The Gauge-Invariant Higgs Mechanism Is Incomplete: From Angular Momentum to Flux Quantization

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March 7, 2026

Abstract

Gauge-invariant reformulations of the Higgs mechanism, such as Struyve's treatment of the Abelian Higgs model, eliminate gauge-dependent quantities in favor of local gauge-invariant variables: the Higgs magnitude $\rho = |\phi|$ and the composite vector field $B_\mu = A_\mu + \frac{1}{e}\partial_\mu\theta$. While mathematically equivalent to the standard theory for local calculations, these formulations suffer from a fundamental incompleteness that exactly parallels the well-known incompleteness of the Madelung hydrodynamic formulation of quantum mechanics. Just as the Madelung equations require an externally imposed quantization condition $m\oint \mathbf{v}\cdot d\mathbf{r} + e\Phi = 2\pi n$ to enforce wave function single-valuedness, Struyve's formulation requires an analogous condition $\oint_C B_\mu dx^\mu = \Phi + \frac{2\pi n}{e}$. This condition involves the global circulation of B_μ and the magnetic flux Φ; it cannot be derived from the local equations of motion for ρ and B_μ. We demonstrate that this incompleteness already manifests for a single Higgs boson in a state with orbital angular momentum, where the circulation of B_μ around the axis must be quantized even in the absence of vortices. Moreover, the problem persists and deepens for exotic topological defects such as Nielsen-Olesen vortices, where flux quantization requires the same externally imposed condition. Our analysis suggests that the gauge potential A_μ and the phase θ remain indispensable for a complete, self-contained description of the Higgs mechanism, carrying essential global and topological information that cannot be fully encoded in local gauge-invariant composites.

1 Introduction

The Higgs mechanism is a cornerstone of the Standard Model, explaining how gauge bosons and fermions acquire mass through the spontaneous breaking of gauge symmetry [3, 5]. However, the status of gauge symmetry as a mere mathematical redundancy has led to persistent philosophical and physical questions about whether the mechanism can be understood without invoking gauge-dependent quantities [2, 8].

To address these questions, gauge-invariant reformulations of the Abelian Higgs model have been developed. As shown by Higgs [6] and Kibble [7], and articulated in detail by Struyve [12, 13], one can decompose the complex scalar field $\phi = \frac{\rho}{\sqrt{2}}e^{i\theta}$ and define a gauge-invariant vector field $B_\mu = A_\mu + \frac{1}{e}\partial_\mu\theta$. The resulting Lagrangian is expressed entirely in terms of the local, gauge-invariant variables ρ and B_μ, and the ground state is unique. This formulation achieves mass generation without invoking spontaneous symmetry breaking, and it appears to provide a fully gauge-invariant description of the theory.

The existence of such formulations raises a profound question: Is the gauge potential A_μ merely a convenient mathematical tool, or is it a physically real field? The Aharonov-Bohm (AB) effect famously provides a test case for this question in quantum mechanics [1]. Gauge-invariant explanations of the AB effect, which attribute the phase shift to the magnetic flux $\Phi = \oint \mathbf{A} \cdot d\mathbf{r}$, are successful in predicting the final outcome. However, as argued in [4], they fail to explain the *process* of phase accumulation in the generalized AB effect, where the flux varies in time. This leads to a no-go theorem: any purely gauge-invariant account cannot provide a complete explanation of the effect.

In this paper, we develop a parallel argument for Struyve's gauge-invariant reformulation of the Higgs mechanism. Our central thesis is that this formulation suffers from a fundamental incompleteness that exactly mirrors the well-known incompleteness of the Madelung hydrodynamic formulation of quantum mechanics [9]. Just as the Madelung equations require an externally imposed quantization condition to enforce wave function single-valuedness and recover quantized angular momentum, Struyve's formulation requires an analogous condition on the circulation of B_μ and the magnetic flux Φ. This condition is not derivable from the local equations of motion; it must be imposed externally, revealing that the formulation is not a complete physical theory.

