Spacetime Models Without Points: The Case of Noncommutative Geometry

Enrico Maresca[1,2]

[1]Department of Civilisations and Forms of Knowledge, University of Pisa, via Pasquale Paoli 15, Pisa, 56126, Italy.
[2]Department of Humanities and Philosophy, University of Florence, via della Pergola 58/60, Florence, 50121, Italy.

Contributing authors: enrico.maresca@phd.unipi.it;

Abstract

Over the past decades, noncommutative geometry (NCG) has emerged as a prominent research project in quantum gravity (QG). This accounts for non-classical effects by introducing a noncommutative parameter and systematically deforming the relativistic spatiotemporal framework. In doing so, NCG challenges the definability of core geometric structures, including spacetime points (the so-called *pointlessness problem*). As a result, it threatens to undermine the foundations and viability of NCG as a candidate approach to QG. In this paper, I examine the pointlessness problem and characterise it as a breakdown of classical dualities between algebraic and geometric frameworks. This yields both the proliferation of inequivalent notions of points and the absence of standard differential-geometric models for noncommutative theories. By contrast, I argue that novel dualities can be recovered in the noncommutative setting, allowing for the identification of more sophisticated geometric models. I conclude that pointlessness alone is insufficient to support claims of non-spatiotemporality in NCG without further qualification.

Keywords: Noncommutative geometry, Algebra, Points, Pointlessness, Dualities

1 Introduction

Noncommutative geometry (NCG) has been a growing research field in mathematics and theoretical physics over the last decades. Key approaches aim to identify novel

mathematical structures by suitably modifying known commutative algebras.[1] These modifications introduce a noncommutative parameter into commutative algebras by deforming the composition rules between their elements. Physically, this deformation represents non-classical effects that are expected to become relevant at or beyond the Planck scale. The novel structure described by NCG is termed *noncommutative spacetime* (NCST).

In this context, NCG has been recently developed not only as a proposal to extend the Standard Model of particle physics and include gravitational interaction (see, e.g., [5, 6]), but also as a potential approach to quantum gravity (QG) (see, e.g., [7–9]). However, introducing a noncommutative parameter into existing theories radically alters the standard geometric framework offered by special relativity (SR) and general relativity (GR). Core geometric objects and physical structures become ill-defined due to the elimination of more fundamental spatiotemporal notions, including spatiotemporal points. This feature of many noncommutative theories, known as *pointlessness* or *point-freedom*, raises concerns about how to reconcile this novel picture with the definability of core physical structures that, in standard treatments, necessarily depend on spacetime points. Failure to address this issue in applications of NCG to QG indicates disappearance of spacetime.

Of course, pointlessness is not exclusive to NCG: notable examples include quantum mechanics (QM) and manifold anti-realist analyses of GR.[2] Nevertheless, within the specific context of QG, it threatens to undermine the foundations of candidate theories of NCST in two main ways: by challenging the formal coherence of their mathematical constructions, and by eliminating interpretative postulates tied to notions as localisation and pointwise evaluation of field quantities. For example, [16] emphasises that geometric points are undefinable in the noncommutative context: the failure of a number of classical dualities connecting algebraic and geometric constructions indicates the breakdown of the classical picture.

Similarly, Huggett, Lizzi, and Menon argue that NCG suffers from a *pointlessness problem* [15]: noncommutativity is incompatible with the standard notion of sharp localisation of events within arbitrarily small spacetime regions. The mathematical and physical significance of small spacetime regions becomes unclear in the noncommutative regime, indicating the need for an algebraic rather than geometric perspective as a potential solution: "[i]n noncommutative field theories, fields-first interpretations are the only game in town" (4697). The authors view pointlessness in NCG as motivation for "a fundamental metaphysics that eschews the concept of arbitrary localisability" (*ivi*). Moreover, on their view, NCG motivates efforts to recover

---

[1]The mathematical description of these modifications, albeit diversified, can be classified within two major approaches: the quantum group approach and the spectral triple approach. For the former, see, e.g., [1, 2]. For the latter, see the *locus classicus* [3], as well as [4].

[2]This point is emphasised, e.g., in [10, ch. 8]. For more recent discussions, see [11–14], as well as [15].

physical spacetime from a noncommutative algebra, thus circumventing issues related to the disappearance of spacetime.[3]

In this paper, I provide a thorough analysis of the pointlessness problem in NCG. Noncommutativity addresses various results against the sharp localisability of physical events in spacetime dating back to the 1930s; however, it also invalidates crucial mathematical relationships between algebraic and geometric treatments of spacetime physics that otherwise hold in the commutative domain. Given this, the paper addresses the question of whether NCG genuinely suffers from pointlessness in a mathematical sense. I argue that this question, as generally stated, is ill-posed: its answer depends on specific metatheoretical and interpretative assumptions, as well as on particular reconstructions of the pointlessness problem. These assumptions can be questioned in order to mitigate Huggett, Lizzi, and Menon's radical conclusion that only an algebraic approach to NCG is viable.

Specifically, I argue that the violation of classical algebraic-geometric dualities caused by noncommutativisation does not raise any serious concern or unexpected result for the foundations of NCST theories. Instead, it encourages research into new duality results that bridge the gap between the noncommutative algebraic picture and a (possibly novel) geometric one. In this sense, I propose a more charitable interpretation of pointlessness: the failure of commutative correlations between definitions of points does not preclude the possibility of identifying alternative definitions and constructions for the noncommutative regime; these new definitions serve the same functions of classical points within the appropriate domain. Put differently, rejecting standard differential geometry does not undermine efforts to discover new geometric structures that can be dual to a given noncommutative algebra and that, at the same time, contain new objects fulfilling the same relevant functions as geometric points. These novel geometric structures, if identified, replace classical geometric points and restore crucial dependence relations between elements of the NCST theory.

To support my claims, the paper is organized as follows. In Section 2, I trace the pointlessness problem back to an operationalist argument against localisability from 1936. Here, noncommutativity offers a suitable framework for eliminating spatiotemporal points; moreover, it serves as a basis for developing an appropriate formalism that incorporates this fundamental limitation. In Section 3, I refine the conclusion of this argument and classify different types of pointlessness. This distinction helps to clarify various instances of the pointlessness problem in NCST theories. In Section

---

[3]This problem points to the elimination of spatiotemporal structures from the ontology of the theory in question, replaced instead by novel, non-spatiotemporal degrees of freedom. The disappearance of spacetime is a central concern raised by several candidate QG theories. For philosophical discussions, see, e.g., [17–20]. This issue is typically analysed alongside that of spacetime emergence, often proposed as a strategy for recovering the missing spatiotemporal structures and features [21, 22]. However, the present paper defends NCST approaches against the charge of spacetime disappearance. Accordingly, I set aside the related question of spacetime emergence: if the central claim advanced here is correct, such a response becomes unnecessary. At most, one might argue that only *specific* features (those deemed necessary for spatiotemporality) need to emerge in an appropriate limit. These could include, for instance, particular separability conditions (see Section 5). On this view, their absence would call for context-dependent solutions, potentially diverging from those developed within other QG approaches: for this line of argument, see [23]. Nevertheless, as argued below, such features may be merely contingent rather than essential to spatiotemporality, and thus their emergence is not required. Spacetime points, arguably, provide a relevant example. A detailed examination of this possibility is left for future work.

4, I examine the issue in the commutative context, focusing on classical algebraic-geometric dualities and the relationship between alternative definitions of points. Then, in Section 5, I analyse the implications of introducing noncommutativity for the breakdown of these commutative relations. Finally, in Section 6, I respond to these issues and revise the conclusion of the argument for mathematical pointlessness.

2 The General Problem with Points

From a historical perspective, theories of quantum spacetime "without points" first gained attention due to a series of arguments against spatiotemporal localisation developed between the 1930s and early 1990s.[4] These arguments aimed to demonstrate that, under suitable operationalist assumptions,[5] the accuracy of localisation procedures for events in spacetime is limited at high-energy scales [28]. Bronstein's [29] argument serves as prototype for constructing such arguments. The core assumptions are:

(B1) GR provides an accurate description of spacetime geometry at all scales.
(B2) For any target system and desired observable, one can always identify a probe adequate to measure it.

While (B1) allows the extension of GR's domain of applicability, particularly to high energies, (B2) introduces a strict operationalist methodology: a probing procedure is considered adequate only if it yields a precise value for the measured quantity.

In Bronstein's setup, the relevant quantities are the Christoffel symbols associated with a spacetime region, and the corresponding probe is a massless quantum particle. Minimising the combined quantum mechanical and gravitational uncertainties carried by the probe necessitates increasing its energy, and therefore its mass. In regimes of strong gravity, however, GR predicts that spacetime reacts to this increase in mass: large-curvature effects lead to the formation of an event horizon, which prevents any information about the Christoffel symbols from reaching the observer. This paradoxical conclusion undermines the operationalist premise. Conversely, if the Schwarzschild radius of the resulting mini black hole is treated as an uncrossable boundary, the outcome of the measurement becomes uncertain, again in violation of the operationalist methodology.

Such Bronstein-like paradoxes arise from the conjunction of three assumptions (operationalism, strong gravity, and the unrestricted validity of GR). The root of the issue lies in the incompatibility between the quantum nature of the probe and the classical description of spacetime. Doplicher, Fredenhagen, and Roberts [26] extend this reasoning by showing that even when the probe is classical and the target system is quantum, sharp localisation of the latter in spacetime remains operationally impossible.

---

[4]Relevant examples of such localisation arguments include, for example, [24] and [25, 26].

[5]Importantly, these operationalist assumptions echoed Bridgman's operationalism: spacetime geometry is physical if and only if, in principle, one can identify suitable operations and protocols for measuring the associated quantities. For a detailed analysis of the role of traditional operationalism in the development of the notion of quantum spacetime, see [27]. Furthermore, note that this condition can be relaxed: see, e.g., the definition of *tempered operationalism* in [15, 4708–4711], which the authors explicitly apply to the problem of spacetime disappearance in NCG.

This family of arguments has far-reaching implications for our definition of spacetime. If spacetime points are conceived as possible locations of events, and an operationalist framework consistent with (B2) is adopted, then beyond a certain energy scale, the very notion of spacetime points becomes operationally meaningless. Consequently, they should be excluded from the theory's ontology. The paradox can be avoided only by abandoning at least one of the core premises, though the argument itself does not specify which one. There are four main options:[6]

1. *Reject (B1)*: Limit the domain of GR to low-energy regimes. This necessitates developing a new theory of spacetime for higher-energy scales that accounts for expected non-classical effects.
2. *Reject (B2):* Abandon operationalism entirely as theoretically untenable. In this case, alternative principles would be required to distinguish physical from non-physical structures.
3. *Restrict operationalism:* Instead of discarding operationalism entirely, limit its applicability to low-energy regimes where the paradox does not arise. At Planckian scales, where classical concepts are expected to break down, operationalism would no longer apply (but see, e.g., [31]).
4. *Introduce quantum probes:* As proposed by Doplicher, Fredenhagen, and Roberts, one can revise the setting by requiring the probes themselves to be quantum systems. This approach circumvents Bronstein's paradox and calls for further development.

