Vector Mereology

Abstract

This paper outlines a notion of parthood for vectors, which has come to the fore in discussions of quantum mechanics. We show that existing accounts fail to generalise in a way that preserves transitivity. We thus present a new more general account that preserves transitivity and is applicable to quantum systems but may have applicability beyond quantum mechanics, including classical vector quantities and waves.

1. Introduction

Mereology faces a range of conceptual challenges when extended to the quantum domain. Entangled particles are often taken to constitute a whole that, in some sense, exceeds the mere sum of its parts. This paper addresses a closely related problem that is raised by quantum mechanics but is not exclusively quantum. Specifically, we examine the part–whole relation as it pertains to entities that can be described by vectors. In quantum mechanics, the properties of physical systems are represented by vectors in a Hilbert space, which can be linearly combined to form superpositions. In classical mechanics, entities such as waves and fields can similarly be described with vectors (real or complex valued), which can also exist in superpositions.

When entities are in a superposition, there seems a clear sense in which the components of that superposition are parts of the composite entity. If a photon is in a superposition of *here* and *there*, it seems to have a part *here* and a part *there*. If an electron is in a superposition of spin up and spin down, it seems to have a component that is spin up and a component that is spin down, where these components can be understood in terms of parthood. However, the parthood relation is complicated by the fact that these 'parts' interfere with each other when they combine. Adding two non-zero vectors results in something that is not wholly one or the other, nor a disjoint pair, but a new vector with properties that are different to those of its components. To repeat a well-used example, the equal superposition $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ gives an equal probability of obtaining $|0\rangle$ or $|1\rangle$, but it is demonstrably different to the mixed state $\hat{\rho} = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$, which also gives an equal likelihood of obtaining $|0\rangle$ or $|1\rangle$ but without interference between the components (in fact, $\hat{\rho}$ must be written as a density operator because it cannot be represented as a vector in the same Hilbert space as $|\psi\rangle$).

Call parthood for physical entities represented by vectors (in quantum mechanics at least) 'vector parthood'. To date, two approaches to vector parthood have been proposed: one by Saunders (2010) and another that appears to have been independently developed (along broadly similar lines) by both Pashby (2013) and Wilhelm (2025).[1] Saunders' notion of vector parthood relies on the idea that we can decompose the universal state vector into orthogonal components that can be thought of as parts that have no parts in common. While Saunders' parthood relation

[1] We appreciate, but set aside, a broader body of work on mereology in the context of quantum mechanics (see, e.g., Calosi, Fano & Tarozzi (2011), Calosi (2025), Carroll & Singh (2021), Ney (2021), Smith & Brogaard (2002)). While this is important work, it does not focus on the specific notion of vector parthood that interests us here.

is proposed in the context of Everettian branching worlds, Pashby's and Wilhem's are proposed for quantum state vectors in the position basis. The aim of this paper is twofold. First, we argue that we have reason to be unsatisfied with these two existing proposals for vector parthood, since they are restricted to specific domains (Everettian worlds and positional degrees of freedom respectively), and are only transitive with respect to a preferred basis, a condition we have reason to reject. Second, we propose an alternative approach to vector parthood that is more general and unconditionally transitive. This has applications to quantum mechanics but also, potentially, to other domains.

The paper proceeds as follows. First, we outline the target of analysis, along with the conditions under which we deem an analysis of this type successful. We go on to introduce Saunders' proposed notion of vector parthood, along with Wilhem's more recent proposal (§2). We then provide examples showing that these proposed parthood relations are not generally transitive without a preferred basis and discuss the shortcomings of such a restriction (§3). We then propose a revised notion of vector parthood that captures the spirit of the original proposals, meets common requirements of a mereology *and* applies to vector quantities generally, without any need for a preferred basis (§5). Finally, we close with some general reflections on vector parthood for applications outside of quantum mechanics (§6).

Before we get going, a word on notation: throughout this paper we will be using P to identify the parthood relation and $\hat{P}$ to identify a projection operator (in fact, to avoid confusion, we will include 'hats' above all operators).

2. Preliminaries

It is important to clarify the target of our discussion. We are not giving an account of parthood for vectors in a purely mathematical sense. Rather, we are interested in the notion of parthood as it applies to physical entities that can be represented by vectors. Not all such entities are in our sights. We are interested in the physical entities described by quantum mechanics. For, as Wilhelm notes, physicists and philosophers often talk about parthood for quantum states, such as wave-functions, and yet it is not entirely clear what notion of parthood is at play.

We don't think the target of our analysis is exclusively quantum: there do appear to be entities represented by vectors outside of quantum mechanics for which a notion of parthood seems appropriate. The clearest examples are classical waves, particularly when they are polarised. Take for instance seismic shear waves – waves that induce transverse motion through solid earth. When the wave hits an anisotropic medium, such as rock with parallel cracks, and the orientation of those cracks is different to the polarisation of the wave, the wave can split into two: a fast-moving wave parallel to the cracks and a slower moving wave perpendicular to the cracks that lags behind it. The energy from the original wave is split into two perpendicularly polarised, effectively non-interfering waves. Not only can we causally trace the energy and polarisation of the successor waves back to the original wave, we can also isolate the components of the original wave associated with each successor wave. Are the successor waves generated by different parts of the original wave? Our intuitions would suggest so. But then the question arises as to how we should specify the relevant notion of parthood. We will return to this question later.

For now, we focus exclusively on the quantum case. We will often talk just of parthood for vectors or vector parthood to describe parthood for such entities, to avoid the cumbersome locution 'parthood for physical entities represented by vectors'. What do we want from an account of vector parthood? At a minimum, such an account will specify the conditions under which, for physical entities x and y that are both represented by vectors in quantum mechanics, x is a part of y. Such an account should provide *adequate coverage*: all of the cases that should be classified as cases of parthood should be classified as such by the relevant specification.

Further to this, the specification should also deliver a notion of vector parthood that is *physically meaningful*. At a minimum, a notion of vector parthood is physically meaningful, if it classifies as parts all and only those entities that are treated as physical in the context of quantum theory. We think the things that are classified as parts should also behave in a manner that reflects their roles in quantum theory. Here it is useful to have an example: one interesting aspect of quantum theory is that it seems that the components of a quantum state vector can constructively or destructively interfere with one another. The presence of some parts can effectively bolster or suppress the presence of others. We think this should be reflected in a specification of vector parthood, insofar as the entities that are classified as parts by that specification do indeed interfere in this way.

3. Vector mereology

We turn now to some existing attempts to specify a notion of vector parthood. The question of vector parthood is particularly pertinent in Everettian Quantum Mechanics (EQM), where entire worlds exist in superposition. The universal state vector can be decomposed into these component worlds, so it seems natural to think of each world as part of some larger universe (or maybe, each universe is part of some larger multiverse). These distinct parts are conceptually easy to identify after macroscopic branching. If Alice opens Schrödinger's box and sees a live cat, she is in one world, if she sees a dead cat, she is in another. The two worlds have effectively separated but both remain part of the set of worlds described by the universal state vector.

Saunders (2010) is motivated to develop a notion of vector parthood because he argues that there is a sense in which Everettian worlds were distinct (albeit qualitatively identical) entities even before branching. Saunders uses a consistent histories approach to show that we can identify a set of orthogonal vectors that provide a time extended representation of the history of each world, stretching from the distant past to the distant future, what he calls *maximal branch vectors*. Moreover, he claims that these maximal branch vectors can be thought of as numerically distinct because they do not share any parts in common.

When Saunders argues that branch vectors share no parts in common, he does so on the basis of his own proposed vector mereology after (rightly) identifying that there is no agreed approach in the literature. His specific purpose is to understand parthood with respect to branch vectors, which are not widely used outside (or even inside) EQM, so it is worth briefly reviewing these vectors before considering his proposal.

Branch vectors are objects described in the consistent histories approach to quantum mechanics (for the seminal paper, see Griffiths (1984)). They are Heisenberg picture vectors that represent a complete 'cradle to grave' history of a system rather than a snapshot of the physical state at a moment in time. The details will not be critical, except to say that the branch vectors are derived by applying a chain operator constructed from a sequence of time-dependent projection operators that represent the system having different sets of (mutually exclusive and exhaustive) properties throughout the history.

Because it is possible to use different sets of projection operators (both at a time and across time), there can be many possible sets or 'families' of chain operators, each one representing a different way of 'carving up' the same sequence of events.

The chain operators and branch vectors can also be coarse-grained by summing the finer grained operators and vectors. For Everettians, it is natural to coarse-grain into branch vectors that represent relatively distinct quasi-classical worlds, although these worlds are only approximate and emergent entities with no fundamental status in the physics.

While branch vectors represent something quite different to Schrödinger state vectors, they are both Hilbert space vectors. In fact, the superposition of all branch vectors gives the Heisenberg state and the Schrödinger state vector is just the Heisenberg state time evolved forward.