Crucially, this incompleteness is not limited to exotic topological configurations like vortices. It already manifests for a single Higgs boson in a state with orbital angular momentum, such as a p-wave state. In such a state, the circulation of B_μ around the axis must be quantized, even though ρ can be everywhere non-zero and smooth. Struyve's local equations cannot enforce this quantization; it must be imposed externally.

2 The Abelian Higgs Model and Its Gauge-Invariant Reformulation

2.1 The Standard Formulation

The Abelian Higgs model describes a complex scalar field ϕ coupled to a $U(1)$ gauge field A_μ. The Lagrangian density is [11]:

$$\mathcal{L} = (D_\mu\phi)^*(D^\mu\phi) - V(\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{1}$$

where $D_\mu\phi = (\partial_\mu + ieA_\mu)\phi$ is the covariant derivative, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and the Higgs potential is

$$V(\phi) = \lambda\left(|\phi|^2 - \frac{v^2}{2}\right)^2, \tag{2}$$

with $\lambda > 0$. The potential's minimum is not at $\phi = 0$ but on the circle $|\phi| = v/\sqrt{2}$. Choosing a specific vacuum state breaks the $U(1)$ gauge symmetry, and expanding around this vacuum reveals that the gauge boson acquires a mass $m_A = ev$, while the remaining scalar degree of freedom becomes the massive Higgs boson.

2.2 Struyve's Gauge-Invariant Reformulation

As detailed by Struyve [12, 13], one can reformulate the Abelian Higgs model using only gauge-invariant variables.

We begin by decomposing the complex scalar field into its magnitude and phase:

$$\phi(x) = \frac{\rho(x)}{\sqrt{2}} e^{i\theta(x)}, \quad \rho(x) = \sqrt{2|\phi(x)|^2}. \tag{3}$$

Under a general gauge transformation,

$$\phi(x) \to e^{i\alpha(x)}\phi(x), \quad A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x), \tag{4}$$

the phase transforms inhomogeneously: $\theta(x) \to \theta(x) + \alpha(x)$. This transformation property is the key to constructing gauge-invariant quantities. The central idea is to combine A_μ with the gradient of θ to form a gauge-invariant vector field:

$$B_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu \theta(x). \tag{5}$$

To verify gauge invariance, apply the transformations above:

$$B_\mu \to \left(A_\mu - \frac{1}{e}\partial_\mu \alpha\right) + \frac{1}{e}\partial_\mu(\theta + \alpha) = A_\mu + \frac{1}{e}\partial_\mu \theta = B_\mu. \tag{6}$$

Thus, B_μ is indeed gauge-invariant by construction.

The physical interpretation of B_μ becomes transparent when we consider a specific gauge choice. The unitary gauge is defined by the condition $\theta(x) = 0$, which eliminates the phase degree of freedom. This gauge is achieved by choosing the gauge transformation parameter

$$\alpha(x) = -\theta(x), \tag{7}$$

where $\theta(x)$ is the original phase field. Under this transformation:

$$\phi(x) \to \frac{\rho(x)}{\sqrt{2}} e^{i(\theta(x)-\theta(x))} = \frac{\rho(x)}{\sqrt{2}}, \tag{8}$$

$$A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu \theta(x). \tag{9}$$

In this unitary gauge, the gauge field becomes precisely $A_\mu^{\text{unitary}} = A_\mu + \frac{1}{e}\partial_\mu\theta$. Comparing with (5), we see that

$$B_\mu(x) = A_\mu^{\text{unitary}}(x). \tag{10}$$

That is, B_μ is exactly the gauge field in the unitary gauge, but expressed in a gauge-invariant manner without having fixed the gauge. The magnitude field ρ remains unchanged, as it is gauge-invariant from the outset.

