The Problem of Extra-Mathematical Explanation*

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Abstract

We sometimes appeal to pure mathematical facts to explain physical phenomena. But it is unclear how a pure mathematical fact could possibly explain something physical. I develop a concrete formulation of this problem informed by the view that explanations of physical phenomena are backed by relations of dependence. I show that prominent analyses of mathematical explanations of physical phenomena fail to provide satisfying solutions to the problem, and I develop a novel analysis that succeeds in their place. I claim that a mathematical explanation of a physical phenomenon works by establishing that its target shares dependencies with a fact that is independently-understood. My analysis dissolves the problem and enjoys many other benefits besides.

1. Introduction

Scientific explanations are often expressed in mathematical terms: mathematical objects and properties are invaluable for representing physical phenomena and their dependencies. However, we sometimes invoke *pure* mathematical facts, i.e. theorems that do not stand for anything physical, to explain physical phenomena. In such explanations, mathematics appears to feature in an explanatory, rather than merely representational, capacity. Call this practice *extra-mathematical explanation*.

Extra-mathematical explanation has attracted much interest over the last twenty years. One reason is that, if mathematical facts explain physical phenomena, then, by inference to best explanation, we arguably have empirical evidence for the existence of mathematical objects. If successful,

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this *explanatory indispensability argument* has profound consequences for the philosophy of mathematics.[1] Another reason is that extra-mathematical explanations are thought to be non-causal, and so have featured in debates over the nature and diversity of scientific explanation.[2] To adjudicate these debates, we require an impartial understanding of extra-mathematical explanation, towards which much recent work contributes.[3]

This paper develops and defends a novel analysis of extra-mathematical explanation on the basis of its capacity to solve a certain philosophical problem. The problem is to account for how pure mathematical facts could possibly explain physical phenomena, given certain plausible assumptions about explanation and mathematics. We can encapsulate the problem as the following (apparently) inconsistent triad:

(1) Explanations disclose what their target phenomena depend on.

(2) There are extra-mathematical explanations.

(3) Physical phenomena are not mathematically dependent.

Each of (1)–(3) is independently plausible. (1) is a point of broad agreement in the philosophy of explanation; (2) is supported by a wealth of compelling examples; and (3) flows from standard assumptions about mathematics and its relation to the physical world. Yet they seem incompatible. If by (1) the function of an explanation is to disclose the dependencies of its target, and by (2) there are explanations of physical phenomena in terms of pure mathematical facts, then pure mathematical facts sometimes disclose the dependencies of physical phenomena. But, by hypothesis, pure mathematical facts do not stand for anything physical, so the dependencies they disclose must be mathematical in some way, contrary to (3). Call this the *problem of extra-mathematical explanation*.

In §2 I expand on (1)–(3) and provide the prima facie case for each. In §3 I consider influential analyses of extra-mathematical explanation, all of which give up either (1) or (3) and face serious

[1] See e.g. Baker 2005 for this line of argument. See e.g. Melia 2000 and Yablo 2012 for detractors.
[2] See e.g. contributions to Reutlinger & Saatsi 2018.
[3] See e.g. Bangu 2021; Baron 2019, 2020, 2024; Baron et al. 2017; Knowles 2021b; Knowles & Saatsi 2021; Lange 2017; Leng 2021; Pincock 2015; Povich 2021.

difficulties as a result, thus failing to provide a satisfying solution to the problem. This sets the stage for my positive proposal, which I develop and defend in §4. I propose that, for a given extra-mathematical explanation, the role of the pure mathematical fact is to reveal that the target physical phenomenon P and some independently-understood fact F must co-occur by mathematical necessity. From this mathematical relationship, and the fact that P and F are both contingent facts about the same physical situation, we can infer that P and F depend on the same (physical) things. This allows us to grasp what P depends on through our prior understanding of F. On this analysis, the function of extra-mathematical explanations is to disclose the dependencies of physical phenomena, affirming (1); their success turns on the mathematicality of the theorems they invoke, affirming (2); and yet there is nothing mathematical about the dependencies they disclose, consistent with (3). My analysis dissolves the tension between (1)–(3), providing a satisfying solution to the problem of extra-mathematical explanation. In §5 I draw out the broader philosophical consequences of my analysis.

2. The Problem

2.1. Backing realism

Here I provide the prima facie case for (1). Let me first pin down some terminology. I use 'explanation' as a success term. The *target phenomenon* is the phenomenon to be explained. The *explanandum* is the representation of the target phenomenon. The *explanans* is what is adduced to account for the target phenomenon—typically a set of interpreted sentences—including a generalisation that links to the explanandum, which I call the *operative generalisation*.

The relationship between explanans and explanandum, in virtue of which the explanation succeeds, is *explanatory relevance*, the illumination of which is the central aim for a theory of explanation. One attractive approach takes it to be a matter of exhibiting a dependence relation between the target phenomenon and features of its situation alluded to by the explanans. On this view, explanations are 'backed' by relations of dependence, which are taken to be objective, mind-independent, irreflexive, and asymmetric. Call this theory *backing realism*.

While specific elaborations differ, backing realism is a point of broad agreement in the philosophy of scientific explanation.[4] One elaboration—the *counterfactual-interventionist theory* of causal explanation (Woodward 2003)—shows great promise for furnishing a fully general theory of scientific explanation. On this theory, an explanation reveals causal dependence by providing information about what would be different with respect to the target phenomenon were certain features of its physical situation intervened on in specific ways. This theory is remarkably successful at capturing the core features of causal explanation. For example:

Value Why are we so interested in causal explanation? Because knowledge of causal dependence facilitates control of our environment, which clearly has practical value.

Gradability When is one explanation better than another? When it provides more of the relevant kind of counterfactual information, or presents it in a way that makes it easier to grasp or use. This accounts for a range of intuitions about comparative explanatory power (Ylikoski & Kuorikoski 2010).

Understanding What is *explanatory understanding* and how is it related to explanation? The theory illuminates the cognitive and practical dimensions of explanatory understanding. The cognitive dimension is a matter of *grasping* a phenomenon's dependencies, which we can think of in terms of possessing a suitable mental model of them (Dellsén 2020). The practical dimension concerns a set of associated abilities through which the cognitive achievement is manifest, including the ability to draw the right kind of counterfactual inferences, control the phenomenon using targeted interventions, frame and explore new but related problems, and so on.

Epistemology How do we know about instances of causal dependence? Some are directly established through suitable experiments: interventions are actually performed

[4]For affirmations and developments, see e.g. Audi 2012; Kim 1994; Strevens 2008; Woodward 2003.

and their effects observed. For many others, we proceed indirectly, by using independently confirmed theory to derive generalisations that guide relevant counterfactual reasoning.

Directionality Why can we explain the period of a pendulum in terms of its length but not vice versa? Because we can intervene directly on the length of a pendulum to control its period, but we cannot intervene directly on the period to control its length. Thus, the direction of causal dependence is manifest in facts about intervention and control.