These four options are largely compatible with each other.[7] Historically, different approaches to QG have responded to the localisation arguments by endorsing various combinations of these options. Notably, the NCST approaches of the early 1990s embraced pointlessness and sought to resolve the paradox through a combination of options 1 and 3. In this context, noncommutativity provides a mathematically rigorous framework for eliminating points. Moreover, the structure of quantum mechanical phase space offered a familiar and useful model for introducing a fundamental bound via a noncommutative product.

3 The Pointlessness Problem

The incorporation of pointlessness into a noncommutative theory of QG, while natural, raises significant concerns due to the breakdown of various mathematical structures. Most notably, it raises the issue of replacing the familiar four-dimensional smooth

---

[6]A fifth option is to argue that, beyond the horizon, the target system remains sharply localised regardless of our ability to access that information. This view is incompatible with operationalism, and therefore with (B2): any realistic observer would be unable to access the information after preparing the experiment. Still, one might argue that black hole evaporation does support such a possible solution. The evaporation time is approximately equal to $(m/M_P)^3 10^{-39} s$, so a mini black hole, like the one produced in Bronstein-like scenarios, with mass $m \approx M_P$, would actually evaporate very quickly. The viability of this solution then hinges on whether the evaporation is unitary or entails an information loss scenario: see [30]. Moreover, the formation of a singularity due to large curvature effects would still disturb the position of the target system within the event horizon, thereby introducing additional uncertainty.

[7]The only exception is, of course, options 2 and 3: the complete rejection of operationalism at all scales automatically precludes its restriction to low-energy regimes

metric framework with a new geometric structure, compatible with a novel noncommutative parameter. It has been argued, in fact, that NCG challenges the very notion of point. In this paper, I examine this issue from the perspective of mathematical physics.[8]

Points are frequently introduced as geometric primitives or as basic set-theoretic elements upon which additional structure is defined. A more refined treatment, primarily developed within algebraic geometry, imposes two conditions for the definability of points in an algebraic framework.

The first is a *syntactic criterion*:

(AP1) Points are definable in a theory T within an algebraic framework if there exists a suitable set of predicates $\Phi_1, \ldots, \Phi_n$, all expressible in the algebraic language of the framework, such that $(\exists x)(\Phi_1(x) \wedge \cdots \wedge \Phi_n(x))$ is derivable from T.[9]

Typical properties captured by these predicates include: being a pure state of the algebra; being an equivalence class of one-dimensional irreducible representations of the algebra; or being a maximal ideal of the algebra (Section 4.1). The element that satisfies all these predicates can be denoted by an individual constant s, such that $\Phi_1(s) \wedge \cdots \wedge \Phi_n(s)$ is derivable in the algebraic theory T. This s is called an *algebraic point*.

The second is a *semantic criterion*:

(AP2) Given an algebraic theory T, there exists at least one algebraic model M of T (i.e., a model formulated within an algebraic framework) whose domain includes a structure $\bar{t}$ such that $M \vDash \Phi_1(t) \wedge \cdots \wedge \Phi_n(t)$.

In other words, (AP2) requires that a model of the theory exist in which there is an element $\bar{t} \in \bar{\Phi}_1 \cap \cdots \cap \bar{\Phi}_n$, where $\bar{\Phi}_1, \ldots, \bar{\Phi}_n$ are now interpreted as relations within the model.[10]

An algebraic theory T fails to define algebraic points if either: (i) none of the predicates $\Phi_1, \ldots, \Phi_n$ is expressible in its algebraic language, or the relevant conjunctive formula is not derivable, violating (AP1); or (ii) no model of T exists that contains a structure satisfying the conjunction of those predicates, violating (AP2). Either case indicates that the theory T exhibits *algebraic pointlessness*.

Analogous conditions can be formulated for *geometric points*. (GP1) thus indicates a syntactic criterion for geometric points, those which are definable in a geometric theory T, and (GP2) indicates a semantic criterion for geometric points. Relevant

---

[8]It is worth emphasising that the status of spacetime points has been extensively discussed in the recent literature. Notable work stemmed from the interpretation of spatiotemporal symmetries and diffeomorphism invariance, especially in debates concerning Einstein's hole argument. These have been mainly concerned with the subtle relationship between spacetime points and coordinate systems in relativistic theories: see, e.g., [32]. In the rest of this paper, I will not discuss these contributions. Indeed, NCG offers an entirely coordinate-free perspective on pointlessness. In this context, identification and labelling problems are irrelevant for both the claim I defend and the arguments I examine.

[9]In this paper, I use the term *expressible* in its technical meaning: a predicate is expressible in a language if it can be defined within that language. In other words, there is a logical formula, written in that language, that captures the same meaning as the concerned predicate. I use the term *framework* to designate a language enriched with all the results, techniques, and definitions employed to build theories and study the definable structures formulated in that language.

[10]Note that (AP2) depends on the possibility of identifying a suitable definition of algebraic point in terms of expressible predicates. Consequently, if such a definition cannot be provided, then any assessment of (AP2) becomes impossible: the criterion is simply meaningless for the theory in question.

predicates, in this case, include: being a primitive set-theoretic constituent of a topological space (where the topology is conceived as an additional structure); being an adimensional primitive element.

Full mathematical pointlessness generalises this notion to include geometric frameworks. It arises when the relevant properties of points cannot be expressed in algebraic or geometric language, violating both (AP1) and (GP1), or when there exists no algebraic or geometric models of the theory that include suitable point-like structures, violating both (AP2) and (GP2). In this context, one may then speak of elimination of geometric points and failure of standard geometric constructions to describe the relevant spaces.

It is important to note that the examples given for point predicates in (AP1) rely on the classical (commutative) case. Huggett, Lizzi, and Menon [15] extend the scope of these commutative results to reveal the implications of noncommutativity for physical geometry.[11] Mathematical pointlessness, a fortiori, gives rise to *physical pointlessness*: the inability to define spacetime points as physical entities.

Accordingly, the *pointlessness problem* is defined as the inability to reconstruct the notion of a mathematical (algebraic or geometric) or spacetime point within a given theory. A theory that exhibits this problem raises two distinct concerns.

First, pointlessness challenges the formal coherence of current mathematical constructions. Eliminating geometric (or algebraic) points can render many foundational structures ill-defined, including differentiable manifolds. Mathematical theories that rely on point-based structures are thereby excluded from noncommutative descriptions of the domain of applicability of the theory. In response, new point-free mathematical frameworks must be identified and developed to replace standard ones.

Second, pointlessness may conflict with current interpretative postulates. For example, Teller characterises field configurations as assignments of specific values to determinable quantities at each spacetime point ([33, 95]; see also [34, 148–152]). On this view, a field quantity is defined only if its value can be assigned to each spacetime point.[12] Similarly, to say that a system is located somewhere in spacetime often entails specifying the smallest region it occupies: possibly, a point. Thus, physical pointlessness demands the revision of these interpretative principles in order to preserve essential theoretical roles within affected physical theories.

Finally, in the context of QG, pointlessness raises the issue of the disappearance of spacetime. Classically, spacetime is composed of spacetime points, conceived as possible locations of events, and its structure is built from relations among these points, such as separability and distance. The pointlessness problem, therefore, challenges the standard picture of spacetime at the relevant scales. As discussed below, the issue stems

---

[11]*Physical geometry* indicates a geometric structure that purports to represent a certain physical context of investigation. The representation relation is specified by a suitable interpretation; it can be extremely complicated and possibly depend on the specific investigative context to which the geometric theory applies. In this paper, I do not endorse any specific account of physical representation or interpretation. Indeed, the prime topic of this paper is *mathematical* pointlessness, as opposed to *physical* pointlessness. Of course, mathematical pointlessness is definitely motivated by physical concerns, yet it is not conditioned by any choice of a specific account of physical representation.

[12]Despite this, Teller's principle is largely outdated. Moreover, the necessity of points in QFT for expressing field quantities does not directly imply the existence of spatiotemporal points in QG, nor does the converse hold.

from the extension of commutative constructions into the noncommutative domain: an extension that, in many cases, contradicts the principles of NCG.

In this paper, I focus on mathematical pointlessness. As will become clear in the following discussion, assessing whether mathematical pointlessness obtains, and evaluating the issues it raises, requires a careful examination of several controversial issues. To be sure, within a NCST theory, mathematical pointlessness, if established, would entail physical pointlessness: if the relevant algebra lacks points, then no point can be spatiotemporal. However, this does not mean that physical pointlessness is entirely reducible to purely mathematical considerations.

More specifically, in what follows I challenge the claim that mathematical pointlessness arises in noncommutative settings. Consequently, critics of noncommutative approaches to QG must provide independent arguments in support of physical pointlessness. The presence or absence of points as mathematically defined entities is, strictly speaking, independent of their role as physically significant objects within a theory. For this reason, although physical considerations motivate many of the claims advanced in this paper, a systematic examination of physical pointlessness will be left to future work.

4 Algebraic-Geometric Dualities

Textbook presentations of GR describe spacetime as a four-dimensional, semi-Riemannian manifold equipped with appropriate structures. The manifold functions as a model of spacetime within GR, being defined, together with an appropriate metric, as a solution to Einstein field equations modulo diffeomorphism invariance. This mathematical framework draws on geometric insight and employs the language, results, and techniques of differential geometry to describe the possible models of the theory. Differential geometry is indeed expressive enough to characterise the relevant models: I refer to this as the *geometric picture of spacetime*.

The geometric picture can be extended beyond differential geometry to include more general geometric structures as viable spacetime models. This extension allows for the incorporation of additional features and structures that are not expressible within standard differential geometry, yet can still qualify as legitimate models of the theory. Relevant examples include locales, quantales, toposes, non-Hausdorff topological spaces, and similar constructions. I refer to this broader framework as the *generalised geometric picture of spacetime*.

