Assessing POVMs (and povms)

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For many decades the development of the quantum theory and its applications were guided by the framework laid out in von Neumann's *Mathematische Grundlagen der Quantenmechanik* (1932). The spectacular success of the theory was seemingly strong testimony to the resilience of this framework. In recent decades, however, a number of researchers working under the banner headlined "POVMs" have claimed that the von Neumann framework needs to be expanded in various ways. Some researchers claim that in order to be empirically adequate the theory has to recognize observables represented by maximally symmetric but nonselfadjoint operators. Other researchers, working under the same banner, stick with selfadjoint observables but claim that realistic quantum measurements which are "unsharp" have to be described not in terms of projection operators but "effects"—positive bounded (and, thus, selfadjoint) operators that are not projections. A critical examination of this program reaffirms the resilience of the von Neumann framework.

1 The old and the new orthodoxies

Within a few years of the publication of John von Neumann's *Mathematische Grundlagen der Quantenmechanik* (1932) a consensus developed regarding the mathematical framework of quantum theory; in particular, the mathematical apparatus to be used to formulate the theory, an account of which variables in the apparatus represent "observables" and the class of operators which are the bearers of quantum probabilities, how expectation values of measurements of observables are computed, etc. We will refer to this consensus as the "von Neumann orthodoxy," without any implication that von Neumann himself would agree in all respects with the version we will elaborate below.[1]

[1] In fact a disagreement is noted in Section 2.5.

Some decades later there arose a heterodoxy, advocating not root and branch revision of the old orthodoxy but an expansion which allows for more liberal notions of what counts as an observable and what objects in the theory bear probabilities. If widespread nodding assent amounts to orthodoxy then this heterodoxy has become the new orthodoxy. We will refer to it as the "POVM/povm orthodoxy". We are not outright heretics from the new orthodoxy, but we are skeptical of some of its claims. If the alleged inadequacies of the old orthodoxy were indeed as serious as the new orthodoxy claims, we find it implausible that they could have been overlooked for so long by so many acute researchers. And when we query those who give their nodding assent to the new orthodoxy we often find that the nodders cannot provide convincing reasons for their assent.[2] At the very least the new orthodoxy deserves a skeptical look. And if the skepticism proves to be baseless then the new orthodoxy stands on firmer ground.

This new orthodoxy was constructed by the efforts of many researchers, and naturally their motivations for and conceptions of what constitutes the new framework for QM differ. We will not attempt to do justice to this fascinating and complex thicket of ideas, but will focus on the central claims of what we identify as the two main strands of the new orthodoxy. Before attending to the new, it will be helpful to have before us a sketch of the main tenets of the old orthodoxy.

2 The von Neumann orthodoxy

2.1 Basic assumptions

There are various ways to present the old orthodoxy. We will work with the algebraic formulation because, by bringing the algebra of observables to the fore, it puts it into sharp relief with the new orthodoxy. Three principal elements are used to characterize a quantum system: a von Neumann algebra $\mathfrak{M}$ of observables acting on a Hilbert space $\mathcal{H}$ along with a set of admissible states on $\mathfrak{M}$. In ordinary non-relativistic QM (sans superselection rules) it is assumed that $\mathfrak{M}$ is $\mathfrak{B}(\mathcal{H})$, the algebra of all bounded operators acting on $\mathcal{H}$. More exotic algebras are encountered in relativistic QFT. It is often presupposed that $\mathcal{H}$ is a separable Hilbert space over the complex numbers. I

[2]Other than their desire to stand with those in the know. Skepticism with regard to the new orthodoxy is rare; for two exceptions see Uffink (1994) and Fleming (2000, 2013).

will follow this practice here since I do not want to deal with the complications that arise when non-separable spaces are used, and likewise I do not want to be drawn into the issue of why complex Hilbert spaces are to be preferred to real spaces.[3]

A key assumption of the von Neumann orthodoxy is that the observables characterizing a quantum system are represented by the selfadjoint elements of $\mathfrak{M}$. When such an A has a discrete spectrum the possible values of the observable are given by the eigenvalues of A. The notion of quantum "state" is used in the algebraic sense of a normed positive linear functional $\omega : \mathfrak{M} \rightarrow \mathbb{C}$. We will follow standard practice in assuming that the admissible states are normal, where ω is normal just in case there is a unique density operator ϱ_ω (a positive, trace-class operator with $Tr(\rho_\omega) = 1$) such that $\omega(A) = Tr(\varrho_\omega A)$ for $A \in \mathfrak{M}$. Equivalently, the normal states are the states that are countably additive on any family of mutually orthogonal projections in $\mathfrak{M}$. The expression $\omega(A)$, for selfadjoint $A \in \mathfrak{M}$, can be read as asserting that when the system is in state ω the expectation value of a measurement of the observable which A represents is $\omega(A)$. This reading is motivated by the way in which probabilities are handled in the von Neumann orthodoxy.

Among the observable elements of $\mathfrak{M}$ are the projection operators (aka Yes-No operators) $\mathcal{P}(\mathfrak{M})$, where $P \in \mathcal{P}(\mathfrak{M})$ is a selfadjoint operator such that $P^2 = P$ (eigenvalues 0, 1).[4] $\mathcal{P}(\mathfrak{M})$ forms a lattice under the partial order relation $\preceq$ where $P_1 \preceq P_2$ iff $P_2 - P_1$ is a positive operator, i.e. $\omega(P_2 - P_1) \geq 0$ for all normal ω.[5] That is to say, $\mathcal{P}(\mathfrak{M})$ is closed under taking the meet $P_1 \wedge P_2$ (greatest lower bound) and join $P_1 \vee P_2$ (least upper bound) of elements of $\mathcal{P}(\mathfrak{M})$. With negation (or complementation) defined by $\neg P := I - P := P^\perp$ the operation $P \mapsto P^\perp$ has the properties: $(P^\perp)^\perp = P$, $P \wedge P^\perp = \mathbf{0}$, and $P_1 \preceq P_2$ implies $P_2^\perp \preceq P_1^\perp$, which is to say that $\mathcal{P}(\mathfrak{M})$ is an orthocomplemented lattice.

[3] For some opinions on the latter issue see Earman (2024).

[4] Some writers distinguish between orthogonal projections and oblique projections, the former being bounded self-adjoint operators such that $P^2 = P$ (what we call projections simpliciter), and the latter being symmetric but not necessarily self-adjoint operators such that $P^2 = P$. Throughout we will stick to our usage.

[5] Equivalently, (i) $Ran(P_1) \subseteq Ran(P_2)$ or (ii) $((P_2 - P_1)\psi, \psi) \geq 0$ for all $\psi \in \mathcal{H}$ with $(\cdot\, , \cdot)$ the inner product on $\mathcal{H}$.

2.2 Orthodox quantum probability

In the von Neumann orthodoxy quantum probabilities are assigned to projections in the algebra, and quantum probability theory becomes the study of probability measures on the projection lattice $\mathcal{P}(\mathfrak{M})$ (see Hamhalter 2003). In more detail, a quantum probability measure is a map $pr : \mathcal{P}(\mathfrak{M}) \to [0, 1]$, such that

(A1) $pr(I) = 1$.

(A 2) $pr(P_1 + P_2) = pr(P_1) + pr(P_2)$ when $P_1, P_2 \in \mathcal{P}(\mathfrak{M})$ are orthogonal (and, thus, $P_1 + P_2 = P_1 \vee P_2$).

The requirement of countable additivity strengthens (A2) to

(A3) $pr(\sum_{n=1}^{\infty} P_n) = \sum_{n=1}^{\infty} pr(P_n)$ for any countable family $\{P_n\} \in \mathcal{P}(\mathfrak{M})$ of mutually orthogonal projections.[6]

Gleason's theorem shows that, if $\dim(\mathcal{H}) > 2$, any countably additive pr on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ extends uniquely to a normal state ω on $\mathfrak{B}(\mathcal{H})$. This theorem has been subsequently generalized from $\mathfrak{B}(\mathcal{H})$ to cover any von Neumann algebra $\mathfrak{M}$ with no Type I_2 summands. In the other direction a normal state ω on $\mathfrak{M}$ induces a countably additive $pr(P) = \omega(P)$ on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$.

A fellow traveler with the decision to assign quantum probabilities to elements of $\mathcal{P}(\mathfrak{M})$ and only these elements of $\mathfrak{M}$ is the idea that quantum measurements are projective measurements, either measurements of individual projections or joint measurements of jointly measurable (= mutually commuting) projections. Given the pre-measurement state of the system the theory should give a means of computing the probabilities of outcomes of the measurements of observables, and given that probabilities are assigned only to projections such a means is possible only if measurements are projective. The new orthodoxy abandons both of these fellow travelers. This abandonment necessitates a change in the rule that the old orthodoxy uses to update states on the outcomes of measurements.

[6] The limit in $\sum_{n=1}^{\infty} P_n$ is taken in the strong operator topology.

2.3 Projection-valued measures and positive operator-valued measures

Here we meet some of the principal actors in our story, PVMs and POVMs.[7]

With X a topological space, $B(X)$ the σ-algebra of Borel sets of X, and $\mathcal{H}$ a (separable) Hilbert space, a $(X,\ B(X),\ \mathcal{H})$ *POVM* (positive operator-valued measure) is a map $E()$ from $B(X)$ to "effects" $\mathcal{E}(\mathcal{H})$, i.e. positive and bounded linear operators acting on $\mathcal{H}$ such that for any countable family of disjoint sets $\{\Delta_n\} \in B(X)$, $E(\bigcup_{n=1}^{\infty} \Delta_n) = \sum_{n=1}^{\infty} E(\Delta_n)$.[8] Since a positive and bounded operator is necessarily selfadjoint (Reed and Simon 1980, p. 195), the $E(\Delta)$ are selfadjoint operators. The POVM is *normalized* if $E(X) = I$ and $E(\emptyset) = \mathbf{0}$. (From here on normalization will be assumed.) A POVM is *orthogonal* if $E(\Delta_1)E(\Delta_2) = \mathbf{0}$ whenever $\Delta_1 \cap \Delta_2 = \emptyset$. And it is *commutative* if $E(\Delta_1)E(\Delta_2) = E(\Delta_2)E(\Delta_1)$ for all $\Delta_1, \Delta_2 \in B(X)$.

A $(X,\ B(X),\ \mathcal{H})$ *PVM* (positive-valued measure) is a $(X,\ B(X),\ \mathcal{H})$ POVM that is both orthogonal and normalized, which implies that $E(\Delta)^2 = E(\Delta)$ for all $\Delta \in B(X)$, i.e. $E(\Delta) \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ for all $\Delta \in B(X)$.[9] A prime example of a PVM is the spectral measure of a selfadjoint operator with $X = \mathbb{R}$.[10] A *proper POVM* is a POVM that is not a PVM. A POVM may be proper either because the $E(\Delta)$ are not projections or because the orthogonality condition fails.

Under the assumption that observables are represented by selfadjoint operators[11] the von Neumann orthodoxy assigns probabilities to outcomes of measurements of an observable by using the spectral theorem and the Born rule.

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[7] For in depth treatments the reader is referred to Beneduci (2020) and Akheiser and Glazman (1993).

[8] $\mathcal{E}(\mathcal{H})$ can also be written as $\mathcal{E}(\mathfrak{B}(\mathcal{H}))$, the effect algebra associated with the von Neumann algebra $\mathfrak{B}(\mathcal{H})$ of bounded operators acting on $\mathcal{H}$. The structure of $\mathcal{E}(\mathfrak{M}(\mathcal{H}))$, the effect algebra associated with a general von Neumann algebra $\mathfrak{M}(\mathcal{H}) \subseteq \mathfrak{B}(\mathcal{H})$ is discussed below in Section 6.2.