Given these riches, the promise for a fully general theory of scientific explanation is great. If non-causal explanations can be understood in similar terms, the riches will spill over. No wonder there is a small industry dedicated to generalising the theory.[5]

Exactly how to generalise the theory, and how far it will take us, is still up for debate. Nevertheless, the prospects look very good. Consider an explanation of why a certain object is fragile in terms of its micro-physical makeup. Such explanations track a non-causal dependence relation—*constitutive dependence*—where the target phenomenon is a feature of the whole object, and what it depends on are the constituents and their mode of organisation. Like causation, constitutive dependence is objective, irreflexive, and asymmetric, and its asymmetry is manifest in facts about intervention and control. The benefits of the counterfactual-interventionist theory seem to carry straight across to these explanations, despite the evident differences between the kind of dependence involved (Ylikoski 2013).

Constitutive dependence is one among many so-called 'building' relations: non-causal relations through which certain portions of reality 'make up' other portions of reality (Bennett 2017). Another is *grounding*, through which more fundamental portions of reality generate less fundamental portions. Plausibly, the success in extending the counterfactual-interventionist theory to constitutive explanations can be replicated for those involving building relations in general, because of the features that building relations share. The building blocks are characteristically more

[5] See e.g. Bokulich 2011; Povich 2018; and Ylikoski 2013 for its application to non-causal explanations in science.

specific, lower-level, and concrete than what they build. Changes to the latter have to go via changes to the former. So the directionality of dependence will be manifest in facts about intervention and control, which can be captured by counterfactual information about how changes to the building blocks ramify in what is built. Thus, the counterfactual-interventionist theory takes us a considerable way towards a fully general theory of scientific explanation.[6]

2.2. Extra-Mathematical Explanation

(2) is supported by a wealth of compelling examples. Here are two.

Cicadas North-American periodical cicadas emerge from their larval form every 13 or 17 years. Ecological factors (e.g. the temperature of soil) restrict their life-cycle periods to 12–18 years. But why 13 and 17 years specifically? For Darwinian reasons, we assume the cicadas have periods that maximise the number of years between their co-emergence with nearby periodical predators, which are likely to have periods of less than 12 years. Suppose there are nearby periodical predators with periods of each length from 2 to 12 years. The number of years between co-emergence for two periodical organisms of x-year and y-year periods is maximised when x and y are *coprime*. And it is a theorem of number theory that a number p is coprime with all $n < p$ if and only if p is prime. It follows that the cicadas have 13- and 17-year life-cycles.[7]

Bridges The town of Königsberg in 1735 consisted of four landmasses connected by seven bridges. None of the townsfolk in 1735 succeeded in crossing each bridge sequentially without re-crossing any of them. Assume they were systematic and competent in their efforts to make this *Eulerian crossing*. Why did none of them succeed? The layout of the town corresponds to a certain connected graph, where all four nodes are of *odd degree*—they have an odd number of edges contiguous with them. It is a theorem of graph theory that a connected graph admits an Eulerian circuit iff it

[6] See Schaffer 2016 and Wilson 2018 for accounts of grounding explanations along these lines.

[7] Baker 2005 introduced this example to the literature. The assumption that there are predator periods of each length from 2 to 12 years is hefty, but a stronger mathematical generalisation shows that the prime periods follow when only 2- and 3-year predator periods are assumed (Baker 2016).

has either exactly zero or exactly two nodes of odd degree. Thus, the townsfolk failed.[8]

There is something distinctively mathematical about these explanations. This is not just a matter of the mathematics being indispensable to the explanation. For example, a certain mathematical function might be indispensable for representing the causal dependencies of a given phenomenon, but we wouldn't normally think that the function itself plays an explanatory role. The role of the function is discharged as soon as we grasp its physical significance.

By contrast, to say that an explanation of a physical phenomena is *extra-mathematical* is to say that the role of the pure mathematical fact within it goes beyond a purely representational one. How exactly it goes beyond is (clearly!) up for debate. However, we can say that the role of the pure mathematical fact is discharged by grasping its *mathematical* significance, alongside any physical significance it might have.

Consider, for example, the number-theoretic generalisation in *Cicadas*. It relates two properties of numbers, *being coprime with all smaller integers* and *being prime*, each of which clearly has physical significance. Indeed, given that numbers are invoked as representations of time-periods in this explanation, we understand that time periods can instantiate physical correlates of these mathematical properties.

However, the role of the number-theoretic generalisation is not discharged by simply grasping the physical significance of the mathematical properties it concerns. We must also grasp the mathematical relationship between these properties: if something instantiates one of these properties, then it instantiates the other, by mathematical necessity. Of course, this tells us something about the physical situation. It tells us that the physical correlates of these properties must co-occur by mathematical necessity, but this is no physical relationship. The strength of the modal connection far outstrips the modal strength of any physical relation, which would only hold with physical necessity, i.e. necessity relative to worlds where the laws of physics hold.

Thus, the number-theoretic generalisation tells us that, given that the cicadas have life-cycle periods of between 12 and 18 years that minimise intersection with all periods shorter than 12,

[8]Pincock 2007 introduced this example to the literature on extra-mathematical explanation.

they *had* to have 13- or 17-year periods by mathematical necessity, since 13 and 17 are the only primes between 12 and 18. While the relata are features of the physical situation, the relationship between them is mathematical, and *Cicadas* succeeds in part via our grasp of this mathematical relationship. Thus, the role of the operative generalisation in *Cicadas* is not discharged by our grasp of its physical significance alone; we must also grasp its mathematical significance. The same can be said about *Bridges*. Thus, we have prima facie support for (2).

2.3. Mathematics and dependence

We have seen that (1) and (2) are prima facie attractive. Together, they imply that extra-mathematical explanations disclose the dependencies of their target phenomena by pointing to a pure mathematical generalisation whose explanatory function is only discharged once its purely mathematical significance is grasped. But this suggests that the pure mathematical generalisation fulfils one of two functions. Either it relates the target physical phenomenon to a pure mathematical fact on which it depends, or it relates the target physical phenomenon to another physical phenomenon that mathematically necessitates it, characterising the dependence relation itself as mathematical. Either way, it follows that some physical phenomena are mathematically dependent, and so (3) is false. But there are good independent reasons to accept (3).

Consider first the idea that a physical phenomenon can depend on a pure mathematical fact. It is standardly assumed that mathematical facts concern abstract mathematical objects, which are non-spatiotemporal and non-causal. This rules out physical phenomena being caused by mathematical facts. But might physical phenomena depend on mathematical facts non-causally? Some claim that the nature of mathematical objects speaks against their making a physical difference of any kind. For example, it has been said that, 'if all the objects in the mathematical realm suddenly disappeared, nothing would change in the physical world' (Balaguer 1998: 132). More recently, it has been suggested that their apparent 'remoteness' poses a 'bar in principle' to the notion that physical things might counterfactually depend on mathematical facts (Lange 2021: 49). However, remoteness is something only spatial objects enjoy, and blinking out of existence is a something only temporal objects can do, so these intuitions seem based on considerations that apply primarily to physical objects (Baker 2003; Paseau & Baker 2023: 44–47).

Even so, there are still prima facie reasons for doubt. For familiar cases of physical dependence, such as causal or constitutive dependence, there are relatively uncontroversial examples that can be established directly via intervention and observation. These can serve as a foundation of our understanding of these relations, and our knowledge of where they obtain in more speculative cases. By contrast, there are no uncontroversial examples of physical-on-mathematical dependence, and certainly none that can be verified via intervention and observation. Accordingly, we have comparatively little idea of what physical-on-mathematical dependence would be, how it might behave, and how to identify cases of it. This provides prima facie reason to doubt that physical phenomena depend on mathematical facts.