In parallel, the same models of GR can also be formulated within an algebraic framework, where spacetime structure is described in terms of algebras of fields defined over the manifold. I refer to this approach as the *algebraic picture of spacetime*. These constructions are often presented in terms of a duality result, commonly referred to as *algebraic-geometric duality*, which connects geometric structures to a specific class of algebraic theories that are equivalent to them.[13] As discussed below, the precise

---

[13]It should be noted that algebraic-geometric dualities are *mathematical* results. As such, most of the literature on dualities between *physical* theories does not apply straightforwardly here. Rather, a relevant point of contention may be whether certain choices between dual mathematical frameworks can enhance understanding of the investigated physical domain, or the extent to which such choices reveal underlying ontological commitments. Due to length constraints, I leave this issue open to future exploration.

nature of the equivalence established by such dualities is a crucial aspect of their interpretation.

Classical algebraic-geometric dualities apply to differentiable manifolds and enable the recovery of an algebraic picture on two levels. In Section 4.1, I examine local applications: here, dualities facilitate the definition of geometric points in terms of specific algebraic structures, thereby bridging the gap between criteria (AP1) and (GP1). In Section 4.2, I then consider global applications: in these cases, algebraic-geometric dualities relate the models of candidate theories of spacetime in the two pictures, thus bridging the gap between (AP2) and (GP2).

4.1 Local Dualities

In the commutative domain, algebraic points can be identified in several distinct ways. Each approach amounts to specifying a predicate Φ_i that appears in (AP1), is expressible within an algebraic framework, and is saturated by the logical constants of the theory. As discussed below, examples of such predicates include: being a pure state, being an element of the structure space, and being a maximal ideal of the algebra. At the same time, geometric points can be defined as the fundamental constituents of an appropriate topological space upon which the algebra acts. The precise relationship between algebraic and geometric points is captured by a number of classical dualities. These dualities provide systematic methods for reconstructing algebraic points from geometric ones, and vice versa, thereby establishing derivation procedures between the two frameworks (corresponding to principles (AP1) and (GP1)).

The derivation of algebraic points from an underlying topological space is a standard and relatively straightforward construction. Given a topological space X, and the algebra $C(X)$ of continuous functions on it (with pointwise operations), each geometric point $x \in X$ corresponds to an *evaluation map* $\epsilon_x : f \mapsto f(x)$. The associated algebraic point is then characterised as a homomorphism $C(X) \to \mathbb{R}$ (or $\mathbb{C}$). The set of all such evaluation homomorphisms, denoted by $|C(X)|$, is called the *dual space of $\mathbb{R}$-points* (respectively, $\mathbb{C}$-*points*) *of* $C(X)$.[14]

This construction admits generalisation: in broader contexts, algebraic points can be identified as the *pure states* of an algebra $\mathcal{A}$. This definition does not presuppose prior knowledge of an underlying topological space. States are specific linear functionals on $\mathcal{A}$, and pure states are those that cannot be expressed as convex combinations of other states. Assume now that $\mathcal{A}$ is commutative: this assumption will hold for the rest of the section. Then, the previous construction can be inverted by defining pure states as δ-functions peaked at individual points of a manifold. For any $x \in X$, one defines $\delta_x : \mathcal{A} \to \mathbb{C}$, where $\delta_x(f) = f(x)$. Pure states are thus in bijective correspondence with geometric points: each geometric point gives rise to a pure state that evaluates the elements of $\mathcal{A}$ at that point, while each pure state abstracts the corresponding geometric information.

Moreover, algebraic points can also be reconstructed using *zero sets*, i.e., subsets of the space where elements of the algebra vanish. [11] suggests this approach in

---

[14]In general, this dual space can contain either more or fewer evaluation maps than the underlying geometric points. This issue is resolved by stipulating that the algebra be *geometric* (as in the case of $C(X)$); that is, the intersection of the kernels of all the $\mathbb{R}$-points is trivial: see [35].

the context of regular topological spaces, where for any closed set and any point outside that set, the two can be separated by disjoint open neighbourhoods. Regularity implies that the collection of zero sets forms a basis for the closed sets in the topology of X. Each geometric point corresponds to a unique family of non-empty zero sets: specifically, a nested sequence of such sets whose intersection converges to a single geometric point.[15]

The ideals of $\mathcal{A}$ correspond to families of zero sets, and in particular, *maximal ideals* correspond to the geometric points at which all elements of the ideal vanish. This leads to another definition of algebraic points: as maximal ideals of $\mathcal{A}$. For any $x \in X$, the set $\mathfrak{m}_x = \{f \in \mathcal{A} | f(x) = 0\}$ is a maximal ideal of $\mathcal{A}$. Conversely, every maximal ideal $\mathfrak{m}$ of $\mathcal{A}$ can be expressed as $\mathfrak{m}_x$ for some geometric point x of the underlying space.[16] In fact, each $\mathfrak{m}_x$ is a kernel of the pure state δ_x, and this allows one to define a functional $\mathcal{A} \to \mathcal{A}/\mathfrak{m}$ from the $\mathbb{C}$-algebra to its quotient, which is isomorphic to $\mathbb{C}$. Geometric points are then reconstructed from the vanishing loci of these functionals, i.e., the points at which all elements of the corresponding maximal ideal vanish.[17]

Finally, algebraic points can also be defined as equivalence classes of irreducible representations of $\mathcal{A}$. An irreducible representation of $\mathcal{A}$ is a pair (X, π), where only trivial subspaces of X are invariant under π. Two such representations are considered equivalent if they are related by a unitary transformation. The *structure space of* $\mathcal{A}$, denoted $\hat{\mathcal{A}} = \operatorname{Irrep}(\mathcal{A})/\sim$, consists of equivalence classes of its irreducible representations.

If $\mathcal{A}$ is unital, the structure space is compact and Hausdorff, and one can show that irreducible representations (up to unitary equivalence) correspond to pure states of the algebra, and thus to the underlying geometric points. Moreover, each irreducible representation encodes information about the algebra at a specific point, often in the form of a matrix algebra or an infinite-dimensional Hilbert space. In the commutative case, Schur's lemma implies that every irreducible representation is one-dimensional, reflecting the fact that each point contains complete information about how the algebra acts.

Importantly, these three definitions (via pure states, maximal ideals, and irreducible representations) are all equivalent in the commutative setting. They yield isomorphic spaces of geometric points, provided that appropriate topologies are defined. For instance, each pure state of a commutative C^*-algebra $\mathcal{A}$ gives rise, via the GNS construction, to an irreducible representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ is the algebra of bounded operators on the Hilbert space $\mathcal{H}$. Irreducibility follows from

---

[15]By regularity, if an element of $\mathcal{A}$ vanishes on an open region of X, then it also vanishes on the closure of that region. Furthermore, note that Rynasiewicz's construction of zero sets is equivalent to that proposed in [36], which is itself based on [35]. Indeed, the former uniquely determines the $\mathbb{R}$-points of the latter, and vice versa: see [36, 311, ft. 18]. Finally, note that [35] advocates for a different construction of points using prime ideals rather than maximal ideals.

[16]This is very common in algebraic geometry, where constructions often use the Gelfand-Kolmogorov theorem or Hilbert's *Nullstellensatz*, when X is the affine variety that corresponds to the algebra under investigation.

[17]For example, in algebraic geometry, $\mathbb{C}[x_1, \ldots, x_n]$ is the ring of polynomials over $\mathbb{C}$. Each geometric point is identified by its coordinates $(a_1, \ldots, a_n)$, and bijectively associated with the maximal ideal consisting of all polynomials in the ring that vanish at $(a_1, \ldots, a_n)$.

the purity of the state. The kernel of π is then a maximal ideal of $\mathcal{A}$.[18] This equivalence ensures that geometric points can be reconstructed from algebraic data regardless of the specific local duality used. In this precise sense, geometric points are *well-defined*.

These local dualities establish a systematic correspondence between the basic constituents of algebraic and geometric theories, coherently bridging alternative definitions of algebraic and geometric points. In the commutative case, they guarantee that satisfaction of an algebraic criterion entails satisfaction of the corresponding geometric definition: geometric points are thereby defined within the dual (geometric) theory. As a result, local algebraic-geometric dualities offer a principled answer to the question of how points should be defined in both frameworks, by establishing a consistent translation procedure between them.

4.2 Global Dualities

Global algebraic-geometric dualities relate the semantic categories of the relevant theories. For any given theory, its *semantic category* consists of all its models (as formulated in either the geometric or algebraic framework) and the homomorphisms between them (i.e., the morphisms of the category).[19] These global dualities ensure that each geometric model (i.e., solution) of the theory can be translated into an equivalent algebraic model, and vice versa. Therefore, they bridge the gap between conditions (AP2) and (GP2).

To illustrate, consider the categories $\mathbf{T_{C,H}}$ and $\mathbf{C^*Alg_c}$. The former includes compact Hausdorff topological spaces as objects and homeomorphisms as morphisms. The latter has commutative unital C^*-algebras as objects, and algebra homomorphisms as morphisms. A duality between these categories is a correspondence between their categorical structures: specifically, how the morphisms connect the respective objects.

More generally, two categories are *dual* if they are connected by two contravariant functors, whose composition is naturally isomorphic to the identity functor.[20] Each category thus "mirrors" the other, in the sense that they have isomorphic structures, but with arrows reversed. The *Gelfand duality theorem* establishes precisely such an equivalence: the category $\mathbf{C^*Alg_c}$ is dual to $\mathbf{T_{C,H}}$. That is, every commutative unital C^*-algebra can be realised as a C^*-algebra of continuous complex-valued *functions* on some compact Hausdorff space. This result provides a precise correspondence between commutative algebras and topological spaces, contingent on the identification of the appropriate topological structure.[21]

---

[18]Specifically, it is a *primitive ideal* of $\mathcal{A}$, i.e., the kernel of an irreducible representation. Primitive ideals offer coarser geometric information, since they group together irreducible representations with the same kernel. Therefore, irreducible representations sharing the same kernel correspond to the same primitive ideal. One can show that if $\mathcal{A}$ is commutative, primitive ideals coincide with maximal ideals and induce a unitary equivalence relation on irreducible representations: $\hat{\mathcal{A}}$ is the space of the irreducible representations generated by pure states, modulo identification of representations with the same kernel.

[19]This construction is discussed in detail in [37].

[20]When discussing duality in category theory, I will sometimes use the phrase *categorical equivalence*. The reader should keep in mind that if one category is dual to another, then it is equivalent to the opposite of the latter (i.e., the category with all arrows reversed), and vice versa. Therefore, I will not dwell on this otherwise important distinction, since the two situations are mathematically intertranslatable.