[9] For $\Delta \in B(X)$ define $\overline{\Delta} := X - \Delta$. Then $X = \overline{\Delta} \cup \Delta$ and $\overline{\Delta} \cap \Delta = \emptyset$. So by normalization $E(X) = I = E(\overline{\Delta} \cup \Delta) = E(\overline{\Delta}) + E(\Delta)$ and $E(\overline{\Delta}) = I - E(\Delta)$. By orthogonality $E(\Delta)E(\overline{\Delta}) = 0 = E(\Delta)(I - E(\Delta)) = E(\Delta) - E(\Delta)^2$ and, thus, $E(\Delta)^2 = E(\Delta)$.

[10] Of course, this can be generalized to include $X = \mathbb{R}^n$

[11] When we speak of probabilities of outcomes of measurements of an operator A it should be understood that what is meant is the probabilities of outcomes of the observable that A represents.

2.4 The spectral theorem and the Born rule

Probabilities are assigned to measurement outcomes via the Born rule. The application of the Born rule to outcomes of measurements of selfadjoint operators uses the spectral theorem, which demonstrates that selfadjoint operators are in one-one correspondence with PVMs. The explanation takes a detour through the notion of a resolution of the identity. An orthogonal resolution of the identity (ORI) $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is a non-decreasing map $\lambda \mapsto E_\lambda \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ where the E_λ are right continuous and $s - \lim_{\lambda\to-\infty} E_\lambda = 0$ and $s - \lim_{\lambda\to+\infty} E_\lambda = I$.[12] There is a bijective correspondence between PVMs and ORIs (Blank et al. 1994, 5.1.7 Corollary). A PVM $E()$ for ($\mathbb{R}$, $B(\mathbb{R})$, $\mathcal{H}$) determines an ORI given by $E_\lambda := E((-\infty,\lambda])$; and conversely, an ORI $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ determines a PVM $E()$ such that $E_\lambda = E((-\infty,\lambda])$. To illustrate, consider a selfadjoint A with a purely discrete spectrum $\{a_k\}$ where there is a least a_k. Renumber the a_k, if necessary, so that they are ordered by size, $a_1 < a_2 < a_3, \ldots$. Then $E^A_\lambda = \sum\limits_{n=1} \chi_{(-\infty,\lambda)}(a_n)P_n$, where P_n is the projection onto the eigenspace of a_n, with the first moment of E^A_λ giving the representation $A = \sum a_n P_n$.

In the case of $\mathfrak{M} = \mathfrak{B}(\mathcal{H})$ the spectral theorem (Akhiezer and Glazman 1993, Vol. II, Theorem 1, p. 36) shows that to every selfadjoint A acting on $\mathcal{H}$ there corresponds a unique ORI $\{E^A_\lambda\}_{\lambda\in\mathbb{R}}$ yielding an integral representation of A:

$$A = \int_{\mathbb{R}} \lambda dE^A_\lambda. \tag{1}$$

Conversely, for every ORI $\{E_\lambda\}$ there corresponds a unique selfadjoint A defined by (1), the domain $D(A)$ consisting of all $\xi \in \mathcal{H}$ such that $\int_{\mathbb{R}} \lambda^2 d(E^A_\lambda \xi, \xi) < \infty$ with the lhs of the inequality equal to $||A\xi||^2$.[13] In view of the bijection between PVMs and ORIs, the spectral theorem ensures a one-one correspondence between PVMs and selfadjoint operators. From the functional calculus it follows from (1) that

[12] The non-decreasing condition implies that $E_{\lambda_1}E_{\lambda_2} = E_{\lambda_2}E_{\lambda_1} = E_{\lambda_1}$ for $\lambda_1 < \lambda_2$. For an interval $\Delta = [\lambda', \lambda''] \subset \mathbb{R}$ define $E(\Delta) := E_{\lambda''} - E_{\lambda'}$. Then $E(\Delta_1)E(\Delta_2) = E(\Delta_1 \cap \Delta_2)$. Thus, $E(\Delta_1)E(\Delta_2) = 0$ if $\Delta_1 \cap \Delta_2 = \emptyset$, justifying the appellation orthogonal resolution of the identity.

[13] (ψ, ξ) denotes the inner product of vectors $\psi, \xi \in \mathcal{H}$.

$$f(A) = \int_{\mathbb{R}} f(\lambda) dE^A_\lambda \tag{2}$$

for Lebesgue measurable f and, in particular,

$$A^2 = \int_{\mathbb{R}} \lambda^2 dE^A_\lambda. \tag{3}$$

The Born probability assignment rule for selfadjoint operators can now be given concrete form:

> For a selfadjoint operator A with corresponding PVM $E^A() : B(\mathbb{R}) \to \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ and spectrum $\sigma(A) \subseteq \mathbb{R}$, the probability that a measurement of A when the system is in (normal) state ω yields a value lying in $\Delta \in B(\sigma(A))$ is $\omega(E^A(\Delta)) = Tr(\rho_\omega E^A(\Delta))$. (*)

In particular, if the system is in state ω then the probability measure that governs to outcomes of Yes-No measurements of projections is $pr(P) := \omega(P) = Tr(\rho_\omega P)$, $P \in \mathcal{P}(\mathfrak{M})$. The success of (*) is witnessed by massive empirical evidence.

POVMers agree that, together with quantum states, normalized PVMs generate probability measures. If $E()$ is a $(X,\ B(X),\ \mathcal{H})$ PVM and ω is a normal state then $P(\Delta) := \omega(E(\Delta))$, $\Delta \in B(X)$, is a countably additive probability measure on $(X,\ B(X))$. But they hasten to add that this isn't the only way to generate such probability measures on $(X,\ B(X))$. If $E()$ is a $(X,\ B(X),\ \mathcal{H})$ normalized POVM and ω is a normal state then again $\widetilde{P}(\Delta) := \omega(E(\Delta))$, $\Delta \in B(X)$, is also a countably additive probability measure on $(X,\ B(X))$ even if the POVM is non-orthogonal and the $E(\Delta)$ are not projections. This is an uncontroversial mathematical fact. But its application to QM requires the development of an alternative to the standard formalism of quantum probability on the projection lattice (Section 2.2). This theory of "generalized probability" assignments of non-projection effects and its physical interpretation will be discussed below. One can also wonder whether, just as PVMs are in one-one correspondence with selfadjoint operators, there is an important class of operators whose members are in one-one correspondence with POVMs such that POVMs play a role in assigning Born rule probabilities that is analogous to the role played by PVMs in assigning probabilities to selfadjoint operators. We shall see.

2.5 Updating

In the von Neumann formalism there are rules for updating states and probabilities on an obtained measurement result:

> *Lüders state-updating rule.* Let ω be a normal state on a von Neumann algebra $\mathfrak{M}$ with no Type I_2 summands. If ω is the state of the system prior to the measurement of $Q \in \mathcal{P}(\mathfrak{M})$ where $\omega(Q) \neq 0$, and if and the measurement of Q yields a Yes answer then the immediate post measurement state is the normal state $\omega_Q(A) := \dfrac{\omega(QAQ)}{\omega(Q)}$ for all $A \in \mathfrak{M}$.

The Lüders state-updating rule together with the Born rule gives an updating of pre-measurement probability pr on $\mathcal{P}(\mathfrak{M})$ to post-measurement probability pr_Q; namely,

> *Lüders probability-updating rule.* Let ω be a normal state on a von Neumann algebra $\mathfrak{M}$ with no Type I_2 summands. If ω is the state of the system prior to the measurement, giving a pre-measurement probability $pr(\bullet) = \omega(\bullet)$, and a measurement of $Q \in \mathcal{P}(\mathfrak{M})$ where $\omega(Q) \neq 0$ yields a Yes answer, then the post-measurement probability pr_Q is $pr_Q(P) := \dfrac{\omega(QPQ)}{pr(P)}$ for all $P \in \mathcal{P}$.

The numerator is written as $\omega(QPQ)$ rather than $pr(QPQ)$ since if P and Q do not commute $QPQ \notin \mathcal{P}(\mathfrak{M})$ and, thus, in the von Neumann orthodoxy it does not receive a probability. If we want to interpret $pr_Q(P)$ as the probability of P conditional on Q, denoted by $pr(P//Q)$, there is an independent justification for this rule deriving from the requirement that $pr(P//Q)$ exhibits the analog of a property of conditional probability assignments on an abelian algebra:

Theorem (Cassinelli and Zanghi 1983). Let $\mathfrak{M}$ be a von Neumann algebra without Type I_2 summands and let pr be a countably additive measure on $\mathcal{P}(\mathfrak{M})$. Then for $Q \in \mathcal{P}(\mathfrak{M})$ such that $pr(Q) \neq 0$ there is a unique probability measure $pr(\bullet//F)$ on $\mathcal{P}(\mathfrak{M})$ such that $pr(P//Q) = \dfrac{\Pr(P)}{\Pr(Q)}$ for any $P \in \mathcal{P}(\mathfrak{M})$ such that $P \preceq Q$; namely, $pr(P//Q) := \dfrac{\omega(QPQ)}{\omega(Q)}$, where ω is the unique normal state that extends pr to $\mathfrak{M}$.

Here we must note that von Neumann would have dissented from one aspect of what we are calling the von Neumann orthodoxy. Consider the case of a selfadjoint A with a purely discrete spectrum. The spectral decomposition of A takes the form $A = \sum_i a_i P_i$ where the a_i are eigenvalues of A and, thus, the P_i are projections onto the corresponding eigenspaces. When the a_i are non-degenerate and the P_i are rank one projections, von Neumann would have agreed with the Lüders state updating rule which sends the pre-measurement state to an eigenstate of A, an instance of the infamous projection postulate aka state vector reduction (details below). But when the a_i are degenerate and the P_i are multidimensional he would have dissented. In particular, the Lüders rule preserves a superposition of eigenstates belonging to the eigenspace of an a_i; to the contrary, von Neumann thought that measurement would break the eigenvalue degeneracy, fail to preserve the superposition, and result in some particular post-measurement eigenstate in the eigenspace of a_i (see von Neumann 1955, pp. 347ff). Hegerfeldt and Mayato (2012) developed a protocol to distinguish between the von Neumann and the Lüders reduction rules, and preliminary experimental results favor the Lüders rule (see Kumar et al. 2016).

Von Neumann's proposal is best seen as directed to algebras of observables containing rank-one projections. Type III algebras, for example, not only do not contain any rank-one projections but they also do not contain any finite dimensional projections. The infinite dimensional eigenvalue degeneracy cannot be broken. For these reasons we will stick with the Lüders rule and, with apologies to von Neumann, will continue to refer to the orthodoxy with the inclusion of the Lüders rule as the von Neumann orthodoxy.

For future reference it will be useful to go into more detail here about state reduction or, as we would prefer to say, state updating. And for this purpose it is helpful to introduce the concept of a filter for normal states: $F_\varphi \in \mathcal{P}(\mathfrak{M})$ is a filter for normal state φ on the algebra $\mathfrak{M}$ just in case for any normal state ω, if $\omega(F_\varphi) \neq 0$ then $\dfrac{\omega(F_\varphi A F_\varphi)}{\omega(F_\varphi)} = \varphi(A)$ for all $A \in \mathfrak{M}$. In the case of non-degenerate eigenvalues, if P_i is a rank one projection onto the ray spanned by the eigenvector v_i for eigenvalue a_i then P_i is a filter for the for the normal vector state corresponding to v_i. Thus, if the pre-measurement state ω assigns a non-zero probability $\omega(P_i)$ to P_i, Lüders updating on a positive outcome of measuring P_i produces a post-measurement state ω_{P_i} corresponding to the eigenstate v_i. Here we apply the generally acknowledged interpretation principle that is the innocuous eigenvector-to-eigenvalue half of the eigenvalue-eigenvector link; namely, if the state of a system is an eigenstate v_i of A then the system possess the eigenvalue a_i for v_i.[14]

This may seem a long winded way of arriving at common wisdom, but it is good to know what assumptions lie behind this wisdom. Incidentally, with all of this in place $\omega(P_i)$, which in the first instance is the probability that a measurement of P_i yields a Yes answer, also serves as the probability that in the post-measurement state the system possess the eigenvalue a_i. When the eigenvalue a_i is degenerate the multi-dimensional projection P_i no longer serves as a filter for any state corresponding to an eigenstate of a_i. But Lüders updating the pre-measurement state ω on P_i results in a post-measurement state ω_{P_i} that assigns probability 1 to P_i, the projection onto the eigenspace of a_i. And here we propose the natural extension of the eigenvector-to-eigenvalue principle for the non-degenerate case; namely, when the state of the system assigns probability 1 to the projection onto the eigenspace of eigenvalue a_i then, regardless of whether or not a_i is a degenerate eigenvalue, the system possesses the eigenvalue a_i of A.