Now consider the idea that a physical phenomenon can depend on another physical phenomenon that mathematically necessitates it. Consider, for example, the fact that there are seven pens on my desk. This fact mathematically necessitates the fact that there are three pens on my desk and a further four pens on my desk. But we wouldn't want to say that the latter fact depends on the former fact, at least not in the sense relevant to explanation, for entirely general reasons. First, mathematical necessitation is too intimate: we cannot distinguish changes to one relata from changes to the other. Second, relations of mathematical necessitation can be symmetrical, as they are in this case, in which case it seems there is no principled reason to take dependence to run in one direction rather than the other. These are prima facie reasons to doubt that physical phenomena depend on what mathematically necessitates them, at least in the sense relevant to explanation.

We thus have a presumption in favour of (3). It is defeasible, to be sure; but it is important to be clear on what it would take to defeat it. In the previous section, I outlined two compelling examples of extra-mathematical explanation. Might such examples alone defeat (3)? Unfortunately not. The above considerations show that we have neither an understanding of how pure mathematical facts might explain physical phenomena, nor a sense of how to reliably identify when they do. The fact that there are nevertheless examples of scientific explanation where pure mathematical facts *seem* explanatory of physical phenomena is the heart of the problem I am developing here. If we take such examples seriously, we require an understanding of how such explanations reveal the explanatory significance of pure mathematical facts. This is what it would take to defeat (3), and it is precisely what analyses of extra-mathematical explanation seek to provide. In the following section, I will argue that they have hitherto failed in this regard.

3. Extant Analyses

Here I discuss three broad philosophical approaches to extra-mathematical explanation. The first contravenes (1), insisting that extra-mathematical explanations work by establishing the (conditioned) necessity of their target phenomena. The second contravenes (3), positing *sui generis* physical-on-mathematical dependence. The third contravenes (3), suggesting that extra-mathematical explanations represent mathematical dependence relations between high-level physical features. I will argue that each camp fails to illuminate the nature and value of extra-mathematical explanations, and so fails to provide a satisfying solution to the problem they pose. My discussion is not exhaustive. It focuses on these approaches because they are prominent, and because their relation to (1) and (3) is relatively clear.[9] Nor is my discussion conclusive. I hold out no hope of convincing proponents of these analyses to abandon them; I hope only to highlight reasons to seek a different approach.

3.1. The Mathematical-Consequence Analysis

Several authors have argued that a given extra-mathematical explanation works by showing that its explanandum is a logical consequence of a mathematical description of the physical situation. For Mary Leng (2021), *Cicadas* works by showing that, when the axioms of arithmetic are suitably interpreted in terms of the physical situation, the cicadas' life-cycle periods can be deduced by

[9]Notable omissions include Bangu 2021, who takes mathematics to play an *explicatory* role in mathematical explanation, and Lyon 2012, who takes mathematics to play a *programming* role à la Jackson and Pettit 1990. How these proposals relate to (1) and (3) is not obvious, so discussion of them would be lengthy and distracting.

applying the number-theoretic generalisation. Other elaborations of this approach differ in terms of how the conditions on the physical situation and the logical relationship are spelled out (see Baron 2019, 2024; Lange 2017). Call this the *mathematical-consequence analysis*.

Proponents of this approach tend to say that extra-mathematical explanations are backed by certain modal relations. For example, Marc Lange (2017) and Leng (2021) invoke relations of necessitation: the instantiation of certain structural features in the physical situation (attributed by the mathematical description) mean the target phenomenon *had to occur* by mathematical necessity.[10] This is the elaboration I discuss here, but the concerns I raise will apply equally to elaborations that remain at the level of logical relations between representations.[11]

The mathematical-consequence analysis contravenes (1). Relations of logical consequence are not relations of dependence, and neither are the relations of necessitation they track. It is, moreover, not clear what such relations have to do with explanation. The point can be illustrated by noting that derivations of explananda can often be performed in reverse. One can derive the height of a flagpole from the length of its shadow (along with the angle of the sun), and the former is indeed (nomically) necessitated by the latter. But explanation runs in the opposite direction! Corresponding examples for *Bridges* and *Cicadas* are easily obtained.[12]

Bizarro-Bridges The town of Königsberg in Bizarro world has four landmasses connected to each other by six bridges. The townsfolk regularly succeed in crossing each of the bridges in a row without re-crossing any. Why does the town have either exactly zero or exactly two landmasses contiguous to an odd number of bridges? A connected graph permits an Eulerian circuit iff it has either exactly zero or exactly two nodes of odd degree. So, given that Bizarro-Königsberg permits Eulerian crossings, it must be that it has either exactly zero or exactly two landmasses contiguous to an odd number of bridges.

This is no explanation, but proponents of the mathematical-consequence analysis have trouble say-

[10] See also Baron's 2024 appeal to a relation of *incompatibility*.

[11] For example, Baron 2019 appeals to a relation of *information containment* between propositions, remaining neutral on its metaphysical significance.

[12] Craver & Povich 2017 develop this genre of counterexample.

ing why. Leng (2021: 10424, fn.5) bites the bullet and accepts that extra-mathematical explanations can be symmetrical. I consider this too great a cost. To my mind, asymmetry is so core to explanation that denying it amounts to changing the subject.

Lange (2017: 42–43) attempts to dodge the bullet by invoking a further condition on extra-mathematical explanation: that the explanandum be necessitated by *essential* features of the physical situation. For Lange, having actually been crossed is not an essential feature of the arrangement of bridges, so *Bizarro-Bridges* is no explanation. Unfortunately, this proposal runs into trouble. First, note that we can run a Bizarro explanation that, rather than appealing to the fact that the bridges have actually been crossed, appeals only to the fact that the relevant crossing *can* be made. And, were an Eulerian crossing not possible, we would be dealing with a different arrangement of bridges. So, the question arises: is the permissibility of an Eulerian crossing an essential feature of the physical situation? If it is, then Lange's proposal fails to generalise. But, if it isn't, it's not at all clear why not.[13]

To save Lange's proposal, there must be principled reasons for ruling in *having (neither) exactly zero (n)or exactly two landmasses contiguous to an odd number of bridges* as essential features of bridge systems, and ruling out *(not) permitting an Eulerian crossing* as inessential. Given that essence is hyperintentional, the fact that both sets of properties are equally necessitated by the various layouts of bridge systems does not decide the matter, so principled reasons may yet be provided. However, it is worth noting how much work they would have to do. They would not only have to support a principled distinction between essential and inessential necessary features of physical situations; they would have to draw this distinction in such a way that systematically classifies extra-mathematical explanations as explanatory and their Bizarro counterparts as non-explanatory. Further, the factors that distinguish the essential and inessential features would have to be factors that we are in general sensitive to when appreciating extra-mathematical explanations and distinguishing them from their Bizarro counterparts. To my mind, this is a tall order, and I'm not optimistic about meeting it.[14]

[13] See Pincock 2023: §4.2.2 for discussion.
[14] See Craver & Povich 2017; Lange 2018; and Povich 2020 for related discussion.