[21]Note that unitality is necessary for the algebra to be dual to a *compact* Hausdorff space. Still, many spacetime models are noncompact, but only *locally* compact. This requires refining said duality by abandoning the unitality of the dual algebra. In particular, a locally compact Hausdorff space is dual to the commutative algebra of continuous functions vanishing at infinity defined on it. This algebra is non-unital. Nonetheless, note that the main interpretative points raised in this paper still apply to this refined case: the

A similar result holds between the categories **SmoothMan** and **SmoothAlg**. The former includes smooth manifolds as objects, with diffeomorphisms as morphisms; the latter consists of smooth algebras (e.g., $C^\infty(M)$) with algebra homomorphisms. Although it is straightforward to associate a smooth algebra to a given manifold, characterising a manifold structure from an arbitrary commutative algebra is a more delicate task. Building on [35], [36] demonstrate that **SmoothMan** is dual to **SmoothAlg**.

This duality can be extended to general relativistic models by introducing *Einstein algebras*. Geroch motivated this approach by proposing that, "by a judicious choice of definitions, the entire content of general relativity can be so formulated that the underlying manifold plays practically no role" [38, 271]. Each solution of the Einstein field equations corresponds to an Einstein algebra: an algebraic structure defined as a pair $(\mathcal{E}, g)$, where $\mathcal{E}$ is a smooth algebra and g is a Lorentzian metric on $\mathcal{E}$. These structures offer an algebraic formulation of relativistic spacetime.

However, extending global duality results to Einstein algebras faces significant challenges. Geroch's original definition, in fact, merely reconstructs the standard algebra of scalar functions over a smooth manifold;[22] it does not provide a characterisation theorem that uniquely identifies Einstein algebras independently of such manifolds. This lack of definitional autonomy exacerbates any effort to establish a proper duality theorem.

Suppose, for the sake of the argument, that **GR** is the category of relativistic spacetime geometries (with isometries as morphisms), and **EA** is the category of Einstein algebras (with Einstein algebra homomorphisms as morphisms). Drawing on Geroch's work, Rosenstock, Barrett, and Weatherall seek to establish a duality between these two categories, thereby formalising the relationship between geometric and algebraic formulations of GR. If such a duality can be rigorously proven, then both frameworks share the same mathematical structure and, consequently, the same capacity to model physical scenarios.[23]

Notably, their work also raises critical questions about how to interpret such results. Although categorical equivalence suggests that the two theories are somewhat equivalent (again, with arrows reversed), its precise meaning remains unsettled. The idea is that if two semantic categories are dual, then the associated theories should be inter-translatable and verify the same theorems within their respective signatures, despite these theories being formulated in different languages (algebraic or geometric).[24] In this sense, the geometry of smooth manifolds could be translated into algebraic form

---

duality relation still breaks down once noncommutativity is introduced. For this reason, I will not discuss the locally compact/nonunital case further: it is trivially an instance of the broader following conclusions.

[22] In particular, note that such an algebra is not even a C^*-algebra. Rather, if the manifold M is compact, the algebra of smooth functions on M forms merely a dense $*$-subalgebra of the C^*-algebra of continuous functions on M.

[23] The duality between spacetime solutions and Einstein algebras was initially explored by philosophers as a case for spacetime relationalism. In this context, Einstein algebras represented possible configurations of matter. The duality implies that general relativistic predictions, formulated in the language of differentiable manifolds and tensors, can be recovered within a purely algebraic framework. Unlike substantivalism, this algebraic framework does not require a commitment to spacetime points and is therefore preferable. Specifically, Einstein algebras have been proposed as a means to avoid the well-known hole argument [10], though this application remains contested (see, e.g., [11]).

[24] Equivalently, their semantic categories are related: one is isomorphic to the opposite of the other.

(and vice versa) without any loss of mathematical content. From a semantic perspective, the differences between algebraic and geometric domains become irrelevant: any statement verified by a geometric model should likewise be verified by its algebraic counterpart.

Nevertheless, Rosenstock, Barrett, and Weatherall stop short of proving that the equivalence between **EA** and **GR** is a genuine translation in the logical sense. Specifically, they do not establish the existence of a reconstrual map $f : \Sigma_1 \to \Sigma_2$ between signatures such that, for any theory T_1 in signature Σ_1 and any sentence ψ in Σ_1, if $T_1 \vdash \psi$, then $T_2 \vdash f(\psi)$, where T_2 is a theory in Σ_2. That is, they do not demonstrate that the duality is an intertranslatability result in the model-theoretic sense. In general, categorical equivalence does not entail intertranslatability: the duality may be insufficient to reconstruct one theory from the other.

This limitation is circumvented in the case of Gelfand duality, which satisfies a stronger criterion known as *definable categorical equivalence* [39, 59]. Under this condition, the functors between categories are constrained by definability conditions: they explicitly preserve the internal structure of the objects involved. The equivalence is thus not merely structural: it is a *reconstruction*, allowing one to recover the elements and relations of one signature from the other.

Definable categorical equivalence therefore justifies claims of intertranslatability, yet someone may still demand an even stronger condition: namely, that the duality entails a reduction to a common core structure. This view is motivated by the central principle of category theory that an object be fully determined by its morphisms. As Spivak puts it:

> an object in a category is not only (epistemologically) *known by* its relationships with other objects; it (ontologically) *is* what it is by virtue of its relationships with other objects. In a categorical model, the knowing of an object and the being of an object are essentially identical. ([40, 392]; emphasis in the original)

This ontological reduction is stronger than its epistemological counterpart: morphisms do not merely describe an object's properties via relations; they *constitute* the object. Accordingly, algebraic-geometric dualities may be interpreted as follows: an algebraic model (as an object in the semantic category of an algebraic theory) is reduced to the structure O_1 of its morphisms; similarly, a geometric model is reduced to a structure O_2. The duality then asserts that O_1 and O_2 are isomorphic: the two models are distinct representations of the *same* underlying structure. Any specification of that realisation is thus redundant with respect to the structural information shared between them.

This structuralist interpretation has been influential. Bain [13], for example, argues that one of the major advantages of categorical approaches over set-theoretic ones lies in the elimination of surplus structure, i.e., intrinsic features of objects not captured by their morphisms. On this view, the duality between **GR** and **EA** favours the algebraic formulation: it is as expressive as the geometric one, while avoiding traditional metaphysical issues such as the hole argument. One could then reject ontological commitment to spacetime points and regard algebraic models as superior in terms of parsimony and conceptual clarity.

By contrast, Hudetz cautions against identifying objects with their morphisms [39]. Certain mathematical objects, such as vector spaces, exhibit internal properties that are not fully captured by their morphisms. In such cases, categorical formulations may undermine, rather than enhance, their representational power.[25] A more cautious stance, advocated by Weatherall [42], is to interpret algebraic-geometric dualities in terms of their *epistemological value*. From this perspective, establishing a duality, and knowing the mathematical properties of one of the involved categories, can yield new insight into the structure of the other, even in the absence of a full ontological reduction.

According to Weatherall, categorical equivalence is best understood as a heuristic tool for analysing pre-existing formal relationships: what he calls the *Rosenstock heuristic*. Dualities, on this view, do not necessarily indicate ontological equivalence, but instead support the assessment and clarification of formal connections already of some interest. As Weatherall puts it:

> [the duality between **GR** and **EA**] shows more clearly how one can think of a relativistic spacetime as a means of encoding, or representing, nothing more or less than the possible ways in which matter may be (spatio-temporally) configured and the relations between those possible configurations, precisely as Einstein algebras do. [42, 431]

Nevertheless, he acknowledges that this duality may not preserve the distinction between the underlying objects. Categorical dualities cannot serve as definitive equivalence results, but should be used as heuristic instruments for probing the structure and interrelations of theoretical frameworks, enhancing, rather than resolving, our understanding of the theories involved.

5 The Failure of Classical Dualities in NCG

As discussed, classical dualities establish connections between the algebraic and geometric pictures in several significant ways: locally, they clarify the relationship between different definitions of algebraic and geometric points; globally, they relate the algebraic models of a theory to their geometric counterparts, and vice versa. Although the exact interpretation of these global dualities remains ambiguous, the underlying mathematical results offer valuable insights for theoretical physics, particularly in the investigation of spacetime.

However, the results have limited scope: they do not generalise to the noncommutative domain. All classical dualities fundamentally rely on the assumption that the

---

[25]Admittedly, Spivack's analysis raises concerns. The ontological reduction of objects to morphisms may depend on the specific category to which an object belongs, that is, on its context. Yet Spivack provides no argument against this possibility. One can speculate that a Benacerraf-style argument could substantiate the reduction claim: internal properties of the objects, as alternative realisations of a shared morphism structure, are ultimately irrelevant. However, such context-dependence undermines any clear sense in which the two structures can be compared in the first place. A more refined argument is therefore required. Notably, Dewar has responded to Hudetz with two counterpoints [41]: first, that category theory can recover internal properties by leveraging the full expressive power of morphisms (the *conservative* answer); second, that category theory treats objects as kinds rather than as individual instances (the *radical* answer). Consequently, if categorical equivalence entails theoretical equivalence, and the theories in question admit appropriate category-theoretic representation, then classical dualities suggest that the theories are representationally equivalent: dual algebraic and geometric models offer alternative representations of the same family of targets. Whether Dewar's response supports the structuralist position or introduces new difficulties remains an open question.

relevant algebra is commutative. Abandoning this assumption raises the challenge of formulating new kinds of dualities that are applicable in the noncommutative setting. This is non-trivial, since classical dualities reflect a precise alignment between algebraic and geometric structures: an alignment that is expected to break down in the noncommutative case.

To illustrate, consider the algebra $M_n(\mathbb{C})$ of $n \times n$ complex matrices. In this context, pure states correspond to rank-one projectors, defined as $\phi_A : B \mapsto \mathrm{Tr}(AB)$. Via the GNS construction, each pure state yields an irreducible representation, which assigns additional representation-theoretic content to each point: specifically, an n^2-dimensional matrix algebra. Yet all these irreducible representations are unitarily equivalent to the identity representation on $\mathbb{C}^n$, i.e., $\pi : M_n(\mathbb{C}) \to \mathrm{End}(\mathbb{C}^n)$, where $\pi(A) = A$.[26] Consequently, the structure space of $M_n(\mathbb{C})$ collapses to a single point. Thus, although algebraic points may be defined either as pure states or as elements of the structure space, the corresponding geometric constructions fail to align due to the role of unitary equivalence.

