Cognitive artifacts and their affordances in mathematical practice: Cluster algebras as a case study

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Abstract

This paper investigates the role that various kinds of signs, taken to be *cognitive artifacts*, play in contemporary mathematical practice. We examine two informationally equivalent ways of presenting mathematical content in light of some of their uses in the recently developed theory of *cluster algebras*. Following a short overview of the concept of cognitive artifacts as it pertains to mathematical practice, we present the conceptual frameworks of *affordances* and *responsiveness types* which are used in the analysis of the case study. These frameworks are employed to articulate the sorts of epistemic operations that the given cognitive artifacts support. We find that in the case of cluster algebras, the cognitive artifacts in question offer different opportunities for *computational offloading*, which effectively reduces cognitive load with the aid of external media. In addition, we provide a brief introduction to the theory of cluster algebras, albeit at a restricted level of generality, including some of the initial motivations which resulted in the formulation of this theory.

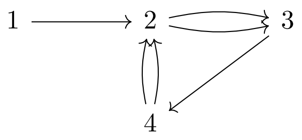
Keywords: Mathematical practice, Cognitive artifacts, Affordances, Diagrams, Matrices, Quivers

1 Introduction

In recent decades there has been growing interest in the epistemic and cognitive roles that various kinds of signs play in mathematical practice. Some themes which have been addressed in the philosophical literature are the fruitfulness of diagrams (Carter, 2019; Starikova, 2012; Eckes and Giardino, 2018) and their justificatory role in mathematical proofs (De Toffoli, 2023; De Toffoli and Giardino, 2014; Manders, 2008), the design of notations in logic and in number theory (Schlimm, 2025; Waszek and Schlimm, 2021) and the cognitive support that signs provide in mathematical reasoning (De Cruz and De Smedt, 2013; De Toffoli, 2017; Giardino, 2018; Johansen, 2024). Notably, these accounts all feature case studies from historical or contemporary mathematical practices which yield new insights into the practice and motivates further investigation into the *actual uses* of mathematical signs. This line of inquiry is advocated in particular by De Toffoli (2023), who, in proposing how we may go about evaluating the effectiveness of diagrams in mathematical practice, writes that we should consider how they facilitate "extracting information, carrying out specific inferences, and performing calculations—all functions that mathematicians generally recognize and accept" (De Toffoli, 2023, p. 2). While our focus in this paper is not limited to diagrams, but considers mathematical signs more broadly, this proposal nevertheless captures essentially what we aim to do in making a comparative study of signs which appear in the recently developed theory of *cluster algebras*. In this theory, two different concepts are used alternatively to present the main object of study, which is a special kind of commutative ring that has a certain combinatorial structure. These concepts are certain kinds of *matrices* and *quivers*, which is another name for directed graphs (see Figure 1).

$$\begin{bmatrix} 0 & 37 & -6 & 1 \\ -5 & 0 & -4 & -2 \\ 11 & 9 & 0 & -1 \\ -1 & 2 & 8 & 0 \end{bmatrix}$$

B



Q

Fig. 1: An inscription of a 4×4 integer matrix B and of a quiver Q with four vertices

The roles of signs in mathematical practice has been analyzed using conceptual frameworks developed in other fields, namely, in cognitive science, in order to articulate the roles that these signs play. Two concepts that have been employed for this purpose, which we will also make use of here, are *cognitive artifacts* and *affordances*. The former refers to objects that are used for the purpose of aiding cognition; the latter describes the possible actions or moves that may be performed for any given physical object. We consider physical objects to be taken in a broad sense so that this category includes material objects ranging from everyday items, like coffee mugs and bookshelves, to objects which arise in the aid of cognitive processes, such as an inscription or a notational form. It has been argued that cognitive artifacts are at times influential in shaping mathematical practices and in this way play a role in advancing mathematical theories (Johansen, 2024). Moreover, the concept of affordances has been used to explicate the kinds of material and epistemic constraints which make some mathematical signs an effective representational tool (Giardino, 2018). In this paper, we use the conceptual framing of cognitive artifacts and affordances to articulate some ways in which matrices and quivers, through the signs associated to them, support reasoning in the specific mathematical domain of cluster algebras. Insofar as a sign facilitates reasoning, we will consider it to be a cognitive artifact. We refer throughout the paper to the *inscriptions* of matrices and quivers as *cognitive artifacts* to maintain focus on their role in supporting cognition through externalizing reasoning.

What is of interest in the case of cluster algebras is how information is *read off* from the matrix or quiver, respectively, and how they support performing a mathematical operation called *mutation* which is central to the theory. The cognitive artifacts associated to matrices and quivers are in a certain sense *informationally equivalent* and yield corresponding results when mutation is performed. Adopting a naïve view, it would seem that having just one of them available would suffice, if they are indeed equivalent. We observe in contrast that both types of cognitive artifacts are regularly used within the context of the theory for the purposes just described. Considering the *cognitive cost* associated with learning to use a notational system proficiently, we aim to investigate which kinds of epistemic gains are achieved through having both kinds of cognitive artifacts available. This is done in addition to investigating the epistemic roles that these cognitive artifacts play in the practice.

In a paper concerning the use of diagrams in contemporary mathematical practice, De Toffoli (2017) examines several uses of commutative diagrams in homological algebra—a branch of algebraic topology—and argues that these sorts of diagrams constitute a "good" mathematical notation because they are highly tractable. That is, they enable mathematicians to perform specific operations and carry out certain inferences which yield mathematically valid results. Her account proposes three criteria—*expressiveness*, *calculability*, and *transparency*—for evaluating what she refers to as the "effectiveness" of a mathematical notation.[1] The criteria of calculability (concerning the possible use of a notation in performing epistemic operations) and transparency (concerning the possibility to exploit cognitive abilities and acquired expertise when using a notation) are especially useful for investigating the roles that matrices and quivers play in cluster algebra theory. We find that the notations associated to these concepts are mathematically tractable, which allows for making certain inferences and performing the mutation operation. In contrast to De Toffoli's (2017) account, which examines a single form of notation, we compare two different kinds of cognitive artifacts which are used for some of the same purposes within their mathematical context. Through making this comparative study, we are in a position to describe some trade-offs that may be associated with these different types of cognitive artifacts in view of their uses in cluster algebra theory.

Comparative studies of notation and signs in the philosophical literature have dealt with, for instance, the respective notations of Boole and Frege which were aimed at supporting problem-solving in logic. Waszek and Schlimm (2021) argue that Boole and Frege held differing views on the possibility, and also the desirability, of formulating a systematic method for solving problems in logic by means of employing a logical calculus, and therefore had distinct aims for what their notational systems should achieve. This comparative work takes the perspectives of the actors as central to the

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[1] By De Toffoli's (2017) account, a commutative diagram constitutes a legitimate notational system and can therefore be evaluated using these criteria.

philosophical analysis. By contrast, our analysis focuses on the use of two different concepts and their related cognitive artifacts which appear within the *same* practice for at least some of the same purposes. Even so, the views of practitioners do play a role in this case, too. In particular, it has been claimed by practitioners working on cluster algebras that quivers—as compared with matrices—have the advantage of being "visible." An additional aim of this paper is to make sense of this claim.

The paper is structured as follows. Section 2 presents the conceptual framework, drawing upon the concepts of *cognitive artifacts* and *affordances*, along with a few selected parts of a framework proposed by Manders (1999), that is used to analyze the case study. Section 3 presents some historical and mathematical details of the theory of cluster algebras which are relevant for the analysis of the case. Section 4 compares matrices and quivers, taken as cognitive artifacts, with the aim to articulate how they differ in terms of the cognitive support they provide for reading off certain information and performing mutations. Section 5 offers several concluding remarks.

2 Cognitive artifacts and their affordances

The term *cognitive artifacts* refers to "physical objects made by humans for the purpose of aiding, enhancing, or improving cognition" (Hutchins, 1999, p. 126). Examples of cognitive artifacts include a map, a digital calculator, and a notepad, all of which fit the criterion of being made to support cognition. More explicitly, a map supports memory and planning, and a digital calculator supports mathematical computation. A notepad supports these functions as well by affording the possibility for symbols, drawings, and natural language expressions to be inscribed on it.