Saunders argues that branch vectors share no parts in common. He arrives at this conclusion on the basis of a proposed vector mereology that he claims fulfils the basic tenets of parthood, namely reflexivity, transitivity and asymmetry. Specifically, he proposes that vector $|\gamma\rangle$ is part of vector $|\beta\rangle$, which we will denote $P_V(|\gamma\rangle, |\beta\rangle)$, when it satisfies the condition:

Vector Part: $P_V(|\gamma\rangle, |\beta\rangle)$ iff either (a) $\gamma = \beta$ or (b) there exists a non-zero $|\delta\rangle$ such that $\langle\delta|\gamma\rangle = 0$, and $|\gamma\rangle + |\delta\rangle = |\beta\rangle$.

In other words, the branch vector $|\gamma\rangle$ is part of $|\beta\rangle$ just in case they represent the same thing or there exists some non-zero branch vector $|\delta\rangle$ that is orthogonal to $|\gamma\rangle$ such that $|\gamma\rangle + |\delta\rangle = |\beta\rangle$. In quantum mechanics, the kets would usually represent vectors with unit length and would have complex coefficients (amplitudes) before $|\gamma\rangle$ and $|\delta\rangle$, but for brevity we will follow Saunders and assume the vectors may have magnitudes other than one, even when they are not shown explicitly.

A very similar notion of parthood is proposed by Pashby (2013) and (apparently independently) Wilhelm (2025). The two are broadly similar but we will focus on Wilhelm's formulation as the more detailed of the two.[2] Wilhelm uses the term projective part, which is defined as follows:

Projective Part: A vector $|\gamma\rangle$ is part of $|\beta\rangle$ iff there exists a projection operator $\hat{P}_S$ such that $\hat{P}_S|\beta\rangle = |\gamma\rangle$.[3]

By defining $|\delta\rangle = (\hat{I} - \hat{P}_S)|\beta\rangle$ we can see that Wilhelm's projective part condition aligns closely with Saunders' vector part conditions for any non-zero $|\delta\rangle$.[4] There are some nuanced differences between the approaches that we will discuss later.

It is easy to see the appeal of the proposed definitions of vector parthood. Orthogonal components of a vector $|\gamma\rangle$ have no extension in any direction that is orthogonal to $|\gamma\rangle$. So if $\langle\delta|\gamma\rangle = 0$ then the $|\gamma\rangle$ vector has no extension in the direction of the $|\delta\rangle$ vector. Colloquially, we might take that to mean that there is no part of the vector $|\gamma\rangle$ that is "in" the vector $|\delta\rangle$. But,

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2 Pashby (2013) claims "that in quantum theory the spatial parts of a quantum object $|\psi\rangle$ may be given in terms of the subspaces of $\mathcal{H}$ associated with the spectral decomposition of the position observable Q" (p. 1142).

3 More fully: for all non-zero $|x\rangle$ and $|y\rangle$ in $\mathcal{H}$, $|x\rangle$ is part of $|y\rangle$ iff there exists an $S \subseteq \mathbb{N}$ such that $\hat{P}_S|y\rangle = |x\rangle$, where $\hat{P}_S$ is a projection operator, projecting onto the subspace defined by some set of basis vectors indexed by S.

4 Because $\langle\delta|\gamma\rangle = \hat{P}_S(I - \hat{P}_S)\langle\beta|\beta\rangle = (\hat{P}_S - \hat{P}_S)\langle\beta|\beta\rangle = 0$ and $|\gamma\rangle + |\delta\rangle = \hat{P}_S|\beta\rangle + (I - \hat{P}_S)|\beta\rangle = |\beta\rangle$.

as we will see in the next section, the feature of vectors that makes them so useful for describing quantum mechanical states – superposition – will lead us to doubt that orthogonal vectors are really as distinct as our pre-theoretical intuitions suggest.

4. Transitivity

In this section, we outline some reasons for going beyond both Saunders' and Wilhelm's definitions of vector parthood. Our worry is that the approaches are not fully general: they don't allow us to apply the notion of parthood throughout quantum mechanics in the way that would achieve adequate coverage (as discussed in §2). In both cases, the accounts apply only to specific kinds of vectors in specific preferred bases. While it is possible that a vector parthood relation should be so restricted, there is nothing in the underlying physics that would single out these vectors or these bases as unique. As we will show, one could generalise the accounts of vector parthood given, but doing so results in a non-transitive relation. Both Saunders and Wilhem agree, as do we, that any notion of vector parthood should ideally accord with the standard (i.e. classical) axioms of mereology (Calosi et al. (2011) make a similar point). If a classical mereology is truly desired, then only a transitive parthood relation will do. After detailing our concerns in this section, we propose our alternative in §5.

4.1. A Preferred Basis

To demonstrate the problem, we'll first consider Wilhelm's proof of the transitivity of Projective Part. Here's a simplified version of the proof. Take a vector $|y\rangle$ that can be described with a minimal set of orthogonal basis vectors $\mathcal{B} = \{|b_i\rangle\}$. Then consider a projection operator $\hat{P}_S$, which projects $|y\rangle$ onto a coordinate subspace – that is, a subspace spanned by a subset of the basis vectors in $\mathcal{B}$.[5] According to Wilhelm's projective part relation, $P(|x\rangle, |y\rangle)$ iff there exists a $\hat{P}_S$ such that $\hat{P}_S|y\rangle = |x\rangle$. In other words, $|x\rangle = \sum c_j|b_j\rangle$ is a part of $|y\rangle = \sum c_i|b_i\rangle$ when $\{c_j|b_j\rangle\} \subseteq \{c_i|b_i\rangle\}$. Put simply, each of the terms used to describe $|x\rangle$ must either be the same as those used to describe $|y\rangle$ or set to zero.

Now, consider the following projections, where all vectors are non-zero:

$$\hat{P}_{S'}|\beta\rangle = |\gamma\rangle$$
$$\hat{P}_{S''}|\epsilon\rangle = |\beta\rangle$$

[5] That is, the subspace is spanned by a set of basis vectors $\{|b_j\rangle\}$ where $\{|b_j\rangle\} \subseteq \{|b_i\rangle\}$. Wilhelm provides a lemma for the fact that $\{|b_j\rangle\} \subseteq \{|b_i\rangle\}$ based on his definition of $\hat{P}_S$, but we will simplify the explanation by making it an assumption.

Because $\hat{P}_{S'}$ projects onto a coordinate subspace of $\hat{P}_{S''}$, they will commute and so we also get:

$$\hat{P}_{S'}\hat{P}_{S''}|\epsilon\rangle = |\gamma\rangle$$

Given the above, it follows from the definition of a projective part that: $|\gamma\rangle$ is part of $|\beta\rangle$; $|\beta\rangle$ is part $|\epsilon\rangle$; and $|\gamma\rangle$ is part of $|\epsilon\rangle$ (because $\hat{P}_{S'}\hat{P}_{S''}$ is also a projection operator). Accordingly, the relation is transitive.

In laying out the proof, one particular aspect of Wilhelm's account becomes clear. $\hat{P}_{S'}$ and $\hat{P}_{S''}$ must both project onto subspaces that are spanned by some subset of the basis vectors in $\mathcal{B}$. But then the demonstration of transitivity for Projective Part only holds relative to a choice of basis vectors. Wilhelm singles out the position basis. If we use only this preferred basis when applying Projective Part, then transitivity holds. But, as we'll show in the next section, as soon as we allow for arbitrary selection of basis vectors, and thus give up on any preferred basis, transitivity fails. When the basis vectors for each projection operator are not orthogonal or the same, the projection operators will not commute and so the conclusion above that $\hat{P}_{S'}\hat{P}_{S''}|\epsilon\rangle = |\gamma\rangle$ will not (in general) hold.

A similar issue arises for Saunders. He does not provide a proof of transitivity, so we will not dwell on the detail, but transitivity would entail if $P_V(|\gamma\rangle, |\beta\rangle)$ and $P_V(|\beta\rangle, |\delta\rangle)$, then $P_V(|\gamma\rangle, |\delta\rangle)$. Given the context, the most charitable reading of Saunders' definition is that the vectors in the relation must be either branch vectors representing Everettian worlds (or more correctly, world histories) or superpositions of such worlds. The latter is required to make sense of exactly what a branch vector might be a part *of*. If this reading is correct, then it is easy to see that P_V is transitive – if branch vector $|\gamma\rangle$ is part of some superposition of branch vectors $|\beta\rangle$, and $|\beta\rangle$ is itself part of a larger superposition of branch vectors $|\delta\rangle$, then $|\gamma\rangle$ will also be part of $|\delta\rangle$. But this notion of parthood again assumes there is a preferred set of bases, in this case bases that are picked out by decoherence, potentially at some time in the far distant future. But there is broad agreement among modern Everettians that these worlds are just effective and approximate features of the universe, not anything fundamental to the physics. If we give up this restriction then, as we'll see below, transitivity fails.