Moreover, $M_n(\mathbb{C})$ has only trivial ideals: the zero ideal and the whole algebra.[27] Neither qualifies as a (two-sided) maximal ideal. We are then faced with a dilemma: either we relax the definition and admit one-sided maximal ideals, collapsing the associated geometry to a single algebraic point, or we retain the original definition and conclude that no algebraic point exists. In either case, the different definitions fail to converge. While $M_n(\mathbb{C})$ does admit both algebraic and geometric points under various constructions, their interrelation becomes intricate and *construction-dependent*.

More generally, local classical dualities may fail in several ways in the noncommutative setting. First, a given noncommutative algebra may admit no pure states or maximal ideals.[28]. Second, pure states may not coincide with the maximal ideals. Third, maximal ideals may not correspond to primitive ideals, i.e., kernels of irreducible representations. Fourth, the equivalence classes of irreducible representations may differ from primitive ideals. These divergences underscore the necessity of case-by-case specification of both algebraic and geometric definitions of points in the noncommutative context. In light of this, abstract claims such as "Noncommutative algebras have no points" are ill-formulated, unless they first identify a specific definition of "point" and justify its privileged status. I argue that any claim based on such a privileged definition is methodologically suspect and conceptually unfounded.

Similarly, global commutative dualities typically fail in noncommutative cases. In particular, the geometric models associated with noncommutative algebras often fail to be Hausdorff. For instance, both the space of primitive ideals and the structure space are generally Kolmogorov (i.e., T_0) spaces, but not Hausdorff (or even T_1). This issue does not arise only in NCG: even in commutative contexts, non-Hausdorff spaces already violate Gelfand duality. As [43] discuss, this raises the question of whether such spaces qualify as genuine general relativistic models.

Indeed, Geroch and Horowitz briefly note that the Hausdorff property is essential for a manifold to be spatiotemporal in the standard sense [44, 218]. Its failure implies

---

[26]Similarly, there is only one primitive ideal, the zero ideal: see fn. 18.
[27]This and the uniqueness of the irreducible representation (up to unitary equivalence) both follow from the fact that $M_n(\mathbb{C})$ is a simple algebra.
[28]One notable example is the algebra generated by position and momentum operators in QM.

that certain pointwise physical events become indistinguishable.[29] Wu and Weatherall further argue that if geometric points are to be defined in terms of local interactions of matter fields,[30] then a breakdown of the Hausdorff condition introduces a new kind of asymmetry: one can encode geometric information into a suitable algebra by "forgetting" fine-grained geometric distinctions and working with a "coarser" space, but the full reconstruction process fails. The algebra alone becomes insufficient to recover the geometry, and the resulting indistinguishability implies that geometric points may possess an ontological status irreducible to matter-field configurations.

Noncommutative algebras force us to confront these issues.[31] One could take a conservative approach and argue that only those geometric structures that admit commutative algebraic-geometric duals are acceptable representations of spacetime. But this position is unsatisfactory. The **EA − GR** duality, which was originally formulated to provide an algebraic characterisation of general relativistic spacetimes, remains vague and contested. Moreover, insisting on this restriction risks begging the question. One could argue: general relativistic models describe spacetime and satisfy certain geometric properties; hence, any model lacking those properties cannot represent spacetime. However, the fact that the former possess such properties is actually a *consequence* of them being selected as models of spacetime: it cannot serve as justification for excluding alternatives.

Alternatively, one might accept noncommutative algebras as legitimate physical structures in certain domains and investigate the novel geometries they describe. I contend that this is a more promising path, as indicated by NCST approaches to QG. A more detailed discussion of this perspective is offered in Section 6.3.[32]

---

[29]Incidentally, in the commutative setting the notions of point and event are tightly intertwined. Events are typically taken to occur at spacetime points and are physically individuated, for instance, by timelike worldlines passing through them. Conversely, in the noncommutative setting, the introduction of novel separation properties may give rise tto what might be called an *eventlessness problem*: namely, the potential absence of events in NCST. This difficulty could amplify the foundational concerns already raised by pointlessness. In particular, mathematical pointlessness suggests that the notion of event, if retained at all, would have to be distinguished from that of a spacetime point, yet it is far from clear how such a notion should then be understood in the noncommutative framework. What, then, would count as an event in NCST? Due to space limitations, I leave this related but distinct issue for future investigation. For present purposes, I simply note that any putative eventlessness in the noncommutative regime might be mitigated through an appropriate reconceptualisation of the relevant physical notions; in practice, this would require a careful and well-motivated choice of the interpretative postulates applied to the domain under consideration.

[30]This is central to the *deflationary view of manifolds*: manifolds are merely structures encoding information about possible configurations of matter. Manifold points then represent locations where these configurations may coincide or differ. This interpretation is challenged by vacuum solutions in GR: in the absence of fields, the manifold cannot be defined solely by matter content, unless the deflationary view admits highly idealised scenarios, such as those involving only test matter.

[31]Notably, most of these issues become urgent from a global perspective, which is common in the field. For example, consider Milnor's famous exercise: prove that for any manifold M, $\mathrm{Hom}(C^\infty(M, \mathbb{R}), \mathbb{R}) \cong M$. Similarly, in dealing with fiber bundles over a manifold M, it is common to consider only the algebra of global sections, neglecting local ones. However, local structures are generally just as important (possibly more so) than global ones. This holds for many geometric structures relevant to classical algebraic-geometric dualities and suggests that certain failures may stem from an unexamined global/local distinction.

[32]For now, it suffices to observe that indistinguishability (as expressed by a failure of the Hausdorff property) is indeed expected in a NCST theory due to position uncertainty. This may prompt the exploration of new separation properties suitable to non-classical physical regimes at appropriate scales.

6 Putting the "Geometry" Back in NCG

The violation of commutative algebraic-geometric dualities raises significant concerns about the relationship between algebraic and geometric pictures in the noncommutative domain. Lizzi, for instance, interprets this violation as indication that geometric points cease to exist in this context [16]. However, this interpretation prompts a series of deeper philosophical and mathematical questions: What is a space without points? What then becomes of geometry? How can one conceive of a geometry devoid of points? Is such a geometry even well-defined?

More precisely, an argument against the viability of the geometric picture in the noncommutative domain can be reconstructed as follows:

1. Let a commutative algebraic-geometric duality be interpreted as I (i.e., in terms of intertranslatability, ontological reduction to common structure, or via the Rosenstock heuristic: see Section 4.2).[33]
2. Suppose that this duality, so interpreted, extends to the noncommutative domain.
3. This duality, interpreted as I, is violated in the noncommutative domain.
4. *From 2 and 3:* either the algebraic or the geometric picture is ill-defined.
5. The algebraic picture is well-defined: the content of the noncommutative theory can be fully and exclusively expressed by an algebraic model.
6. *From 4 and 5:* the geometric picture is ill-defined, i.e., the noncommutative algebra has no well-defined dual geometric model.

In this section, I critically assess this argument and contend that it is ultimately inconclusive. To demonstrate this, in Section 6.1 I examine premise 5, which, in a sense, represents the "standard view" in NCG. Then, in Section 6.2, I discuss premise 2 and its role within the overall argument. I argue that this argument relies on an improper extension of classical (commutative) results and vague inferential steps. The only premise I accept unconditionally is premise 3, as previously discussed in Section 5. Finally, in Section 6.3, I explore possible responses to this challenge.

6.1 The Algebra-First Approach

In the noncommutative domain, the primacy of the algebraic over the geometric picture, as asserted in premise 5, requires substantial supporting arguments. One key motivation for this primacy is methodological: noncommutative theories are formulated as modifications of commutative *algebras*, often through deformation or quantisation techniques. This choice is primarily strategic, since algebras are generally more "tractable" and "easier" to manipulate than geometric theories. In this regard, much of the research in NCG is devoted to understanding how geometric and topological information can be encoded within suitable algebras (a question inherited from algebraic geometry), and how certain algebraic manipulations preserve or reflect geometric content.

---

[33]While the difficulties surrounding the local construction of geometric points might cast doubt on the viability of a noncommutative geometric theory, I take the main challenge to arise from the global perspective on algebraic-geometric dualities.

For example, Connes characterises noncommutative geometry in terms of so-called (noncommutative) *spectral triples* [45]. A spectral triple $(\mathcal{A}, H, D)$ consists of a (noncommutative) involutive algebra $\mathcal{A}$, faithfully represented on a Hilbert space H, together with a (generalised) Dirac operator D acting on the algebra. This framework reformulates key geometric structures in purely algebraic terms.[34] Classical (i.e., commutative) geometry is then recovered when $\mathcal{A}$ is commutative, while NCG arises by replacing the ordinary pointwise product with a noncommutative one, thereby deforming the underlying structure. Crucially, Connes explicitly introduces Dirac operators in order to encode the metric structure of a geometric space in the structure of derivations of the associated algebras [46]. In his approach, a spectral triple fully encapsulates the geometric and topological information of the space, while also admitting natural generalisations when the algebra is noncommutative. In other words, a suitable algebra equipped with a Dirac operator guarantees the translatability of geometric information into an algebraic framework.[35]

Additionally, commutative algebraic models are only dual to differential geometric ones, whereas noncommutativity arguably affords greater expressive power. Specifically, noncommutative algebras allow to characterise models that lack dual structures in the narrower geometric picture. Put differently, the algebraic framework, encompassing both commutative and noncommutative contexts, is strictly more expressive than the geometric one: whereas noncommutative theories can be formulated algebraically, structures that fall outside the domain of differential geometry are excluded from the classical geometric picture.

It is also important to note that new dualities may emerge in the noncommutative case (and indeed some have been found: see Section 6.3), but their corresponding geometric models may be difficult to characterise. The failure of commutative dualities suggests that these new models lie beyond the scope of differential geometry,

---

[34]Specifically, the Dirac operator is necessary for the reconstruction of a metric structure, alongside the topological structure. This operator is unbounded, self-adjoint, and has compact resolvent; it is employed to define a metric on the space of states of the C^*-algebra $\mathcal{A}$. Importantly, the Dirac operator facilitates the description of the Riemannian structure on the manifold in algebraic terms via Connes' reconstruction theorem, as detailed in [46]. However, note that Connes' theorem applies to metrics with Euclidean, rather than Lorentzian, signature. This might entail limitations to the algebraic programme, while still licensing topological considerations, such as those concerning the existence of points.