3.2. The Mathematical-Dependence Analysis

According to a number of theorists, extra-mathematical explanations work by locating mathematical facts among what their target physical phenomena depend on. On this view, extra-mathematical explanations are backed by objective relations of physical-on-mathematical dependence, in clear violation of (3). Call this the *mathematical-dependence analysis*. Elaborations of this view differ in how they characterise relations of physical-on-mathematical dependence.[15] However the dependence is characterised, the view faces the serious epistemological challenge of accounting for how we are able to establish that something physical depends on something mathematical.

For example, one elaboration invokes the counterfactual-interventionist theory but drops the requirement that explanatory counterfactuals concern possible interventions.[16] On this view, mathematical explanations track physical-on-mathematical dependence by conveying information about how certain physical phenomena would change, were the properties of certain mathematical objects to change. Thus, *Cicadas* works by revealing that, had 13 and 17 not been prime, the cicadas would have evolved to have different life-cycle periods. The epistemological challenge here is to account for how we establish counterfactuals such as these.

Many balk at the idea of non-trivially true counterfactuals with mathematically impossible antecedents, since contradictions can easily be inferred from mathematical impossibilities, and (by explosion) anything can be inferred from a contradiction.[17] On the other hand, a case can be made that mathematicians routinely consider what would be the case were certain mathematical facts to be otherwise (Reutlinger et al. 2022: §5). So perhaps we need a good way of dealing with such counterfactuals, even if we currently lack one. This dispute is far from resolved.[18]

However, even if we accept that counterfactual reasoning forms an important part of mathematical practice, this does little to help in our present predicament. To begin with, it does not decide

[15] Baron 2020; Baron et al. 2017; and Povich 2021 characterise them in counterfactual terms. Pincock 2015 characterises them in terms of instantiation and relative abstractness.

[16] See Baron 2020; Baron et al. 2017; Povich 2021.

[17] Williamson 2007: 172 observes this independently of the literature on extra-mathematical explanation. See Baron et al. 2017: 2–12 for an attempt to deal with the issue; see Kasirzadeh 2021: §3.1.1–§3.1.2 for replies.

[18] See Kocurek 2021 for a good overview of the debate over counter-possible reasoning.

the issue of what exactly we are considering when we consider a counter-mathematical situation. When we consider a situation in which 13 is composite, for example, are we really considering a situation in which one and the same number has different factors, or are we simply considering a different number within in a different structure? If the latter, then the relevant counterfactuals do not track relations of dependence in which the number 13 participates. If the former, then the identity of the number 13 is not due to its distinctive mathematical properties. But then wherein lies its identity?

Perhaps 13 has an essence that persists across the envisioned change. Many metaphysicians appeal to essence to make sense of certain relations of metaphysical dependence, and to ground associated counterfactuals (e.g. Fine 1994; Koslicki 2012). However, I am at a loss as to how we might discern the essence of a mathematical object, and so delineate which counterfactual changes to it are possible and which aren't. The obvious option is to treat the mathematical properties used to define a mathematical object as essential. But that would rule out precisely the kind of counter-mathematical we are being asked to consider.[19]

Even if we can make sense of same-object counter-mathematical changes, and consideration of them forms a substantial part of mathematical practice, this does little to help us discern what would be true of *physical things* under such changes. There are two relevant questions here. First, why think that the envisioned changes will have any effect on the physical world at all? As discussed in §2.2, we have prima facie reason to think that such changes will leave the physical world as it is. Second, why think that they will have the specific effects we are being asked to believe they will? The widespread applicability of mathematical concepts to physical phenomena suggests that, if counter-mathematical changes do ramify in the physical world, they do so at a cosmic scale.

Sam Baron, Mark Colyvan, and David Ripley (2017) address these worries by claiming that, when we evaluate counterfactuals of any kind, we stipulate both what is varied and what is held fixed. Thus, by stipulation, we can hold fixed the fact that 13 and 17 measure time periods in years,

[19] Perhaps the identity of 13 is given independently of its properties, via some self-individuating constituent or 'bare particular'. There are good reasons to resist this account of mathematical identity, however. See Builes 2022: §2 and Assadian & Fraser 2025: §4.

and let their relation to other parts of physical reality vary. With the right stipulations in place, it will be true that, were 13 and 17 composite, time would be different such that it would not be advantageous for the cicadas to have 13- and 17-year life-cycle periods. However, dependence cannot be established by stipulation. To illustrate, suppose that my shoes and my shirt are both red. Why are my shoes red? If we stipulate that the sameness in colour remains under counterfactual changes to the colour of my shirt, then the colour of my shoes counterfactually varies with the colour of my shirt. We should not conclude from this that the colour of my shoes depends on the colour of my shirt (Knowles 2021a: §3; Lange 2021: 49).[20]

Might we appeal to the success of extra-mathematical explanations themselves to secure knowledge of physical-on-mathematical dependence? Christopher Pincock (2015: 878) suggests that the novelty and informativeness of an extra-mathematical explanation might indicate an instance of physical-on-mathematical dependence. It is difficult to see how this could work without begging the question. Only on the presumption that extra-mathematical explanations seek to establish physical-on-mathematical dependence could indications of their success amount to evidence of physical-on-mathematical dependence. But, absent a plausible epistemology of physical-on-mathematical dependence, we should not understand mathematical explanations this way.[21]

3.3. The Structural-Dependence Analysis

A number of theorists have suggested that, for a given extra-mathematical explanation, the mathematical generalisation represents the non-causal dependence of the target phenomenon on some high-level, structural feature of the physical situation, where the dependence is typically spelled out

[20] Do the above concerns only apply to the platonistic conception of mathematics? Lange (2021) argues that Aristotelian Realism, according to which mathematics is the study of mathematical properties of physical systems (see Franklin 2008), fares better. On this view, he claims, since no morphism mediates between numbers and physical objects, 'there is no opportunity for the morphism to break down, rather than the physical explanandum be different, under the key counterfactual antecedent' (2021: 50). But one and the same physical system can change its mathematical properties, so the relationship between numbers and physical objects can still break down. Further, even physically-instantiated numbers cannot be intervened on, so it remains unclear how we know what the physical outcomes of counter-mathematical changes would be. Perhaps Aristotelian metaphysics has the resources to address these problems, but demonstrating this would require more work.

[21] See Kasirzadeh 2021; Knowles 2021a; and Kuorikoski 2021 for further criticisms of the mathematical-dependence analysis.

in counterfactual terms.[22] Call this the *structural-dependence analysis*. Since the relationship established by the operative generalisation is a mathematical one, this proposal contravenes (3). The problem with this proposal is that the mathematical generalisations featuring in extra-mathematical explanations are typically symmetrical, so it is difficult to see why mathematical dependence should run in one direction rather than another.

One might attempt to address this concern in the spirit of the counterfactual-interventionist theory. While the idea of intervening on a high-level structural property is strained, one can instead appeal to a broader notion of change. The idea is that, for example, one can fix whether or not the bridges of Königsberg permit an Eulerian crossing by changing the particular layout of the bridges, but one cannot fix the particular layout of the bridges by changing whether or not the system as a whole permits an Eulerian crossing (see Jansson & Saatsi 2019: §4). Thus, the argument goes, the direction of non-causal dependence is manifest in a way that is analogous to causal explanations, even if the notion of an intervention requires broadening.