Why is this basis-dependence a problem? Well, we think that a definition of vector parthood should not hold only relative to a specific choice of basis vectors. The basis vectors we choose to work with are a matter of convenience or convention and do not represent any deep division in the world. Without the freedom to change bases, the only vectors that could count as a part of another would be those aligning with the selected set basis vectors, which are ultimately not physical. It seems akin to restricting the classical parthood relation to objects in our own inertial reference frame even though relativity tells us there is no preferred reference frame.

Recall that we want a notion of vector parthood that is physically meaningful. Saunders' and Wilhelm's accounts do not seem to satisfy this desideratum. These accounts of parthood seem to elevate the choice of basis beyond that of mere convention into a physically meaningful distinction. For the choice seems to have significant implications for how a system is divided into parts. A vector $|\gamma\rangle$ could only be a part of $|\beta\rangle$ in the position basis (Wilhelm) or the decoherence basis (Saunders), but not any others. It is difficult to believe that these, or any particular sets of bases, should hold some special place in mereology that others cannot. We think it better to retain the conventionality of the basis choice, and so we seek an account of vector parthood that does not invest this choice with metaphysical importance.

4.2. Transitivity Counterexamples

Having outlined the problem for Wilhelm and Saunders in general terms, we now turn to demonstrating that in the absence of a preferred basis neither definition is transitive. Our counterexamples are aimed at suitably generalised versions of these definitions: namely, versions that allow for an arbitrary choice of basis. The counterexamples are not designed to show that their definitions always fail to be transitive as we grant that, in the presence of a preferred basis they likely are.

As a warm-up, let us consider simple Euclidean vectors, which are easier to visualise than complex Hilbert space vectors. Imagine the vector represents something like the polarisation of a seismic shear wave, as we discussed earlier. Euclidean vector spaces are special cases of Hilbert spaces in which we restrict ourselves to real, finite dimensional spaces where the inner product is the standard dot product. So, for Euclidean vectors we could rewrite Saunders definition of a vector part as:

Euclidean Vector Part: $P_E(\vec{a}, \vec{c})$ iff either (a) $a = c$ or (b) there exists a non-zero $\vec{b}$ such that $\vec{a} \cdot \vec{b} = 0$ and $\vec{a} + \vec{b} = \vec{c}$.

To determine if P_E is transitive, let us first note that transitivity would be the requirement that:

$$\forall x \forall y \forall z (P_E(x, y) \wedge P_E(y, z) \rightarrow P_E(x, z))$$

Now consider a situation in which we have non-zero, perpendicular vectors $\vec{a}$ and $\vec{b}$ that are of equal magnitude and add to form vector $\vec{c}$ (so $\vec{a}$ and $\vec{b}$ are parts of $\vec{c}$ by the above definition). Further, let us assume that $\vec{c}$ is perpendicular to another vector $\vec{d}$ of equal magnitude and the two add to form $\vec{e}$ (so $\vec{c}$ and $\vec{d}$ are parts of $\vec{e}$ by the above definition). This situation is depicted in Figure 1.

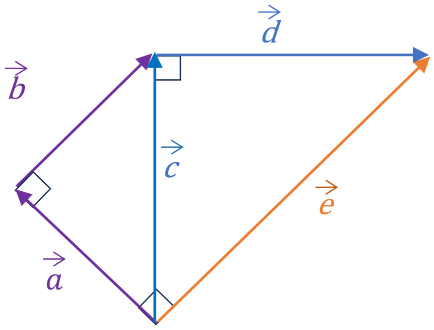

Figure 1: Euclidean vectors

In this example, simple geometry tells us that vectors $\vec{a}$ and $\vec{e}$ will be perpendicular to each other. So, according to our definition of a Euclidean vector part:

- $P_E(\vec{a},\vec{c})$ because $\vec{a}\cdot\vec{b}=0$ and $\vec{a}+\vec{b}=\vec{c}$
- $P_E(\vec{c},\vec{e})$ because $\vec{c}\cdot\vec{d}=0$ and $\vec{c}+\vec{d}=\vec{e}$
- But $\sim P_E(\vec{a},\vec{e})$ because $\vec{a}\cdot\vec{e}=0$, so there can be no vector $\vec{f}$ such that $\vec{a}\cdot\vec{f}=0$ and $\vec{a}+\vec{f}=\vec{e}$.[6]

The last point provides our first counterexample to the transitivity of a generalised Euclidean version of Saunders' proposed parthood relation and makes it clear why it is so. Our parthood relation P_E tells us that a vector's parts are orthogonal 'splits' of the original vector. But the parts (the orthogonal components) are rotated compared with the original. If we then split again, the parts of the parts are rotated by some additional amount (imagine splitting each orthogonal part of a seismic wave a second time). In Figure 1, we have split the original vector $\vec{e}$ such that one of the component parts $\vec{c}$ is rotated by $\pi/4$ compared with the original, and then split $\vec{c}$ in a similar way so that one of the component parts $\vec{a}$ is now orthogonal to $\vec{e}$. Consequently, $\vec{a}$ is not part of $\vec{e}$ and transitivity does not hold.

We come to the same conclusion if we use Wilhelm's definition of a projective part. In that case, $\vec{a}$ is the projection of $\vec{c}$ onto a basis vector pointing in the $\vec{a}$ direction, so $\vec{a}$ is part of $\vec{c}$ (noting that there is also an orthogonal complement pointing in the $\vec{b}$ direction). But if the choice of basis vectors is arbitrary, we could equally say that $\vec{c}$ is part of $\vec{e}$ because $\vec{c}$ is the

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[6] Otherwise, we have a contradiction because $\vec{a}\cdot\vec{e}=\vec{a}\cdot(\vec{a}+\vec{f})=\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{f}=\|\vec{a}\|^2+0=\|\vec{a}\|^2\neq 0$. That is, the sum of two orthogonal non-zero vectors (in this case $\vec{a}$ and $\vec{f}$) can never be a vector that is orthogonal to either one of them (in this case $\vec{e}$).

projection of $\vec{e}$ onto a basis vector pointing in the $\vec{c}$ direction (with an orthogonal complement pointing in the $\vec{d}$ direction). Transitivity fails because the projection of $\vec{e}$ onto any vector in the $\vec{a}$ direction gives zero (a null vector), so $\vec{a}$ is not part of $\vec{e}$.

Next, while neither Saunders nor Wilhelm claim their parthood relation can be applied to Schrödinger picture spin states, applying the parthood relation to such simple state vectors is still instructive because it can easily be generalised to more complex vectors.

Consider the states $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. According to Saunders' vector part relation P_V (assuming it applies to non-normalised Schrödinger state vectors):

- $P_V\left(\frac{1}{2}|-\rangle, \frac{1}{\sqrt{2}}|0\rangle\right)$ because $\frac{1}{4}\langle +|-\rangle = 0$ and $\frac{1}{2}|+\rangle + \frac{1}{2}|-\rangle = \frac{1}{\sqrt{2}}|0\rangle$
- $P_V\left(\frac{1}{\sqrt{2}}|0\rangle, |+\rangle\right)$ because $\frac{1}{2}\langle 0|1\rangle = 0$ and $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = |+\rangle$
- But, $\sim P_V\left(\frac{1}{2}|-\rangle, |+\rangle\right)$ because there is no non-zero $|\delta\rangle$ such that $\langle \delta|-\rangle = 0$ and $|\delta\rangle + \frac{1}{2}|-\rangle = |+\rangle$.

So, for much the same reason as the Euclidean example, we are led to conclude that P_V is not transitive. It is a short step from here to branch vectors. To provide a simple but concrete example, consider a branch vector for a single particle whose state evolves from t_0 to t_1 via the Hadamard operator $\hat{H}$ such that:

$$t_0: |\psi_0\rangle = |0\rangle$$
$$t_1: |\psi_1\rangle = \hat{H}|0\rangle = |+\rangle$$

While Everettian 'worlds' might usually refer to some set of states that are effectively macroscopically independent of each other, this single particle example will suffice to demonstrate how branch vectors generally are not transitive under P_V. For our single particle, consider the following set of projection operators, with the superscript denoting time:

$$\hat{P}_+^{(0)} = |+\rangle\langle +| \qquad \hat{P}_0^{(1)} = |0\rangle\langle 0|$$
$$\hat{P}_-^{(0)} = |-\rangle\langle -| \qquad \hat{P}_1^{(1)} = |1\rangle\langle 1|$$

Relative to these projectors, there are exactly two consistent histories: $|+\rangle$ at t_0 and $|0\rangle$ at t_1; and $|-\rangle$ at t_0 and $|1\rangle$ at t_1. The associated branch vectors are:[7]

$$|\beta\rangle = C_\beta|\psi_0\rangle = \hat{P}_0^{(1)}\,\hat{H}\,\hat{P}_+^{(0)}|0\rangle = \frac{1}{\sqrt{2}}|0\rangle$$

$$|\delta\rangle = C_\delta|\psi_0\rangle = \hat{P}_1^{(1)}\,\hat{H}\,\hat{P}_-^{(0)}|0\rangle = \frac{1}{\sqrt{2}}|1\rangle$$

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[7] The branch vectors are consistent because $\langle \beta|\epsilon\rangle = \frac{1}{2}\langle 0|1\rangle = 0$.

