriate indices. The second and third postulates then imply

$$F = S_{\mathrm{cl}}, \qquad F^i = R^i{}_\alpha C^\alpha, \tag{146}$$

respectively, while the classical master equation gives a set of equations inductively constraining the coefficients of higher degree. In the minimal sector the leading such equations are

$$\begin{aligned} 0 &= \frac{\delta R^i{}_\alpha}{\delta \varphi^j} R^j{}_\beta - \frac{\delta R^i{}_\beta}{\delta \varphi^j} R^j{}_\alpha + R^i{}_\gamma F^\gamma{}_{\alpha\beta} - F^{ij}{}_{\alpha\beta} \frac{\delta S_{\mathrm{cl}}}{\delta \varphi^j}, \\ 0 &= \frac{\delta F^\gamma{}_{\rho\sigma}}{\delta \varphi^i} R^i{}_\tau + \frac{\delta F^\gamma{}_{\tau\rho}}{\delta \varphi^i} R^i{}_\sigma + \frac{\delta F^\gamma{}_{\sigma\tau}}{\delta \varphi^i} R^i{}_\rho \\ &\quad + F^\gamma{}_{\rho\beta} F^\beta{}_{\sigma\tau} + F^\gamma{}_{\tau\beta} F^\beta{}_{\rho\sigma} + F^\gamma{}_{\sigma\beta} F^\beta{}_{\tau\rho} \\ &\quad + F^{\gamma j}{}_{\rho\sigma\tau} \frac{\delta S_{\mathrm{cl}}}{\delta \varphi^j}, \end{aligned} \tag{147}$$

where by ghost number considerations the coefficients are of the form

$$\begin{gathered} F^\gamma = \frac{1}{2!} F^\gamma{}_{\alpha\beta} C^\alpha C^\beta, \qquad F^{ij} = \frac{1}{2!} F^{ij}{}_{\alpha\beta} C^\alpha C^\beta, \\ F^{\gamma j} = \frac{1}{3!} F^{\gamma j}{}_{\rho\sigma\tau} C^\rho C^\sigma C^\tau, \end{gathered} \tag{148}$$

with the coefficients on the right totally antisymmetric in the lower indices.

For example, in the case of Yang–Mills theory, we have

$$R^a{}_{\mu c}(x, y) = \delta^a{}_c \frac{\partial}{\partial x^\mu} \delta(x, y) + g f^a{}_{bc} A^b_\mu(x) \delta(x, y). \tag{149}$$

The commutator of these Noether identities is

$$\frac{\delta R^a{}_{\mu c}}{\delta A^b_\nu} R^b{}_{\nu d} - \frac{\delta R^a{}_{\mu d}}{\delta A^b_\nu} R^b{}_{\nu c} = -g R^a{}_{\mu b} f^b{}_{cd}, \tag{150}$$

and so the leading part of the classical master equation is

$$0 = -g R^a{}_{\mu b} f^b{}_{cd} + R^a{}_{\mu b} F^b{}_{cd} - F^a{}_\mu{}^b{}_{\nu cd} \frac{\delta S_{\mathrm{cl}}}{\delta A^b_\nu}, \tag{151}$$

which can be solved by taking $F^{b}{}_{cd} = gf^{b}{}_{cd}$ and $F^{a}{}_{\mu}{}^{b}{}_{\nu cd} = 0$. With these choices, the next equation is

$$0 = g^2 f^{a}{}_{bc} f^{c}{}_{de} + g^2 f^{a}{}_{ec} f^{c}{}_{bd} + g^2 f^{a}{}_{dc} f^{c}{}_{eb} + F^{ac}{}_{\mu bde} \frac{\delta S_{\mathrm{cl}}}{\delta A^{c}_{\mu}}. \tag{152}$$

The first three terms vanish by the Jacobi identity, and the last vanishes if we take $F^{ac}{}_{\mu bde} = 0$. The full classical master equation is then satisfied by taking $F^{A_1 \cdots A_p} = 0$ for $p > 1$, and we recover the minimal sector of the Lagrangian of §5.2.

More generally, suppose that $S_{\mathrm{cl}}(\varphi)$ is the sort of action to which BRST quantization applies. Then we introduce a set of ghosts C^a $(a = 1, \ldots, m)$ along with fields $\bar{C}_a$ and B_a, giving a set of fields Φ^A $(A = 1, \ldots, n + 3m)$, and there is a transformation $\Phi \mapsto \epsilon\, \delta\Phi$ with ϵ odd such that

$$\delta^2 = 0, \qquad \delta S_{\mathrm{cl}} = 0, \qquad \frac{\delta}{\delta \Phi^A} \delta \Phi^A = 0. \tag{153}$$

These equations imply that

$$S(\Phi, \Phi^{\ddagger}) = S_{\mathrm{cl}}(\varphi) - (\delta \Phi^A)\, \Phi^{\ddagger}_A \tag{154}$$

satisfies the classical and quantum master equations.