The Lüders reduction rule, no less than the von Neumann reduction rule, generates the notoriously contentious measurement problem in QM. The problem can be avoided by rejecting state reduction rules. But then one is left without an account of state preparation, the only extant account of which invokes the concept of a filter along with a state reduction rule. The procedure for preparing a normal state φ is to measure a filter F_φ for φ until a Yes answer is obtained, in which case conclude by Lüders rule that the post-measurement state is φ. In a sense, accounting for state preparation is

[14] The other, controversial, half of the eigenvalue-eigenvector link asserts that only if the state of a system is an eigenstate v_i of A with eigenvalue a_i does the system possess the eigenvalue a_i. It is this half that generates the notorious measurement problem.

even more problematic than accounting for outcomes of measurements made on a prepared system. Among the normal states only the pure ones possess filters. Since some von Neumann algebras do not admit normal pure states (e.g. the Type III local algebras encountered in relativistic field theory) this raises the issue of how state preparation is possible in such instances. This is not the venue to discuss such issues.[15]

3 How POVMs arise in the von Neumann apparatus

3.1 Measurement

One way POVMs arise is by projecting a PVM down to a smaller Hilbert space. With $\mathcal{P}(\mathfrak{M})$ the projection lattice of $\mathfrak{M}$, choose an $P \in \mathcal{P}(\mathfrak{M})$ such that $\mathbf{0} \preceq P \preceq I_{\mathcal{H}}$, and let $\mathcal{H}^- := P\mathcal{H}$. If $E()$ is a $(X,\ B(X), \mathcal{H})$ PVM then $E^-() := PE()P$ is a normalized $(X,\ B(X),\ \mathcal{H}^-)$ POVM, and generally it is a proper POVM.[16]

Mathematically at least, this is the generic way in which POVMs arise, as shown by Naimark's dilation theorem.

> *Naimark's dilation theorem* (Akhiezer and Glazman 1993, Vol. II, p. 124; Beneduci 2020). Let $E()$ be a normalized $(X,\ B(X),\ \mathcal{H})$ POVM. Then there is an extended Hilbert space $\mathcal{H}^+ \supset \mathcal{H}$ and a $(X,\ B(X),\ \mathcal{H}^+)$ PVM $E^+()$ such that $E(\Delta)\psi = P^+E^+(\Delta)\psi$ for all $\psi \in \mathcal{H}$ and all $\Delta \in B(X)$, where $P^+ \in \mathcal{P}(\mathfrak{M})$ is the projection such that $P^+\mathcal{H}^+ = \mathcal{H}$.[17] The dilation $E^+()$ can be chosen to be minimal in the sense that $\mathcal{H}^+$ is the closure of the span of $\{E^+(\Delta)\psi : \psi \in \mathcal{H},\ \Delta \in B(X)\}$, in which case the dilation is unique up to unitary equivalence.

Needless to say, there is no guarantee that a Naimark dilation is physically realized in a natural way whenever POVMs arise. One natural application is provided by the measurement context. Let $E()$ be a $(X,\ B(X),\ \mathcal{H})$ PVM

[15] For some opinions on these matters see Earman and Ruetsche (2020).

[16] Instead of $PE()P$ we can write $PE()$ since P acts as the identity on $\mathcal{H}^-$.

[17] Again $P^+E^+(\Delta)\psi = P^+E^+(\Delta)P^+\psi$ for all $\psi \in \mathcal{H}$ since P^+ acts as the identity on $\mathcal{H}$.

for a system with algebra $\mathfrak{M}$ acting on $\mathcal{H}$. Suppose that a Yes-No measurement of $P \in \mathcal{P}(\mathfrak{M})$ returns a Yes answer. Then as discussed above, the von Neumann orthodoxy posits that if the pre-measurement state is ω then the post-measurement state ω_P applied to the elements of the PVM gives $\omega_P(E(\Delta)) = \dfrac{\omega(PE(\Delta)P)}{\omega(P)}$, $\Delta \in B(X)$. When P and the $E(\Delta)$ do not commute $E^-() := PE()P$ is a proper $(X, B(X), \mathcal{H}^-)$ POVM with $\mathcal{H}^- = P\mathcal{H}$. So effects that are not projections and POVMs that are not PVMs show up in calculations used in the von Neumann orthodoxy. But this is not embarrassment to the non Neumann orthodoxy; for these objects are not viewed as having any ontological significance, but are merely way stations on the road to a final result that is couched in terms of quantities to which the orthodoxy assigns physical significance.

3.2 Imperfect measurements

When laboratory instruments are unable to deliver the idealized measurements assumed in the von Neumann nomenclature, supplementary assumptions are needed to account for the actual measurement results. Such is the case when the instrumentation is unable to make arbitrarily fine discriminations or when noise in the instrument leads to misreporting of results. In these cases the use of plausible supplementary assumptions to produce the observed measurement results also lead to the emergence of POVMs/povms. But that is the point—these gadgets are emergent phenomena, much more akin to epiphenomena than explainers. More groundwork is needed before the case for this reading can be made. Details will be provided in Sections 7.1-7.3.

4 The new orthodoxy

The new orthodoxy contains two related but distinct strands, each of which poses a challenge to the von Neumann orthodoxy. The first—POVM land—proposes a small but nevertheless significant expansion of the von Neumann notion of quantum observables to include observables represented by maximally symmetric but non-self adjoint operators.[18] The second—povm

[18] Roberts (2018) argues for a much more expansive notion of observable. Here we are concerned only with the implications of an expansion of the von Neumann orthodoxy for

land—has two related substrands. The first substrand accepts that obsevables are represented by selfadjoint operators but claims that in order to treat non-idealized "unsharp" measurements, projective measurements need to be supplemented by measurements of non-projection but nevertheless selfadjoint operators ("effects"), necessitating probability assignments to non-projections. The second substrand conjures with povms—a finite or countable resolutions of the identity by means of effects. We examine these various strands and substrands in turn.

4.1 Observables in QM

Despite its importance, "observable" remains a vague concept in QM, but the intended reference is to quantities which play important roles in the theory and which, in principle, are measurable. Given the universally acknowledged ansatz that observables are represented in the theory by linear operators, three questions arise. First, which class of linear operators represents observables? Second, how are we to know which observable quantity an operator in this class represents? Third, how do we design an experiment to measure this quantity? With regard to the last question it would be asking too much to require a blueprint that an engineering firm could use to produce a laboratory apparatus to measure said quantity. But unless some concrete guidance is given, QM remains a symbolic mathematical apparatus with no ties to laboratory phenomena.

The von Neumann orthodoxy asserts an answer to the first question: observables are represented by selfadjoint operators. Without stretching the language, a PVM can be deemed to constitute an observable since the spectral theorem for selfadjoint operators provides a well motivated one-one correspondence between PVMs and selfadjoint operators. And the orthodoxy also gives us a handle on the second and third questions. Which quantity a selfadjoint operator represents can sometimes be divined by its symmetry/covariance properties. This is in fact the means by which the self–adjoint operators representing position and momentum in non-relativistic QM are

the status of POVMs/povms. For the main branch of POVM/povm enthusiasts no such expansion is needed since they accept that quantum observables are represented by self-adjoint operators. For POVMs in the original proper sense the most relevant expansion is to maximally symmetric operators since for these operators POVMs take over the role that PVMs play for self-adjoint operators (with some technical caveats to be explained below). It is this case that will be the focus of attention here.

determined.[19] And in the case of position, for example, knowing that the members of the position PVM given by applying the spectral theorem are the spectral projections $P(\Delta)$, $\Delta \subset \mathbb{R}$, of the position operator Q tells us that the operationalization of a measurement process for $P(\Delta)$ must involve designing detectors that register when the particle is present in the spectral range Δ.

Can a similar story be told for POVMs? To start the story, do non-PVM POVMs correspond to some class of linear operators more inclusive than the class of selfadjoint operators? And, if so, can the rest of the story be filled in in a similar way for elements of the more inclusive class operators? These questions will be taken up in the following subsections.

4.2 POVMs, GRIs, and symmetric and maximally symmetric operators

POVMers refer to POVMs as generalized observables, and they propose to extend the Born rule to these generalized observables. Given the universally accepted ansatz that quantum observables are represented by linear operators on Hilbert space and the need to find such representations for observables in order to do calculations it is natural to ask what class of operators represent POVM observables.

A symmetric linear (aka Hermitian) operator A with dense domain $D(A)$ is selfadjoint just in case it is maximally symmetric, i.e. it cannot be extended as a symmetric operator to a larger domain, and furthermore $D(A) = D(A^{\dagger})$. The least radical but, nevertheless, physically significant, expansion of the von Neumann orthodoxy, where observables are represented by selfadjoint operators, would be to include maximally symmetric but non-selfadjoint operators. Before examining the question of whether such an expansion is required for the empirical adequacy of quantum theory, let's understand the connection with the issue of PVMs vs. POVMs.

Just as a PVM has the companion notion of an ORI, so a POVM has the companion notion of a generalized resolution of the identity (GRI), a one-parameter family of right-continuous operators $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, such that when $\lambda_2 > \lambda_1$ the difference $E_{\lambda_2} - E_{\lambda_1} \in \mathcal{E}(\mathcal{H})$), $s - \lim_{\lambda\to-\infty} E_\lambda = 0$, and $s - \lim_{\lambda\to+\infty} E_\lambda = I$ (Akheiser and Glazman 1993, Vol. 2, p. 121). As an example of how a GRI arises from a ORI, let E_λ be an ORI for the Hilbert

[19] Wightman (1962) showed that a similar strategy can be followed in relativistic QM.

space $\mathcal{H}$ and let $\mathcal{H}^-$ be a proper subspace of $\mathcal{H}$ with P the projection of $\mathcal{H}$ onto $\mathcal{H}^-$, $P\mathcal{H} = \mathcal{H}^-$. Then $E_\lambda^- := PE_\lambda$ is a GRI for $\mathcal{H}^-$. This example is more than just an example, for just as every POVM arises from restricting a PVM for a Hilbert space to a subspace, so every GRI arises from restricting an ORI to a subspace (Akheiser and Glazman 1993, Vol. 2, p. 124). And in analogy with the natural bijection between ORIs and PVMs there is a natural bijection between GRIs and POVMs: a POVM $E()$ for ($\mathbb{R}$, $B(\mathbb{R})$, $\mathcal{H}$) determines an GRI given by $E_\lambda := E((-\infty, \lambda])$; and conversely, an GRI $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ determines a POVM $E()$ such that $E_\lambda = E((-\infty, \lambda])$. [Reference?]

The connection between maximally symmetric but non-selfadjoint operators and POVMs/GRIs is more subtle than the relation between selfadjoint operators and PVMs/ORIs. If A is an arbitrary symmetric operator on $\mathcal{H}$, an integral representation of the strong form (1) with $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ a generalized resolution of the identity may not be possible, and ditto for (3). However, weakened forms of (1) and (3) can hold. Extend the symmetric operator A on $\mathcal{H}$ to a selfadjoint A^+ on $\mathcal{H}^+ \supset \mathcal{H}$ (such an extension always exists but need not be unique). With $E_\lambda^{A^+}$ the ORI corresponding to A^+ and P^+ the projection such that $P^+\mathcal{H}^+ = \mathcal{H}$, define the generalized resolution of the identity $E_\lambda^A := P^+E_\lambda^{A^+}$. Then with the help of the Naimark dilation theorem it can be shown that for $\xi \in D(A)$ and $\psi \in \mathcal{H}$

$$(A\xi, \psi) = \int_{\mathbb{R}} \lambda d(E_\lambda^A \xi, \psi) \tag{1'}$$

and

$$||A\xi||^2 = \int_{\mathbb{R}} \lambda^2 d(E_\lambda^A \xi, \xi). \tag{3'}$$

If A is a symmetric operator on $\mathcal{H}$ and E_λ^A is a generalized resolution of the identity such that (1′) and (3′) hold for arbitrary $\xi \in D(A)$ and $\psi \in \mathcal{H}$ then E_λ^A is said to be a generalized spectral function of A. Akhiezer and Glazman show that this method of obtaining generalized spectral functions is completely general in that every generalized spectral function of a symmetric A has the form $E_\lambda^A := P^+E_\lambda^{A^+}$ for some selfadjoint extension A^+ of A (Akhiezer and Glazman 1993, Vol. II, Theorem 2, pp. 129-130).