The separate branch vectors sum to the time evolved Heisenberg state $|\epsilon\rangle = |+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. Therefore, it is clear that $P_V(|\beta\rangle, |\epsilon\rangle)$.

Then consider the individual branch vector $|\beta\rangle$. In the corresponding 'world' β, the states of the system (suppressing the $\frac{1}{\sqrt{2}}$ coefficient for the magnitude of the vector) are:

$$t_0: |\beta_0\rangle = |+\rangle$$
$$t_1: |\beta_1\rangle = |0\rangle$$

Now, if transitivity applies, we should see that any vector parts of $|\beta\rangle$ are also vector parts of $|\epsilon\rangle$. Assuming we are free to choose any way of 'carving' a branch vector into parts (a matter we will return to shortly), let us apply the following projection operators:

$$\hat{P}_0^{(0)} = |0\rangle\langle 0| \qquad \hat{P}_+^{(1)} = |+\rangle\langle +|$$
$$\hat{P}_1^{(0)} = |1\rangle\langle 1| \qquad \hat{P}_-^{(1)} = |-\rangle\langle -|$$

Relative to these projectors, the branch vectors associated with β are (this time, restoring the $\frac{1}{\sqrt{2}}$ coefficient for the relative magnitude of the 'world' as a whole):

$$|\lambda\rangle = C_\lambda|\gamma_0\rangle = \hat{P}_+^{(1)}\,\hat{H}\,\hat{P}_0^{(0)}\frac{1}{\sqrt{2}}|+\rangle = \frac{1}{2}|+\rangle$$
$$|\gamma\rangle = C_\gamma|\gamma_0\rangle = \hat{P}_-^{(1)}\,\hat{H}\,\hat{P}_1^{(0)}\frac{1}{\sqrt{2}}|+\rangle = \frac{1}{2}|-\rangle$$

Applying similar reasoning to the Schrödinger state example, we can quickly conclude that $P_V(|\gamma\rangle, |\beta\rangle)$, $P_V(|\beta\rangle, |\epsilon\rangle)$ but $\sim P_V(|\gamma\rangle, |\epsilon\rangle)$. This conclusion reflects a controversial premise of the consistent histories approach: that counterfactual reasoning can only be undertaken for histories within the same family (Griffiths, 1999; 2011). The different sets of projection operators above produce different families of histories, so it may come as no surprise that transitivity fails when we consider parts from different families. The non-classical logic of the consistent histories approach has been criticised (for instance by Maudlin, 2011), but more importantly, in §5 we provide an alternative definition of parthood that is generally transitive and free of the limitations of the consistent histories approach.

Wilhelm's projective part condition is similarly not transitive if we remove the fixed basis requirement. The reasoning follows a similar pattern to that above, which for brevity we will not repeat again.

5. Transitive Vector Parthood

Recall that the original condition from Saunders was that $P_V(|\gamma\rangle, |\beta\rangle)$ iff either (a) $\gamma = \beta$ or (b) there exists a non-zero $|\delta\rangle$ such that $\langle\delta|\gamma\rangle = 0$, and $|\gamma\rangle + |\delta\rangle = |\beta\rangle$. As discussed, this relation is not generally transitive without further restrictions on the vectors. One simple proposition to introduce transitivity would be to remove the requirement that parts be mutually orthogonal, leaving only the requirement that $P_{V'}(|\beta\rangle, |\gamma\rangle)$ iff there exists a $|\delta\rangle$ such that $|\beta\rangle + |\delta\rangle = |\gamma\rangle$. However, every vector would be part of every other vector because there is always *some* vector that we can add to $|\beta\rangle$ to make $|\gamma\rangle$, namely the difference $|\delta\rangle = |\gamma\rangle - |\beta\rangle$. Such a condition would be too permissive if we are interested in vectors that represent physical entities that can be 'divided' (actually or notionally) into their parts.

To maintain the spirit of both Saunders' and Wilhelm's proposal, let us recognise that the physical entities of interest are those that we think of as being composed of orthogonal components, like quantum superpositions, positions in space, or even classical polarised waves. Furthermore, let us assume that those components can themselves be divided into further orthogonal components. To create a suitable notion of vector parthood that is also transitive, let us start by taking the original condition P_V to be a sufficient condition for vector parthood. That is, a vector $|\beta_i\rangle$ is part of a vector $|\beta_{i-1}\rangle$ if they are the same or if there is an orthogonal, non-zero vector $|\beta_i'\rangle$ such that $|\beta_i\rangle + |\beta_i'\rangle = |\beta_{i-1}\rangle$. We can then allow that parts can themselves be decomposed into parts and require that the relation be transitive. The transitivity of this new parthood relation would entail that if $|\beta_2\rangle$ is part of $|\beta_1\rangle$, and $|\beta_1\rangle$ is part of $|\beta_0\rangle$, then $|\beta_2\rangle$ is part of $|\beta_0\rangle$. Putting the conditions together, $|\beta_2\rangle$ is part of $|\beta_0\rangle$ if $|\beta_2\rangle + |\beta_2'\rangle = |\beta_1\rangle$ and $|\beta_1\rangle + |\beta_1'\rangle = |\beta_0\rangle$, which we can simplify to $|\beta_2\rangle + |\beta_2'\rangle + |\beta_1'\rangle = |\beta_0\rangle$.

Roughly, we want a parthood condition to specify that any vector $|\beta_i\rangle$ will be a part of vector $|\beta_0\rangle$ if we can arrive at $|\beta_i\rangle$ through some series of orthogonal decompositions of $|\beta_0\rangle$. That is, for two entities represented by vectors:

(Transitive) Vector Part: $P_{TV}(|\gamma\rangle, |\beta_0\rangle)$ iff both are non-zero vectors in the same Hilbert space and either:

(a) $|\gamma\rangle = |\beta_0\rangle$; or

(b) there exists some integer $n>0$ such that there is an orthogonal decomposition of $|\beta_0\rangle$ in which $|\gamma\rangle = |\beta_n\rangle$,

where an orthogonal decomposition of $|\beta_0\rangle$ is any set of vectors $\{|\beta_j\rangle\}$ and $\{|\beta'_j\rangle\}$ where $\langle\beta_j|\beta'_j\rangle = 0$ and $|\beta_j\rangle + |\beta'_j\rangle = |\beta_{j-1}\rangle$ for $j \in \{1,2 \ldots, n\}$.

We can simplify condition (b) further by considering which vectors meet the condition. The Appendix demonstrates that condition (b) will hold if and only if $|\gamma\rangle$ is smaller than $|\beta_0\rangle$. On that basis, our parthood condition can be substantially simplified to:

(Transitive) Vector Part*: $P_{TV^*}(|\gamma\rangle, |\beta\rangle)$ iff both vectors are in the same Hilbert space and either (a) $|\gamma\rangle = |\beta\rangle$ or (b) $0 < \||\gamma\rangle\| < \||\beta\rangle\|$.

P_{TV} maintains the spirit of the original parthood condition P_V by requiring parts to be obtained through orthogonal splitting of a vector but also ensures transitivity because parts of $|\gamma\rangle$ will also be parts of $|\beta\rangle$. Take our earlier Schrödinger state example, in which $P\left(\frac{1}{2}|-\rangle, \frac{1}{\sqrt{2}}|0\rangle\right)$ and $P\left(\frac{1}{\sqrt{2}}|0\rangle, |+\rangle\right)$. Under the simplified P_{TV^*} condition, the relation is now transitive, implying that $P\left(\frac{1}{2}|-\rangle, |+\rangle\right)$, which holds because:

$$0 < \left\|\frac{1}{2}|-\rangle\right\| < \left\|\frac{1}{\sqrt{2}}|0\rangle\right\| < \||+\rangle\|$$

P_{TV} meets the standard requirements on parthood: it is transitive, reflexive and anti-symmetric. First: reflexivity holds in virtue of clause (a) in the definition. For any $|\gamma\rangle$, $P_{TV}(|\gamma\rangle,|\gamma\rangle)$ because $|\gamma\rangle = |\gamma\rangle$. Second: transitivity holds because we can concatenate chains of orthogonal splits. Thus, suppose $P_{TV}(|\alpha\rangle,|\gamma\rangle)$ and $P_{TV}(|\gamma\rangle,|\delta\rangle)$ where $|\alpha\rangle \neq |\gamma\rangle \neq |\delta\rangle$. Then there is a chain of orthogonal decomposition from $|\delta\rangle$ to $|\gamma\rangle$, and a chain of orthogonal decomposition from $|\gamma\rangle$ to $|\alpha\rangle$. Because we can join these two chains to form a new chain, it follows that there's a chain of decomposition from $|\delta\rangle$ to $|\alpha\rangle$ and so it follows that $P_{TV}(|\alpha\rangle,|\delta\rangle)$.