A cognitive artifact serves to provide some amount of organization or scaffolding of information which supports the processing of this information.[2] In this view, inscriptions in general could be considered as cognitive artifacts in their own right, if they are brought about in order to support cognition. For many people, performing a task which is cognitively demanding is often accompanied by the act of writing something down. This can be viewed as an instance of "computational offloading" (see Scaife and Rogers, 1996), which we take here simply to mean reducing cognitive load through the aid of external media. In mathematics, inscriptions of many different sorts are used to support cognition by offloading cognitive processing demands. For instance, a written arrangement of symbols describing a system of equations or a drawing of a Euclidean diagram may be viewed as examples of cognitive artifacts used in mathematics.

2.1 Cognitive artifacts in mathematical practice

Cognitive artifacts in mathematical practice come in many forms, for instance: tools for performing calculations, such as an abacus or a counting board (discussed, e.g., by Johansen, 2024; Macbeth, 2014), and diagrams of various sorts, including those used in knot theory (De Toffoli and Giardino, 2014) and in homological algebra (De Toffoli, 2017). It has been argued that signs constituting some particular mathematical

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[2] Johansen and Misfeldt (2015) take a similar approach in their use of the cognitive artifact concept, which they examine in connection with the concept of *semiotic scaffolding* introduced by J. Hoffmeyer in the 2000s which, roughly put, attempts to articulate how external structure of some sort can support cognition.

notation are not just an arbitrary feature of the practice, but are "always introduced and used in a context" (Schlimm, 2025, p. 50). One reason for why the study of mathematical signs as cognitive artifacts matters is due to the fact that humans as finite agents are subject to cognitive constraints which inform certain aspects of the practice. For instance, signs facilitate in making inferences and contribute to computational efficiency, and thereby "shape our mathematics quite deeply, constraining which mathematical objects end up being salient for us" (Waszek, 2024, p. 2973). This theme of constraints will return when we consider the concept of affordances and what can be said about responses to external stimuli.

Studying the uses of cognitive artifacts is a way to generate understanding of their epistemic and cognitive significance in mathematical practices. For instance, Johansen (2024) considers various cognitive artifacts used in making arithmetic calculations. He argues that the ability to perform calculations, e.g., using clay tokens or written tables, plays several different roles in mathematical practices. On the one hand, calculating is partly a practical skill that sometimes involves "sequences of physical (epistemic) actions" and, moreover, that "the same basic operation can be accomplished using different tools (mental or material) and different practices" (Johansen, 2024, p. 156).[3] On the other hand, performing calculations is, as he argues, not *only* a practical skill, since calculating has at times played a significant role in theory development.

Some philosophical work examines more recent mathematical examples. In particular, De Toffoli (2017) considers the role of commutative diagrams—which she argues constitute an effective notational system—in homological algebra. In this domain, there is a commonly employed technique called 'diagram chasing' which is a proof strategy that makes use of commutative diagrams to organize and keep track of information. De Toffoli argues that diagram chasing facilitates reasoning and is a rigorous practice. Furthermore, her account shows that the effectiveness of commutative diagrams has to do with how they support externalizing reasoning which make them a perspicuous and tractable mathematical notation. She remarks that, in general, "the effectiveness of a notation depends on its uses" (De Toffoli, 2017, p. 163), which may be intended from the outset of creating the notation, or may be developed incidentally later on. One factor contributing to the effectiveness of commutative diagrams is captured by the criterion of *calculability*, which describes "the possibility of using the notation to perform specific operations with epistemic aims" (De Toffoli, 2017, p. 160). This criterion highlights the kinds of manipulations that a notation may support, which, as we will see below, appears to be closely linked to certain kinds of *affordances*.

As stated in the introduction, we examine different kinds of cognitive artifacts used in the theory of cluster algebras, namely inscriptions of certain types of matrices and certain types of quivers (see Fig. 1 for illustration). There is a precise sense in which the types of matrices and quivers considered in this case may be said to be equivalent in terms of the information which they encode (see Section 3.6). Comparing some of the ways in which these cognitive artifacts are used within the context of the

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[3] Kirsh and Maglio (1994) introduce the term *epistemic action* to denote those (physical) actions which have as their main purpose improving cognition by reducing various kinds of factors related to mental computations, such as memory, the number of steps involved, or the probability of error in making, e.g., a calculation. This term denotes phenomena similar to what we have described using the terminology of *computational offloading*.

theory reveals important differences between them in terms of (i) how they support computational offloading and (ii) the types of sign manipulations that one performs in carrying out the mutation operation. In order to articulate these differences, we turn to the concept of *affordances* for describing the possible actions that can be taken in using these cognitive artifacts.

2.2 Affordances and responsiveness types

The concept of *affordances*, coined by J. J. Gibson (1904–1979), describes the possible moves or actions which may be taken, given the immediate physical surrounds of an individual. In his 1979 work (reprinted in 2015) *The Ecological Approach to Visual Perception*, Gibson writes that, "[t]he *affordances* of the environment are what it *offers* the animal, what it *provides* or *furnishes*, either for good or ill" (Gibson, 2015, p. 119; italics in original). This includes both the "natural" environment—e.g., deserts, forests, and mountains—but also the artificial one, including objects such as a pen, which affords "trace-making" (Gibson, 2015, p. 125), or a chair, which affords sitting.

While Gibson's primary focus is on the affordances which are directly linked with visual perception, Zhang and Patel (2006) point out that his theory can be adapted to accommodate other types of affordances, for instance, what they call *perceptional affordances* and *cognitive affordances*. They suggest that the former type are often given by spatial mappings, for instance, the controls for a stove-top burner may take the same layout as the burners themselves. The latter type are provided by particular conventions, for instance, the color of a traffic light indicates an action, e.g., a red light means to stop.[4] In their framework, an affordance can be comprised of some combination of the different types, which they call a *mixed affordance*. For example, a shoelace which affords tying is in their view a mixed affordance because it is a conjunction of both a physical property, and is thus a physical affordance, and the knowledge of tying a knot in a certain way, which is a cognitive affordance in their terminology.

The concept of affordances has been adapted to describe aspects of mathematical practice. Giardino (2018) considers so-called *representational affordances*, which denote the affordances that are offered by a cognitive tool, which is essentially a form of cognitive artifact. This use of the affordance concept for studying cognitive tools used in mathematics is motivated by the observation that "mathematics makes use of *signs* that are to be *perceived* and *acted upon*" (Giardino, 2018, p. 347; italics in original). As Giardino points out, there have to be certain constraints defined on these cognitive tools, i.e., mathematical signs, in order for them to be used effectively since not all of the actions which are afforded by some sign are legitimate mathematical operations. She introduces the concept of affordances with the aim of explicating how these constraints are defined. A key point is that some constraints are brought about by the material features of the cognitive tool and others by the specific use that these tools are put to. She argues that in the case of knot diagrams, which are used in branches

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[4] They specify that cognitive affordances are given by "cultural conventions" (Zhang and Patel, 2006, p. 339), although it is not quite clear what this means in their framework. We understand that such conventions are learned through experience, and thus contribute to some background knowledge of the perceiver, since this category of affordances is seemingly meant in contrast to perceptual affordances which are somehow immediately perceived.

of topology, different ways of notating a given knot yield different sets of affordances which affect how readily certain possible mathematical operations may be observed.

In a similar vein, we employ the concept of affordances to articulate differences between the cognitive artifacts used in cluster algebras which become apparent when comparing their actual use in reading off information and performing specific kinds of computations using the signs. As we will see, there are substantial differences in how information is encoded in each kind of cognitive artifact, which has an impact on the act of searching for and parsing this information. Moreover, the signs yield different opportunities for manipulation, which influences how they support computations.