[35]Notably, Huggett, Lizzi, and Menon [15] raise an additional challenge to the definability of certain geometric structures within the spectral triple approach. In their argument, a structured set is deemed *definable* if and only if it is invariant under an appropriate family of automorphisms. In the commutative setting, one can define a notion of distance in two distinct yet equivalent contexts: as a Riemannian distance on a manifold and as a topological distance (in the sense of [45]) on the set of pure states of the corresponding algebra. The authors show that any set of elements separated by a distance less than or equal to a given value in one framework (e.g., within the space of pure states) corresponds precisely to a set of elements at the same distance in the other framework (e.g., within the manifold). This equivalence ensures that sets of elements lying within a specified distance can be defined in either representation. However, the introduction of noncommutativity breaks this equivalence, as the homeomorphism condition between the space of pure states and the manifold no longer holds. As a result, Huggett, Lizzi, and Menon demonstrate that no set of manifold elements lying within a fixed distance can be defined within the corresponding noncommutative algebra. Consequently, "our conception of localisation is definable in a theory whose domain of discourse is a spectral triple only if the algebra is commutative" [15, 4707]. A fortiori, manifold points (understood as elements belonging to intervals of arbitrarily small size) are excluded from this noncommutative algebraic framework as non-definable entities. Importantly, this argument applies compellingly to Connes' spectral triple approach. Unlike cases involving algebraic-geometric dualities, it does not rely on commutative results but rather exposes an intrinsic limitation of the formalism itself. Nonetheless, it does not extend to all conceivable definitions of manifold points.

necessitating more advanced or alternative mathematical frameworks for their formulation and analysis. Moreover, their exclusion from the familiar geometric picture raises questions about their intended applications, particularly in investigations of QG. In other words, the legitimacy of non-differential geometric models for representing spacetime physics can be questioned, regardless of the specific physical regime under consideration, possibly due to principled expectations about the nature of the latter's geometry.

Finally, the algebraic picture seems indispensable in NCG. The bulk of research has been conducted within an algebraic framework, and current formalisms may be inadequate to capture genuinely noncommutative features outside this algebraic framework, especially within a geometric picture. Accordingly, the notion of geometric pointlessness poses a significant challenge: without suitable adjustments, the geometric formalism may be ill-defined.

The primacy of the algebraic picture in NCG is sometimes termed the *field-first approach* [15], or more broadly the *algebra-first approach*. According to this view, algebraic objects and structures constitute the fundamental ontology of the noncommutative theory. Similarly, models are formulated primarily (and indeed exclusively) within an algebraic framework. The algebraicist is then tasked with either deriving an appropriate, possibly extended, geometric picture from this fundamental algebraic ontology, or demonstrating that such a geometric picture is impossible to construct.

Notably, this articulation of the algebra-first approach is sufficient to support premise 5, but it remains incomplete without further refinement. To illustrate, consider the following features of the spectral triple formalism.

First, in this approach, the algebra and the Dirac operator alone are insufficient to fully specify a spectral triple, whether commutative or noncommutative, and thus to encode the full geometric information. The triple also requires a *representation* of the algebra on a Hilbert space. This representation indicates how the abstract algebraic elements are realised as transformations acting on the states of the underlying space. In this sense, representations mediate the translation of algebraic data into geometric information by bridging the gap between the two pictures.

Second, much of NCG research is not limited to the formulation of noncommutative spectral triples; a significant portion employs functional analysis and Hilbert space structures (i.e., geometric and topological information) to characterise relevant spectral triples. While these geometric structures differ from those of classical differential geometry (the latter being usually associated with the more familiar notion of geometry as spacetime geometry), they nonetheless compel the algebraicist to distinguish which geometric structures can be discarded and which must be retained.

Importantly, the role of representations can be generalised even to classical dualities: the reconstruction of a geometric model is mediated by the so-called *Gelfand representation*, though this is but one among many alternatives [12]. Every Einstein algebra, for instance, can be associated with multiple geometric structures tailored to different purposes, thereby challenging strict algebraic literalism.[36] This discloses the

---

[36] *Algebraic literalism* is the view that a specific representation of an algebra is privileged, in that it is the only one carrying physical meaning.

possibility, similar to the previous case, that a noncommutative algebra could correspond to a novel geometric structure through the specification of an alternative, yet physically significant, representation.[37]

6.2 Problems of Interpretation

For the sake of the argument, let us accept premise 5 as given. As anticipated, this is not the only problematic assumption in the argument against the geometric picture. While the debate over the choice of interpretation in premise 1 has already been addressed, two particularly thorny steps remain: the generalisation to the noncommutative domain (premise 2) and the inference drawn from the breakdown of the classical dualities (step 4).

As discussed, classical algebraic-geometric dualities link distinct constructions within the two pictures; informally, these constructions are different realisations of the same content expressed in alternative frameworks. The construction of noncommutative theories based on their commutative counterparts naturally suggests that these dualities might point to novel relations to be established in the noncommutative domain. The hope is to extend classical results to this broader context. However, as demonstrated, this conflicts with premise 3: classical dualities fail in the noncommutative setting and thus cannot be generalised. In other words, the commutativity of the algebra is a necessary condition for their validity. Opponents of the geometric picture therefore leverage this failure of generalisation to question the definability of noncommutative geometric structures.

I contend that this inference raises serious issues. First, extending commutative results to the noncommutative domain may serve as heuristics, but it is logically inadmissible without further justification. One would require additional arguments proving that a more general form of the dualities holds, such that classical results become special cases within a broader framework. Since this is not the case, classical dualities cannot be expected to hold without commutativity. Therefore, applying commutative results to the noncommutative case is simply a logical error that undermines the inference.

Second, the algebraicist's argument employs classical algebraic-geometric dualities in an unspecified and ambiguous manner. Crucially, these dualities are mathematical results linking different descriptions of models of a single theory. The full import of these dualities in the argument can only be clarified by specifying the nature of the link between the models. An interpretation is intended to provide such clarification, for instance by facilitating the reconstruction of dual models. However, the correct interpretation is contested (see Section 4.2). Consequently, there is no universally accepted interpretation of these dualities that the algebraicist can leverage to support

---

[37]To illustrate, consider an Einstein algebra $\mathcal{E}$. The Geroch representation maps $\mathcal{E}$ to $C^\infty(M)$, the algebra of smooth, real-valued functions on a manifold M contained within the spectrum of $\mathcal{E}$. The result is a *Geroch ringed space* $(M, C^\infty(M))$ that is isomorphic to a general relativistic solution formulated in tensorial terms. Algebraic-geometric duality entails a precise correspondence between Einstein algebras and their Geroch representations. However, as Bain notes, while every general relativistic model is isomorphic to a Geroch ringed space, the converse does not hold if one varies the representation of $\mathcal{E}$. An Einstein algebra may yield more general geometric structures under alternative representations. [47] provides several illustrative examples showing that these alternative concrete representations of the same Einstein algebra can bear physical content.

the argument against the geometric picture. A fortiori, a hypothetical extension of this interpretation to the noncommutative domain (as premise 2 requires) only compounds the problem: it extends a contentious and unclear result to a domain that is itself poorly understood, while assuming it can serve as a valid premise for argumentation.

The inference from premises 2 and 3 to step 4 (namely, the conclusion that one of the two pictures must be ill-defined in the noncommutative domain) inherits all these difficulties. It assumes the extendibility of classical dualities (which is false) and their unambiguous interpretation (which is lacking), in order to conclude a disjunctive statement (either the algebraic picture is ill-defined or the geometric picture is) based on the failure of these assumptions (premise 3). From the foregoing, it follows that this inference is invalid due to the falsity or ambiguity of its premises. Consequently, it cannot be used to argue that either picture is untenable.

At this stage, one might set aside the extensibility issue and speculate that one of the three interpretations of dualities (linguistic, structural, or epistemological, discussed in Section 4.2) could still support step 4. After all, the failure of any one of these interpretations might suffice to question the classical link. Nevertheless, I maintain that this strategy offers no real advantage to the proponent of the argument: the violation of the interpreted classical dualities does not undermine the tenability of the dual pictures and therefore cannot legitimately serve as a premise. To substantiate this claim, I examine each interpretation in turn, indicating which conclusions legitimately follow from their failure and which ones do not.

Duality as inter-translatability assumes that two corresponding models are well-defined; it provides a procedure for reconstructing one from the other coherently, respecting morphism structures. Failing to establish such inter-translatability implies that the concerned models, though each well-defined in isolation, cannot be reconstructed from one another. For example, in the noncommutative domain, the failure of inter-translatability means that given an algebraic model, the corresponding geometric model cannot be recovered from it; yet the geometric model itself may nonetheless be perfectly well-defined. Thus, noncommutativity does not eliminate models in either picture. Instead, it motivates the search for novel reconstruction relations, or suggests extending one or both pictures to include new types of models, connected to noncommutative algebras by novel inter-translation schemes.

Duality as ontological reduction to shared structure posits a deeper connection between dual models. Its breakdown offers a seemingly stronger case for the algebraicist. Suppose the relevant dualities, interpreted thus, are broken, and that the algebraic model is privileged (according to premise 5): then no geometric realisation of the underlying structure can be identified. Under this interpretation, the algebraicist could argue that the geometric picture is ill-defined (conclusion 6), severing its relation of ontological reduction to the basic structure.

However, this use of ontological reduction is misguided. The failure of the duality does not imply that reduction relations are severed; ontological reduction was never the primary feature of classical dualities, but rather a means to shared structure. Classical dualities indicated the *existence* and *uniqueness* of this structure despite alternative realisations in the two pictures. On this view, the breakdown of commutative dualities amounts to a failure of the mutual *relation* between algebraic and geometric models: if

an algebraic model reduces to structure O_1, and the corresponding geometric model to O_2, then O_1 is not isomorphic to O_2. This does not mean that O_2 should be rejected; rather, it means that the two ontologies fail to be alternative concrete realisations of the same underlying structure.

Finally, duality as an epistemological tool plays only a supplementary role, providing additional indication of an already established inter-theoretic relation. This supplementary status implies that its breakdown is relatively harmless to the definability of models. Specifically, the inability to infer features of one model from its dual simply indicates that alternative epistemological methods must be employed: knowledge of algebraic models cannot be facilitated by knowledge of geometric models, nor vice versa. Importantly, this breakdown does not affect more fundamental relations nor the definability of each model independently: each remains well-defined by virtue of the same prior relations that grounded the epistemological use of duality in the commutative setting.

6.3 Noncommutative Algebraic-Geometric Dualities

Up to this point, I have argued that the aforementioned algebraicist argument is inconclusive: the content of its premises is either too general, insufficiently specified, or simply incorrect. While I take the failure of this argument to settle the question of the in-principle viability of a geometric picture in the noncommutative domain (particularly from the standpoint of classical dualities), the algebraicist may remain dissatisfied with this conclusion. Specifically, she might maintain that a proper vindication of the geometric picture also requires the explicit indication of candidate geometric models that could serve as dual structures to noncommutative algebras. Consequently, the opponent of the algebraicist argument (in this case, myself) may be pressed to offer some concrete examples.