Third: anti-symmetry. For this to hold, then for any $|\alpha\rangle,|\gamma\rangle$, if $P_{TV}(|\alpha\rangle,|\gamma\rangle)$ and $P_{TV}(|\gamma\rangle,|\alpha\rangle)$, then $|\alpha\rangle = |\gamma\rangle$. Suppose that $P_{TV}(|\alpha\rangle,|\gamma\rangle)$ and $P_{TV}(|\gamma\rangle,|\alpha\rangle)$ and $|\alpha\rangle \neq |\gamma\rangle$. If $P_{TV}(|\alpha\rangle,|\gamma\rangle)$ then because $|\alpha\rangle \neq |\gamma\rangle$, there must be a chain of orthogonal decompositions from $|\gamma\rangle$ to $|\alpha\rangle$. By P_{TV^*} we know that $\||\alpha\rangle\| < \||\gamma\rangle\|$. If $P_{TV}(|\gamma\rangle,|\alpha\rangle)$ then because $|\gamma\rangle \neq |\alpha\rangle$, there must be a chain of orthogonal decompositions from $|\alpha\rangle$ to $|\gamma\rangle$. But then by P_{TV^*} we know that $\||\gamma\rangle\| < \||\alpha\rangle\|$. But $\||\alpha\rangle\| < \||\gamma\rangle\|$ and $\||\gamma\rangle\| < \||\alpha\rangle\|$ is a contradiction and so it cannot be that $P_{TV}(|\alpha\rangle,|\gamma\rangle)$ and $P_{TV}(|\gamma\rangle,|\alpha\rangle)$ and $|\alpha\rangle \neq |\gamma\rangle$.

What kind of mereology does $P_{TV}(|\alpha\rangle,|\gamma\rangle)$ support? To specify a mereology, we start by adding definitions for overlap, underlap, proper parthood and fusion (assuming all vectors are in the same Hilbert space):

Overlap: $O(|\alpha\rangle,|\gamma\rangle)$ iff there exists a non-zero $|\delta\rangle$ such that $P_{TV}(|\delta\rangle,|\alpha\rangle)$ and $P_{TV}(|\delta\rangle,|\gamma\rangle)$.

Underlap: $U(|\alpha\rangle,|\gamma\rangle)$ iff there exists a non-zero $|\delta\rangle$ such that $P_{TV}(|\alpha\rangle,|\delta\rangle)$ and $P_{TV}(|\gamma\rangle,|\delta\rangle)$.

Proper-parthood: $PP_{TV}(|\alpha\rangle,|\gamma\rangle) =_{def} P_{TV}(|\alpha\rangle,|\gamma\rangle)$ and $|\alpha\rangle \neq |\gamma\rangle$.

For fusion, start with the following definition:

Fusion: $Fz\phi w =_{def} \forall w(\phi w \longrightarrow Pwz) \,\&\, \forall v(Pvz \longrightarrow \exists w(\phi w \,\&\, Ovw))$

Where $Fz\phi w$ means that z is a fusion of every w such that ϕw (where we can think of ϕ roughly as the membership conditions of a set). The first half of the principle says that any w that is a ϕ is a part of z; the second half of the principle says that anything that is part of z overlaps with one of the ϕ. This basic notion of fusion needs to be restricted. The reason for this is because of a unique feature of any mereology defined on P_{TV}:

Universal Overlap: for any pair of non-zero vectors in the same Hilbert space ($|\alpha\rangle$,$|\gamma\rangle$), $\mathrm{O}(|\alpha\rangle,|\gamma\rangle)$.

Two vectors overlap when they have a part in common. But we also know from P_{TV^*} that when $0 < \||\gamma\rangle\| < \||\beta\rangle\|$, then $P_{TV}(|\gamma\rangle, |\beta\rangle)$. So all it takes for two vectors $|\alpha\rangle$, $|\gamma\rangle$ to overlap, is that there is some vector $|\delta\rangle$ where $\||\delta\rangle\| < \||\alpha\rangle\|$ and $\||\delta\rangle\| < \||\gamma\rangle\|$. For any pair of non-zero vectors in the same vector space, we can always find such a $|\delta\rangle$. We just need to take a non-zero vector and scale it down so that its norm is smaller than $\||\alpha\rangle\|$ and $\||\gamma\rangle\|$, which is always possible in a Hilbert space.

Because of universal overlap, the same set of vectors can be members of two different fusions. Take two orthogonal vectors $|\alpha\rangle$ and $|\gamma\rangle$ and the set containing just them ϕ. The vector $|\delta\rangle$ that is their sum $|\alpha\rangle + |\gamma\rangle$ satisfies the first condition of Fusion: everything in ϕ is part of $|\delta\rangle$. The second condition is also satisfied because of universal overlap: every part of $|\delta\rangle$ will overlap some member of ϕ because any two vectors overlap. But now take any other vector $|\beta\rangle \neq |\delta\rangle$ where $\||\beta\rangle\| > \||\alpha\rangle\|$ and $\||\beta\rangle\| > \||\gamma\rangle\|$. Everything in ϕ is part of $|\beta\rangle$ and, again, because of universal overlap, every part of $|\beta\rangle$ will overlap with every member of ϕ.

Fusion is thus not a very interesting notion for P_{TV}. Every vector effectively ends up being a fusion of every vector smaller than it, so there is no uniqueness, which is considered one of the defining features of fusions. It's also the case that fusion isn't tracking what we take to be the physically important aspect of our notion of parthood, namely the way it tracks orthogonal decomposition. We can define a better notion:

Orthogonal Fusion: For any set ϕ of non-zero vectors in the same Hilbert space, where the first member is any arbitrary vector and all other members are orthogonal to the vector sum of all previous members, there exists a vector $|\gamma\rangle$ such that:

(i) $|\gamma\rangle$ is the vector sum of all members of ϕ

(ii) Every member of ϕ is part of $|\gamma\rangle$: $\forall w(w \in \phi \rightarrow \mathrm{P}_{TV}(w, z))$

(iii) Every part of $|\gamma\rangle$ overlaps some member of ϕ: $\forall v(\mathrm{P}_{TV}(v, z) \rightarrow \exists w(w \in \phi \; \& \; \mathrm{O}vw)$

P_{TV} satisfies orthogonal fusion.[8] Orthogonal fusion also gets us uniqueness: for any set of mutually orthogonal vectors, there is only one vector produced by summing them.

Universal overlap also rules out weak and strong supplementation. Strong supplementation says that if x is not part of y, then there is some part of x that does not overlap with y. Weak supplementation says that if x is a proper part of y then there is a part of y that does not overlap with x. Both are ruled out because all vectors overlap. While weak and strong supplementation are ruled out, a nearby complementation principle is entailed by P_{TV}:

Orthogonal Complementation: if $P_{TV}(|\alpha\rangle,|\gamma\rangle)$ and $|\alpha\rangle \neq |\gamma\rangle$, then there exists a set of vectors $\{|\delta_j\rangle\}$, where $j \in \{1,2 \ldots, n\}$, such that $|\alpha\rangle + \sum_1^n |\delta_j\rangle = |\gamma\rangle$, $\langle\alpha|\delta_1\rangle = 0$ and $\langle\delta_j|\delta_{j+1}\rangle = 0$ for $j < n$.

That is, every proper part has a complement: a set of vectors it combines with to produce the whole, where each vector is orthogonal to the sum of all those before it.

In sum, then, the mereology for P_{TV} involves P_{TV} as a partial order, standard notions for overlap, underlap and proper part, plus Orthogonal Fusion and Orthogonal Complementation. This mereology lies somewhere between ground mereology and classical extensional mereology, which includes a notion of unrestricted fusion and strong supplementation. Now, it is important to note that P_{TV} only relates vectors in the same Hilbert space. One way to prevent universal overlap would be to extend the picture to cover vectors in different Hilbert spaces (representing, say, different non-interacting properties). Then universal overlap won't hold. For there is no guarantee for $|\alpha\rangle$ in one Hilbert space, and $|\gamma\rangle$ in another that there is a chain of orthogonal decompositions leading from both $|\alpha\rangle$ and $|\gamma\rangle$ to a common element $|\delta\rangle$.