Unfortunately, this approach won't work. If the operative generalisation in *Bridges* represents a relation of non-causal dependence, the identified difference-maker is not a fact about the particular layout of the bridges, but the high-level structural fact that the bridges have neither exactly zero nor exactly two landmasses contiguous to an odd number of bridges. This is no less general or structural than the fact that the bridges do not permit an Eulerian crossing. If changing the bridge system so that it permits an Eulerian crossing does not fix its particular arrangement, then neither does changing the bridge system so that it has either exactly zero or exactly two landmasses contiguous to an odd number of bridges. Indeed, changing either fact must go via a change to the particular arrangement of the bridges. So, rather than one fact depending on the another, both facts constitutively depend on the lower-level facts concerning the particular arrangement.

A key theoretical role played by the notion of an intervention in the counterfactual-interventionist theory is to rule out cases where an attempt to control Y by manipulating X proceeds by manipulating a common difference-maker of X and Y. That is exactly what is going on here: we control

[22] See e.g. Jansson & Saatsi 2019; Knowles & Saatsi 2021.

whether or not the bridge system has exactly zero or exactly two landmasses contiguous to an odd number of bridges by manipulating the particular way in which the bridges and landmasses are arranged. But then, by manipulating the particular arrangement in this way, we equally control whether or not the bridge system permits an Eulerian crossing. Thus, between the facts connected by the operative generalisation, no direction of dependence is evident.

4. The Solution

We have seen that (1)–(3) seem contradictory and yet independently attractive (§2), and that rejecting any one of them incurs considerable costs (§3). In this section, I develop a novel analysis that reveals (1)–(3) to be consistent after all.

4.1. The iso-dependence analysis

When two facts depend on exactly the same things, I will say that they are *iso-dependent* with each other. I will say that one phenomenon A is *independently-understood* relative to an explanation if, prior to our acceptance of the explanation, we grasp some dependencies of A that are relevant to that explanation. For example, if an explanation discloses the evolutionary causes of some phenomenon P, but we grasp prior to our acceptance of this explanation what the evolutionary causes of F are, then F is independently-understood. Here is my analysis of extra-mathematical explanation.

> **The iso-dependence analysis**
> For any extra-mathematical explanation of target phenomenon P, the explanans is explanatorily relevant to the explanandum in virtue of its identifying an independently-understood fact F concerning P's physical situation, where F is iso-dependent with P.

The basic idea is that the pure mathematical generalisation allows us to leverage our grasp of what F depends on to improve our understanding of P. Allow me to illustrate. For *Cicadas*, P is the fact that North-American periodical cicadas have either 13- or 17-year life-cycle periods, and the context is such that we are interested in causal-evolutionary dependencies. Here, P is not independently-understood: we do not grasp its causal-evolutionary dependencies. The explanans then provides three substantive hypotheses about the physical situation.

(i) The cicadas' life-cycle periods are between 12 and 18 years.

(ii) The cicadas have maximised the number of years between co-emergence with all nearby periodical predators.

(iii) There are nearby periodical predators with periods of each length from 2 to 12 years.

Note the epistemic status of each: (i) and (ii) are justified by scientific theory; and, in the absence of direct evidence for (iii), it stands and falls on the basis of how well it accounts for P. Together, these hypotheses imply that the cicadas have life-cycle periods of between 12 and 18 years that minimise intersection with each period of 12 or less. This is our fact F. Prior to accepting the explanation, we have a good grasp of F's causal-evolutionary dependencies: it depends on a process of natural selection involving acts of predation by the hypothesised predators, along with whichever background ecological conditions result in the life-cycle periods being restricted to between 12 and 18 years, such as the temperature of the soil. Thus, F is independently-understood.

Finally, the number-theoretic generalisation states the material equivalence of two mathematical properties. In the number-theoretic model of the physical situation, these mathematical properties represent properties of time periods measured in years. The first (x maximises its LCM with all $n < x$) represents *minimising intersection with all shorter time periods*. The second (x is prime) represents *being prime-numbered*. Thus, the generalisation establishes a mathematical relationship between these physical properties: they co-occur, by mathematical necessity.

F obtains and we have a good understanding of why. That is, we know the property *minimising intersection with all shorter time periods* is instantiated by the cicadas' life-cycle periods and that the cicadas' life-cycle periods also lie between 12 and 18 years, and we know what these facts depend on. The number-theoretic generalisation then tells us that, in any situation where F obtains, the cicadas' life-cycle periods must be prime, and so P must obtain. Thus, we learn that F and P, both contingent features of the same physical situation, must co-occur by mathematical necessity. But that means whatever brings it about or makes it the case that F must thereby bring it about or make it the case that P, and whatever brings it about or makes it the case that not-F must thereby bring it about or make it the case that not-P. In other words, F is iso-dependent with P.

How does this improve our understanding of P? We grasp (prior to accepting the explanation) that F is the outcome of a certain process of natural selection. By accepting the explanation, we learn that P is iso-dependent with F, and so P is the outcome of the very same process of natural selection as F. We move from not grasping P's causal-evolutionary dependencies to grasping P's causal-evolutionary dependencies. *Cicadas* discloses what P depends on, and thus explains it.

Now consider *Bridges*. Here, we can take P to be that all competent and systematic attempts at Eulerian crossings of the bridges of Königsberg in 1735 failed. The context is such that we are interested in whether there are facts about the way the bridges were arranged that might explain this failure. In this context, P is not independently-understood, since prior to accepting the explanation we do not grasp how it might depend on the way the bridges are arranged. The explanans then gives us two substantive claims about the physical situation:

(i) The bridge-system has a particular graph-theoretic arrangement.

(ii) The bridge-system has neither exactly zero nor exactly two landmasses contiguous to an odd number of bridges.

Both (i) and (ii) can be gleaned immediately from the graph-theoretic depiction of the bridge-system. (ii) is our fact F, whose dependencies are grasped prior to accepting the explanation: F constitutively depends on facts about the particular bridges, landmasses, and their mode of arrangement. In this context, F is independently-understood.

The graph-theoretic generalisation then establishes the material equivalence of two mathematical properties and their negations. The negation of the first property (x does not permit an Eulerian circuit) represents P while the negation of the second property (x has neither exactly zero nor exactly two nodes of odd degree) represents F. This tells us that F and P, while both contingent features of the physical situation, must co-occur by mathematical necessity. Thus, F is iso-dependent with P.

Bridges tells us that P is iso-dependent with the independently-understood fact F. We thereby learn that the fact that no one was able to make an Eulerian crossing of the bridges depends on facts concerning the bridge-system's constituents and their particular mode of arrangement, specifically the very same facts on which F depends. *Bridges* discloses what P depends on, and thus explains it.

4.2. Clarifications

It is worth clarifying my analysis in response to some possible concerns. First, one might object that F and P necessarily co-occurring does not imply that they are iso-dependent. Explanatory dependence is hyperintensional, so we should not expect it to track with necessary co-occurrence. However, I do not move simply from the claim that F and P necessarily co-occur to the claim that they are iso-dependent. I appeal also to the fact that F and P are both contingent facts about the same physical situation. This renders it clear that F and P are iso-dependent. Compare: that Sam is 6ft tall and that Sam is 182.88cm tall are contingent facts about the same individual that co-occur by mathematical necessity, and it is clear that they are iso-dependent.