We now express the entire Lagrangian in terms of ρ and B_μ. The covariant derivative becomes

$$D_\mu \phi = \frac{1}{\sqrt{2}} e^{i\theta} \left(\partial_\mu \rho + ie\rho B_\mu\right). \tag{11}$$

The field strength simplifies because the θ terms cancel:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu \left(B_\nu - \frac{1}{e}\partial_\nu \theta\right) - \partial_\nu \left(B_\mu - \frac{1}{e}\partial_\mu \theta\right) = \partial_\mu B_\nu - \partial_\nu B_\mu \equiv B_{\mu\nu}. \tag{12}$$

Substituting these expressions into the original Lagrangian (1) yields

$$\mathcal{L}_{\mathrm{GI}} = \frac{1}{2}(\partial_\mu \rho)^2 - \frac{\lambda}{4}\left(\rho^2 - v^2\right)^2 + \frac{1}{2}e^2\rho^2 B_\mu B^\mu - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}, \tag{13}$$

The Lagrangian $\mathcal{L}_{\mathrm{GI}}$ depends only on the gauge-invariant fields $\rho(x)$ and $B_\mu(x)$. The ground state of this theory is unique: $\rho = v$ and $B_\mu = 0$. Expanding around this ground state, $\rho = v + \eta$, the term $\frac{1}{2}e^2\rho^2 B_\mu B^\mu$ generates a mass $m_B = ev$ for the B_μ field, while η represents the massive Higgs boson. The phase θ has been completely eliminated from the formalism, having been absorbed into the definition of B_μ.

This reformulation is mathematically equivalent to the original theory for all local, perturbative calculations. It demonstrates that mass generation can be described without invoking spontaneous symmetry breaking and without gauge-dependent quantities. However, as we shall demonstrate in the following sections, this reformulation is not a complete physical theory. The elimination of the phase θ comes at a cost: the global and topological information carried by θ must be re-introduced through global constraints that cannot be derived from the local equations of motion for ρ and B_μ.

3 The Parallel with Madelung's Hydrodynamic Formulation

3.1 Madelung's Formulation of Quantum Mechanics

In Madelung's hydrodynamic formulation [9], the wave function of a charged particle in an electromagnetic field is written in polar form, $\psi = \sqrt{\rho}e^{iS}$ (with $\hbar = 1$). This defines two real fields: the probability density $\rho = |\psi|^2$ and the phase S. The velocity field is derived from the phase and the vector potential $\mathbf{A}$ as

$$\mathbf{v} = \frac{1}{m}(\nabla S - e\mathbf{A}). \tag{14}$$

Substituting this form into the Schrödinger equation for a particle of mass m and charge e in an external electromagnetic potential $(A_0, \mathbf{A})$ yields two coupled real equations.

The first is the continuity equation, which governs the conservation of probability:

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{15}$$

The second is a Hamilton-Jacobi-type equation for the velocity field $\mathbf{v}$:

$$m\left(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right) = -\nabla Q + e(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{16}$$

where $\mathbf{E} = -\nabla A_0 - \partial_t \mathbf{A}$ and $\mathbf{B} = \nabla \times \mathbf{A}$ are the electromagnetic fields, and the quantum potential Q is given by

$$Q = -\frac{1}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} = -\frac{1}{4m}\left(\frac{\nabla^2\rho}{\rho} - \frac{1}{2}\frac{(\nabla\rho)^2}{\rho^2}\right). \tag{17}$$

These two equations, (15) and (16), are local and involve only gauge-invariant quantities ρ and $\mathbf{v}$.