This opens the door to classical mereology. However, we don't pursue this line because we think that universal overlap, and its corollary: the failure of supplementation, are desirable for a mereology of quantum entities represented by vectors in a single Hilbert space. In classical situations, systems can be cleanly separated into parts in a way that respects supplementation. Because of features like superposition and entanglement, the world of quantum mechanics is deeply holistic and resists the same clean separation. The universal overlap that we get from

[8] Consider an arbitrary ϕ where $\phi = \{|\alpha\rangle, |\beta\rangle\}$ and $\langle\alpha|\beta\rangle = 0$. There is a $|\gamma\rangle = |\alpha\rangle + |\beta\rangle$. $|\gamma\rangle$ is a vector sum of $|\alpha\rangle$ and $|\beta\rangle$, thereby satisfying condition (i). Because $|\gamma\rangle = |\alpha\rangle + |\beta\rangle$ and $\langle\alpha|\beta\rangle = 0$, there is a chain of orthogonal decomposition leading from $|\gamma\rangle$ to both $|\alpha\rangle$ and $|\beta\rangle$, thus establishing that every member of ϕ is part of $|\gamma\rangle$, thus satisfying condition (ii). Universal overlap forces condition (iii).

P_{TV} allows the holism of quantum mechanics to be reflected in vector parthood. This is a good thing. It also potentially calls into question a notion of parthood that delivers supplementation (as Wilhelm's does).

Before we turn to some general discussion of P_{TV} it is worth heading off a few objections. The first worry, roughly put, is that we have reduced vector parthood to a *mere* magnitude ordering. But a mere magnitude ordering seems hardly the right thing to qualify as a parthood relation. Suppose one held in general that x is part of y when x is smaller than y. This would be madness: cats would be parts of cows!

Our notion of parthood does make use of magnitude comparisons, but it is not a mere magnitude ordering. This is where the restriction to a vector space becomes important. It is only when x is smaller than y and they are in the *same vector space* that parthood holds. When two vectors are in the same vector space, they both specify values for each degree of freedom of the combined system, although some of those values may be zero. Taking a step back from the quantum case, consider as an analogy parthood for concentric spheres, which represent, say, the progressively larger shells of a star. In that case, it is perfectly plausible that we should say that x is part of y just in case x is in the same physical space as y (more precisely, has the same centre point) but is smaller than y. The parthood relation is based on a magnitude relation (the radius of the sphere) but the restriction to a physical location means that we would never regard a part of one star as part of another.

Another concern may be that the "less than" relation allows vectors that only differ by magnitude to be part of each other, for instance $\frac{1}{2}|\beta\rangle$ would be part of $|\beta\rangle$. But, the objection might go, vectors that only differ by magnitude are usually regarded as the same vector in quantum mechanics. However, this is a somewhat artificial result of the normalisation process, whereby any pure state vector that does not have unit length (say after measurement) is normalised to ensure that the chance of *some* result is 1. But quantum mechanical vectors with different sizes, like classical vectors with different sizes, are not considered identical in most circumstances, particularly when addressing the components of a superposition. For instance, the states $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and $|\psi'\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$ are certainly not identical even though the only difference between them is the magnitude of their component vectors.

A final concern may be that the magnitude relation we have proposed effectively severs any relation with vector structure, such as interference effects and superpositions. There are two responses to this objection. The most obvious is that we should not expect the part-whole relation to be vector-like, any more than we would expect the part-whole relation for a pile of rocks to be rock-like. But more pointedly, the fact that vector parthood depends only on magnitude is exactly *because* we are working with vectors. The interesting feature of vector parthood that sets it apart from non-vector parthood is that vectors combine to form superpositions rather than mereological sums. When a vector is added to another, the parts do not retain a distinct identity but merge together to form a new vector. Importantly, when two entities represented by identical vectors are combined, the original parts subsequently become indistinguishable (imagine, for instance, two identical seismic waves combining). Because vector components with the same direction, phase and amplitude cannot be distinguished mathematically or physically, vector parts do not have uniquely identifiable parts of their own. Consequently, if the parthood relation is transitive, every vector will be part of every larger vector. This relation differs from non-vector parthood where weak supplementation holds because (exotic examples aside) for any small part x of y, there will always be some larger part of y that does not contain x as a part.

6. General Reflections

Our view is that our transitive notion of vector parthood provides a better picture of how a vector's orthogonal components can naturally be thought of as parts than existing options. We can see this by returning to our desiderata in §2. First, the account does better regarding adequate coverage. Because our definition requires no preferred basis, it captures parthood relations between quantum states quite generally, not just under a choice of basis. More than that, so far as we can tell, it captures everything one might consider a part in a quantum context. Second, the account is physically meaningful. None of the entities that satisfy the proposed definition seem objectionable from a physical perspective, and it is hard to see what might be left out of our specification. The definition also does not require any distinctions that the underlying physical theory (quantum mechanics) would usually treat as arbitrary.

The account also does a good job of capturing entities that are fit to play the role of parts in quantum mechanics. Our proposal takes the mathematical components of a state vector to straightforwardly represent the physical components of the entity the vector represents. By contrast, both Saunders' and Wilhelm's accounts only allow for a restricted set of vector components to count as vector parts.

As we have seen, while Saunders' and Wilhelm's approaches could be generalised, they would cease to be transitive because they would not allow entities represented by orthogonal vectors to be part of each other. This feature is, in fact, an important theoretical benefit of P_{TV} because it is exactly what we see with physical objects represented with vector quantities. One of the most interesting and well documented aspects of quantum mechanics is that some objects can be split into components that can be split again to produce objects that seem classically impossible. One of the best know is when vertically polarised light is passed through a diagonal filter and then a horizontal filter. If photons obeyed our classical intuitions, we might expect that some light is (possibly) transmitted through the diagonal filter but no light at all should pass through the final horizontal filter. But the light is transmitted through both because the diagonal filter breaks the vertically polarised light into its diagonal parts (one transmitted, one not) and then the (now) diagonally polarised component that made it through is split into its horizontal and vertical components by the final filter (with only the former transmitted).

There is a perfectly natural sense in which the light that emerges from the horizontal filter should be considered a part of the original light. That is, the filters demonstrate that the vertically polarised photons had a horizontally polarised component all along. We should not be surprised by this conclusion, since it is also supported by the mundane mathematical equivalence (where $|V\rangle$, $|H\rangle$ represent vertical and horizontal polarisation and $|+\rangle$ and $|-\rangle$ represent diagonal polarisations):

$$|V\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}} = \frac{(|V\rangle + |H\rangle) + (|V\rangle - |H\rangle)}{2}$$

That is, if we are to regard the components of a vector as its parts, without privileging any particular decomposition (other than requiring the parts to be smaller than the whole), we are led to conclude that even components that cancel each other should be included as valid parts. Non-vector parthood differs because the parts do not interfere with each other, so there is no sense in which the parts could cancel when combined into a whole.

Wilhem's account has no way to accommodate this example, not only because polarisation is not a positional degree of freedom, but also because it requires projections in two different bases. Saunders' account allows that a time extended *branch vector* with horizontally polarised light at one time could be part of a superposition of branch vectors that had vertically polarised light at an earlier time, but only if the beam is actually split in some Everettian branch. If there is no world in which the beam is split, then every branch vector would contain only vertically polarised light, and horizontally polarised light would not appear in any consistent history at any time. The conclusion seems at odds with the clear intuition that we should be able to identify an entity's parts without having to actually separate it into those parts.

In addition to applications in EQM, the proposed vector mereology may find a role in other contexts. If the world is fundamentally constituted by quantum entities that are best represented as vectors (or more generally tensors), then any notion of parthood applied at the fundamental level should almost certainly be a vector (or tensor) mereology.[9] Even outside of the quantum domain, vectors play an important role in classical physics, with many physical entities better represented as vectors than as scalar quantities. Fields can be composed of vector quantities at each point, and objects in those fields represented as perturbations of those vector quantities.

Even for relatively simple waves, questions about what (if anything) might constitute a *part* of a wave remain relatively unexplored. One of the clearest analogies to quantum superpositions are, as indicated earlier, polarised classical waves like seismic shear waves. We can represent the polarisation of the waves as vectors with amplitudes and orientations. After splitting, the original wave becomes two smaller waves, whose amplitude squared sums to the amplitude squared of the original wave. In this case, the notion of vector parthood we have developed would (appropriately) indicate that the entity described by the original vector had parts that were in 1-1 correspondence with the entities described by the two resultant vectors.

The vector parthood relation might even play a role in identifying parts of more common classical waves without a polarity, such as sound or water waves. Just as quantum wave functions can be described with vectors, so too can functions of classical sinusoidal waves. By taking the Fourier transform of linear combinations of sinusoidal waves, we can describe them as functions in a (possibly infinite dimensional) frequency space. In turn, each individual frequency can be represented as a vector orthogonal to all others. Consider whether the note C should be considered part of the chord C major. When the notes of C major (C, E and G) are played together they are represented by a single wave function, which is a superposition of notes at the three different frequencies. Because the chords can be decomposed into orthogonal frequency vectors (representing the individual notes), our vector parthood relation would indicate (again, appropriately) that the note C would be a part of the C major chord.