Beyond the different types of affordances just described, which focus on the possible moves available for a given object, it is also relevant to consider the information which can be safely ignored when using a cognitive artifact to carry out some chain of reasoning. We refer to Manders' (1999) framework of *responsiveness* and *indifference* types, which takes root in the observation that the limited cognitive resources of the human mind constrain our capacity to pay attention and respond to external stimuli. In this view, what contributes to a conceptual advance in mathematics is not only about the type of *response* which is enabled by a given form of presentation, but also about the strategic constraints of the presentation which *restrict* attention. Such restriction occurs through an *indifference to* response, which Manders argues plays a crucial role in reasoning. Thus, such indifferences, although they are easily overlooked, should not be taken for granted when considering the efficacy of a given mathematical representation.

Manders' main example focuses on the conceptual advances offered by Cartesian geometry, which he claims offers a "modified" use of artifacts which constrains the types of responses required in several strategic ways. The geometrical methods introduced by Descartes allowed for algebraic responses to geometrical problems dating back to the time of Euclid. Many such problems can be translated into systems of equations which make it possible to respond to a given problem using algebraic methods. This way of formulating geometrical problems is advantageous for several reasons, including eliminating the prior need to respond to case distinctions, which allows one to then proceed uniformly using algebraic methods. Manders argues that Descartes' algebraic method offers advanced conceptual resources for dealing with geometrical problems which were previously addressed using the diagram- and text-based approach found in Euclid's *Elements*.

We use Manders' framework in conjunction with the concept of affordances to suggest some advantages and disadvantages that may be associated with the uses of matrices and quivers in the theory of cluster algebras.

3 Cluster algebras

Work on cluster algebras was initiated by Sergey Fomin (1958–) and Andrei Zelevinsky (1953–2013) in the year 2000 with the motivation to study some specific connections that had appeared in Lie theory in the early 1990s.[5] In their attempt to understand these connections from an algebraic and combinatorial point of view, Fomin and Zelevinsky began to study *binomial exchange relations* of the form

$$xx' = M_1 + M_2 \tag{1}$$

where x and x' are elements of the field of rational functions $\mathcal{F} = \mathbb{Q}(x_1, ..., x_n)$ in n variables, and M_1 and M_2 are monomials which are given in terms of the elements of $\mathcal{F}$.[6] They report that relations of this form "played a crucial role" in trying to understand the connections which they were interested in and that "[i]t was the desire to explain the ubiquity of these relations and to place them in a proper context that led us to the concept of cluster algebras" (Fomin and Zelevinsky, 2002, p. 499). This concept turned out to be fruitful in a variety of mathematical and physical domains, which has translated into a flurry of activity within cluster algebra theory during the past quarter century.

A surprising feature of this case is that many of the domains in which cluster algebra theory has found traction have seemingly little *a priori* connection to Fomin and Zelevinsky's initial aims. The theory of cluster algebras thus constitutes another example in the history of mathematics in which what started out as a tool for investigating some specific problem or target domain developed into a theory with its own set of questions and which came to be studied in its own right.[7] The connections between cluster algebras and other areas of mathematics have even been characterized by those working on the theory as "deep" in a number of places.[8] The rapid growth of this theory has been emphasized in a survey article published in the *Bulletin of the American Mathematical Society*:

> ... the theory of cluster algebras has since [its conception] taken on a life of its own, as connections and applications have been discovered to diverse areas of mathematics, including quiver representations, Teichmüller theory, tropical geometry, integrable systems, and Poisson geometry. (Williams, 2014, p. 1)

This list could be extended further to include areas such as category theory and some parts of mathematical physics. Given the fruitfulness of cluster algebra theory, it is of both philosophical and historical interest to attempt to understand the conceptual innovations and the results which were obtained via this theory. The immense productivity in this area makes these general questions extend far beyond the scope of this paper. The modest treatment presented here focuses for the sake of simplicity on

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[5] The connection that Fomin and Zelevinsky were interested in concerned dual canonical bases of quantum groups and the theory of total positivity, which was first investigated by G. Lusztig (1990) and, independently, by M. Kashiwara (1991). See especially the introduction of Fomin and Zelevinsky (2002) and Williams (2014) for historical background.

[6] Further specifications about the terms appearing in the binomial exchange relation are relevant to the theory, although not a main focus of the analysis of this paper. First, the variables x and x' are algebraically independent. Second, the monomials M_1 and M_2 are without common divisors and are expressed in terms of the elements of the same particular subset of $\mathcal{F}$. Additionally, one could relax the conditions defined on $\mathcal{F}$ so that it is merely *isomorphic*, but not necessarily *identical*, to $\mathbb{Q}(x_1, ..., x_n)$. See Fomin and Zelevinsky (2002) for more details.

[7] Another instance of this type of phenomenon is the development of category theory. This case is analyzed for its historical and philosophical significance by Krömer (2007) in his book *Tool and Object*, which provides a broad overview of the developments which resulted in the formulation of the theory.

[8] See, e.g., Felikson, Shapiro and Tumarkin (2012), Fomin (2010), and Leclerc and Williams (2014).

a quite restricted version of the theory. We highlight, in particular, a combinatorial operation called *mutation* which is central to the theory.[9] The term “mutation” is used at times to refer to different sorts of processes. In this paper, we focus on two “categories” of mutations: (i) *cluster mutations* and (ii) *matrix mutations* or *quiver mutations.*

3.1 Cluster algebras: fundamental ideas

A cluster algebra is a commutative ring that is equipped with a set of distinguished generators called *cluster variables.*[10] Generating sets appear with some frequency in abstract algebra as a convenient way of presenting a particular algebraic structure, such as a group or a ring. For instance, the set of integers $\mathbb{Z}$ together with the operation of addition forms a group $(\mathbb{Z}, +)$ which is finitely generated by the set $\{1\}$. This group is generated by other finite sets as well, e.g., by $\{22, 13\}$, so $\{1\}$ is not unique in this regard.

In the case of cluster algebras, the generating set is constructed through the combinatorial operation of *mutation.* This construction process begins with some particular collection of cluster variables, called a *cluster.* One “mutates” the cluster to obtain new cluster variables and new clusters. A *cluster mutation* involves exchanging precisely one of the cluster variables with a new one that gets constructed from the other cluster variables belonging to the cluster. This yields a new cluster which is identical to the one that has been mutated with the single exception of the cluster variable that has been exchanged. Iteratively mutating all clusters obtained in this way produces the remaining cluster variables.[11] The following example illustrates how the cluster mutation procedure unfolds.

3.2 An elementary example: the A_2 case

One of the most elementary examples of a cluster algebra appears in what we will refer to as the A_2 case.[12] We consider the initial cluster (x_1, x_2) which contains the cluster variables x_1 and x_2. Cluster mutation begins with choosing an index, in this case either ‘1’ or ‘2’. The chosen index is referred to as the “direction” in which one mutates. For instance, mutation of the cluster (x_1, x_2) in *direction* 1 involves exchanging x_1 with a new cluster variable x_1', which yields a new cluster (x_1', x_2). The structure of each new cluster variable obtained via cluster mutation is given by some exchange relation

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[9] This paper considers cluster algebras of geometric type which are defined by a skew-symmetric matrix, or, equivalently, by a quiver, and are without coefficients.

[10] The cluster variables are algebraically independent variables in a field $\mathcal{F}$ which is isomorphic to the field of rational functions in n variables, denoted by $\mathbb{Q}(x_1, ..., x_n)$. We may think of the cluster variables as ratios of polynomials $f(x_1, ..., x_n)/g(x_1, ..., x_n)$ in n independent variables, where $f, g \in \mathcal{F}$.

[11] In general, a cluster algebra could be infinite, meaning that it has infinitely many cluster variables and clusters. If this is the case, the mutation procedure continues indefinitely to yield new cluster variables which are distinct from the ones already obtained. In some cases, a cluster algebra is finite, meaning that it has only finitely many cluster variables and clusters.

[12] More precisely, this example is of a cluster algebra of Dynkin type A_2 with trivial coefficients. The name derives from the A_2 Dynkin diagram consisting of two connected nodes: $\bullet \!-\!\!-\!\!-\! \bullet$. Giving the edge in the diagram a direction transforms it into an A_2 quiver, which incidentally encodes the mutation rule in this case.

like the one shown in (1). In this example, x_1' is given by the relation

$$x_1 x_1' = x_2 + 1. \tag{2}$$

We will later see where this particular relation comes from; for now, the aim is only to show the construction of new cluster variables. Dividing both sides of (2) by x_1 yields a rational expression for x_1' which is given in terms of x_1 and x_2. We have now obtained the cluster $(\frac{x_2+1}{x_1}, x_2)$, which we call the *mutation of* (x_1, x_2) in direction 1. It is sometimes convenient to denote mutation in direction k by the symbol μ_k. We then write $\mu_1(x_1, x_2) = (\frac{x_2+1}{x_1}, x_2)$.