3.2 The Quantization Condition in Madelung Theory

The wave function must be single-valued. This imposes a condition on the phase S: for any closed loop C,

$$\oint_C \nabla S \cdot d\mathbf{r} = 2\pi n, \quad n \in \mathbb{Z}. \tag{18}$$

In terms of the velocity field $\mathbf{v}$ and the vector potential, this becomes

$$m \oint_C \mathbf{v} \cdot d\mathbf{r} + e \oint_C \mathbf{A} \cdot d\mathbf{r} = 2\pi n. \tag{19}$$

Defining the magnetic flux $\Phi = \oint_C \mathbf{A} \cdot d\mathbf{r}$, we have

$$m \oint_C \mathbf{v} \cdot d\mathbf{r} = 2\pi n - e\Phi. \tag{20}$$

This condition is not derivable from the local Madelung equations. The equations admit solutions for any smooth $\mathbf{v}$, but only those satisfying (20) correspond to single-valued wave functions. Crucially, the condition involves two global quantities: the circulation of $\mathbf{v}$ around C and the magnetic flux Φ. Neither can be determined from local data along the path alone. This is the sense in which the Madelung formulation is incomplete: it requires an externally imposed, global constraint to recover the full content of quantum mechanics [17, 18, 14, 15].

3.3 The Quantization Condition in Struyve's Formulation of the Higgs Mechanism

In Struyve's gauge-invariant reformulation [12], the fundamental variables are ρ and $B_\mu = A_\mu + \frac{1}{e}\partial_\mu\theta$. Their equations of motion, derived from $\mathcal{L}_{\text{GI}}$, are local:

$$\Box\rho + \lambda\rho(\rho^2 - v^2) - e^2 B_\mu B^\mu \rho = 0, \tag{21}$$

$$\partial^\nu B_{\mu\nu} + e^2\rho^2 B_\mu = 0. \tag{22}$$

The Higgs field ϕ must be single-valued. This requires that for any closed loop C,

$$\oint_C \partial_\mu\theta \, dx^\mu = 2\pi n, \quad n \in \mathbb{Z}. \tag{23}$$

Expressing this in terms of B_μ and A_μ:

$$\oint_C B_\mu dx^\mu = \oint_C A_\mu dx^\mu + \frac{1}{e}\oint_C \partial_\mu\theta dx^\mu = \Phi + \frac{2\pi n}{e}, \tag{24}$$

where $\Phi = \oint_C A_\mu dx^\mu$ is the magnetic flux through a surface bounded by C.

Equation (24) is the exact analogue of (20). It is not derivable from the local equations (21)-(22). The local equations admit solutions for any smooth B_μ, but only those satisfying (24) correspond to single-valued Higgs fields. Moreover, the condition involves two global quantities: the circulation of B_μ around C and the magnetic flux Φ. Neither is locally determined.

3.4 The Structural Parallel

Aspect	Madelung (QM)	Struyve (Higgs)
Fundamental variables	ρ, $\mathbf{v} = \frac{1}{m}(\nabla S - e\mathbf{A})$	ρ, $B_\mu = A_\mu + \frac{1}{e}\partial_\mu\theta$
Local equations	Eqs. (15)-(16)	Eqs. (21)-(22)
Single-valuedness condition	$m\oint \mathbf{v}\cdot d\mathbf{r} + e\Phi = 2\pi n$	$\oint B_\mu dx^\mu = \Phi + \frac{2\pi n}{e}$
global quantities involved	$\oint \mathbf{v}\cdot d\mathbf{r}$, Φ	$\oint B_\mu dx^\mu$, Φ
Derivable from local equations?	No	No
Status	Externally imposed	Externally imposed

Table 1: Parallel between Madelung's formulation of QM and Struyve's formulation of the Abelian Higgs model.

The above structural parallel is exact. Both formulations achieve local dynamics at the cost of losing the ability to enforce global topological constraints. The quantization condition must be added by hand, and it inherently involves global information.

4 Incompleteness Without Topology: A Single Higgs Boson with Angular Momentum

A common misconception in discussions of gauge-invariant reformulations is that problems related to the phase θ only arise in topologically non-trivial configurations involving vortices or defects. In this section, we demonstrate that this is false. Struyve's formulation suffers from a fundamental incompleteness even for the simplest possible excitation of the Higgs field: a single Higgs boson propagating in empty space with definite orbital angular momentum. This example serves as a crucial bridge between the abstract quantization condition derived in the previous section and the topological defects to be discussed in the next section. It reveals that the incompleteness is not about exotic topology but about the very nature of phase in quantum theory.