While these applications for a vector mereology are merely tentative candidates at this stage, they do suggest a promising direction for further inquiry.

Appendix: Splitting

Throughout this appendix, all vectors are non-zero vectors in a Hilbert space $\mathcal{H}$.

Definition A.1 (Orthogonal decomposition): If (i) $\langle\lambda|\lambda'\rangle = 0$, and (ii) $|\lambda\rangle + |\lambda'\rangle = |\epsilon\rangle$, we say that the pair of vectors $|\lambda\rangle, |\lambda'\rangle$ are an orthogonal decomposition of $|\epsilon\rangle$.

[9] Except perhaps in de Broglie-Bohm pilot wave theory, which holds a very classical view of the nature of fundamental particles.

Definition A.2 (Vector Part): Denote by $|\beta_1\rangle$, $|\beta'_1\rangle$ an arbitrary orthogonal decomposition of $|\beta\rangle$. Similarly, for each $j = 1,2,...,$ let $|\beta_j\rangle$, $|\beta'_j\rangle$ be an arbitrary orthogonal decomposition of $|\beta_{j-1}\rangle$. We say a vector $|\gamma\rangle$ is a vector part of $|\beta\rangle$ if and only if either:

(a) $|\gamma\rangle = |\beta\rangle$; or

(b) $|\gamma\rangle \neq |\beta\rangle$ but $|\gamma\rangle = |\beta_n\rangle$ for some choice of the orthogonal decompositions above and some finite $n \in \{1,2 ...\}$

Theorem A.3: Condition (b) of Definition A.2 is satisfied if and only if $\||\gamma\rangle\| < \||\beta\rangle\|$.

Proof. (⇒)

Assume that condition (b) in the vector part definition is satisfied. This means that $|\gamma\rangle \neq |\beta\rangle$ but $|\gamma\rangle = |\beta_n\rangle$ for some finite $n \in \{1, 2, \dots\}$. Using Definition A.2 and condition (ii) from Definition A.1, we can write $|\beta\rangle = |\beta_1\rangle + |\beta_1'\rangle$ which gives us that:

$$\begin{aligned}\||\beta\rangle\|^2 &= \langle\beta|\beta\rangle \\ &= (\langle\beta_1'| + \langle\beta_1|)(|\beta_1'\rangle + |\beta_1\rangle) \\ &= \langle\beta_1'|\beta_1'\rangle + 2\langle\beta_1|\beta_1'\rangle + \langle\beta_1|\beta_1\rangle \\ &= \langle\beta_1'|\beta_1'\rangle + 0 + \langle\beta_1|\beta_1\rangle \\ &= \||\beta_1'\rangle\|^2 + \||\beta_1\rangle\|^2\end{aligned}$$

where we have used Condition (i) of Definition A.1 in the fourth equality. By the same logic, for all $j \in \{1, 2, \dots\}$ we have that $\||\beta_{i-1}\rangle\|^2 = \||\beta_i\rangle\|^2 + \||\beta_i'\rangle\|^2$. Iteratively applying this equation, we have that:

$$\||\beta\rangle\|^2 = \sum_{j=1}^{n} \left\||\beta_j'\rangle\right\|^2 + \||\beta_n\rangle\|^2 \tag{1}$$

Also note that:

$$\begin{aligned}|\beta\rangle &= |\beta_1'\rangle + |\beta_1\rangle \\ &= |\beta_1'\rangle + |\beta_2'\rangle + |\beta_2\rangle \\ &= |\beta_1'\rangle + |\beta_2'\rangle + |\beta_3'\rangle + |\beta_3\rangle \\ &\vdots \\ &= \sum_{j=1}^{n} |\beta_j'\rangle + |\beta_n\rangle\end{aligned}$$

Since $|\beta_n\rangle \neq |\beta\rangle$, there exists some $j \in \{1, \dots, n\}$ such that $|\beta_j'\rangle \neq 0$ and so $\left\||\beta_j'\rangle\right\|^2 > 0$. This, together with Equation (1), implies that $\||\beta_n\rangle\| < \||\beta\rangle\|$, so $\||\gamma\rangle\| < \||\beta\rangle\|$.

Proof. (⇐)

The proof proceeds in three parts:

- Part 1 (global phase) shows that a vector part $|\gamma\rangle$ can have an arbitrary global phase and norm vanishingly close to $|\beta\rangle$.
- Part 2 (direction) shows that a vector part $|\gamma\rangle$ can have the same global phase but arbitrary direction and a norm vanishingly close to $|\beta\rangle$.
- Part 3 (magnitude) shows that a vector part $|\gamma\rangle$ with the same global phase and direction can have an arbitrarily smaller norm than $|\beta\rangle$.

When applied successively, these parts entail that any non-zero vector can be orthogonally decomposed into any other smaller non-zero vector with arbitrary phase, direction and norm. Because these attributes fully exhaust the features of a Hilbert space vector, if it is the case that $0 < \||\gamma\rangle\| < \||\beta\rangle\|$, then condition (b) must be satisfied.

Note that when $|\gamma\rangle$ and $|\beta\rangle$ have the same global phase and direction, only Part 3 of the proof is required. When they have the same global phase but different direction, only Parts 2 and 3 are required. When they have a different global phase, all three parts are required (when the direction is the same, Part 2 is required because $|\gamma\rangle$ is orthogonal to $|\beta\rangle$ in Part 1).

Part 1: Global phase

Goal: To show that some vector $|\gamma\rangle$, which is part of a series of n orthogonal decompositions of non-zero vector $|\beta\rangle$, can have an arbitrary global phase $e^{i\alpha}$ and a norm vanishingly close to $|\beta\rangle$ (but may have a different direction).

Proof: Construct an orthonormal basis $\{|e_1\rangle, |e_2\rangle\}$, such that:

- $|e_1\rangle = \frac{|\beta\rangle}{\||\beta\rangle\|}$
- $|e_2\rangle$ is equal to a unit vector orthogonal to $|\beta\rangle$ with phase $e^{i\alpha}$, i.e. $|e_2\rangle = e^{i\alpha}|\beta'\rangle$ where $\||\beta'\rangle\| = 1$ and $\langle\beta|\beta'\rangle = 0$

For some natural number $n \geq 2$, construct a set of vectors $\{|\beta_j\rangle\}$ such that:

$$|\beta_j\rangle := b\cos^j(\theta_n)\,[\cos(j\theta_n)\,|e_1\rangle + \sin(j\theta_n)\,|e_2\rangle]$$

where $\theta_n = \frac{\pi}{2n}$ and $b = \||\beta_0\rangle\|$, for all $j \in \{1,2,\dots,n\}$.

Also construct a set of vectors $\{|\beta_j'\rangle\}$ such that $|\beta_j'\rangle = |\beta_{j-1}\rangle - |\beta_j\rangle$. As demonstrated in the lemma below, it will always be the case that $\langle\beta_j|\beta_j'\rangle = 0$. When $j = n$ (that is, after n steps in the orthogonal decomposition):

$$\begin{aligned}|\beta_n\rangle &= b\cos^n(\theta_n)\,[\cos(n\theta_n)\,|e_1\rangle + \sin(n\theta_n)\,|e_2\rangle]\\ &= b\cos^n\left(\frac{\pi}{2n}\right)\left[\cos\left(\frac{\pi}{2}\right)|e_1\rangle + \sin\left(\frac{\pi}{2}\right)|e_2\rangle\right]\\ &= b\cos^n\left(\frac{\pi}{2n}\right)|e_2\rangle\end{aligned}$$

As $n \to \infty$, $\cos^n\left(\frac{\pi}{2n}\right) \to 1$, so $\lim_{n\to\infty} |\beta_n\rangle = b|e_2\rangle$.

Consequently:

- Because $|\beta_j'\rangle + |\beta_j\rangle = |\beta_{j-1}\rangle$ and $\langle\beta_j|\beta_j'\rangle = 0$, we can say that $|\beta_n\rangle$ is part of a series of orthogonal decompositions of $|\beta\rangle$.
- Because $|\beta_n\rangle$ is a positive real multiple of $|e_2\rangle$, it has the same phase, which in the original $|\beta'\rangle$ basis is a global phase of $e^{i\alpha}$.
- Because $\lim_{n\to\infty} |\beta_n\rangle = b|e_2\rangle$ and $b = \||\beta_0\rangle\|$, we can say that in the limit, $|\beta_n\rangle$ is vanishingly close in norm to $|\beta\rangle$.