The next step is to mutate the cluster $(\frac{x_2+1}{x_1}, x_2)$ which we have just obtained in some other direction than 1. For this, we choose a different index. In this example, the only choice remaining is index 2. We mutate $(\frac{x_2+1}{x_1}, x_2)$ in direction 2 to obtain a new cluster variable x_2' given by

$$x_2 x_2' = \tfrac{x_2+1}{x_1} + 1 \tag{3}$$

which we can observe has a form similar to (2). Dividing both sides of this equality by x_2 and simplifying the resulting expression yields that $x_2' = \frac{x_2+x_1+1}{x_2 x_1}$. We now have the cluster $(\frac{x_2+1}{x_1}, \frac{x_2+x_1+1}{x_2 x_1})$ in addition to the two previous clusters. Continuing in this way of performing cluster mutations with alternating directions, we obtain yet another cluster variable $x_1'' = \frac{x_1+1}{x_2}$ by mutating $(\frac{x_2+1}{x_1}, \frac{x_2+x_1+1}{x_2 x_1})$ in direction 1. Mutating the cluster $(\frac{x_1+1}{x_2}, \frac{x_2+x_1+1}{x_2 x_1})$ thus obtained in direction 2 yields $(\frac{x_1+1}{x_2}, x_1)$. Finally, mutating this new cluster in direction 1 yields the cluster (x_2, x_1) whose cluster variables are identical to those belonging to the initial cluster. It turns out that additional cluster mutations will not yield any further cluster variables which are distinct from the ones we have already obtained. The cluster algebra is thus generated by the set

$$S = \{x_1, x_2, \tfrac{x_2+1}{x_1}, \tfrac{x_1+1}{x_2}, \tfrac{x_1+x_2+1}{x_1 x_2}\},$$

which is obtained by taking the non-disjoint union of all the clusters.

3.3 Exchange relations

It may still seem unclear why the cluster variables take the specific form that they do. Where do the relations in (2) and (3) come from? These are derived from a so-called *exchange relation* in combination with either an *exchange matrix* or a *quiver* which encodes some combinatorial data. The way in which an exchange relation is formulated varies depending upon whether the combinatorial data is encoded by a matrix or by a quiver.[13] We will return to exchange relations used with quivers in Section 3.5. When

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[13] There are further considerations, namely the degree of generality that one is working in, which inform the expression of an exchange relation. If one is working with (non-trivial) coefficients, there is another formulation used. Here we consider only cluster algebras with trivial coefficients, sometimes called "coefficient free" cluster algebras (see Felikson, Shapiro and Tumarkin, 2012).

working with matrices, the exchange relation takes the form

$$x_k x'_k = \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}} \ , \tag{4}$$

where the $b_{ik} \in \mathbb{Z}$ are entries of an $n \times n$ matrix $B = (b_{ij})$, with n being the number of cluster variables in each cluster, and the x_i belong to the cluster that is being mutated in direction k. Furthermore, the empty product is taken to be equal to 1.

By examining this exchange relation (4) we see that the exchange matrix B encodes the values of the exponents appearing on the right-hand side of the equality. In the A_2 case, the exchange matrix which is used to construct x'_1 from the initial cluster variables is the 2×2 matrix

$$B = \begin{bmatrix} 0 & 1 \\ \text{-1} & 0 \end{bmatrix} . \tag{5}$$

When we mutate in direction $k = 1$, we consider the entries b_{i1} in the *first column*, i.e., $b_{11} = 0$ and $b_{21} = -1$. The first product in (4) is empty since no entries in this column are positive. This yields a '1'. The second product in the relation becomes $x_2^{-(-1)} = x_2$. This gives the particular exchange relation $x_1 x'_1 = 1 + x_2$ which corresponds to what we saw in (2). From this procedure, we obtain the variable $x'_1 = \frac{x_2+1}{x_1}$. If we mutate (x_1, x_2) in direction $k = 2$, we instead consider the entries $b_{12} = 1$ and $b_{22} = 0$ in the *second column* of the matrix. This produces the term $x'_2 = \frac{x_1+1}{x_2}$. The other cluster variables in the A_2 case are produced in a similar manner, but using possibly different matrices from B, as given in (5), which are related to B via the operation of *matrix mutation* (see Section 3.7).

In general, the construction of the cluster variables transpires with the exchange relations, in their explicit form, being determined by an exchange matrix together with the general exchange relation (4). At each step of the mutation process, as it has just been described, one must also perform a *matrix mutation*, which yields a new exchange matrix. Before seeing how this procedure works, we consider how exchange relations are used in combination with quivers, as well as the correspondence between matrices and quivers. To do this, we present the definition of a quiver followed by an example given for illustration.

3.4 Quivers

We begin with the formal definition: a quiver Q is a quadruple (Q_0, Q_1, s, t) consisting of a set Q_0 of *vertices*, a set Q_1 of *arrows* and two maps $s : Q_1 \to Q_0$ and $t : Q_1 \to Q_0$ called the *source* and *target*, respectively. Given an arrow $\alpha \in Q_1$, the source sends α to the vertex $s(\alpha) \in Q_0$ at which α originates and the target sends α to the vertex $t(\alpha) \in Q_0$ at which α terminates.[14]

A quiver is thus given by its vertices and arrows, along with its source and target maps. In practice, a quiver is often shown by a diagram (Fig. 2). In such a diagram, the arrows are shown by the symbol "⟶" and the vertices by their numerical labels, or by some other kind of symbol, often a dot '•'. The diagram uses *spatial indexing*

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[14] The quivers dealt with here will all be *finite*, meaning that their set of vertices and set of arrows are both finite.

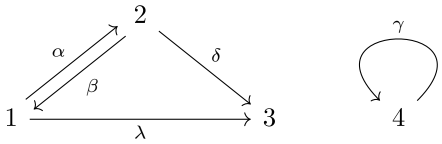


Fig. 2: A quiver with vertices $Q_0 = \{1, 2, 3, 4\}$ and arrows $Q_1 = \{\alpha, \beta, \gamma, \delta, \lambda\}$, and the source and target maps given by $s(\alpha) = 1$, $s(\beta) = 2$, $s(\gamma) = 4$, $s(\delta) = 2$, $s(\lambda) = 1$, $t(\alpha) = 2$, $t(\beta) = 1$, $t(\gamma) = 4$, $t(\delta) = 3$, and $t(\lambda) = 3$

to show the particular relations between arrows and vertices, which may otherwise be presented as a list of source and target maps like what is given in the caption of Figure 2. These maps specify the appearance of the diagram, which is then used for *reading off* certain kinds of information. The spatial indexing represents, for instance, that the source of the arrow denoted by 'λ' is '1' and its target is '3,' since the tail and tip of 'λ' are co-located with '1' and '3', respectively, in the diagram.[15]

This use of symbols and indexing offers particular opportunities for reading off certain information from the quiver, in particular, the *multiplicity* of arrows between pairs of vertices and the *direction* of the arrows.[16] This information is relevant for constructing cluster variables because it provides another way of encoding the values of the exponents that will appear in new cluster variables. In the following section, we revisit the topic of exchange relations to see how cluster mutation proceeds when working with quivers.

Before doing so, we observe that there are some kinds of quivers which cannot be used in the construction of new cluster variables. These are quivers which contain *loops* or 2-*cycles*, both of which are illustrated in Fig. 2. The arrow γ is a loop since the source and target of γ map coincide, i.e., they both map γ to the same vertex. The arrows α and β form a 2-cycle, which formally is a pair (ϵ, σ) of arrows such that $s(\epsilon) = t(\sigma)$ and $s(\sigma) = t(\epsilon)$. For the remainder of the paper, we consider only quivers for which neither of these types of situations occurs, unless otherwise indicated.