In general the correspondence between generalized spectral functions and symmetric operators is loose: a symmetric operator may correspond to a set of spectral functions, and a spectral function may correspond to some set of symmetric operators. However, the connection between maximally symmetric operators and generalized spectral functions is simpler and tighter:

> *Theorem* (Akhiezer and Glazman 1993, Vol. II, Theorem 2, p. 135). A symmetric A has a unique generalized spectral function iff it is maximally symmetric. And the POVM corresponding to the unique generalized spectral function of A is a PVM iff A is selfadjoint.

Changing terminology a bit, say that a POVM $E^A()$ implements a symmetric operator A with dense domain just in case the generalized resolution of the identity E^A_λ corresponding to $E^A()$ is a generalized spectral function for A (i.e. (1′) and (3′) hold). Restating the Theorem to bring POVMs to the fore, if A is a maximally symmetric operator with dense domain then there is a unique POVM $E^A()$ that implements it; and $E^A()$ is a PVM iff A is selfadjoint. Going from a non-PVM POVM $E()$ to a maximally symmetric operator is more delicate. For there is no guarantee that the set of vectors for which the integral $\int_{\mathbb{R}} \lambda^2 d(E_\lambda \xi, \xi)$, with E_λ the generalized resolution of the identity corresponding to $E()$, converges is dense in $\mathcal{H}$; indeed, there are POVMs for which the integral does not converge for *any* non-zero vector in $\mathcal{H}$ (Akhiezer and Glazman 1993, Vol. II, p. 132). In these cases the POVM is not the implementer for any symmetric operator much less a maximally symmetric one. So the uniqueness result, such as it is, has to be stated in conditional form: if a POVM implements a maximally symmetric A then A is the only maximally symmetric operator it implements. Note that condition (3′) is crucial to these results; if it fails uniqueness in going from maximally symmetric operators to POVM implementers or vice versa is lost. More on this below.

We are now in a position to formulate a Born rule for maximally symmetric operators that parallels the Born rule (*) for selfadjoint operators:

> Let A be a maximally symmetric operator and
> let $E^A() : B(\mathbb{R}) \to \mathcal{E}(\mathcal{H})$ be the unique
> POVM implementing A. Then when the system is (**)
> in (normal) state ω the probability that a
> measurement of A yields a value lying in
> $\Delta \in B(\sigma(A))$ is $\omega(E^A(\Delta)) = Tr(\rho_\omega E^A(\Delta))$.

Using (**) takes us beyond orthodox von Neumann quantum probability on the projection lattice. The generalized probability needed to accommodate (**) is discussed below in Sections 6.3 and 6.5. Of course, (**) is an idle rule unless there are applications of QM that require us to recognize observables that are represented by maximally symmetric but non-selfadjoint operators.

5 A need for maximally symmetric but non-self adjoint operators?

5.1 *The time-of-arrival problem in QM*: event-time operators

A bank of detectors surrounds a potential well with a particle trapped inside. The particle is released, and after an elapse of time one of the detectors "clicks," signaling the arrival of the particle at the detector location. The experiment is repeated over and over with the particle prepared in identical initial states, and the arrival time statistics are recorded. QM is in big trouble if it cannot predict the arrival time distribution.

One idea for treating this and similar event-time problems within the von Neumann orthodoxy is by analogy with position. The spectral decomposition of the selfadjoint position operator Q for a particle gives a means of computing the spatial distribution of the clicking of detectors at a specified time. So an appropriate selfadjoint event-time operator T_e will, it is hoped, provide the temporal distribution of the event e of the clicking of a detector at a specified spatial location.

The difficulty is that there is no such selfadjoint event-time operator T_e that answers to seemingly reasonable demands. This no-go result is a consequence of what is known as Pauli's theorem, which shows that there cannot be a selfadjoint T_e with the following properties:

(i) $\sigma(T_e) = \mathbb{R}$.

(ii) the Hamiltonian H generating the time evolution of the system is bounded from below.

(iii) with $U(t) = \exp(-i\hbar tH)$, $-\infty < t < +\infty$, the time evolution operator generated by H, $U(-\tau)T_e(\Delta)U(\tau) = T_e(\Delta + \tau)$, for all

Borel subsets Δ of the time axis $\mathbb{R}$ and all real τ, where $\Delta + \tau = \{t : t - \tau \in \Delta\}$.

Condition (i) is needed for time of occurrence measurements performed on the entire time axis. Condition (ii) is expected to obtain for any physically realistic system. Condition (iii) is an expression of the homogeneity of time. In terms of the spectral projections $P^{T_e}(\Delta)$ of the presumed selfadjoint T_e, (iii) is equivalently expressed as: (iv) $U(-\tau)P^{T_e}(\Delta)U(\tau) = P^{T_e}(\Delta+\tau)$. Here it helps to think in terms of the Heisenberg state ψ since the time distribution of the e events does not change with time. $\psi_\tau = U(\tau)\psi$ is then the time-translate of the Heisenberg state, i.e. ψ_τ is the same as the state ψ prepared at some time t except that ψ_τ is prepared at $t + \tau$ (see Srinvas and Vijayalakshimi 1981,pp. 182-183). So the probability $(U(\tau)\psi, P^{T_e}(\Delta)U(\tau)\psi) = (\psi, U(-\tau)P^{T_e}(\Delta)U(\tau)\psi)$ of occurrence of the event e in the time interval Δ should be the same as the probability $(\psi, P^{T_e}(\Delta - \tau)\psi)$ of the occurrence of e in the interval $\Delta - \tau$. Equating the two probabilities and requiring the equality holds for all ψ yields (iv).

There are a variety of no-go results, some equivalent to Pauli's theorem while others are stronger; for an overview see (Pashby 2014, Ch. 4 and Srinvas and Vijayalakshimi 1981). Of course, these no-go results do not apply if the desired operator is maximally symmetric but not selfadjoint. An example of a maximally symmetric but non-selfadjoint candidate for a time-of-arrival operator (applicable to a freely propagating particle) is the Aharonov-Bohm (1961) operator $T_{AB} := \frac{1}{2}(\frac{m}{P}Q - Q\frac{m}{P})$ where Q and P are respectively the position and momentum operators in the Schrödinger representation.

More generally, symmetry conditions applicable to the time-of-arrival experimental set up can be used to single out a POVM and corresponding maximally symmetric T_e.[20] If T_e were selfadjoint we would be done since then the expectation values of the spectral projections of T_e supply the sought-after time distribution of the event e. But if T_e is merely maximally symmetric more work needs to be done to obtain the time distribution. How this can be accomplished is discussed in Pashby (2014, Ch. 8) and Brunetti and Fredenhagen (2002). I will not pursue the details here since the no-go result

[20] Werner (1986, 1987) constructs "screen observables," POVMs covariant under the translation group of a spacelike hyperplane representing the detector screen. Symmetry conditions applicable to the arrival of particles at points on the screen can be used to single out one such screen observable.

under discussion does not apply to experimental arrangements that satisfy the schematics of the thought experiment that opened this section. Those schematics, I claim, apply to most time-of-arrival experiments.

In that thought experiment, time of occurrence measurements are *not* performed on the entire time axis. Without loss of generality, take the time of release of the particle from the potential well trap to be $t = 0$. Then the time of occurrence measurements are performed on the *positive* real axis, and what holds is not condition (i) but (i′) $\sigma(T_e) = \mathbb{R}^+$. Correspondingly, the relevant covariance requirement is not (iii) but (iii′) demanding covariance with respect to the contraction of the group $U(t)$, $-\infty < t < +\infty$, to the semi-group with parameter $0 < t < +\infty$. It remains to be seen whether or not a no-go result for a selfadjoint T_e can be obtained from the conditions (i′), (ii), and (iii′). Even if the answer is in the affirmative there is no compelling reason to modify the von Neumann orthodoxy unless it is a given that the time-of-arrival problem must be treated by means of an event-time operator, whether selfadjoint or maximally symmetric.

5.2 Alternative treatments of the time-of-arrival problem

There are other means of treating the time-of-arrival problem that are available within the von Neumann orthodoxy and that do not try to define an event-time operator, whether selfadjoint or merely maximally symmetric. For example, the quantum current operator has been used to treat the time-of-arrival problem. In the simple one-dimensional case the probability density $\Pi(\tau)$ for time-of-arrival for a particle of mass m released at time $t = 0$ at position $x = 0$ to arrive at $x = L$ is defined to be $\Pi_{QC}(\tau) := \dfrac{\hbar}{m}\,\mathrm{Im}[\psi^*(L,\tau)\partial_x\psi^*(L,\tau)]$ where $\psi(x,\tau)$ is the solution to the Schrödinger equation for initial wave function $\psi(x,0)$ of the particle. And there are a number of competing proposals of the same ilk; for example, there is the distribution $\Pi_{SC}(\tau)$ based on the semi-classical approximation of the arrival time for a classical particle yielding $\Pi_{SC}(\tau) := \dfrac{mL}{\hbar\tau^2}|\widetilde{\psi}(\dfrac{mL}{\hbar\tau})|^2$ where $\widetilde{\psi}(k) := \dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\psi(x,0)e^{-ikx}dx$ is the Fourier transform of $\psi(x,0)$ (see see Das and Dürr (2019) and Das and Struyve (2021) for a critical discussion of these and other proposals).

What these proposals have in common with one another and with attempts to define a selfadjoint or maximally symmetric time-of-arrival operator is that they yield apparatus-independent predictions. This is strange since what the burden of the demand for empirical adequacy places on QM is that it predicts the observed time distribution of clicks of the particle detector, and this distribution will certainly not be apparatus-independent. And in principle there is a means of predicting said distribution in conventional textbook QM. A crude toy model for a detector suffices to illustrate the prospects and problems.[21]

Again simplify to the one-dimensional case. To repeat, a particle trapped in a potential well centered at $x = 0$ is released at $t = 0$. A particle detector is situated some distance $x > 0$ from the departure point $x = 0$ of the particle. The task is to derive the time-of-arrival distribution at the location of the detector. Break this task down into two steps. The first is to derive the probability distribution over $0 > t > +\infty$ for the detector to click, i.e. to go from an unexcited state to an excited state. Since we are only interested in the first passage time, we set to zero the click-time probability after the first click. This first step is straightforward in principle but messy in the details. Assume that the particle + detector forms an isolated system and that it is described in a tensor product Hilbert space $\mathcal{H}_{par} \otimes \mathcal{H}_{\det}$. Write the Hamiltonian for this system in the form $H_{tot} = H_{par} \otimes I + H_{int} + I \otimes H_{\det}$, where H_{par} generates the evolution of the particle in the absence of the detector, $H_{\det}$ generates the evolution of the detector in the absence of the particle, and H_{int} describes the particle-detector interaction. Assume that initially the particle and detector states are uncorrelated so that the initial wave function for the particle + detector system takes the form $\Psi(x, 0) = \psi_{par}(x, 0) \otimes \psi_{\det}(x, 0)$, where $\psi_{par}(x, 0)$ has no support for $x > 0$ and $\psi_{\det}(x, 0)$ is an unexcited state of the detector. With P_{exc} the projection onto the subspace $\mathcal{H}_{exc} \subset \mathcal{H}_{\det}$ of excited (click) states and $\Psi(x, \tau)$ the Schrödinger evolute of $\Psi(x, 0)$ under the Hamiltonian H_{tot}, the proposal for defining the temporal distribution over $0 < t < +\infty$ of device clicks is to set the probability $\Pr(\tau)$ for a click at time $t = \tau > 0$ as $\Pr(\tau) := (\Psi(x, \tau), I \otimes P_{exc}\Psi(x, \tau))$, unless $\tau > \tau_c$ with τ_c the least $\tau > 0$ such that $\Pr(\tau_c) = 1$, in which case $\Pr(\tau) \equiv 0$.[22]

[21] For a much more sophisticated model see Tumulka (2022).