4.1 Physical Description of a Higgs Boson with Angular Momentum

After spontaneous symmetry breaking, the Higgs field is expanded around its vacuum expectation value. In the unitary gauge ($\theta = 0$), we have $\phi = \frac{1}{\sqrt{2}}(v + h)$, where $h(x)$ is the physical Higgs field whose quanta are particles. Like any quantum field in three-dimensional space, the Higgs field can be excited into states with definite orbital angular momentum.

Consider a single Higgs boson in a p-wave state with azimuthal quantum number $m = 1$. In a first-quantized description, its wave function has the form

$$\psi_h(\mathbf{r}) = f(r,\theta)e^{i\phi}, \tag{25}$$

where (r, θ, ϕ) are spherical coordinates and $f(r, \theta)$ is a smooth, non-vanishing function (e.g., a Gaussian wave packet centered away from the origin). Returning to the original field variables before gauge fixing, this configuration corresponds to a Higgs field $\phi = \frac{1}{\sqrt{2}}(v + h)e^{i\theta}$ where the phase θ winds by 2π around the z-axis.

Crucially, this configuration has the following properties:

- The magnitude $\rho = |\phi|$ is everywhere non-zero and smooth. There are no vortices, no zeros, and no singularities.
- The domain is simply-connected: $\mathbb{R}^3$ is topologically trivial.
- The phase winding is not associated with any topological defect—it is a feature of the quantum state itself.

Despite the absence of any topological defects, the single-valuedness of the Higgs field ϕ imposes a non-trivial condition. Since $\phi = \frac{\rho}{\sqrt{2}}e^{i\theta}$ must be single-valued, the phase θ must satisfy

$$\oint_C \partial_\mu \theta \, dx^\mu = 2\pi, \tag{26}$$

for any closed loop C that encircles the z-axis. For loops that do not encircle the axis, the integral vanishes.

4.2 The Quantization Condition in Terms of Struyve's Variables

In Struyve's formulation, the phase θ has been eliminated as a fundamental variable, absorbed into the definition of $B_\mu = A_\mu + \frac{1}{e}\partial_\mu\theta$. The single-valuedness condition on ϕ must therefore be expressible—if at all—in terms of these remaining variables.

Recall from Section 3.3 that the general quantization condition derived from single-valuedness is

$$\oint_C B_\mu dx^\mu = \Phi + \frac{2\pi n}{e}, \tag{27}$$

where $\Phi = \oint_C A_\mu dx^\mu$ is the magnetic flux through a surface bounded by C. For our p-wave Higgs boson in the absence of magnetic fields, we have $\Phi = 0$ and $n = 1$. Condition (27) therefore reduces to

$$\oint_C B_\mu dx^\mu = \frac{2\pi}{e}. \tag{28}$$

This is the quantization condition that any configuration corresponding to a physical p-wave Higgs boson must satisfy. It is a specific instance of the general condition derived earlier, now applied to a concrete physical state.

4.3 Why Struyve's Local Equations Cannot Enforce This Condition

The equations of motion for ρ and B_μ derived from $\mathcal{L}_{\text{GI}}$, (21) and (22), are local partial differential equations. These equations involve only the values of $\rho(x)$ and $B_\mu(x)$ and their derivatives at each spacetime point. They contain no information whatsoever about integrals of B_μ around closed loops. Consequently, they admit solutions for any smooth assignment of B_μ, including those where the circulation $\oint_C B_\mu dx^\mu$ takes arbitrary real values.

One could, in principle, prepare initial data for ρ and B_μ corresponding to a wave packet with azimuthal dependence $e^{i\alpha\phi}$ for any real α. The local equations would evolve this data consistently, producing a solution that appears perfectly well-behaved at every point. Yet such a solution would correspond to a multi-valued Higgs field ϕ if α is not an integer, and would therefore be unphysical. The local equations cannot distinguish between the physical case $\alpha = 1$ satisfying (28) and the unphysical case $\alpha = 0.7$ that violates it.