Second, where some facts F and P are iso-dependent, aren't they just the same fact represented differently? I lack a background theory of facts to settle this issue, and providing one would go well beyond the scope of this paper. Thankfully, the success of the iso-dependence analysis does not hang in the balance. If F and P are identical, F may still be independently-understood while P is not, since we may be unaware that F and P are identical by virtue of their different modes of presentation. If F and P are distinct, then perhaps we are owed an account of how distinct facts can be iso-dependent. But one would hope that a theory of facts that distinguished between F and P, despite their necessary co-occurrence, would provide such an account.

Third, the iso-dependence analysis only applies to explanations invoking generalisations in biconditional form. Might there be some extra-mathematical explanations that invoke generalisations in conditional form, identifying some F that necessitates P but is not necessitated by P in turn? In such cases, we could not infer that F and P share all their dependencies, since P may occur without F. A perusal of the literature suggests that the most compelling cases of extra-mathematical explanation invoke biconditional generalisations, or can be reconstructed so that they do, so there may not be a problem here. Nevertheless, there is a simple fix. Given that F and P are contingent facts about the same physical situation, the fact that F necessitates P is enough to ensure that they share some dependencies. Whatever brings it about or makes it the case that F will thereby bring it about or make it the case that P. So, if F is independently-understood, then showing that F necessitates P will supply understanding of P by facilitating our grasp of the dependencies it shares with F. To avoid the issue, we could couch the analysis in terms of the notion of *semi*-iso-dependence, which is a matter of sharing *some* dependencies, and specify that our prior understanding of F must be a matter of grasping the dependencies it shares with P. For simplicity, I will leave this complication aside.

Fourth, some countenance extra-mathematical explanations whose target phenomena are necessary. For example, Lange (2017: 419; 2018: 87) says that, where P follows from fact F via a mathematical proof, we can explain why the conditional fact *if F then P* obtains by pointing out that it is mathematically necessary. It is true that the target phenomena of certain extra-mathematical explanations are expressed in conditional form. For example, we might ask why, if a bridge system has neither exactly zero nor exactly two landmasses contiguous to an odd number of bridges, one cannot make an Eulerian crossing of it. And, the argument goes, the graph-theoretic generalisation is a good answer because it shows that this conditional is mathematically necessary. However, even in such cases, I argue that the reason it is satisfying to learn that the conditional is mathematically necessary is that it reveals that the consequent is iso-dependent with the independently-understood antecedent. Given our prior grasp of why the antecedent obtains, learning that the consequent is mathematically necessitated by it shows that there is no further puzzle as to why the consequent obtains. Our understanding of F is already sufficient for understanding P if F.

Fifth, one may worry that there are perfectly good extra-mathematical explanations where we do not have an independent understanding of the relevant F. There are a three different cases to consider here. One is where F is a fundamental fact, and so not dependent on anything. Here, it would appear there is no independent-understanding to leverage. If such cases exist, they can be captured as limiting cases of the iso-dependence analysis. When confronted with some P that puzzles us, we want to know what it depends on. Learning that it depends on nothing is a perfectly good way to assuage our puzzlement. In this way, we can take grasping a fact's *in*dependence (i.e. its fundamentality) as a limiting case of understanding. Thus, learning that P is iso-dependent with some fundamental F is to gain understanding of P.

Another case is where F is non-fundamental, but we grasp very little about what it depends on. For example, Mark Colyvan (1998: 321–322; 2001 49–50) discusses an explanation of why there always exists a pair of equatorial antipodal points on Earth of the same temperature. The explanation appeals to the fact that temperature is continuous as a function of position, which mathematically necessitates the existence of the pair. Here, F would be the fact that temperature is a continuous function of position. The worry is that we know very little about what this fact depends on. However, I would argue that we know enough. We know that it has *something* to do with how kinetic energy is transferred through materials via countless microscopic collisions. By contrast, prior to the explanation, we have *no idea* why there are always two equatorial antipodal points of the same temperature. By learning that this latter fact is mathematically necessitated by the former, we learn that the existence of the antipodal points depends on features of the physical processes through which kinetic energy is transferred. While schematic, this understanding is still a significant improvement on our initial epistemic standing. The iso-dependence analysis does not require that we have a particularly *good* understanding of F prior to the explanation; it only requires is that we have an *independent* understanding of it. Even where this understanding is minimal, if it facilitates an improved understanding of P, the explanation is a success.

The final case to consider is where we grasp nothing of the relevant dependencies of F. Suppose we are interested in the causal provenance of features of the bridges of Königsberg. Suppose that the city planners purposely designed the bridges so that each of the four landmasses were contiguous to an odd number of bridges, for no other reason than that they liked odd numbers. But suppose we know nothing about this. Then it seems that there is a perfectly good extra-mathematical explanation of why an Eulerian-crossing of the bridges is not possible that deploys the graph-theoretic generalisation to show that the target phenomenon has precisely the same causal provenance as the fact that the bridges have neither exactly zero nor exactly two landmasses contiguous to an odd number of bridges, even if we are not in a position to recognise this explanation. This case suggests that the explanatory significance of the graph-theoretic generalisation is nothing to do with whether we grasp the dependencies of F.

Responding to this case requires some delicacy. There are two uses of 'explain' associated with backing realism. On one, one fact explains another just in case the latter depends on the former. In this sense, it is true that the intentions of the planners explain why an Eulerian crossing is not possible, regardless of what we know. On the other use of 'explain', however, one thing explains another just in case the former discloses the dependencies of the latter to an audience. According to the iso-dependence analysis, pure mathematical facts explain only in this latter sense, and it is only this kind of explanatory significance that is relativised to our epistemic situation. Relative to the hypothetical 'we' within the scope of the example, the target phenomenon is caused by the planners' decisions even though we know nothing about this; but there is no extra-mathematical explanation that discloses this because there is no independent-understanding of the fact F for the graph-theoretic generalisation to transpose. However, relative to the 'we' who are considering the example from outside, there is an extra-mathematical explanation, but only because we are privy to what F depends on, and so the graph-theoretic generalisation has something to transpose.

Sixth, if the explanatory power of an extra-mathematical explanation is due to an imbalance in our understanding of F and P, then doesn't that explanatory power vanish once we accept the explanation and the imbalance is redressed? No. There remains an important difference in *how* we grasp the dependencies of F and P, even after we accept the explanation. Our understanding of F is independent of the explanation. Perhaps it is obvious what F depends on (as is the case in *Bridges*); or perhaps we have some prior information that pertains specifically to what F depends on (as is the case in *Cicadas*). By contrast, our understanding of P is achieved only *through* our recognition of the relationship it bears to F. Of course, we could kick away the ladder and take away with us the mere propositional knowledge of what P depends on, and that would be to achieve a minimal improvement of our understanding of P. However, as I emphasised in §2.1, explanatory understanding is often richer than this, involving the grasping of a mental model of a phenomenon's dependencies manifest by a set of inferential abilities. An extra-mathematical explanation allows us to exploit our grasp of F as a means of understanding P, and the explanation retains this function even once it is accepted.