[22] This proposal doesn't assign click-time probabilities directly to time intervals $(a, b) \subseteq (0, +\infty)$, but if desired such probabilities can be assigned by averaging $\Pr(\tau)$ over time

One problem with the proposal is that the distribution $\Pr(\tau)$ may fail to normalize, which occurs if $\mathcal{L} := \lim_{\tau \to +\infty}(\Psi(x,\tau), I \otimes P_{exc}\Psi(x,\tau)) < 1$. This can be solved formally by adding to $\Pr(\tau)$ the normalization constant $(1 - \mathcal{L})$. But such a solution may not be justified, which brings us to the second step. To plausibly equate the click-time distribution with an arrival time distribution, the device should, in some suitable sense, count as a good detector. That $\mathcal{L} < 1$ may be an indication that the detector lacks sufficient sensitivity to serve as a good detector. Additionally, a good detector should have a low spontaneous excitation probability, calculated from the Schrödinger evolute $\widetilde{\Psi}(x,\tau)$ of $\Psi(x,0)$ under the Hamiltonian with the interaction Hamiltonian H_{int} omitted from H_{tot}. Doing the messy calculations needed to resolve these and other issues about the suitability of the detector makes one yearn for a clean device-independent version of time-of-arrival. Besides, one might argue, unless the calculated time distribution of clicks of the detector for some set of reasonable initial particle wave functions $\psi_{par}(x,0)$ and initial detector states $\psi_{\text{det}}(x,0)$ approximates the consensus estimates of the various apparatus-free methods of defining time-of-arrival, the detector can be dismissed as unsuitable to its purpose. The alternative point of view holds that apparatus-free methods of defining time-of-arrival are empirically irrelevant if they do not yield results consonant with the observed click-time distribution for detectors that laboratory scientists have come to rely on in other contexts.

How the negotiation between these opposing points of view can be brought to a happy resolution is not our concern. The relevant point is that the negotiation between these two points of view can be conducted entirely within the von Neumann orthodoxy. In particular, in keeping with the von Neumann orthodoxy the proposed time-of-arrival probabilities are defined by probabilities assigned to elements of the projection lattice.

6 Effects, unsharp measurements, generalized quantum probability, povms and all that

With the qualifications and caveats mentioned above, POVMS take over the role for maximally symmetric (but non-selfadjoint) operators that PVMs play for selfadjoint operators. Thus, one might expect that the POVM enthusi-

intervals (a, b).

asts would concentrate on making the case for the need to employ maximally symmetric operators in the applications of QM. Instead the authors most associated with the new orthodoxy focus on making the case that in order to treat non-idealized "unsharp" measurements, projective measurements need to be supplemented by measurements of non-projection but nevertheless self-adjoint operators ("effects"). There is a sprawling literature conjuring with effects, and I aim only to deal with selected aspects. As an unsavory introduction let's meet a perversion—povms and pvms.

6.1 povms and pvms

Many writers, especially in quantum information and quantum computing, use the terms 'POVM' and 'PVM' not to denote POVM/GRI and PVM/ORI respectively, at least not in their original official senses detailed above; rather what we get are dumbed down versions thereof. To avoid confusion we will refer to the dumbed down versions as povms/gris and pvms/oris. A pvm/ori (the distinction between the two is usually ignored) for a Hilbert space $\mathcal{H}$ is a set $\{P_k : k \in K\}$, with a finite or countable index set K, of mutually orthogonal projections such that $\sum_{k \in K} P_k = I_{\mathcal{H}}$. A povm/gri (again the distinction is ignored) is a set $\{E_k : k \in K\}$, also with index set K finite or countably infinite, of effects $E_k \leq I_{\mathcal{H}}$ that need not be projections or pairwise commuting, but they are required to sum to the identity $\sum_{k \in K} E_k = I_{\mathcal{H}}$. This represents nothing short of a perversion; it severs the connection to a body of results that give mathematical and physical significance to POVMs/GRIs and PVMs/ORIs.

To add insult to injury, in an abuse of language, povms/gris and pvms/oris are not uncommonly referred to as observables or generalized observables. A motivation for calling povms/gris "generalized observables" might seem to flow from the claim that every povm/ori "is realizable, at least in principle, as a generalized measurement" (Barnett 2024, p 95). The reader may be surprised to learn what a "generalized measurement" is. An object system with Hilbert space $\mathcal{H}$ initially in a pure state corresponding to the vector $\psi \in \mathcal{H}$ is adjoined to an ancillary system with Hilbert space $\mathcal{H}'$ prepared in some pure state $\varphi_0 \in \mathcal{H}'$, and the composite object system + ancillary system state $\psi \otimes \varphi_0 \in \mathcal{H} \otimes \mathcal{H}'$ is subjected to a unitary transformation $\psi \otimes \varphi_0 \mapsto U(\psi \otimes \varphi_0)$ to establish an entanglement between the subsystem states. For any povm/gri $\{E_k : k \in K\}$ of $\mathcal{H}$ with $E_k \in \mathcal{E}(\mathcal{H}))$ it is possible to choose $\mathcal{H}'$, φ_0, and U such that there is a pvm/ori $\{P_k : k \in K\}$ of $\mathcal{H}'$ with $P_k \in \mathcal{P}(\mathfrak{B}(\mathcal{H}'))$ such that $(\psi, E_k\psi) = (U(\psi \otimes \varphi_0), (I \otimes P_k)U(\psi \otimes \varphi_0))$ for all $k \in K$, which is taken to show that the povm/gri can be "realized as a von Neumann measurement in our extended state space" (Barnett 2024, p. 97).[23] What is shown is that the initial ancillary system state φ_0 and the object-system-ancillary-system interaction can be arranged so that the generalized probability $(\psi, E_k\psi)$ of the object system effect E_k in the initial object system state ψ is equal to the probability $(U(\psi \otimes \varphi_0), (I \otimes P_k)U(\psi \otimes \varphi_0))$ that a measurement of the ancillary system projection P_k operator in the evolved state $U(\psi \otimes \varphi_0)$ will return a positive answer.[24] This is a purely formal result. In Section 7.3 it will be seen that there is a sense in which the Stern-Gerlach experiment can be regarded as an example of a physical realization of the formal result.

Call this a measurement if you will, but thus far there has been no measurement with an outcome, either sharp or unsharp. So what information is conveyed by the fact that such a "generalized measurement" of the povm/gri $\{E_k : k \in K\}$ has been made? Nothing that is not already conveyed by the information that as a result of the interaction the object system-ancillary system is in state $U(\psi \otimes \varphi_0)$.

One can continue the scenario by supposing that a sharp projective measurement of some $P_{k^*} \in \{P_k : k \in K\}$ is made on the ancillary system. It can be posited that a Yes result of this sharp measurement "triggers" the corresponding unsharp result $E_{k^*} \in \{E_k : k \in K\}$ for for the object system. On the von Neumann orthodoxy the implications of the projective measurement are gleaned by Lüders updating on the positive result of the P_{k^*} measurement. If the fact that the unsharp E_{k^*} result is triggered is new information then some form of the updating rule for effects discussed in Section 6.6 below should be applied, setting up a testable confrontation between the von Neumann and the POVM/povm orthodoxy. More on this below in Section 7.3.

For observables represented by selfadjoint operators there is no abuse in referring to PVMs/ORIs as observables since there is a bijection between PVMs/ORIs and selfadjoint operators. But for the class of observables rep-

[23] Barnett's discussion illustrates a weaker version of this result in which the projective measurement is made on the composite object-ancilliary system rather than the ancilliary system alone.

[24] The interpretation of the expectation value $(\psi, E_k\psi)$ as a probability is questioned in Section 6.5).

resented by symmetric operators at large, or the smaller class of observables represented by maximally symmetric operators, it is problematic to refer to POVMs/GRIs as observables. From the GRI E_λ corresponding to a POVM construct the symmetric operator A that is the first moment of E_λ (eq. (1)). But this operator may fail to have a dense domain and, indeed, may have a null domain. But supposing that it does have a dense domain, there is a further problem when the class of observables at issue are those represented by maximally symmetric operators: the A in question may fail to be maximally symmetric (conditions (3) and (3′) fail). And even if A is maximally symmetric another issue arises. If POVMs/GRIs are observables then different POVMs/GRIs are different observables, and different observables should be represented by different operators. But the same maximally symmetric A may be the first moment of many different POVMs/GRIs.

The issue already arises for selfadjoint operators, as illustrated by a simple example from Grabowski (1989). Start with a selfadjoint A with purely discrete spectrum $\{a_k\}$ and spectral resolution $A = \sum_k a_k P_k$. Averaging over the projections P_k of the spectral resolution using real semi-positive matrices $\alpha_{jk} \geq 0$, produces the effects $E_j^\alpha := \sum_k \alpha_{jk} P_k$. Provided that the α_{jk}'s all satisfy $\sum_j \alpha_{jk} = 1$ for every k, we have $\sum_j E_j^\alpha = I$ for each α. And provided that λ_j^α satisfy $\sum_j \lambda_j^\alpha \alpha_{jk} = a_k$ we have $\sum_j \lambda_j^\alpha E_j^\alpha = \sum_k a_k P_k$. Although the foregoing doesn't explicitly display the different POVMs/GRIs that all yield A as the first moment, interested readers can back-engineer them for themselves. A more physically interesting example of the one-many relation between maximally symmetric operators and POVMs/GRIs relevant to the time-of-arrival problem is discussed in Fleming (2013); here the POVMs/GRIs are explicitly displayed.

The way to avoid getting snared in this thicket is to stop talking of POVMs/GRIs as observables. And I also recommend not trying to attribute intrinsic significance to POVMs/GRIs and concentrate instead on their instrumental value. That value consists primarily in providing the feedstock for Born rule assignments of probabilities to measurement outcomes (recall the probability assignment rules (*) and (**)). That value turns negative if the feedstock is ambiguous because many different POVMs compete to provide the feedstock. Presented in this way the problem solves itself. In the case of a selfadjoint A in the Grabowski example there is no worry about which non-PVM POVM to feed into the Born rule; the answer is "None of them" since (i) there is a unique PVM/ORI corresponding to A and (ii) the feedstock it supplies to the Born rule (*) is known from many experiments to give empirically correct statistics. The non-PVM POVMs and corresponding non-ORIs exist as mathematical objects, but as far as quantum physics is concerned they have no role to play in generating probability predictions for measurements of the selfadjoint A. For maximally symmetric but non-selfadjoint operators the solution of the ambiguity problem is similar: don't let the problem arise in the first place by imposing conditions ($1'$) and ($3'$) that ensure that the POVMs/GRIs correspond one-one to maximally symmetric operators. Finally we need verify that this ansatz leads to empirically correct probability predictions. Regardless, the point is that in this context POVMs/GRIs are judged by their instrumental value, and the vast majority of them lack any such value.[25]

Returning to the above example of Grabowski, the condition that the E_j^α give a resolution of the identity, $\sum_j E_j^\alpha = I$, is secured, as noted, if $\sum_j \alpha_{jk} = 1$ for every k. This is the case if α_{jk} is a stochastic matrix describing a noisy measuring instrument, a circumstance which will receive more attention below in Section 7.2.