3.5 Exchange relations revisited

Previously we saw the general form that an exchange relation takes when working with matrices (4). Now equipped with the definition of a quiver, we are ready to make sense of the form that an exchange relation takes when working with a quiver Q without loops or 2-cycles. Recalling that Q_1 denotes the set of arrows of Q, we have the relation

$$x_k x'_k = \prod_{\alpha \in Q_1 \,:\, s(\alpha)=k} x_{t(\alpha)} + \prod_{\alpha \in Q_1 \,:\, t(\alpha)=k} x_{s(\alpha)} \ , \tag{6}$$

---

[15] It has been proposed in the cognitive science literature that spatial indexing holds several advantages over symbolic encoding in terms of the ease by which one accesses concepts. In particular, searching for information and recognizing operators is facilitated by spatial indexing (Scaife and Rogers, 1996; see also Cheng, 2016).

[16] We may think of the direction of an arrow as being given by its source and target maps: an arrow goes *from* its source *to* its target.

where s and t denote the source and target maps of Q, and the empty product is again taken to be 1. Just as in (4), the two products on the right-hand side of the equality sign indicate which terms appear in the numerator of the cluster variable that is being constructed.

To see how this works in practice, we return to the A_2 case. The cluster variables which we obtain by mutating the cluster (x_1, x_2) result from working with a quiver that has two vertices with a single arrow going between them, which can be shown diagrammatically as follows: $1 \longrightarrow 2$. We use the combinatorial data presented in this quiver together with the general exchange relation (6) to determine the particular exchange relations for the variables x_1' and x_2' that are obtained via mutation of (x_1, x_2) in directions 1 and 2, respectively.

To see how this works, we mutate (x_1, x_2) first in direction $k = 1$. Considering the first product in (6), we find that

$$\prod_{\alpha \in Q_1 \,:\, s(\alpha)=1} x_{t(\alpha)} = x_2 \tag{7}$$

since vertex 2 is the *target* of an arrow whose *source* is vertex 1. The second product becomes

$$\prod_{\alpha \in Q_1 \,:\, t(\alpha)=1} x_{s(\alpha)} = 1 \tag{8}$$

since there are no arrows whose target is vertex 1, and so the product is empty. Just as we did when working in the context of exchange matrices, we add together x_2 and 1, and divide this by x_1, which yields $x_1' = \frac{x_2+1}{x_1}$. This is identical to what we found before in the context of matrices. The other cluster variables are constructed in a similar way, although at times a quiver different from $1 \longrightarrow 2$ is used in combination with (6).

We observe that the exchange relation in (6) which pertains to quivers does indeed correspond to the one given in (4) which pertains to matrices, although the exponent values are encoded in a different way than is the case when working with an exchange matrix. In an exchange matrix, the exponent values are encoded directly as the numerical values of its entries, whereas in a quiver the exponent values are encoded by the multiplicity and direction of its arrows.

To make this more clear, consider that instead of using the quiver just presented for the A_2 case, we were using a different quiver, for instance, a quiver which has *two* arrows going between vertices 1 and 2, such as $1 \rightrightarrows 2$. If this were the case, then mutating the cluster (x_1, x_2) in direction 1 would yield $x_1' = \frac{x_2 x_2 + 1}{x_1}$, which can be written more concisely as $x_1' = \frac{x_2^2+1}{x_1}$. In other words, the relation (6) tells us to read off the number of arrows from vertex 1 to vertex 2, so that

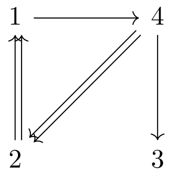
$$\prod_{\alpha \in Q_1 \,:\, s(\alpha)=1} x_{t(\alpha)} = x_2 x_2 \ , \tag{9}$$

instead of what we found in (7). Thus, multiple arrows yield repeated terms. Note that in order to make use of the more concise form, one has to exploit the rule which allows xx to be written as x^2.

We have now seen how cluster mutation proceeds both in the context of exchange matrices and in the context of quivers, and have observed that the results produced in both contexts are identical to one another. We now turn to the correspondence that is directly between matrices and quivers, which is made explicit with the definition of a *signed adjacency matrix.*

3.6 Signed adjacency matrix

Given a quiver Q which does not contain any loops or 2-cycles, we assign a matrix $B(Q)$ which encodes the multiplicity and direction of the arrows between each pair of vertices of the quiver as illustrated in Figure (3).



Q

$$\begin{bmatrix} 0 & -2 & 0 & 1 \\ 2 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 \\ -1 & 2 & 1 & 0 \end{bmatrix}$$

$B(Q)$

Fig. 3: A quiver Q and its signed adjacency matrix $B(Q)$

In this example, Q has four vertices and so $B(Q) = (b_{ij})$ is a 4×4 matrix. The general rule for encoding Q by a so-called *signed adjacency matrix* $B(Q)$ is that the entry b_{ij} of $B(Q)$ records the number of arrows from vertex i to vertex j in Q.

For instance, there are two arrows pointing from vertex '2' to vertex '1', so $b_{21} = 2$. In some sense, the signed adjacency matrix encodes the relations between arrows and vertices doubly, since $b_{ij} = -b_{ji} = l$, where l is the number of arrows $i \longrightarrow j$. That is, entry $b_{12} = -2$ in $B(Q)$ encodes the same information as entry b_{21}. Continuing with the example, we observe there is one arrow going from '1' to '4' so $b_{14} = 1$ and $b_{41} = -1$. There are no arrows mapped between vertices '1' and '3', so the entries b_{13} and b_{31} are both equal to zero. One can check that the other entries of $B(Q)$ correspond to the mapping of arrows to vertices in Q. In particular, the entries b_{ii} along the diagonal are all equal to zero, which corresponds to the fact that there are no loops in the quiver. Notice also that this way of encoding the relations between arrows and vertices as values in a matrix leave it unclear as to how a 2-cycle could be represented in such a matrix.

The definition of a signed adjacency matrix offers a basis for our claim that certain types of matrices and quivers are *informationally equivalent*: either of them can be systematically constructed using the information contained in the other. The types of quiver and matrices dealt with in this paper are therefore interchangeable within the context of cluster algebra theory for the purpose of encoding exponent values to support cluster mutations. Beyond this informational equivalence of exchange matrices and quivers, they also support mutations of the second "category," i.e., matrix mutations and quiver mutations. We will see in the following sections that these operations yield corresponding results.

3.7 Matrix mutation

We have already seen how an exchange matrix encodes the exponent values which are needed for the construction of new cluster variables. In mutating a given cluster, one uses a particular exchange matrix B to determine the exchange relation for that cluster mutation. In performing a mutations generally, one mutates both the clusters *and* the exchange matrices. This latter operation is known as *matrix mutation.* It amounts to computing a new matrix B' using the entries of a given square integer matrix B according to the following relation:

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise.} \end{cases} \tag{10}$$

Performing a matrix mutation occurs through entry-by-entry computations. Just as with cluster mutations, 'k' is the so-called "direction" in which we are mutating. To see how this works in practice, let us mutate the following matrix

$$\begin{bmatrix} 0 & 2 & -1 & 0 \\ -2 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{11}$$

$$B$$

in direction 2. Computing the entries of the second row and column of B', where $i = 2$ or $j = 2$, amounts to simply "changing the sign" of the corresponding entries in B. At this stage, we have the matrix

$$\begin{bmatrix} * & -2 & * & * \\ 2 & 0 & -1 & 0 \\ * & 1 & * & * \\ * & 0 & * & * \end{bmatrix}.$$

Some shortcuts can be found for computing the other entries for which neither i nor j is equal to 2. In particular, if b_{ik} and b_{kj} have opposite sign then the numerator of the rational expression will be equal to zero. In such cases, $b'_{ij} = b_{ij}$. For instance, the computations for b'_{11} and b'_{33} become

$$b'_{11} = 0 + \frac{|2|(-2) + 2|-2|}{2} = 0$$

and

$$b'_{33} = 0 + \frac{|-1|(1) + (-1)|1|}{2} = 0$$

both of which are equal to their corresponding entries in B. In cases where b_{ik} and b_{kj} have the same sign, then $b'_{ij} = b_{ij} + \mathrm{sgn}(b_{ik})b_{ik}b_{kj}$. For instance,

$$b'_{13} = -1 + 2(1) = 1.$$

Proceeding with the computations for the remaining entries yields the matrix

$$\underset{B'}{\begin{bmatrix} 0 & \text{-}2 & 1 & 0 \\ 2 & 0 & \text{-}1 & 0 \\ \text{-}1 & 1 & 0 & 1 \\ 0 & 0 & \text{-}1 & 0 \end{bmatrix}} \tag{12}$$

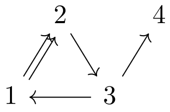
which is a square integer matrix of the same size as B. Notably, the matrix B in this example satisfies the condition that $b_{ij} = -b_{ji}$ which means that one can construct a quiver Q via the signed adjacency matrix definition. In the following section, we will see that mutating this quiver Q results in a quiver Q' which corresponds to B'.