For example, when thinking about how we might intervene on the bridges of Königsberg to render them Eulerian-crossable, it is not at all clear what to do, even if we are aware that the phenomenon constitutively depends on the particular layout of the bridges. By invoking *Bridges*, however, we can think instead about which changes to the bridges yield exactly zero or exactly two landmasses contiguous to an odd number of bridges, in full confidence that they will render an Eulerian crossing possible. Thus, while there is a sense in which, once the explanation is accepted, the imbalance between our understanding of F and P is redressed, there is another important sense in which the imbalance remains, since the understanding of P remains mediated by our understanding of F. The explanatory power of an accepted extra-mathematical explanation remains because the full understanding of P it offers is achieved only through our continued appreciation of the mathematical relationship it bears to F.

Finally, does the iso-dependence analysis underplay the significance of the pure mathematical generalisations at the heart of extra-mathematical explanations? For example, the iso-dependence analysis of *Bridges* places emphasis on our prior knowledge of what the fact that the bridge system has neither exactly two nor exactly zero landmasses contiguous to an odd number of bridges depends on. But, as I have emphasised, this knowledge is obvious in the context of the explanation, and so not particularly interesting. By contrast, the graph theoretic generalisation is an interesting and insightful one, specifying under what particular circumstances a bridge system would and would not permit an Eulerian crossing. But, if the generalisation simply serves as a conduit for this boring knowledge, doesn't that underplay its significance?

No. Through the graph-theoretic generalisation in *Bridges* and our prior (albeit uninteresting) knowledge, we grasp that, were we to have intervened on the particular layout of the bridges in any of the specific ways that would result in there being exactly zero or exactly two landmasses contiguous to an odd number of bridges, some townsfolk would have succeeded in making an Eulerian crossing. This is not obvious or uninteresting, and it can only be seen through appreciation of the pure mathematical generalisation. According to the iso-dependence analysis, the operative generalisations of extra-mathematical explanations reveal the *iso-dependence* of P and F, and this allows us to grasp which particular features of the physical situation make a difference to the target phenomenon (namely, those that make a difference to F). Even if the prior knowledge of what makes a difference to F is obvious and uninteresting, the information that P depends on the very same things is not, and can only be gleaned through appreciation of the relevant mathematical generalisation. This does full justice to the illuminating and insightful mathematical results at the heart of extra-mathematical explanations.

4.3. Diagnosis and dissolution

With the iso-dependence analysis on the table, we are in a position to diagnose and dissolve the problem of extra-mathematical explanation, expressed as the following (apparently) inconsistent triad:

(1) Explanations disclose what their target phenomena depend on.

(2) There are extra-mathematical explanations.

(3) Physical phenomena are not mathematically dependent.

The problem arises because (1) and (2) seem incompatible with (3). But this appearance is based on the assumption that the operative generalisation of an explanation must *represent* the dependence relations that back the explanation. The iso-dependence analysis articulates a more subtle, non-representational role in disclosing dependencies. For a given extra-mathematical explanation, the pure mathematical generalisation represents a mathematical relationship between F and P. Our grasp of what P depends on requires our continued grasp of this mathematical relationship, but this is not because it is in any way part of the dependence relation that backs the explanation. Rather, our grasp of this mathematical relationship discloses what P depends on by allowing us to infer that all and only those changes that make a difference to F make a difference to P, and thereby supporting relevant counterfactual reasoning. As I will now explain, this analysis allows us to retain each of (1)–(3).

Let me start with (2). In §2.2, I said that to call an explanation of a physical phenomenon *extra-mathematical* is to say that it contains a pure mathematical fact whose explanatory role is discharged only by grasping its *mathematical* significance, alongside any physical significance it might have. On my analysis, this is true of the pure mathematical generalisations at the heart of extra-mathematical explanations. Their explanatory role is discharged only once we appreciate the mathematical relationship they establish between the target phenomenon P and the associated fact F, namely that P and F must co-occur by mathematical necessity. Only by appreciating this does our independent-understanding of F carry across to P. Thus, the iso-dependence analysis honours the sense that explanations like *Bridges* and *Cicadas* are distinctively mathematical.

Some may object that the moniker ‘extra-mathematical’ implies that mathematics is responsible for something, which is something the iso-dependence analysis denies. There are two ways of spelling out this objection. First, one could insist that our intuitive responses to extra-mathematical explanations, which incline us towards deploying the moniker in the first place, involve imputing something mathematical to the relations that back them. Taken in this way, the objection goes too far. The contents of our intuitions are not specific enough to warrant this claim. Our intuition is that there is something distinctive about these explanations, and that what sets them apart from other explanations has to do with the fact that the understanding they offer relies essentially on mathematical facts. This is enough to warrant loose talk of mathematical facts explaining physical phenomena. We should not think that such talk articulates a pre-theoretical belief that there are explanatory relations of mathematical dependence in the sense discussed in §3. Indeed, extra-mathematical explanations strike us as particularly interesting and puzzling precisely because it is not immediately clear how they fit into our broader explanatory practices.

Second, the objection could be taken as a claim about how best to categorise explanations: we *ought* to take extra-mathematical explanations to be explanations that involve mathematical backing relations. The guiding principle here is that explanations should be classified in line with the nature of their backing relations. The iso-dependence analysis flouts this principle by classifying extra-mathematical explanations instead in terms of how the explanatory role of their operative generalisations is discharged. The former approach is neater, to be sure, but it comes with costs. One will either have to tackle the problems besetting extant analyses of extra-mathematical explanations discussed in §3, or else deny the existence of extra-mathematical explanations and explain away our intuitions to the contrary. My approach avoids these costs.

Let us turn to (1). The iso-dependence analysis is compatible with the considerations outlined in §2.1. As a result of *Cicadas*, we learn that, were we to have intervened on the cicadas' environment so that there were no periodical predators with life-cycles of between 2 and 12 years, the cicadas would likely not have evolved to have 13- or 17-year periods. As a result of *Bridges*, we learn that, were we to have intervened on a particular pair of landmasses so that they were each no longer contiguous to an odd number of bridges (e.g. by destroying one bridge that connects them), someone would have succeeded in making an Eulerian crossing of the bridges of Königsberg. Because of this, the iso-dependence analysis promises to inherit the benefits enjoyed by the counterfactual-interventionist theory of explanation.

It inherits the compelling account of why explanations are valuable—because they facilitate control of our environment—while also highlighting what makes extra-mathematical explanations valuable in particular. Showing that P is iso-dependent with the independently-understood F can be useful even if our grasp of the mechanisms of dependence are relatively thin, as discussed in the example of equatorial antipodal points. Extra-mathematical explanations can operate at a high level of abstraction from the details concerning dependence relations. For this reason, they are desirable in domains where the details of underlying mechanisms are relatively unknown or not of concern, such as evolutionary biology and cognitive science.