One such proposal is to identify dual geometric models of noncommutative algebras with *q-spaces*: noncommutative algebras equipped with a complete lattice of projectors [48, 49]. In the commutative case, this lattice is atomic: projectors correspond to subregions of a base space, and atoms designate geometric points within that space. More generally, set-theoretic notions are extended to the noncommutative domain by encoding the entire geometric content in the lattice of projectors: families of projectors become *q-sets*, and minimal non-zero projectors are *q-points*. This lattice-based characterisation provides a suitable foundation for a point-free geometric construction, defined via relations among projectors. Open q-sets form a topology (the *q-topology*) on the space that is T_1 (i.e., every q-point is closed) and regular, but not generally Hausdorff. In other words, projective structure systematically replaces set-theoretic structure and extends the scope of commutative notions.

This construction admits various refinements. For example, Patel shows that a noncommutative algebra $\mathcal{A}$ is canonically isomorphic to the algebra $\mathcal{A}(P(\mathcal{A}))$ of q-continuous functions on its set of pure states, equipped with a suitable q-product [50].[38] Nonlocal products are introduced at the lattice level to capture noncommutative effects. Since noncommutativity is entirely encapsulated in certain equivalence

---

[38]Q-continuity and q-product are extensions of commutative concepts, defined in terms of the q-topology of the q-space $P(\mathcal{A})$.

relations between the GNS-representations of $\mathcal{A}$, the topological dual of $\mathcal{A}$ is identified with its pure states, rather than with the spectrum of irreducible representations. This construction effectively extends Gelfand duality into a noncommutative duality between unital C^*-algebras and their q-spaces of pure states.[39]

A different proposal, developed in [51] and [52, ch. 2], constructs a lattice structure in the noncommutative setting by means of topological approximations. The key idea, based on the work of Sorkin, is to cover a topological space with an open covering, and then declare indistinguishable all points that lie within each covering set. The result is a lattice of countable points with a quotient topology, which can be identified with the structure space of certain noncommutative algebras (specifically, approximately finite-dimensional ones).

This lattice differs from the previous approach in that it is not derived from the pure states of the algebra. Furthermore, it discards fine-grained information while still allowing for the definition of appropriate noncommutative algebras of functions. Here, geometric points correspond to elements of the covering, that is, to features retained by the chosen topology. Put differently, noncommutativity induces a topological coarse-graining that disregards irrelevant details for the representation theory of the noncommutative algebra. The original topological space may then be recovered via an inverse system of increasingly fine posets.

Alternatively, Borceux and van Den Bossche demonstrate that noncommutative algebras are dual to *quantales* [53]. These are complete lattices $(Q, \leq)$, equipped with an associative multiplication $\& : Q \times Q \to Q$ that distributes over arbitrary meets and satisfies $a\&a = a = a\&1$ for all $a \in Q$.[40] Points in a quantale are defined as morphisms $f : Q \to \{0, 1\}$ that preserve the lattice structure.[41] A quantale has "enough points" when distinct elements can be separated by some point f, i.e., $a \neq b$ implies that $f(a) \neq f(b)$. Any set X can be equipped with a quantale $\mathcal{O}(X)$ of its subsets, thereby yielding a *quantum space*: the set of topological points (those $x \in X$ such that the closure $\overline{\{x\}}\&X = X$) forms a topological space with an appropriate topology derived from $\mathcal{O}(X)$.

In this framework, Borceux and van den Bossche show that sober quantum spaces are characterised by an isomorphism between their points and those of the associated quantale: the category of sober spaces is thus dual to the category of quantales with enough points.[42] More broadly, the spectra of arbitrary C^*-algebras can be promoted to *quantum spectra* by equipping the pure states with an appropriate family of open subsets. These spectra form quantum spaces whose topological structures are encoded by a quantale, with or without enough points. Consequently, the topological content

---

[39]Notably, these q-spaces are both q-compact and q-Hausdorff.

[40]Intuitively, the operator $\&$ represents a noncommutative intersection of open subsets. This operation, and the related notion of quantale, were first introduced in [54]. If $a\&b = a \wedge b$, then $(Q, \leq)$ is a *locale* describing a possibly coarse-grained topological structure for the spectra of commutative C^*-algebras.

[41]Specifically, a quantale morphism preserves $\vee$ and 1 and satisfies $f(a\&b) \geq f(a)\&f(b)$ for all $a, b \in Q$. Here, $\{0, 1\}$ is also equipped with a quantale structure.

[42][55] compares this construction of points in quantales with the q-points of Giles and Kummer, demonstrating that the two are inequivalent: quantales contain more points than q-points. They then propose an alternative definition of points in a quantale that is equivalent to q-points and captures appropriate separability properties in an extended geometric framework.

and separability properties of arbitrary C^*-algebras can be fully reconstructed from a suitably chosen quantale on their quantum spectra, and vice versa.[43]

These paradigmatic attempts all aim to generalise classical dualities, subsuming the commutative case within broader frameworks of noncommutative or extended dualities. In the commutative domain, these reduce to the familiar duality between commutative unital C^*-algebras and compact Hausdorff spaces. In the noncommutative setting, however, they encompass a richer variety of structures: noncommutative algebras, quantales, noncommutative topological spaces, and quantum spaces.

Notably, the breakdown of compatibility relations between different commutative dualities (as discussed in Section 5) entails that these approaches diverge on a crucial aspect: which structure constitutes the appropriate basis for reconstructing a geometric model. While in the commutative case various structures and topologies are equivalent, in the noncommutative domain they are not: each approach privileges either the set of pure states or the set of irreducible representations. Accordingly, each introduces its own generalised notion of geometric point, defined within a specific geometric framework, and these notions are often nontrivially related to one another.

On the one hand, the existence of multiple approaches suggests a *pluralistic* stance: we should not privilege one construction over the others. Instead, we should explore the relations among them (particularly potential morphisms) and make context-sensitive choices depending on the mathematical problem at hand. In this sense, all claims concerning the existence and definability of geometric structures, including points, must be understood as relative to a specified duality.

On the other hand, none of these alternatives can be fully reduced to the commutative case. In a precise sense, the commutative case is fortuitous: all structures cohere harmoniously, with no ambiguity in definitions or constructions. Such harmony cannot be expected to extend to the more general case. For example, it is evident that classical definitions of points fail to apply to the noncommutative domain: they are simply the wrong notions in that context. Likewise, the elimination of "commutative points" (that is, the failure of some point predicates to be satisfied due to broken duality) is unproblematic for noncommutative constructions, which rely instead on alternative structures that fulfil the same conceptual and theoretical roles. Thus, while it remains possible that some noncommutative theories are genuinely pointless (as in *full mathematical pointlessness*), this is not a *necessary* consequence of noncommutativity itself.

Moreover, all these noncommutative constructions differ from differential-geometric models, insofar as they are defined in terms of lattice structures. While such structures can also be identified in the commutative case, where they are equivalent to standard geometric objects, this equivalence breaks down in the noncommutative domain. This raises the question of whether noncommutative models can be defined within the geometric picture. As we have seen, the answer is generally negative.

---

[43] [56] refines this duality by restricting the geometric models to *Gelfand quantales*: those that are unital, involutive, and satisfy the relation $a = a\&a^*\&a$ for the involution. They prove that arbitrary C^*-algebras are dual to the category of Gelfand quantales, with morphisms preserving both the quantale and involutive structures. Furthermore, they reconstruct a more precise notion of point based on irreducible representations of the Gelfand quantales: see Def. 10.1 in [56].

Nonetheless, this negative conclusion does not favour the algebraicist: once again, the error lies in adopting classical terms from the outset. The geometric picture is merely a special instance of a broader, generalised geometric framework that accommodates models beyond those of differential geometry. These new models include, for example, lattices. Significantly, the ordering relations that define the lattice constructions above are spatial and topological in nature, inherited (by construction) from their commutative limits.[44]

To conclude, noncommutativity enriches our understanding of classical dualities between the algebraic and geometric pictures. These turn out to be special cases of more encompassing dualities between algebraic structures and generalised geometric frameworks.

7 Conclusion

NCG raises crucial questions regarding the definability of key physical and mathematical structures, as well as their applicability as representations of certain physical domains. These structures include differentiable manifolds and compact topological spaces. The elimination of their basic constituents, geometric points, from the ontology of the theory raises the issue of reconstructing these objects within the noncommutative theory. Failing at this enterprise significantly undermines the internal consistency of noncommutative approaches, both as mathematical and physical theories. Moreover, it may preclude any possibility of reconstructing appropriate geometric counterparts to the novel algebraic structures.

In this paper, I have argued that the elimination of mathematical points in noncommutative settings crucially depends on specific construction strategies and commitments about the interpretation of classical algebraic-geometric dualities. These constructions are well-defined in the commutative setting, yet their failure upon introduction of noncommutativity does not necessarily entail inconsistency. In fact, claims of pointlessness in NCG often hinge on illicit extensions of commutative results to the noncommutative domain, thereby resulting in artificial contradictions. Instead, I have contended that the failure is to be expected, and in no way undermines the foundations of NCG. The impossibility of providing geometric models for noncommutative algebras does not follow from the violation of classical dualities. Instead, noncommutativity promotes research into novel geometric structures that differ, in many aspects, from the familiar objects of differential geometry.

In particular, I have first indicated certain localisation arguments in QG as motivations for adopting NCG, as an approach that characteristically lacks points. A specification of the relevant senses of pointlessness reveals that NCG indeed may exhibit algebraic or geometric pointlessness, in two senses: the noncommutative theory can fail to provide a family of satisfiable predicates that are expressible in the framework, or it may lack suitable models. In the commutative setting, pointlessness is avoided in two respects: locally, due to the equivalence of the alternative constructions of algebraic and geometric points; globally, due to the validity of classical dualities. These relations break down in the noncommutative case.

---

[44]In this sense, I understand these lattice structures to differ from analogous constructions in QG, such as causal sets, which are defined in terms of causal relations as links.

An algebraicist may argue that the failure of classical dualities is sufficient to discredit any geometric picture in the noncommutative regime. By contrast, I have argued against this claim. Indeed, the algebraicist argument is inconclusive: it includes false premises and invalid inferences. Moreover, the argument can be counteracted by indicating the existence of alternative dualities for the noncommutative domain. These theorems correlate noncommutative algebras with novel geometric structures, suggesting that the correct duality relation should hold between an algebraic picture and a generalised geometric picture (thereby richer than differential geometry).