From the points above, vector part $|\gamma\rangle = |\beta_n\rangle$ can an arbitrary global phase and, in the limit of n, $|\gamma\rangle$ will be vanishingly close in norm to $|\beta\rangle$.

Lemma: $\langle\boldsymbol{\beta_j}|\boldsymbol{\beta_j'}\rangle = \mathbf{0}$

First, note that:

$$\begin{aligned}\langle\beta_j|\beta_j'\rangle &= \langle\beta_j|\left(|\beta_{j-1}\rangle - |\beta_j\rangle\right)\\ &= \langle\beta_j|\beta_{j-1}\rangle - \langle\beta_j|\beta_j\rangle\end{aligned}$$

To evaluate the first term, expand $|\beta_{j-1}\rangle$ as:

$$|\beta_{j-1}\rangle = b\cos^{j-1}\theta_n\,[\cos([j-1]\theta_n)\,|e_1\rangle + \sin([j-1]\theta_n)\,|e_2\rangle]$$

Because $\langle e_1|e_2\rangle = \langle e_2|e_1\rangle = 0$ and $\langle e_1|e_1\rangle = \langle e_2|e_2\rangle = 1$, the first term expands to:

$$\langle\beta_j|\beta_{j-1}\rangle = b^2\cos^j(\theta_n)\cos^{j-1}(\theta_n)\,[\cos(j\theta_n)\cos([j-1]\theta_n) + \sin(j\theta_n)\sin([j-1]\theta_n)]$$

Using the identity $\cos A\cos B + \sin A\sin B = \cos(A-B)$, we get:

$$\begin{aligned}\langle\beta_j|\beta_{j-1}\rangle &= b^2\cos^j(\theta_n)\cos^{j-1}(\theta_n)\,[\cos(j\theta_n - [j-1]\theta_n)]\\ &= b^2\cos^j(\theta_n)\cos^{j-1}(\theta_n)\cos(\theta_n)\\ &= b^2\cos^{2j}(\theta_n)\end{aligned}$$

Evaluating the second term:

$$\begin{aligned}\langle\beta_j|\beta_j\rangle &= b^2\cos^{2j}(\theta_n)\,[\cos^2(j\theta_n) + \sin^2(j\theta_n)]\\ &= b^2\cos^{2j}(\theta_n)\end{aligned}$$

So:

$$\begin{aligned}\langle\beta_j|\beta_j'\rangle &= \langle\beta_j|\beta_{j-1}\rangle - \langle\beta_j|\beta_j\rangle\\ &= b^2\cos^{2j}(\theta_n) - b^2\cos^{2j}(\theta_n)\\ &= 0\end{aligned}$$

Part 2: Direction

Goal: To show that some vector $|\gamma\rangle$, which is part of a series of n orthogonal decompositions of non-zero vector $|\beta\rangle$ with the same global phase, can have an arbitrary direction and a norm vanishingly close to $|\beta\rangle$.

Proof: Given a vector $|\beta\rangle$ and a unit vector with the same global phase but arbitrary direction $|\alpha\rangle$, construct an orthonormal basis $\{|e_1\rangle, |e_2\rangle\}$ such that:

- $|e_1\rangle = \frac{|\beta\rangle}{\||\beta\rangle\|}$ (setting the global phase to 1 by construction)
- $|e_2\rangle$ is a unit vector in the real span of $|\beta\rangle$ and $|\alpha\rangle$ that is orthogonal to $|\beta\rangle$ and has a positive inner product with $|\alpha\rangle$.

Therefore $|\alpha\rangle$ can be expanded as $|\alpha\rangle = a_1|e_1\rangle + a_2|e_2\rangle$ where a_1 and a_2 are real and positive, and ${a_1}^2 + {a_2}^2 = 1$. Following the same logic as Part 1, for some natural number $n \geq 2$, construct a set of vectors $\{|\beta_j\rangle\}_{j\in\{1,2,\dots,n\}}$ such that:

$$|\beta_j\rangle := b\cos^j(\theta_n)\,(\cos(j\theta_n)\,|e_1\rangle + \sin(j\theta_n)\,|e_2\rangle)$$

where $b = \||\beta\rangle\|$, but this time $\theta_n = \frac{\phi}{n}$, where ϕ is determined by:

$$\begin{aligned}\cos(\phi) &= \frac{|\langle e_1|\alpha\rangle|}{\||e_1\rangle\|\,\||\alpha\rangle\|} \\ &= \frac{a_1}{\sqrt{{a_1}^2 + {a_2}^2}} \\ &= a_1\end{aligned}$$

Construct a set of orthogonal vectors $\{|\beta_j'\rangle\}$ as in Part 1. When $j = n$ (that is, after n steps in the orthogonal decomposition):

$$\begin{aligned}|\beta_n\rangle &= b\cos^n(\theta_n)\,(\cos(n\theta_n)\,|e_1\rangle + \sin(n\theta_n)\,|e_2\rangle) \\ &= b\cos^n\left(\frac{\phi}{n}\right)(\cos(\phi)\,|e_1\rangle + \sin(\phi)\,|e_2\rangle) \\ &= b\cos^n\left(\frac{\phi}{n}\right)(a_1|e_1\rangle + a_2|e_2\rangle) \\ &= b\cos^n\left(\frac{\phi}{n}\right)|\alpha\rangle\end{aligned}$$

As $n \to \infty$, $\cos^n\left(\frac{\phi}{n}\right) \to 1$, so $\lim_{n\to\infty}|\beta_n\rangle = b|\alpha\rangle$.

Consequently, using similar reasoning to Part 1, we can see that:

- $|\beta_n\rangle$ is part of a series of orthogonal decompositions of $|\beta\rangle$.
- $|\beta_n\rangle$ is a positive real multiple of $|\alpha\rangle$, so $|\beta_n\rangle$ has the same (arbitrary) direction as $|\alpha\rangle$.

- $\lim_{n\to\infty} |\beta_n\rangle = b|\alpha\rangle$ and $b = \||\beta\rangle\|$, so in the limit $|\beta_n\rangle$ is vanishingly close in norm to $|\beta\rangle$.

From the points above, vector part $|\gamma\rangle = |\beta_n\rangle$ can have an arbitrary direction and, in the limit of *n*, can be vanishingly close in norm to $|\beta\rangle$.

Part 3: Magnitude

Goal: To show that for any vector $|\beta\rangle$ and smaller vector $|\beta_2\rangle = \lambda|\beta\rangle$, where $0 < \lambda < 1$, that $|\beta_2\rangle$ is part of a series of orthogonal decompositions of $|\beta\rangle$.

Proof: Construct an orthonormal basis $\{|e1\rangle, |e2\rangle\}$, such that:

- $|e_1\rangle = \frac{|\beta\rangle}{\||\beta\rangle\|}$ (setting the global phase to 1 by construction)
- $|e_2\rangle$ is a unit vector orthogonal to $|\beta\rangle$, with the same phase

So:

$$|\beta_2\rangle = b\lambda|e_1\rangle$$

where $b = \||\beta\rangle\|$. Based on the above, we can define the following vectors:

$$|\beta_1\rangle := b[\lambda|e_1\rangle + \sqrt{\lambda(1-\lambda)}\,|e_2\rangle]$$

$$|\beta_1'\rangle := b[(1-\lambda)|e_1\rangle - \sqrt{\lambda(1-\lambda)}|e_2\rangle]$$

$$|\beta_2'\rangle := b[\sqrt{\lambda(1-\lambda)}\,|e_2\rangle]$$

These vectors form a series of orthogonal decompositions of $|\beta\rangle$, because (noting that complex conjugation is not required because λ is real):

$$\langle\beta_1|\beta_1'\rangle = b^2[\lambda(1-\lambda) - \sqrt{\lambda(1-\lambda)}\sqrt{\lambda(1-\lambda)}]$$

$$= b^2[\lambda(1-\lambda) - \lambda(1-\lambda)]$$

$$= 0$$

and it is easy to check the remaining conditions:

$$\langle\beta_2|\beta_2'\rangle = 0$$

$$|\beta\rangle = |\beta_1\rangle + |\beta_1'\rangle$$

$$|\beta_1\rangle = |\beta_2\rangle + |\beta_2'\rangle$$

$$|\beta_2\rangle + |\beta_2'\rangle + |\beta_1'\rangle = |\beta_0\rangle$$

Consequently, using similar reasoning to Part 1 and 2, we can see that:

- $|\beta_2\rangle$ is part of a series of orthogonal decompositions of $|\beta\rangle$.
- $|\beta_2\rangle$ is a real positive multiple of $|\beta\rangle$, so $|\beta_2\rangle$ has the same direction as $|\beta\rangle$.
- $|\beta_2\rangle = \lambda|\beta\rangle$ holds for all $0 < \lambda < 1$, so $|\beta_2\rangle$ can be any vector with a norm smaller than $|\beta\rangle$ with the same phase and direction.

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