Relatedly, the analysis inherits a good account of why extra-mathematical explanations are *better* than nearby non-mathematical alternatives. On the counterfactual-interventionist theory, one explanation is better than another if it provides more of the relevant kind of counterfactual information, or presents the information in a way that makes answering relevant what-if-things-had-been-different questions easier. Consider an alternative explanation of the cicadas' life-cycle periods that just consists of the following statement: a process of natural selection involving acts of predation from nearby predators with 2- to 12-year life-cycles (along with background ecological constraints) resulted in the cicadas having life-cycle periods of 13 or 17 years. Why is this explanation worse than *Cicadas*? The iso-dependence analysis suggests an answer. Though the same dependencies are disclosed, the explanation leaves us much worse off in terms of our ability to answer questions concerning which changes in the cicadas' periods would result from which changes in the predators' periods. By showing how the target phenomenon is mathematically related to the fact that the cicadas have life-cycles of between 12 and 18 years that minimise intersection with each shorter period, *Cicadas* reduces these questions to questions about which outcomes would/would not minimise intersection in the relevant way, which are highly tractable.

These last two points are related to the further benefit of furnishing a compelling account of the relationship between explanation and understanding. On the iso-dependence analysis, extra-mathematical explanations contribute to both the cognitive and practical dimensions of explanatory understanding. By associating P with an independently-understood F, the mathematics not only allows us to infer what the target phenomenon depends on, it opens it up to pre-existing inferential capacities, enhancing our ability to make relevant counterfactual inferences, to frame and explore new but related problems, and so on.

One may worry that the iso-dependence analysis cannot account for the directionality of extra-mathematical explanation. Iso-dependence is runs in both directions, so the kind of inference highlighted by the analysis works in both directions. For example, we can infer what the target phenomenon of *Bizarro-Bridges* depends on by recognising that it is iso-dependent with the fact that the town permits Eulerian crossings, so long as we have an independent grasp of what the latter fact depends on.

The worry is dispelled by recognising that we don't have an independent grasp of what the latter fact depends on, while what the target phenomenon depends on is independently obvious. Thus, in relation to *Bizarro-Bridges*, the fact indicated by the explanans is not independently-understood, and we glean nothing about what the target phenomenon depends on by recognising that it is iso-dependent with this fact. So, the iso-dependence analysis can account for the directionality of extra-mathematical explanation. However, it does not take the directionality of extra-mathematical explanations to be supplied by the directionality of the dependence relations that back them. Directionality is supplied by an asymmetry in how we grasp the dependencies of the two related facts F and P.

Independent support for this account of the directionality of extra-mathematical explanation can be found by considering relevant changes in context (determining which fact is independently-understood), and noting that the directionality changes accordingly. Suppose that, in building the bridges of Bizarro-Königsberg, the architects purposefully made it such that an Eulerian crossing could be made. They did this, not by applying graph theory, but rather by painstakingly testing the result of erecting each new bridge. Suppose this information is conveyed in a context where we are interested in what caused the bridge system to have the structural features it has. In this context, we grasp that the fact that the bridge system permits an Eulerian crossing depends causally on the intentional actions of the architects, since we have been given this information explicitly. By contrast, we lack an independent grasp of what the fact that the bridge-system has exactly zero or exactly two landmasses contiguous to an odd number of bridges causally depends on. Thus, in this context, the former fact is independently-understood while the latter is not. In this context, we can run a satisfying extra-mathematical explanation of why the bridges of Bizarro-Königsberg have exactly zero or exactly two landmasses contiguous to an odd number of bridges. By learning that this phenomenon is iso-dependent with the fact that the bridges permit an Eulerian crossing, we learn that it too was caused by the very same intentional actions of the architects. Thus, we can see that the directionality of mathematical explanations tracks the asymmetry in how we understand the two iso-dependent facts, precisely as the iso-dependence analysis predicts.

The analysis also inherits the plausible epistemology of dependence from the counterfactual-interventionist theory. According to the iso-dependence analysis, extra-mathematical explanations leverage a prior grasp of what one thing depends on to facilitate a grasp of what another thing depends on. In the examples I have discussed, the dependence relations at work were the familiar relations of causal and constitutive dependence, for which we have a plausible epistemology. I am open to the possibility that there are other kinds of dependence suitable for backing extra-mathematical explanations, but there is nothing in the iso-dependence analysis that suggests physical-on-mathematical dependence will be among them. Thus, the iso-dependence analysis is compatible with (3).

The iso-dependence analysis affirms (1) and (2) while remaining compatible with (3), dissolving the problem of extra-mathematical explanation and simultaneously avoiding the problems that arise for extant analyses that reject either (1) or (3). Thus, alone among its rivals, the iso-dependence analysis provides a satisfactory answer to the question 'how *can* mathematical facts explain physical phenomena, given what we tend to think about explanation and mathematics?' The answer is that, while mathematical facts do not make a difference to physical phenomena, they can provide understanding of them by revealing their iso-dependence with other, independently-understood facts.

5. Conclusions

As well as its intrinsic merits, the iso-dependence analysis promises to facilitate dialectically stable progress in long-standing philosophical debates. In §1, I noted two reasons that extra-mathematical explanations have received attention. One concerns the classification of different kinds of scientific explanation. The iso-dependence analysis has interesting consequences for this debate. It exemplifies an approach to the taxonomy of explanations that takes into account not only the nature of the relations that back them, but also the manner in which they disclose their backing relations.

This permits the possibility of kinds of explanation that cut across differences in backing relations. Extra-mathematical explanations are like this, on my view. *Cicadas* is backed by a relation of causal dependence, while *Bridges* is backed by a non-causal relation of constitutive dependence. Yet they are both *extra-mathematical* because they disclose their backing relations in the same distinctively mathematical way. This challenges the received view that extra-mathematical explanations are paradigm cases of non-causal explanation.

Extra-mathematical explanations have also been thought to lend support to a certain kind of argument in favour of mathematical realism. According to the explanatory indispensability argument, we should believe in mathematical objects because their existence and nature explains a wide range of observable phenomena. The iso-dependence analysis makes progress in the debate surrounding this argument by lending support to a particular line of response. The response concedes that mathematics is indispensable to scientific explanation, but denies that this carries ontological commitment to mathematical objects. The idea is that mathematics plays an indispensable role in disclosing certain explanatory features; but this does not imply that mathematical entities or facts about them feature in the explanatory features thereby disclosed.[23] On this way of thinking, mathematics can play its indispensable explanatory role even if mathematical objects turn out to be fictions.

A problem with this response is that it does not seem to recognise the apparent difference between ordinary representational applications of mathematics in scientific explanations, and the apparently explanatory applications found in examples such as *Bridges* and *Cicadas*. According to the response, both kinds of application simply involve mathematical concepts being used to represent certain explanatory physical features. In light of this, it may appear that proponents of this response must ultimately deny that there are extra-mathematical explanations. The iso-dependence analysis overcomes this problem. It spells out how mathematical concepts can play an explanatory role that does not simply involve representing physical things, but without thereby locating anything mathematical in the dependence relations ultimately disclosed. Thus, proponents of the present re-

[23]See in particular Rizza 2011 and Yablo 2012.

sponse can account for the difference between explanatory and non-explanatory applications of mathematics, placing them in a strong dialectical position.

Moreover, because the iso-dependence analysis enjoys advantages over those that would provide pressure towards realism (such as the mathematical-dependence analysis), it places the burden of proof on the shoulders of those who defend the enhanced indispensability argument. They must develop a rival analysis of extra-mathematical explanation that provides pressure towards mathematical realism, and also enjoys at least comparable advantages to those enjoyed by the iso-dependence analysis. Specifically, this rival analysis must provide an equally-satisfying solution to the problem of extra-mathematical explanation.

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