3.8 Quiver mutation

Let Q be a quiver without loops or 2-cycles. *Quiver mutation* of Q in direction k, where k is a vertex in Q, is carried out according to a three-step procedure:

(1) for every path $i \longrightarrow k \longrightarrow j$ in Q, add a new arrow $i \longrightarrow j$;
(2) reverse all arrows α whose *source* or *target* coincides with k, i.e., $s(\alpha) = k$ or $t(\alpha) = k$;
(3) remove both arrows of any 2-cycles which may have arisen in step (1).

This process yields a new quiver Q' with the same vertices as Q, but possibly with different arrows. To see how quiver mutation works in practice, consider the following



Q

which is a quiver constructed from the matrix B in (11) in accordance with the definition of a signed adjacency matrix. We mutate Q in direction 2 following the step-by-step process outlined in (1)–(3) above. This procedure is shown in Figure 4.

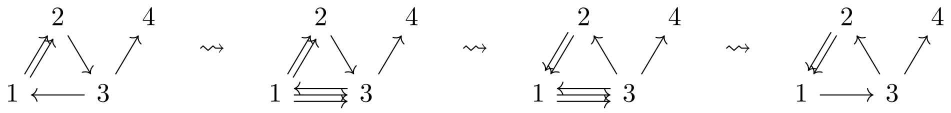
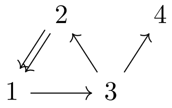


Fig. 4: The step-by-step process of mutating Q in direction 2 to obtain a mutation of Q

In the first step, two arrows $1 \longrightarrow 3$ are added since Q contains two distinct paths $1 \longrightarrow 2 \longrightarrow 3$. This introduces a 2-cycle between ‘1’ and ‘3’ which will later be removed. In the second step, the direction of the arrows which are incident to ‘2’ are reversed. Finally, in the third step, the 2-cycle between ‘1’ and ‘3’ is removed, leaving a single arrow $1 \longrightarrow 3$. This process yields the quiver



Q'

which is called the mutation of Q in direction 2.

As indicated above, Q' is related to B' in (12) via the signed adjacency matrix definition. It turns out that operations of quiver mutation and matrix mutation yield corresponding results when one is operating on a quiver and a matrix which are mutually related via this definition. There are nevertheless important differences in how the operation is performed in terms of the actual steps that are carried out. Some of these differences are discussed in the following section.

4 Comparing forms of presentations

In the previous section, we demonstrated how exchange matrices and quivers are informationally equivalent in the sense given by the signed adjacency matrix definition (Section 3.6) and how these cognitive artifacts yield corresponding results when performing the various kinds of mutations that we have considered. These observations provide a point of comparison for assessing how computational offloading and symbolic manipulations are supported by these different kinds of cognitive artifacts.

We will show that quiver diagrams have several epistemic advantages, in the uses considered here, due to their particular syntax which constrains attention in convenient ways.[17] The visual cues present in a quiver diagram support a higher degree of computational offloading as compared with those found in an exchange matrix. Moreover, we suggest some explanations for why quivers have been described as “visible” and what benefit this purported *visibility* may yield. Exchange matrices, on the other hand, possess the advantage that they directly support a greater level of generality than is achieved by quivers. The increase in generality may be exploited without any need to adjust the rules governing mutation, and so it comes essentially “for free” in this sense. It is possible to achieve a corresponding level of generality in quivers by employing a variation on the quiver concept, but this requires some modification of the concept to be made.[18] In this way, achieving corresponding results using quivers comes with the additional cost of generalizing the concept.

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[17] The term “quiver diagram” has a technical meaning in mathematical physics. For the sake of clarity, we emphasize that we do not mean this term in the technical way that it is used in mathematical physics. Rather, we use this term to refer to a presentation of a quiver by a diagram. Moreover, we use “quiver diagram” rather than simply “quiver” to indicate that we are talking about the *form of presentation* and not the abstract object of a quiver as such.

[18] The “variation” is in the form of a so-called *valued quiver*, in which the arrows each have some number, known as a *weight*, attached.

4.1 Computational offloading via exchange matrices and quivers

As we have seen, both exchange matrices and quivers are used in practice in conjunction with a generalized form of the exchange relation for the purpose of constructing new cluster variables. The role of the exchange matrix or quiver in this type of construction is to encode the combinatorial data, in the form of exponent values, that is needed for determining the new cluster variables. When extracting this data in the process of performing a *cluster mutation*, there are different steps involved depending upon whether one is working with an exchange matrix or with a quiver. In a quiver diagram, the symbols (i.e., numerals) denoting the vertices directly conveys which of the vertices is the relevant one to look at. In this way, the symbols facilitate the search for information by providing a visual cue which indicates the desired information. In an exchange matrix, one exploits the conventions which dictate the general form of a matrix in order to locate the relevant information. That is, one operationalizes the knowledge that an $n \times n$ matrix $B = (b_{ij})$ takes the general form

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}. \tag{13}$$

By convention, the entry b_{ij} is found in the ith row and jth column of the matrix. It is the *spatial layout* of an exchange matrix, taken to be a cognitive artifact such as given in (11) or (12), that indicates the location of the relevant information. The symbols denoting the individual entries of the exchange matrix are parsed through their spatial positioning *relative* to one another, rather than by serving as *manifest labels*, as is the case in quiver diagrams. Quiver diagrams thus possess the epistemic advantage that the retrieval of information, which begins with locating the information that is relevant, is supported manifestly by the symbols. Exchange matrices, in contrast, miss this epistemic advantage since they require that one responds to features of the matrix which are related only *indirectly* to the task of determining an exchange relation.

This dissimilarity between quiver diagrams and exchange matrices is consequential for the opportunities for computational offloading that are supported by these respective cognitive artifacts. To investigate how computational offloading plays out in practice, we briefly recall the steps required for extracting information in each of these two contexts. When working with an exchange matrix B, the exponent values are recorded via the numerical value of the entries b_{ik} of B. To locate these entries, where k is a fixed value and i varies, one searches for the kth column of B, thus exploiting the conventions governing its spatial layout, as illustrated in (13). This *conventional knowledge* is operationalized in locating the kth column by, e.g., counting the columns from left to right until column k is reached. One systematically checks each of its non-zero entries to see whether the value is positive or negative. This information determines the exchange relation by revealing the appropriate terms x_i and their exponents, given by either b_{ik} or $-b_{ik}$, to be entered into the general formula (4).

In a quiver diagram, one locates the relevant vertex by searching for the alphanumeric character, e.g., 'k' or '3', that denotes it. The vertex label provides a visual cue that facilitates this search without requiring that one responds to any additional features of the diagram. The spatial indexing of the diagram—in particular, the co-location of arrow symbols (such as '$\longrightarrow$') with the vertex labels—shows which arrows need to be accounted for in reading off the exchange relation. The orientation of the arrows determines in which of the products in the exchange relation (6) a given term x_i should appear. The multiplicity of arrows determines the value of the exponent that will be assigned to each individual term appearing in the relation, and must therefore be enumerated.