This example illustrates that BV quantization is more general than BRST quantization. Any BV action obtained from a BRST quantization in this way satisfies $F^{ij} = 0$. In this case, say that the theory has a 'closed algebra'; if $F^{ij} \neq 0$, say that the theory has an 'open algebra'. BV quantization applies to open algebras, while BRST quantization leads to theories with closed algebras.

BV quantization has various advantages; here we list three. First, since BRST quantization requires that the gauge algebra close off-shell, it is unsuitable for the quantization of theories in which this is not the case—e.g., supergravity theories.[18]

Second, in BV quantization, we know that gauge invariance at the quantum level is equivalent to satisfaction of the quantum master equation. As such, we know that anomalies are obstructions to satisfaction of this equation—or, put another way, if we know that the equation is satisfied, then we know that we have an anomaly-free QFT. One example of this is in string field theory (roughly, the second-quantised version of string theory), actions for which were constructed and subsequently shown to satisfy the quantum master equation, assuring theorists that the theory was consistent and anomaly-free. (See Krátký et al. (2025) for philosophical discussion of this case.)

A third virtue of BV quantization is the treatment of renormalization that it offers, as we have already seen in §5.3.

6.3.2 Monism

For the monist, there's nothing particularly new about combining antifields and ghosts. The recipe is always the same. Start with an action S on a classical

[18]See Dall'Agata and Zagermann (2021).

configuration space—which will have (higher) ghosts, if it's a (higher) groupoid. Form the derived critical locus of S. Choose a point of this critical locus and a coordinate chart $(\Phi, \Phi^{\ddagger})$ around this point. Add a perturbative deformation $-i\hbar\Delta$, *et voilà*.

7 Conclusion

As discussed in §1, although canonical quantization is known to philosophers, even path integral quantization is rarely discussed in the philosophical literature in any significant detail. In this article, we have clearly elaborated the methodologies behind both positive Hamiltonian and positive Lagrangian (i.e., path integral) quantization schemes, before turning to the main event: a detailed study of *non-positive* Hamiltonian and Lagrangian quantization.

Despite its being the bread-and-butter of quantization in modern theoretical physics, non-positive quantization is essentially unknown to philosophers.[19] Moreover, presentations of non-positive quantization in the physics literature often treat these schemes in an *ad hoc* way, running together conceptually distinct notions such as ghosts and antifields, and e.g. using terms such as 'BRST' as catch-alls for all of the methodologies of non-positive quantization.

Our purpose in this article has been to clarify this terrain in a way which should be useful to both physicists and philosophers. We have distinguished non-positive Hamiltonian and non-positive Lagrangian quantization, and have been explicit about the distinct origins of devices such as ghosts and antifields. We have, moreover, distinguished 'pluralist' and 'monist' approaches to non-positive quantization, and have pinpointed specific approaches to quantization within each of these camps. We have substantiated a common refrain in the high energy physics literature that BV quantization offers the most general and powerful approach to quantization, and have justified this by identifying a number of specific advantages of the BV approach.

Stepping back, this article should be taken as an invitation to philosophers to get their teeth into important and conceptually delicate (and sometimes confusingly presented) aspects of modern theoretical physics—of which quantization is of course but one example. If philosophy of physics is to bear fruits which might be relevant to first-order physics, then it is incumbent on philosophers to study the kinds of issues in which physicists are interested, rather than focussing on simple toy cases (here, e.g., positive Hamiltonian quantization).[20] And in any case, aside from the payoffs of this work for clarity on the first-order physics, it should also open up further interesting philosophical work. For example, authors such as Feintzeig (2024, 2025) have used formal methods (of category theory etc.) to assess the extent to which there is continuity of structure between a classical theory and some quantization thereof; it would of course be inter-

[19] Again, with some exceptions, e.g. Redhead (2002, §8).

[20] We of course don't mean to imply that it is the purpose of philosophy of physics to bear fruits relevant to first-order physics. Nevertheless, *ceteris paribus* we take these fruits to be a good thing.

esting to investigate such questions in the context of the quantization schemes which we have studied in this article.

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