6.2 The effect algebra

The effect algebra $\mathcal{E}(\mathfrak{M})$ associated with a von Neumann algebra $\mathfrak{M}$ is $\{E \in \mathfrak{M}_{sa} : 0 \preceq E \preceq I\}$ where $\mathfrak{M}_{sa}$ is the set of selfadjoint elements of $\mathfrak{M}$ and, as before, $E_1 \preceq E_2$ iff $\omega((E_2 - E_1)) \geq 0$ for all for normal states ω on $\mathfrak{M}$.[26] An element $E \in \mathcal{E}(\mathfrak{M})$ is said to be a proper effect if it not a projection. Some proper effects are said to represent "unsharp" measurements, whereas projections represent sharp measurements. One might want to exclude from the class of effects that correspond to unsharp measurements those of the form $E = aP$, where P is a projection and $a > 0$. Such an effect has eigenvalues a and 0, and may be deemed to correspond to a Yes-No question. There are considerable differences of opinion about exactly which subset of

[25] A treatment of the ambiguity problem that is kinder to POVMs is given in Grabowski (1989).

[26] Equivalently $\mathcal{E}(\mathfrak{M}) = \{E \in \mathfrak{M}_{sa} : \sigma(E) = [0, 1]\}$. The effect algebra is not an algebra in the usual sense since it is not closed under scalar multiplication. But since the term effect algebra has become ingrained in the literature we will continue to follow this usage.

effects to count as "unsharp"; for a discussion of some of the options and references to the literature see Uffink (1994).

Most of the discussion within the new orthodoxy concentrates on the case where $\mathfrak{M} = \mathfrak{B}(\mathcal{H})$, the von Neumann algebra of all bounded operators acting on $\mathcal{H}$, and little attention is devoted to whether results established for this regime can be extended to more general $\mathfrak{M}$s. It is widely accepted[27] that $\mathcal{E}(\mathfrak{M})$ is a lattice under the partial order $\preceq$ relation only if $\mathfrak{M}$ is abelian (which $\mathfrak{M}$ is not, for example, when $\mathfrak{M} = \mathfrak{B}(\mathcal{H})$ and $\dim(\mathcal{H}) > 1$) and, hence $\mathfrak{M}$ describes a classical rather than a quantum system. What has actually been proven in the published literature is that if $\mathfrak{M}_{sa}$ is a lattice under the partial order $\preceq$ relation then $\mathfrak{M}$ is abelian and (Kadison and Ringrose 1991, pp. 186-187). The gap would be filled by showing that if $\mathcal{E}(\mathfrak{M})$ is a lattice under the partial order $\preceq$ relation then so is $\mathfrak{M}_{sa}$. Alexander Wilse (private communication) has filled this gap with an explicit proof. The effect algebra *is* a lattice under an alternative partial order $\preceq_s$(called the spectral order) that agrees with $\preceq$ on projections but in general is coarser than $\preceq$ (see Olson 1971 and de Groote 2005).[28] This mathematically interesting development will not be discussed here since the generalized probability theory the povmers advocate is based on the standard partial order $\preceq$.

6.3 Generalized quantum probability measures

In the presently considered version of the new orthodoxy, quantum probability theory is not the study of quantum probability measures on the projection lattice of a von Neumann algebra $\mathfrak{M}$ but rather the study of "generalized probability measures" on the effect algebra $\mathcal{E}(\mathfrak{M})$ associated with $\mathfrak{M}$. Since $\mathcal{E}(\mathfrak{M})$ does not form a lattice under the partial order $\preceq$ relation, generalized probabilities won't be defined for the meet $E_1 \wedge E_2$ and join $E_1 \vee E_2$ for arbitrary $E_1, E_2 \in \mathcal{E}(\mathfrak{M})$. This is disturbing, but it is the least of the disturbing features of generalized quantum probabilities.

A generalized probability measure for effects is a map $\mathfrak{pr} : \mathcal{E}(\mathfrak{M}) \to [0, 1]$ such that

(A1*) $\mathfrak{pr}(I) = 1$

[27] Including those advocating the need for effects in describing unsharp quantum measurements; see for example Busch et al. (1995, p. 25).

[28] The spectral order $\preceq_s$is defined as follows. For $A, B \in \mathfrak{M}_{sa}$ with corresponding spectral families $E^A = (E^A_\lambda)_{\lambda\in\mathbb{R}}$ and $E^B = (E^B_\lambda)_{\lambda\in\mathbb{R}}$, $A \preceq_s B$ iff $E^A_\lambda \preceq E^B_\lambda$ for all $\lambda \in \mathbb{R}$.

(A2*) $\mathfrak{pr}(E_1+E_2) = \mathfrak{pr}(E_1)+\mathfrak{pr}(E_2)$ for $E_1, E_2 \in \mathcal{E}(\mathfrak{M})$ such that $E_1 + E_2 \preceq I$.

The requirement of countable additivity for effects strengthens (A2*) to

(A3*) $\mathfrak{pr}(\sum_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mathfrak{pr}(E_n)$ for any family $\{E_n\} \in \mathcal{E}(\mathfrak{M})$ such that $\sum_{n=1}^{\infty} E_n \preceq I$.

6.4 A virtue of povms?

The utility of povms/oris is supposedly demonstrated by their ability to give information about the *pre*measurement state of the system not obtainable from pvms. A standard example is the problem of discriminating between pre-measurement non-orthogonal vector states, say, ξ_1 and ξ_2 (see Brun 2017). The pvm/ori $\{P_1, P_2\}$, whose elements are $P_1 = P_{[\xi_1]}$, $P_2 = I - P_{[\xi_1]}$ won't do the job. If the pre-measurement state is the vector state ξ_1 then the standard *pr* probabilities for the pvm/ori are $pr_1 := pr(P_1) = 1$ and $pr_2 := pr(P_2) = 0$; and if the pre-measurement state is ξ_2 then $pr_1 = ||P_1\xi_2|| > 0$ and $pr_2 = 1$. If we get a positive result for P_2 we know the pre-measurement state couldn't have been ξ_1 and, thus, must have been ξ_2. But if we get a positive result for P_1 we cannot tell with certainty what the pre-measurement state was. A similar verdict applies to the pvm/ori $\{P_{[\xi_2]}, I - P_{[\xi_2]}\}$.

By contrast it seems that the povm $\{E_1, E_2, E_3\}$, whose elements are proper effects $E_1 = a(I - P_{[\xi_1]})$, $E_2 = a(I - P_{[\xi_2]})$, $E_3 = I - E_1 - E_2$, with $1/2 \leq a \leq 1$ and a set small enough that $0 \preceq E_3$, does a better job.[29] If the pre-measurement state was ξ_1 the generalized $\mathfrak{pr}$-probabilities for the povm are $\mathfrak{pr}_1 := \mathfrak{pr}(E_1) = 0$, $\mathfrak{pr}_2 := \mathfrak{pr}(E_2) = a(1 - |(\xi_1, \xi_2|^2)$, $\mathfrak{pr}_3 := \mathfrak{pr}(E_3) = \mathfrak{pr}(I - E_1 - E_2) = \mathfrak{pr}(I) - \mathfrak{pr}(E_1) - \mathfrak{pr}(E_2) = 1 - \mathfrak{pr}_2 = 1 - a(1 - |(\xi_1, \xi_2|^2)$. And if the pre-measurement state was ξ_2 the $\mathfrak{pr}$-probabilities are $\mathfrak{pr}_1 := \mathfrak{pr}(E_1) = a(1 - |(\xi_1, \xi_2|^2)$, $\mathfrak{pr}_2 = \mathfrak{pr}(E_2) = 0$, $\mathfrak{pr}_3 := \mathfrak{pr}(E_3) = 1 - \mathfrak{pr}_1 = 1 - a(1 - |(\xi_1, \xi_2|^2)$. If we get a positive result for E_2 we can reason that the pre-measurement state was ξ_1 because if it had been ξ_2 the probability $\mathfrak{pr}(E_2)$ of this result is 0. Similarly, if we get a positive result for E_1 we can

[29] E_1 and E_2 may be deemed to correspond to Yes-No questions and, thus, hardly count as hard core effects representing unsharp measurements.

infer that the pre-measurement state was ξ_2 because if it had been ξ_1 the probability $\mathfrak{pr}(E_1)$ of this result is 0. Only the outcome E_3 fails to determine the pre-measurement state.

It isn't evident how the povm boosters conceive of the measurement of the povm. They sometimes seem to envision a joint measurement of all of the elements of the povms, assuming that one and only one will be the outcome. The corresponding assumption is unproblematic for a pvm/ori, for the elements of the pvm/ori commute so that joint measurability is unproblematic, and since the elements are mutually orthogonal and sum to the identity, one and only one of the elements returns a Yes answer. But these features are lacking in a povm. Alternatively, they might envision the measurement of different elements of the povm on different but identically prepared systems. This is problematic in its own way. If the object is to infer the pre-measurement state of some particular target system, it needs to be assumed that the measurement of an element of the povm on a different system yields the same result as would have been obtained if the measurement had been made on the target system, a questionable assumption since we are dealing with stochastic outcomes. But let's not quibble here since there is a more important point.

Proper effects do not have a magic discriminating power that projections lack; measurements of the projections $P^{\perp}_{[\xi_1]} = I - P_{[\xi_1]}$ and $P^{\perp}_{[\xi_2]} = I - P_{[\xi_2]}$ have the same discriminating power for pre-measurement states as measurements of E_1 and E_2. Of course, since $P^{\perp}_{[\xi_1]}$ and $P^{\perp}_{[\xi_2]}$ are not orthogonal they cannot be elements of a pvm/ori, and so it has not been shown that a pvm/ori can do as good a discrimination job as a povm. But why should one care about that? Why should one be drawn into an artificial contest between povms/gris vs. pvms/oris? There is a strong whiff here of hocus pocus. povms and pvms/oris are corruptions of POVMs/GRIs and PVMs/ORIs, and they lack both the mathematical and physical significance of POVMs/GRIs and PVMs/ORIs. Rather than providing any new physical insights their main function seems to be to provide tools for clever-clogs showmanship involving unclear goals and dubious assumptions.

6.5 Assessing generalized quantum probability

One apparent advantage of the more liberal notion of quantum probability over the effect algebra is that it overcomes a limitation of Gleason's theorem noted above. For the special case where $\mathfrak{M} = \mathfrak{B}(\mathcal{H})$ Busch (2003) shows that, even when $\dim(\mathcal{H}) = 2$, a countably additive generalized probability measure on $\mathcal{E}(\mathfrak{B}(\mathcal{H}))$ extends uniquely to a normal state on $\mathfrak{B}(\mathcal{H})$. I am unaware of a proof that this result generalizes to more types of von Neumann algebras much less to all von Neumann algebras that do not contain any Type I_2 summands.

Another apparent advantage concerns an oddity, noted above, in the von Neumann version of quantum probability theory; namely, the expression for the conditional probability $pr(P//Q) = \dfrac{\omega(QPQ)}{\omega(Q)} = \dfrac{\omega(QPQ)}{pr(Q)}$, $P, Q \in \mathcal{P}(\mathfrak{M})$, cannot be viewed as a ratio of probabilities because the numerator $\omega(QPQ)$, although real valued, is not not treated as a probability since $QPQ \notin \mathcal{P}(\mathfrak{M})$ when the projections P and Q do not commute. The defenders of the von Neumann orthodoxy may reply that when one tries to extend the classical notion of probability, which was developed for abelian algebras, into non-abelian algebras something has to give, the oddity in question being but one example. The advocates of the povm orthodoxy will reply that they can remove the oddity by assigning probabilities to proper effects, such as QPQ, as well as projections, so they can write $\mathfrak{pr}(P//Q) = \dfrac{\omega(QPQ)}{\omega(Q)} = \dfrac{\mathfrak{pr}(QPQ)}{\mathfrak{pr}(Q)}$.