To make this more concrete, let us consider how the quiver Q and the exchange matrix $B(Q)$ appearing in Figure 3 support mutation of the cluster (x_1, x_2, x_3, x_4) in a given direction, say, in direction 4. The aim, as described in Section 3.3 and Section 3.5, is to construct a new variable x_4' in terms of $x_1, x_2, x_3,$ and x_4, according to the relation (4) and (6).

Beginning with the exchange matrix $B(Q)$, we first locate the fourth column of the matrix, since we are mutating in direction 4. The fourth column is given by virtue of its position in relation to the other columns and is found by accounting in some way for its relative spatial positioning. Having located the appropriate column, we survey its entries to find that the entry $b_{14} = 1$ has a positive sign, while the entries $b_{24} = -1$ and $b_{34} = -1$ are negative, and finally, $b_{44} = 0$.

This information determines that the exchange relation is

$$x_4' x_4 = x_1 + x_2^2 x_3 \tag{14}$$

in accordance with the general form (4). Performing a division on both sides by x_4 yields the cluster variable

$$x_4' = \frac{x_1 + x_2^2 x_3}{x_4}, \tag{15}$$

which completes the task.

We now perform the same task with the aid of the quiver Q. First, we locate vertex '4' by observing the corresponding inscribed numeral—an epistemic move which does not depend on parsing the other notational elements of the diagram. We survey the arrows entering and leaving vertex '4' to find which of the terms $x_1, x_2, x_3,$ and x_4 to include in the exchange relation. There is one arrow entering the vertex and three arrows leaving it. The orientation of these arrows reveals the exchange relation

$$x_4' x_4 = x_1 + x_2 x_2 x_3, \tag{16}$$

in accordance with the general form (6). This expression abbreviates to (14) by exploiting that $aa = a^2$. A division by x_4 yields the cluster variable (15).

4.1.1 Responsiveness types and the *transparency* criterion

Both exchange matrices and quivers, taken to be cognitive artifacts, thus support computational offloading, but to varying degrees. To make sense of the differences, we recall Manders' framework of *responsiveness types*. Manders highlights that exercising *indifference to* certain information, as opposed to *responding* to it, plays a significant epistemic role in mathematical reasoning, since the cognitive resources of mathematicians are finite. I claim that quiver diagrams offer elevated opportunities for computational offloading, as compared with exchange matrices, by way of facilitating practitioners in exercising indifference to the features of the diagram which do not play a role in conveying the relevant information. In contrast, when reading off information from an exchange matrix, one has to pay attention to the entries *in relation to other entries*. This ultimately requires practitioners to respond to features of the exchange matrix which do not factor directly into the task of determining the exchange relation, since not all entries are relevant in general for its construction.

A further remark concerns the notion of *transparency* introduced by De Toffoli (2017) for the purpose of evaluating the effectiveness of mathematical notations in practice. In brief, *transparency* refers to the possibility to exploit "our pre-existing cognitive abilities and acquired expertise" (De Toffoli, 2017, p. 160) when using and interpreting a notation, and may otherwise be viewed as evaluating how "intuitive" a notation is to use. Applying this notion to the present case suggests that quiver diagrams constitute a more transparent notation than do exchange matrices, for the specific task that we have examined, in the sense that quiver diagrams support information to be readily parsed by means of the visual system.

The differences described here are subtle, but important given the issue of cognitive constraints that was emphasized earlier. Seemingly modest changes in computational efficiency prove to be significant in the long run. In addition, it is worth highlighting that differences in computational offloading are relative to (i) the users, (ii) the tasks to be performed, and (iii) the means for performing the tasks.[19] Variations in these factors underline a need for an analysis different from the one presented here, which focuses on cluster mutation as performed "by hand" in the way described above.

4.1.2 Variations on the cognitive artifacts

The examples of exchange matrices and quivers considered in Section 3 are fairly simple in the sense that they are parsed with relative ease. It is at times necessary to deal with comparatively "larger" matrices, containing many rows and columns, or "larger" quivers, containing many vertices and arrows. The way that such a cognitive artifact is presented visually may vary accordingly.

In cases of large quivers, it is convenient to denote the multiplicity of arrows in a way different from what we have shown here. Instead of drawing multiple arrows, one can instead denote their multiplicity by a numeral so that, for instance, the quiver shown in Fig. 5(a) is instead presented as shown in Fig. 5(b). Using a numeral for this purpose conserves physical space on the page and supports a possibly higher degree of perspicuity when presenting quivers which contain a large number of arrows. The

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[19] Schlimm (2025) writes extensively on trade-offs between different notations and presents a framework for comparing aspects of mathematical notations which takes into account possible advantages or disadvantages of a notation in terms of not only the users and tasks, but also in terms of the materials used for making inscriptions (e.g., clay or paper), the subject matter, and the historical tradition within which the notation is embedded, among other things. This latter aspect of the historical tradition may influence the perceived familiarity that practitioners have of, for instance, a new notation.

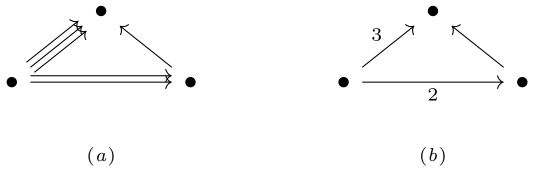

Fig. 5: (a) A diagram of a quiver with the multiplicity of arrows presented with repeated arrows. (b) A diagram of the same quiver as in (a), but with the multiplicity of arrows (greater than 1) indicated instead by a numeral.

numeral form also makes it so that reading off the exponent values encoded by the multiplicity of arrows in a quiver tracks more closely with the procedure for reading off exponent values from an exchange matrix. In Fig. 5(a) we count the number of arrows shown much as one counts tally marks, whereas in in Fig. 5(b), we extract this information from the numeral that is spatially co-located with the '$\longrightarrow$' symbol.[20]

Having seen how these differences play out when reading off information from these cognitive artifacts, we turn to comparing how they support mathematical operations in the form of *matrix mutations* and *quiver mutations*, respectively.

4.2 Mutation and affordances

Quivers and exchange matrices as cognitive artifacts afford different kinds of sign manipulations. Learning to manipulate these cognitive artifacts in a mathematically meaningful way is done through careful training, and to an experienced practitioner, the possibilities for correct sign manipulations—i.e., sign manipulations which respect the rules imposed by the mathematical theory—may be readily perceived. Recalling the different kinds of affordances that were presented in Section 2.2, it appears that those which are most relevant for our purposes are *cognitive affordances* which are given by some kind of learned convention and *representational affordances* which denote the possible actions that are afforded by a cognitive tool. As it turns out, matrices and quivers afford different kinds of sign manipulations which influence how they are used to aid reasoning.

In performing a matrix mutation of a given matrix B, one computes the entries of the mutation B' of B on a case-by-case basis in accordance with the relation (10). These computations, if worked out by hand, do not take place directly in the matrix but are inscribed on some other part of the page. The kinds of sign manipulations performed are ones which respect the rules of arithmetic, so here, too, some background knowledge must be put to use in order for these kinds of cognitive artifacts to be used effectively in practice. For an $n \times n$ matrix, this translates to n^2 computations, although, as we have seen, certain shortcuts can be taken to ease the computational load. These shortcuts depend on further case-branching than what is immediately

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[20] This way of presenting the multiplicity of arrows via a numeral invokes the possibility for ambiguity, since such a character could be mistaken for a label denoting the arrow, as we saw with the Greek letters denoting arrows in Fig. 2. This is a mostly vacuous concern, since it should be clear from context whether a numerical symbol which is spatially co-located with an arrow is a label or represents the multiplicity of arrows going between the relevant pair of vertices.

given by the relation (10), which means a less uniform treatment is applied and therefore a greater degree of *responding-to* is required. More explicitly, the cases that arise depend on whether the entries b_{ij} and b_{ji} have the same sign, i.e., both positive or both negative, or opposite sign. Paying attention to these different cases occupies valuable cognitive resources. As a consequence, it may be the case that matrices provide reduced cognitive support when it comes to carrying out mutations of this sort, as compared with the same type of operation performed on a quiver.