Acknowledgements

References

[1] Majid, S.: Foundations of Quantum Group Theory. Cambridge University Press, Cambridge (1995)

[2] Szabo, R.J.: Quantum field theory on noncommutative spaces. Physics Reports **378**, 207–299 (2003) https://doi.org/10.1016/S0370-1573(03)00059-0

[3] Connes, A., Marcolli, M.: Noncommutative Geometry, Quantum Fields and Motives. American Mathematical Society, Providence (2007)

[4] van Suijlekom, W.D.: Noncommutative Geometry and Particle Physics. Second Edition. Springer, Dordrecht (2025)

[5] Connes, A., Lott, J.: Particle models and noncommutative geometry. Nuclear Physics B (Proc. Suppl.) **18B**, 29–47 (1990) https://doi.org/10.1016/0920-5632(91)90120-4

[6] Chamseddine, A.H., Connes, A., Marcolli, M.: Gravity and the Standard Model with neutrino mixing. Advances in Theoretical and Mathematical Physics **11**(6), 991–1089 (2007) https://doi.org/10.4310/ATMP.2007.v11.n6.a3

[7] Aschieri, P., Castellani, L.: Noncommutative gravity solutions. Journal of Geometry and Physics **60**, 375–393 (2010) https://doi.org/10.1016/j.geomphys.2009.11.009

[8] Marcolli, M.: Noncommutative Cosmology. World Scientific, Singapore (2018)

[9] Franco, N., Eckstein, M.: Noncommutative geometry, Lorentzian structures and causality. Retrieved from https://arxiv.org/abs/1409.1480 (2014)

[10] Earman, J.: World Enough and Spacetime. Cambridge University Press, Cambridge (1989)

[11] Rynasiewicz, R.: Rings, Holes and Substantivalism: On the Program of Leibniz Algebras. Philosophy of Science **59**(4), 572–589 (1992) https://doi.org/10.1086/289696

[12] Bain, J.: Einstein Algebras and the Hole Argument. Philosophy of Science **70**(5), 1073–1085 (2003) https://doi.org/10.1086/377390

[13] Bain, J.: Spacetime Structuralism. In: Dieks, D. (ed.) The Ontology of Spacetime, pp. 37–66. Elsevier, Amsterdam (2006)

[14] Menon, T.: Algebraic Fields and the Dynamical Approach to Physical Geometry. Philosophy of Science **86**(5), 1273–1283 (2019) https://doi.org/10.1086/705508

[15] Huggett, N., Lizzi, F., Menon, T.: Missing the point in noncommutative geometry. Synthese **199**, 4695–4728 (2021) https://doi.org/10.1007/s11229-020-02998-1

[16] Lizzi, F.: Noncommutative Spaces. In: Aschieri, P., Dimitrijević, M., Kulish, P., Lizzi, F., Wess, J. (eds.) Noncommutative Spacetimes. Symmetries in Noncommutative Geometry and Field Theory, pp. 89–110. Springer, Berlin Heidelberg (2009)

[17] Callender, C., Huggett, N.: Introduction. In: Physics Meets Philosophy at the Planck Scale. Contemporary Theories in Quantum Gravity, pp. 1–30. Cambridge University Press, Cambridge (2004)

[18] Carlip, S.: Challenges for emergent gravity. Studies in History and Philosophy of Modern Physics **46**, 200–208 (2014) https://doi.org/10.1016/j.shpsb.2012.11.002

[19] Oriti, D.: Disappearance and emergence of space and time in quantum gravity. Studies in History and Philosophy of Modern Physics **46**, 186–199 (2014) https://doi.org/10.1016/j.shpsb.2013.10.006

[20] Huggett, N., Wüthrich, C.: Out of Nowhere. Oxford University Press, New York (2025)

[21] Crowther, K.: Effective Spacetime. Understanding Emergence in Effective Field Theory and Quantum Gravity. Springer, Cham (2016)

[22] Wüthrich, C.: The emergence of space and time. In: Gibb, S., Findlay Hendry, R., Lancaster, T. (eds.) The Routledge Handbook of Philosophy of Emergence, pp. 315–326. Routledge, New York (2018)

[23] Jaksland, R., Salimkhani, K.: The many problems of spacetime emergence in quantum gravity. British Journal for the Philosophy of Science (2023) https://doi.org/10.1086/727052

[24] Mead, C.A.: Possible Connection Between Gravitation and Fundamental Length. Physical Review **135**(3B), 849–862 (1964) https://doi.org/10.1103/PhysRev.135.B849

[25] Doplicher, S., Fredenhagen, K., Roberts, J.E.: Spacetime quantization induced by classical gravity. Physics Letters B **331**(1-2), 39–44 (1994) https://doi.org/10.1016/0370-2693(94)90940-7

[26] Doplicher, S., Fredenhagen, K., Roberts, J.E.: The quantum structure of spacetime at the Planck scale and quantum fields. Communications in Mathematical Physics **172**, 187–220 (1995) https://doi.org/10.1007/BF02104515

[27] Hagar, A.: Discrete or Continuous? The Quest for Fundamental Length in Modern Physics. Cambridge, Cambridge University Press (2014). https://doi.org/10.1017/CBO9781107477346

[28] Maresca, E.: Noncommutative Geometry and Spacetime: A Historical Reconstruction. Journal of Physics: Conference Series **2948**, 012011 (2025) https://doi.org/10.1088/1742-6596/2948/1/012011

[29] Bronstein, M.: Republication of: Quantum theory of weak gravitational fields. General Relativity and Gravitation **44**, 267–283 (1936/2012) https://doi.org/10.1007/s10714-011-1285-4

[30] Harlow, D.: Jerusalem lectures on black holes and quantum information. Reviews of Modern Physics **88**, 015002 (2016) https://doi.org/10.1103/RevModPhys.88.015002

[31] Jacobs, C.: Does Quantum Gravity Happen at the Planck Scale? Philosophy of Physics **3**(1), 2 (2025) https://doi.org/10.31389/pop.159

[32] Wallace, D.: Who's afraid of coordinate systems? An essay on representation of spacetime structure. Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics **67**, 125–136 (2019) https://doi.org/10.1016/j.shpsb.2017.07.002

[33] Teller, P.: An Interpretive Introduction to Quantum Field Theory. Princeton University Press, Princeton (1995)

[34] Teller, P.: So What *Is* the Quantum Field? In: Kuhlmann, M., Lyre, H., Wayne, A. (eds.) Ontological Aspects of Quantum Field Theory, pp. 145–162. World Scientific, Singapore (2002)

[35] Nestruev, J.: Smooth Manifolds and Observables. Second Edition. Springer, Cham (2003)

[36] Rosenstock, S., Barrett, T.W., Weatherall, J.O.: On Einstein algebras and relativistic spacetimes. Studies in History and Philosophy of Modern Physics **52**, 309–316 (2015) https://doi.org/10.1016/j.shpsb.2015.09.003

[37] Halvorson, H., Tsementzis, D.: Categories of Scientific Theories. In: Landry, E.M. (ed.) Categories for the Working Philosopher, pp. 402–429. Oxford University Press, Oxford (2017)

[38] Geroch, R.: Einstein Algebras. Communications in Mathematical Physics **26**, 271–275 (1972) https://doi.org/10.1007/BF01645521

[39] Hudetz, L.: Definable Categorical Equivalence. Philosophy of Science **86**, 47–75 (2019) https://doi.org/10.1086/701047

[40] Spivak, D.I.: Categories as Mathematical Models. In: Landry, E. (ed.) Categories for the Working Philosopher, pp. 381–401. Oxford University Press, New York (2018)

[41] Dewar, N.: On Internal Structure, Categorical Structure, and Representation. Philosophy of Science **90**, 188–195 (2023) https://doi.org/10.1017/psa.2022.10

[42] Weatherall, J.O.: Why Not Categorical Equivalence? In: Madarász, J., Székely, G. (eds.) Hahnal Andréka and István Németi on Unity Of Science, pp. 427–451. Springer, Cham (2021)

[43] Wu, J., Weatherall, J.: Between a Stone and a Hausdorff Space. British Journal for the Philosophy of Science (forthcoming) https://doi.org/10.1086/728532

[44] Geroch, R., Horowitz, G.T.: Global structure of spacetimes. In: Hawking, S., Israel, S.W. (eds.) General Relativity: An Einstein Centenary Survey, pp. 212–293. Cambridge University Press, Cambridge (1979)

[45] Connes, A.: Noncommutative Geometry. Academic Press, Cambridge (1995)

[46] Connes, A.: On the spectral characterization of manifolds. Journal of Noncommutative Geometry **7**, 1–82 (2013) https://doi.org/10.4171/JNCG/108

[47] Heller, M., Sasin, W.: Sheaves of Einstein Algebras. International Journal of Theoretical Physics **34**(3), 387–398 (1995) https://doi.org/10.1007/BF00671598

[48] Akemann, C.A.: The general Stone-Weierstrass problem. Journal of Functional Analysis **4**(2), 277–294 (1969) https://doi.org/10.1016/0022-1236(69)90015-9

[49] Giles, R., Kummer, H., Sneddon, I.N.: A Non-Commutative Generalization of Topology. Indiana University Mathematics Journal **21**(1), 91–102 (1971) https://doi.org/10.1512/iumj.1972.21.21008

[50] Patel, M.S.: Noncommutative Gelfand Duality and Applications I: The Existence

of invariant subspaces. Retrived from https://arxiv.org/abs/0811.1824 (2006)

[51] Balachandran, A.P., Bimonte, G., Ercolessi, E., Lizzi, F., Sparano, G., Teotonio-Sobrinho, P.: Finite quantum physics and noncommutative geometry. Nuclear Physics B – Proceedings Supplements **37**(3), 20–45 (1995) https://doi.org/10.1016/0920-5632(94)00787-V

[52] Landi, G.: An Introduction to Noncommutative Spaces and Their Geometries. Springer, Heidelberg (1997)

[53] Borceux, F., van Den Bossche, G.: An essay on noncommutative topology. Topology and its Applications **31**(3), 203–223 (1989) https://doi.org/10.1016/0166-8641(89)90018-7

[54] Mulvey, C.J.: &. Supplementi ai Rendiconti del Circolo Matematico di Palermo Series 2 **12**, 99–104 (1986)

[55] Coniglio, M.E., Miraglia, F.: Non-commutative topology and quantales. Studia Logica **65**(2), 223–236 (2000) https://doi.org/10.1023/a:1005267714448

[56] Mulvey, C.J., Wick Pelletier, J.: On the quantisation of points. Journal of Pure and Applied Algebra **159**(2), 231–295 (2001) https://doi.org/10.1016/S0022-4049(00)00059-1