So much for the advantages of the more liberal notion of quantum probability. Now some qualms. The principle of finite additivity for effects requires that $\mathfrak{pr}(P_1+P_2) = \mathfrak{pr}(P_1)+\mathfrak{pr}(P_2)$ for projections P_1 and P_2 with $P_1+P_2 \preceq I$, and that is is so even when P_1 and P_2 are not orthogonal and $P_1P_2 \neq P_2P_1$. Under the von Neumann orthodoxy a probability is not assigned to $P_1 + P_2$ since it is not a projection and $P_1 + P_2 \neq P_1 \vee P_2$. This refusal to assign a probability to $P_1 + P_2$ is backed by the orthodoxy that non-commuting observables are not jointly measurable. If the non-commuting P_1 and P_2 are not jointly measurable and $\mathfrak{pr}$ is given a frequency interpretation there is no way to verify the axiom $\mathfrak{pr}(P_1 + P_2) = \mathfrak{pr}(P_1) + \mathfrak{pr}(P_2)$. The old orthodoxy on non-joint measurability of non-commuting observables is rejected by some proponents of POVM/povm orthodoxy. A trenchant critique of this rejection is to be found in Uffink (1994).[30] I will not pursue this matter here

[30] In particular, Uffink concludes that the claims that a joint unsharp measurement of position and momentum or a pair of non-commuting spin components is possible "rest on the adoption of inappropriate definitions that trivialize the problem."

because there is a more pressing issue about the assignment of generalized probabilities to effects.

A deeper qualm is generated by asking: What are the conventional probability $pr(P)$ for a projection P and the generalized probability $\mathfrak{pr}(E)$ for a proper effect E probabilities *of*? The answers we desire are appropriate filling of the blanks in "$pr(P)$ is the probability that _ _ _ _" (respectively "$\mathfrak{pr}(E)$ is the probability that _ _ _ _") with the specification of a determinate outcome of an operational procedure. The need for a determinate outcome arises whether one is working with a frequentist or a personalist interpretation of probability. The former requires determinate outcomes in order to compute relative frequencies. The latter requires determinate outcomes in order to be able to settle the bets used to show that, on pain of "dutch book," rational credence must satisfy the axioms of probability. And the constraint on any filling is that when the system is in state ω, $pr(P)$ (respectively, $\mathfrak{pr}(E)$) is equal to $\omega(P)$ (respectively, $\omega(E)$). The constraint works both ways: if there is no suitable filling for the blank that meshes with the value $\omega(P)$ of $pr(P)$ (respectively, $\mathfrak{pr}(E)$) then the wisdom of assigning probabilities to projections (respectively, effects) is brought into question.

In the case of projections a suitable filling is obvious: $pr(P)$ is the probability that a measurement of P yields a Yes answer (eigenvalue 1). But for effects that are not projections the solution is not at all obvious. In a case where the effect E has a purely discrete spectrum the spectral resolution of E takes the form $E = \sum\limits_k e_k P_k$ where the e_k are eigenvalues of E and the P_k are mutually orthogonal projections onto the eigenspaces of the e_k. In the simplest case where the effect has the form $E = eP$, with $1 > e > 0$, $\omega(P)$—and not $\omega(E) = e\omega(P)$—is the probability that a measurement of E yields eigenvalue e. This follows directly from the Born rule (*) for selfadjoint operators: the probability that a measurement of $E = eP$ will yield a value lying in the Borel set $\Delta = \{e\}$ is $\omega(P)$. And similarly in the the more general case the probability that a measurement of $E = \sum\limits_k e_k P_k$ will yield one of the non-zero eigenvalues e_k is $\sum\limits_k \omega(P_k)$, not $\omega(E) = \sum\limits_k e_k \omega(P_k)$. (Since the P_k are mutually commuting they are jointly measurable, and since the P_k are pairwise orthogonal $\vee_k P_k = \sum\limits_k P_k$ so that $\omega(\vee_k P_k) = \omega(\sum\limits_k P_k) = \sum\limits_k \omega(P_k)$.)

In sum, from the Born rule (*) $\omega(E)$ for a proper effect E has a perfectly good meaning, not as a probability but as the expectation value of outcomes of measurements of E when the system is in state ω, i.e. $\omega(E)$ is the probability weighted average of the (non-zero) eigenvalues e_k, with the $\omega(P_k)$ supplying the probability weights. The experimentally verified correctness of the Born rule does not preclude regarding the generalized probability $\mathfrak{pr}(E) = \omega(E)$ as a probability; but to so regard it we need the specification of a determinate outcome of an operational procedure of which $\mathfrak{pr}(E)$ is the probability. The discussion in Section 6.1 provides a possible answer, albeit a not very attractive one; namely, $\mathfrak{pr}(E) = \omega(E)$ is the probability that a measurement of a suitable projection operator for an ancillary system would return a positive result if it were measured in some suitable future state unitarily evolved from the composite object system + ancillary state $\omega \otimes \phi_0$, where ϕ_0 is some suitably chosen initial state of the ancillary system.

When the Hilbert space is finite dimensional the spectrum of an effect E will be purely discrete, but when the dimension is infinite the spectrum may be purely continuous, i.e. E has no eigenvalues.[31] What then is the outcome of which $\mathfrak{pr}(E)$ is the probability of obtaining upon a measurement of E? Of course, even in the case of an E with a purely continuous spectrum there is a unique spectral resolution of the selfadjoint E, and we can ask for the probability that a measurement of some spectral projection of E will yield the eigenvalue 1. But that is a different ask, and it is an ask that is answered within the von Neumann orthodoxy.

In sum, the old and new orthodoxies can agree on the values of the probabilities of the projections in the effect algebra $\mathcal{E}(\mathfrak{B}(\mathcal{H}))$. And since by Gleason's theorem, except in the case of $\dim(\mathcal{H}) = 2$, these probabilities determine a unique normal state ω, they can agree on the expectation values $\omega(E)$ assigned to the effects $E \in \mathcal{E}(\mathfrak{B}(\mathcal{H}))$. The new orthodoxy wants to label these values probabilities rather than expectation values. The above considerations make this seem a stretch.

6.6 Updating on effects

If the outcome of an unsharp measurement can be a proper effect then Lüders rule needs to be generalized to handle updating on effects. A rule that appears in the literature is:

Effect updating rule for normal states: Let ω be a normal state

[31] A mathematical example uses the Hilbert space $L^2_{\mathbb{C}}([0,1])$ and the usual position operator Q that acts by multiplication, i.e. $Qf(x) = xf(x)$ for $f(x) \in L^2_{\mathbb{C}}([0,1])$. Q's spectrum is $[0,1]$. One might dismiss such examples as merely mathematical. But particle in a box?

on a von Neumann algebra $\mathfrak{M}$. If ω is the state of the system prior to an unsharp measurement with outcome effect $E \in \mathcal{E}(\mathfrak{M})$ where $\omega(E) \neq 0$ then the post-measurement state is $\omega_E(A) := \dfrac{\omega(E^{1/2}AE^{1/2})}{\omega(E)}$ for all $A \in \mathfrak{M}$.[32]

There are notable features of the rule for updating probabilities that derives from the state updating rule; namely

> *Effect-updating rule for probabilities*: If ω is the pre-measurement state of the system prior to an unsharp measurement with outcome effect $E \in \mathcal{E}(\mathfrak{M})$ where $\omega(E) \neq 0$ and $\mathfrak{pr}(\bullet) = \omega(\bullet)$ is the pre-measurerment probability measure on $\mathcal{E}(\mathfrak{M})$, then the post-measurement probability is $\mathfrak{pr}_E(F) = \dfrac{\omega(E^{1/2}FE^{1/2})}{\omega(E)} = \dfrac{\mathfrak{pr}(E^{1/2}FE^{1/2})}{\mathfrak{pr}(E)}$ for all $F \in \mathcal{E}(\mathfrak{M})$.

Unlike the Lüders projection-updated probability, the effect-updated probability is expressible as a ratio of probabilities assigned to effects, albeit generalized probabilities. But also unlike Lüders updating, there is a difficulty in treating effect-updating $\mathfrak{pr}_E(F)$ as producing a conditional probability, $\mathfrak{pr}(F \wr E)$, the probability of F given E. A necessary condition for such a construal is that $\mathfrak{pr}(F \wr F) = 1$ for all $F \in \mathcal{E}(\mathfrak{M})$. When F is a projection, $F^{1/2}FF^{1/2} = F^2 = F$, so automatically $pr_F(F) = pr(F)/pr(F) = 1$. When F is a proper effect $F^{1/2}FF^{1/2} = F^2$ and $\mathfrak{pr}_F(F) = \mathfrak{pr}(F^2)/\mathfrak{pr}(F)$. So $\mathfrak{pr}_F(F) = 1$ for all $F \in \mathcal{E}(\mathfrak{M})$ only iff $\mathfrak{pr}(F^2) = \mathfrak{pr}(F)$ for all $F \in \mathcal{E}(\mathfrak{M})$, which is to say that $\mathfrak{pr}$ treats effects as if they were projections (see Pashby 2014, Section 7.3.2).

So suppose that $\mathfrak{pr}$ does not treat effects as if they were projections, which is to say that there is an $F \in \mathcal{E}(\mathfrak{M})$ such that $\mathfrak{pr}(F^2) \neq \mathfrak{pr}(F)$. For such a proper effect the proposed effect-updating rule seems at war with itself. For is F is the outcome of an unsharp measurement, the proposed updating rule seems to say that updating on F makes F uncertain since $\mathfrak{pr}_F(F) < 1$ unless F was a certainty prior to measurement (i.e. $\mathfrak{pr}(F) = 1$). But we were told in no uncertain terms that F was the outcome. When and why did certainty about F as the outcome change to uncertainty?

[32] Since E is positive it has a unique square root $E^{1/2}$. For a projection P, $P^{1/2} = P$.

The effect-updating rule can be seen as a generalization of the Lüders rule since it reduces to Lüders rule for projections. But there are many possible generalizations of the Lüders rule from projections to non-projection effects, and the question remains, what justifies the choice of effect-updating rule? The attempted answer has some disturbing features. Let us a agree that an updating rule for effects should take the form of an "operation," a linear completely positive map $\Phi : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H})$ where $\mathcal{D}(\mathcal{H})$ is the convex space of density operators[33] (in their role as representatives of normal states). Expressed in terms of density operators, the effect-updating rule for probabilities given above takes the form $\Phi^E : \rho \to \dfrac{E^{1/2}\rho E^{1/2}}{tr(\rho E)}$. There are many other such maps. Instead of splitting E as $E^{1/2}E^{1/2}$, use a unitary $U : \mathcal{H} \to \mathcal{H}$ to split E as $(UE^{1/2})^\dagger UE^{1/2}$ $(= (E^{1/2})^\dagger U^\dagger UE^{1/2} = E^{1/2}E^{1/2} = E)$, and take the updating to be given by $\Phi^E_U : \rho \to \dfrac{(UE^{1/2})^\dagger \rho UE^{1/2}}{tr(\rho E)}$.[34] Different choices of U give different probability updating rules, which means that, unless additional restrictions are placed on the operation, the post-measurement state is not not specified by the outcome of the effect E.

This is a conclusion that povmers who want to link povms and updating are happy to embrace. The idea is as follows: to specify the post-measurement state of the povm/gri $\{E_i : i \in K\}$ we need to first specify a set of "measurement operators" $\{M_i\}$ where $E_i = M_i^\dagger M_i$ for all $i \in K$. If the pre-measurement state is ρ and the outcome of the povm/gri measurement is E_i then the post-measurement state is $\dfrac{M_i^\dagger \rho M_i}{tr(\rho E)}$. One option for M_i is $E_i^{1/2}$, but equally good is $UE^{1/2}$. The generalized probability $tr(\rho E_i)$ for outcome E_i is unaffected by the choice of U but the updated post-measurement state is affected.

The discussion to this juncture has implicitly conceded that, in the real world of imperfect measurements, measurement outcomes are represented in the theory by non-projection effects and that these effects represent new information that requires updating of the state and resultant probabilities.

[33]If $\rho_1, \rho_2 \in \mathcal{D}(\mathcal{H})$ and $0 < \lambda < 1$ then $\lambda\rho_1 + (1-\lambda)\rho_2 \in \mathcal{D}(\mathcal{H})$.