When mutating a quiver, one follows the three-step procedure outlined in Section 3.8, which amounts to first drawing additional arrows on the diagram, followed by reversing the direction of certain arrows, and finally removing any 2-cycles that may have arisen (illustrated in Fig. 4). At least some of these manipulations can take place without re-drawing the diagram over again each time an arrow is added, removed, or its direction reversed. Quiver diagrams thus support *agglomerative* reasoning, in which one sequentially adds information onto the same form of presentation (see Carter, 2024). This type of reasoning is typically contrasted with *discursive* reasoning, in which one re-writes a transformed expression usually below or next to a prior expression, as one often does when performing, e.g., arithmetical calculations. We observe that both kinds of reasoning are present when performing a quiver mutation. That is, we add new arrows onto the diagram, which is an agglomerative move; we also, for the sake of perspicuity, re-draw the transformed diagram next to the prior one, which is a discursive move. This allows one to show the computation as a concise chain of reasoning.[21]

Comparing these two forms of cognitive artifacts, it is apparent that both quivers and matrices afford discursive reasoning, but only quivers afford agglomerative reasoning. The discursive mode of reasoning allows one to keep track of the sign manipulations which have been performed; the agglomerative mode of reasoning allows for an efficient method in performing an operation. Performing a mutation typically involves discursive reasoning, which allows one to keep a record of each computation. When considering matrix mutation, discursive reasoning allows for keeping track of, e.g., the computation of the entries belonging to the mutated matrix B'. This involves inscribing a series of arithmetic expressions—in principle, one for each of the n^2 entries, as indicated in Section 3.7. In performing a quiver mutation, one has the option to reduce the volume of writing needed to "show" the steps of the operation, since quiver diagrams support agglomerative reasoning. Quiver mutation thus offers a succinct way of keeping track of the relevant information.

In addition, quiver diagrams support practitioners in exercising a greater level of *indifference* since there are not distinct cases to consider in the context of quiver mutation, as there are with matrix mutation. Rather, there is an algorithm that determines which sign manipulations may legitimately occur that does not depend on distinguishing between separate cases. This is an advantage since it eliminates considerations of case-branching and, as such, naturally constrains attention due to the suitability of

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[21] We refer to Macbeth's (2014) distinction between a *description* of a chain of reasoning, as one presents using natural language, and a *display* of a chain of reasoning, such as a computation presented using the Indo-Arabic numeral system. In this terminology, we might say that a quiver mutation as we have presented it here *displays* the computation, rather than merely reporting or describing it. We could argue that a matrix mutation falls under the former category in Macbeth's framework.

applying a uniform treatment in all cases. This may be seen as a conceptual advance in the way that it shapes the response required.

Finally, we remark that one achieves a greater level of generality when using exchange matrices, which comes basically “for free,” by which we mean that performing a mutation of a matrix which is more general than the ones considered here proceeds according to precisely the same procedure as what we have seen in Section 3.7. No adjustment of the mutation procedure is required for this more general setting. The type of matrices presented here are of a less general variety, which we focus on due to their informational equivalence with quivers. There is a way to generalize quivers so that they correspond to the increased generality of the matrices, which implies modifying the quiver mutation procedure accordingly.

4.3 Quivers as ‘visible’

When it comes to their use in the theory of cluster algebras, quivers have been described as “visible” as compared with exchange matrices.[22] We investigate what this claim might mean in view of the uses of quivers and matrices explored in this paper and propose three possible interpretations. These relate to (i) the way in which a quiver is typically presented, i.e., by a diagram; (ii) the type of indexing that quiver diagrams exploit; and (iii) the modes of reasoning supported by quiver diagrams.

The first interpretation is motivated by the observation that a quiver is typically presented by a diagram which is in some sense a *visual* presentation of an abstract mathematical idea. Conversely, the typical way of inscribing a matrix, i.e., as an array, does not usually fall into the category of being a visual presentation. This observation might reveal something about the role that the purported *visibility* of a quiver diagram has in making these an *effective* cognitive artifact. To make sense of this, we recall De Toffoli’s (2017) criterion of transparency which refers to the possibility of using “our pre-existing cognitive abilities and acquired expertise” when faced with the task of interpreting and using a notation correctly. A quiver as a highly tractable notation may be considered as *transparent* in De Toffoli’s sense of the term. That is, our cognitive abilities might be well-suited in general for parsing the information presented in a quiver diagram. In contrast, the same type of data presented in the form of a matrix may be less accessible for the visual system to parse. Matrices as cognitive artifacts may be considered as less “visible” that quivers for this reason. Presenting a quiver in some other form than a diagrammatic one may result in it losing the attribute of visibility, and even perhaps of transparency. This interpretation thus highlights the way of presenting a quiver as a potential main contributor to its visibility.

The second interpretation is related to the first in that it takes the particular syntax of quiver diagrams, especially their use of spatial indexing, to contribute to their visibility. We have argued in Section 4.1 that the spatial indexing of these diagrams facilitates the search for information which is needed, e.g., when performing a cluster mutation. In this instance, the vertex labels indicate where to find the relevant information. We have previously noted that inscriptions of matrices and quivers both make use of specific conventions that practitioners need to exploit in order to extract information from these inscriptions. Moreover, one may see straightaway which are

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[22] B. Keller, personal communication, 14 May 2024.

the relevant vertices and arrows to pay attention to in the quiver diagram, whereas the search for information in a matrix may be less immediate. In this way, the matrix may also be less transparent.

The third interpretation concerns the modes of reasoning which are supported by quiver diagrams, namely that they support reasoning both agglomeratively and discursively. This allows for these diagrams to be particularly tractable for the reasons mentioned in the preceding section and allows one to display the mutation procedure as a concise chain of reasoning (cf. Macbeth, 2014). In this way, one can "see" what are the steps involved in quiver mutation. Of course, one can also see the steps of a matrix mutation if they are written out, but the visibility metaphor breaks down, perhaps due in part to the actual sign manipulations that are involved in this case.

Furthermore, we emphasize that in the mathematical context we have considered here, a quiver diagram is a semantic object and encodes the combinatorial information that is used for constructing new cluster variables when performing a cluster mutation. When performing a quiver mutation, the diagram becomes a syntactic object which supports sign manipulations that can be made without consideration of the meaning of the signs. When the mutation procedure is complete, the quiver diagram is again thought of as a semantic object, which encodes new combinatorial information and is, moreover, related to the prior quiver in a specific way.[23] In contrast, an exchange matrix does not become a syntactic object in quite the same way when performing a matrix mutation. In this case, the sign manipulations take place elsewhere on the page, as observed above, and the results of these symbolic manipulations must be collected into the appropriate matrix form in order for the procedure to terminate successfully. In short, quiver diagrams allow for one to *display* a chain of reasoning, while a matrix allows rather for one to *describe* a chain of reasoning. Displaying the reasoning and, in particular, the inferences that are made in performing a mutation, allows the sequence of steps determined by the operation to be visually accessible, i.e., they are "put before your eyes."

5 Concluding remarks

In this paper, we have analyzed the use of different kinds of cognitive artifacts which appear in the contemporary mathematical theory of cluster algebras using the conceptual framework of *affordances* together with that of *responsiveness types*. The aim has been to understand the role of these cognitive artifacts, namely certain types of matrices and quivers, in light of some specific ways in which they are used within the context of cluster algebras. Our analysis reveals that while the types of matrices and quivers considered here are (i) informationally equivalent and (ii) yield corresponding computational results, there are important differences in terms of how they support computational offloading and sign manipulations. The kinds of trade-offs which we

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[23] The idea that some notational form may change between being a semantic object and being a syntactic object is observed by Johansen (2024), who remarks that in, e.g., performing an arithmetic calculation, "symbols change the function from semantic objects (when we write the problem on the paper) to syntactic objects (when we perform the calculation) and back again (when we read off the result)" (Johansen, 2024, p. 161).

have identified may help to account for why both of these cognitive artifacts are used in the practice.

Finally, we proposed three possible interpretations of what the claim that quivers are "visible" might mean. First, quivers are typically presented diagrammatically. Second, the spatial indexing of quiver diagrams facilitates the search for information in ways which, we have argued, exchange matrices do not. And third, quiver diagrams support both discursive and agglomerative modes of reasoning.

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Declarations

Conflict of interest Not applicable.

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