4.4 Implications for Struyve's Formulation

The example of a single Higgs boson with angular momentum demonstrates conclusively that Struyve's gauge-invariant reformulation is incomplete. The incompleteness is not about exotic topological defects—it is about the most basic quantum property of a single particle: its orbital angular momentum. Any formulation that claims to be a complete physical description of the Higgs mechanism must be able to account for the quantization of angular momentum of its own particle excitations. Struyve's formulation cannot do this without invoking external, global information.

This failure reveals that the formulation has eliminated essential physical information along with the phase θ. The gauge potential A_μ and the phase θ are not mere mathematical redundancies—they carry the global and topological information that makes quantum theory consistent. The next section will show that this problem becomes even more acute when genuine topological defects are considered.

5 Topological Defects and Flux Quantization

The previous section demonstrated that Struyve's gauge-invariant reformulation is incomplete even for a single Higgs boson in a simply-connected region. When we turn to topologically non-trivial configurations—where the phase θ winds around a defect—the incompleteness becomes even more striking. The Abelian Higgs model supports vortex-line solutions, known as Nielsen-Olesen or ANO vortices [10], which carry quantized magnetic flux. Any complete formulation of the Higgs mechanism must be able to account for their properties, including flux quantization. In this section, we show that Struyve's formulation cannot enforce flux quantization from its local equations; the quantization condition must be imposed externally, just as in the Madelung case and as already required for the simple angular-momentum state.

5.1 Flux Quantization in the Standard Abelian Higgs Model

In the standard Abelian Higgs model, vortex solutions arise due to nontrivial winding of the phase of the complex scalar field ϕ. A vortex is characterized by the winding number $n \in \mathbb{Z}$, which measures how many times the phase θ of ϕ wraps around the $U(1)$ vacuum manifold S^1 when traversing a closed loop encircling the vortex. In polar coordinates (r, θ), a static, cylindrically symmetric vortex solution can be expressed as

$$\phi(r,\theta) = \frac{v}{\sqrt{2}} f(r) e^{in\theta}, \qquad A_\theta(r) = \frac{n}{e} g(r), \qquad A_r = A_0 = 0, \tag{29}$$

where $f(r)$ and $g(r)$ are profile functions satisfying $f(0) = 0$, $f(\infty) = 1$, $g(0) = 0$, and $g(\infty) = 1$, and v is the vacuum expectation value of the scalar field. The magnetic flux through a loop encircling the vortex at infinity is

$$\Phi = \oint A_\mu dx^\mu = \frac{2\pi n}{e}. \tag{30}$$

The flux quantization in the standard formulation is a direct consequence of two intertwined physical requirements. First, the Higgs field ϕ must be single-valued, which implies that the phase θ changes by an integer multiple of 2π when going around a closed loop:

$$\oint \partial_\mu \theta \, dx^\mu = 2\pi n. \tag{31}$$

Second, the energy per unit length of the vortex must be finite, which requires that the covariant derivative vanish asymptotically:

$$D_\mu \phi = (\partial_\mu + ieA_\mu)\phi \to 0 \quad \text{as} \quad r \to \infty. \tag{32}$$

For $\phi = \frac{\rho}{\sqrt{2}} e^{i\theta}$ and $\rho \to v$ at infinity, this implies

$$A_\mu \to -\frac{1}{e}\partial_\mu \theta \quad \text{as} \quad r \to \infty. \tag{33}$$

These two conditions together enforce the quantization of the magnetic flux in integer multiples of $2\pi/e$, linking the global topological property of the field configuration to the local dynamics.

5.2 Vortices in Struyve's Gauge-Invariant Variables

Struyve's reformulation eliminates the phase θ from the theory and expresses the dynamics in terms of gauge-invariant variables (ρ, B_μ), where $B_\mu = A_\mu + \frac{1}{e}\partial_\mu \theta$ and $\rho = |\phi|$. The Lagrangian and equations of motion are rewritten entirely in terms of these variables. This approach has the advantage of making the local dynamics manifestly gauge-invariant and removing the need to track the phase explicitly.

However, this elimination of θ comes at a cost. The winding number n, which in the standard formulation encodes the topological sector of the vortex, is no longer represented in (ρ, B_μ). The local equations of motion admit smooth solutions for ρ and B_μ that correspond to arbitrary real values of the circulation $\oint B_\mu dx^\mu$ without enforcing that this integral be quantized. In other words, the local PDEs for (ρ, B_μ) do not contain information about the global topological sector of the original gauge theory.

5.3 Structural Incompleteness and the Need for a Global Constraint

The absence of θ in the local equations implies that flux quantization must be imposed externally. In the standard Abelian Higgs theory, the quantization arises automatically from the requirement of single-valuedness of ϕ and the finite-energy asymptotic condition, without adding any extra postulates. In Struyve's formulation, because the local PDEs do not encode the topological sector, one must supplement the theory with a global flux quantization constraint (30) to recover the physically allowed configurations. Without this additional constraint, the theory admits configurations that are not present in the original Abelian Higgs model, such as vortices with arbitrary non-integer flux. This is structurally analogous to the Wallstrom problem in the Madelung hydrodynamic formulation of quantum mechanics, where local hydrodynamic equations admit arbitrary circulation, but physical wavefunctions require the imposition of single-valuedness as an external quantization condition [18, 4].

Thus, while Struyve's local equations capture the gauge-invariant local dynamics of the fields, they fail to encode the global topological information that distinguishes physically distinct vortex sectors. The necessity of imposing an additional global constraint demonstrates that the theory is incomplete and not fully unified in the sense of the standard Abelian Higgs model, in which topology and dynamics are encoded together in the complex scalar field and its covariant derivative.

5.4 Implications for Struyve's Formulation

In summary, the need for an externally imposed quantization condition to enforce flux quantization reveals a fundamental incompleteness in Struyve's formulation. The local equations for ρ and B_μ underdetermine the physical sector; they admit unphysical solutions with non-integer winding numbers that do not correspond to any legitimate configuration of the original Higgs field. This is not a minor technical issue that can be remedied by a more careful choice of variables. It reflects a deep fact about gauge theories: the phase θ carries essential topological information that cannot be fully encoded in local gauge-invariant composites. Any attempt to eliminate θ must reintroduce its effects through global constraints.

In the standard formulation using ϕ and A_μ, flux quantization emerges automatically from the combination of single-valuedness and the finite-energy condition. No external constraints are needed. The gauge potential A_μ and the phase θ are not redundant; they are the carriers of the topological structure that makes the theory consistent and unified.

The vortex example, together with the single-particle case from Section 4, provides a cumulative argument: Struyve's formulation fails to account for the most basic quantum properties—angular momentum quantization and flux quantization—without external input. It is therefore not a complete physical theory.

6 Conclusion

We have demonstrated that Struyve's gauge-invariant reformulation of the Abelian Higgs model suffers from a fundamental incompleteness that exactly parallels the well-known incompleteness of the Madelung hydrodynamic formulation of quantum mechanics. Both formulations require an externally imposed quantization condition involving global circulations and magnetic flux to enforce single-valuedness of the fundamental fields and to account for basic quantum properties like angular momentum quantization and flux quantization.

This incompleteness is not limited to exotic topological configurations. It already manifests for a single Higgs boson in a state with orbital angular momentum, where the circulation of B_μ around the axis must be quantized even in the absence of vortices. Struyve's local equations cannot enforce this quantization; it must be imposed externally.

The gauge potential A_μ and the phase θ remain indispensable for a complete, self-contained description of the Higgs mechanism. They carry the topological and dynamical information that gauge-invariant formulations must hide in external constraints. The Higgs mechanism, far from eliminating the gauge potentials, provides the most profound arena in which their physical indispensability is made manifest.

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