[34]Kraus' theorem shows that any operation Φ admits an operator sum decomposition $\Phi(\rho) = \sum_i K_i \rho K_i^\dagger$ where the Kraus operators satisfy $\sum_i K_i^\dagger K_i = I$, and conversely Kraus operators generate an operation $\Phi(\rho) = \sum_i K_i \rho K_i^\dagger$. The set of Kraus oprerators generating an operation is not unique.

The following section produces reasons to be skeptical about the necessity of this concession.

7 Fuzzy observables, noisy measurements, and unsharp measurements

7.1 Fuzzy measurements

Both the von Neumann orthodoxy and the new orthodoxy associate observables with *all* Borel subsets of a topological space X (usually $\mathbb{R}$ or $\mathbb{R}^n$). When it comes to actual laboratory measurements this association is in tension with the fact that actual measurement instruments are incapable of arbitrarily precise discriminations. A consequence of this fact is the emergence of "fuzzy observables."

The textbook treatment of position in ordinary non-relativistic QM uses the Hilbert space $\mathcal{H} = L^2_{\mathbb{C}}(\mathbb{R})$ of complex-valued square-integrable functions on $\mathbb{R}$. The position operator Q acts by multiplication on elements $\psi \in \mathcal{H}$, i.e. $(Q\psi)(x) = x\psi(x)$ for $x \in \mathbb{R}$ and $\psi \in \mathcal{H}$. If $||\psi|| = 1$ and the system is in a state corresponding to ψ the probability of finding the particle in $\Delta \subset \mathbb{R}$ is $pr_\psi(\Delta) = (P(\Delta)\psi, \psi)$, where $P()$ is the PVM for Q. But if the particle detector is incapable of reliable detection for intervals Δ such that $|\Delta| \leq \delta$ for some $\delta > 0$ the experimental statistics will not conform to $pr_\psi(\Delta)$ but rather to a $pr^\delta_\psi(\Delta)$ obtained by averaging over some appropriate distribution. Twareque Ali and Emch (1974) show that for a distribution satisfying some reasonable properties, $pr^\delta_\psi(\Delta) = (E^\delta(\Delta)\psi, \psi)$ where $E^\delta()$ is a POVM. The fuzzy position operator is given by $Q^\delta = \int_{\mathbb{R}} \lambda dE^\delta_\lambda$, where E^δ_λ is the generalized resolution of the identity corresponding to the POVM $E^\delta()$. Q^δ also acts by multiplication on $\psi \in \mathcal{H}$. If conditions $(1')$ and $(3')$ are satisfied then Q^δ is a is maximally symmetric, and if E^δ is not a PVM then Q^δ is not selfadjoint. In which case one can say that it is *as if* the maximally symmetric but non-selfadjoint Q^δ is being measured. But only *as if*.[35] What there *is*, for example, is an unreliable measurement of the spectral projection projection $P(\Delta)$ of Q when $|\Delta| \leq \delta$. The befuddled detector, which is incapable of reporting an accurate result at this level of discrimination,

[35] One giveaway here is that there should be a super- or subscript of ψ added to Q^δ, E^δ_λ, and $E^\delta()$ since they are different for different states ψ.

nevertheless outputs a number which, given the postulated distribution, averages over many trials with the same ψ to $pr_\psi^\delta(\Delta) = (E^\delta(\Delta)\psi, \psi)$ rather than to $pr_\psi(\Delta) = (P(\Delta)\psi, \psi)$. Call this a measurement of a non-selfadjoint operator if you will, but be aware that this talk adds nothing other than an unilluminating gloss on what is actually happening.

7.2 Noisy measurements

A related way in which measurements can be less than ideal is that the measurement device is "noisy." This is one of the main cases used to motivate the claim that the "von Neumann description of a measurement is insufficiently general" (Barnett 2024, p. 92). Consider the measurement of a selfadjoint observable A of the target system with purely discrete spectrum $\{a_k : k \in K\}$. Suppose that when the measurement results in a value a_k the imperfect recording device attached to the measuring instrument may register the result as some $a_j \neq a_k$. And suppose that the relation between the recorded and obtained results is modeled as a stochastic process governed by a conditional probability distribution $p(a_j/a_k) := \alpha_{jk}$, giving the probability that a_j is recorded when a_k is the actual outcome.[36] The task for the experimentalist in interpreting data from this noisy instrument is an exercise in statistical inference from the recorded instrument's results to actual results, a task that is easier if the matrix α_{jk} has an inverse, harder if it does not.

Thus far effects and POVMs/povms have not entered the picture. They can be introduced as follows. If $A = \sum_k a_k P_k$, $k \in K$, is the spectral decomposition of A and the pre-measurement state of the system is ω then the probability that the measuring instrument will record a value a_j is $\sum_k \alpha_{jk}\omega(P_k) = \omega(E_j)$ where $E_j := \sum_k \alpha_{jk} P_k$. The E_j are clearly not projections, but just as clearly they are positive bounded operators and, thus, are proper effects. And, assuming that for any input a_k the instrument always records some value or other, i.e. $\sum_j \alpha_{jk} = 1$ for all k. Then since $\sum_k P_k = 1$, $\sum_j E_j = I$ so that $\{E_j : j \in K\}$ is a proper povm.

In the case of fuzzy observables POVMs make an appearance, but it was clear that they do not play a foundational role but emerge from tweaking the von Neumann nomenclature with auxiliary assumptions to accommodate the

[36] This example is adapted from Uffink (1994), also used in Barnett (2024).

inability of laboratory instrumentation to make arbitrarily precise discriminations. Likewise in the present case POVMs/pvms make an appearance but again play no foundational role but emerge from auxiliary assumptions about the stochastic misbehavior of the recording device. One might say that the noisy measurement of the observable $A = \sum_k a_k P_k$, is equivalent to the non-noisy measurement of the POVM/povm observable $\{E_j : j \in K\}$ which returns a value a_j with probability $\omega(E_j)$ when the pre-measurement state is ω. But even leaving aside general complaints about talk of POVM/povm observables (Section 6.1), what is being added here other than a misleading description of the stochastically garbled recording of the outcome of a projective measurement as the veridical reporting of non-projective measurement?

The attempt to use this and other examples to show that the von Neumann account of measurements is insufficiently general does succeed in revealing that the account suffices only for idealized measurements, and that to take into account limitations of fallible laboratory instruments the von Neumann account needs to be supplemented with auxiliary assumptions about the instrumentation in order to account for the statistics these imperfect instruments deliver. The need for such supplementary auxiliary assumptions is not something peculiar to the von Neumann framework. It is typical for a theory of mathematical physics to use, explicitly or implicitly, idealized assumptions of the ability of laboratory instruments to realize the theory's predictions, and supplementary assumptions about the imperfect instrumentation are needed to account for actual measurement results. Moreover, while these examples of fuzzy and noisy measurements show that POVMs/povms make appearances in the supplemented account of the measurements statistics of imperfect measuring instruments, they also reveal that the role that POVMs/povms play in this account is more like that of epiphenomena than explainers.

7.3 Unsharp measurements

The new orthodoxy presents a much more serious challenge to the twin assumptions that measurements are measurements of projection operators and that quantum probabilities are assigned only to elements the projection lattice of an appropriate von Neumann algebra. That challenge is contained in the claim that proper effects are needed to accurately describe realistic measurements which are often "unsharp" and that, perforce, probabilities need to be assigned to elements of the effect algebra. The Stern-Gerlach experiment, often naively described as a measurement of spin, is presented as a prime example (see Busch et al. 1995, Sec. I.1.2).

For present purposes we can take the Hilbert space to be the tensor product $L^2_{\mathbb{C}}(\mathbb{R}^3) \otimes \mathbb{C}^2$ of a wavefunction space $L^2_{\mathbb{C}}(\mathbb{R}^3)$ and a spin space $\mathbb{C}^2$, and the algebra of observables to be $\mathfrak{B}(L^2_{\mathbb{C}}(\mathbb{R}^3)) \otimes \mathfrak{B}(\mathbb{C}^2)$. In the experimental setup silver atoms carrying spin-1/2 are initially in a vector state $\Psi_0 = \phi_0 \otimes \varphi_0$ where ϕ_0 is the initial wavepacket representing the center of mass of the atom, and $\varphi_0 = c_+\varphi_+ + c_-\varphi_-$, $|c_+|^2 + |c_-|^2 = 1$, is a superposition of the eigenstates φ_+ and φ_- respectively of z-spin σ_z with eigenvalues $+\hbar/2$ and $-\hbar/2$. The unitary evolution of the system produces a correlation of the spin degree of freedom with the position of the center of mass of the atom, resulting in the state $\Psi = c_+\phi_+ \otimes \varphi_+ + c_-\phi_- \otimes \varphi_-$. Passage through an inhomogeneous magnetic field deflects the atom upward if it has spin $+\hbar/2$ and downward if it has spin $-\hbar/2$. [Add figure.] The silver atom then either deposits on the upper half of the screen or else on the lower half, resulting in a collapse of the superposition into either an eigenstate of $P_+ \otimes I$ or of $P_- \otimes I$, where these are respectively the projection operators corresponding to the position of the silver atom on the top $(+)$ and bottom $(-)$ halves of the screen. From this one might naively conclude that deposition on the top half implies that the spin is $+\hbar/2$ while deposition on the bottom half implies spin $-\hbar/2$. But as Busch et al. (1995) note, due to the spreading of the wave packet as it passes through the Stern-Gerlach apparatus, there is a non-zero probability that a spin $+\hbar/2$ atom will be deposited on the bottom half of the screen and a non-zero probability that a spin $-\hbar/2$ atom will be deposited on the top half.

How best to describe the upshot of the experiment? The von Neumanners will agree that, due to the spreading of the wave packet, there is no sharp (projective) measurement of spin and that "If there was no definite spin value initially [$|c_\pm|^2 \neq 0$ or 1 in the initial state Ψ_0], there will not be one afterwards" (Busch et al. 1995, p. 9). But they see no need to talk of unsharp measurement of spin. Rather, they say, that there is a sharp (projective) position measurement and that the outcome of this measurement has implications for the subsequent expectation values of spin observables. The implications are obtained by first Lüders-updating the vector state ω^Ψ corresponding to the evolved Ψ just before the $P_\pm \otimes I$ projective measurement of position to obtain the post-measurement state resulting from a positive outcome for $P_\pm \otimes I$, namely $\omega^\Psi_{P_\pm\otimes I}(\bullet) = \dfrac{\omega^\Psi(P_\pm \otimes I \bullet P_\pm \otimes I)}{\omega^\Psi(P_\pm \otimes I)}$, and then using $\omega^\Psi_{P_\pm\otimes I}(\bullet)$ to calculate the expectation values of observables in $\mathfrak{B}(\mathbb{C}^2)$.

The new orthodoxy claims that there *is* a measurement of spin, albeit an unsharp one: "[T]he effects $E_\pm = (\phi_+, P_\pm\phi_+)P_{[\varphi_+]} + (\phi_-, P_\pm\phi_-)P_{[\varphi_-]}$ [where $P_{[\varphi_+]}$ (respectively, $P_{[\varphi_-]}$) is the projection onto the ray spanned by φ_+ (respectively, by φ_-)] constitute the unsharp observable actually measured in this experiment" (Busch et al. 1995, p. 8). E_+ and E_- are proper effects (i.e. $E^2_\pm \neq E_\pm$, since $(\phi_+, P_\pm\phi_+)$ and $(\phi_-, P_\pm\phi_-)$ are neither 0 nor 1 because of the spreading of the wave packet); and $E_+ + E_- = P_{[\varphi_+]} + P_{[\varphi_-]} = I_{\mathbb{C}^2}$ so that $\{E_+, E_-\}$ is a povm/gri for the spin space $\mathbb{C}^2$.

One interpretation of the claim is that $E_\pm$ "constitute the unsharp observable actually measured in [the Stern-Gerlach experiment]" is that the povm $\{E_+, E_-\}$ for spin is subject to a "generalized measurement"; in particular, the experiment provides a physical realization of the abstract mathematical result on measuring povms on an object system by utilizing an ancillary system (recall Section 6.1), wherein the wave packet of the center of mass of the silver atom is used to describe the state of the ancillary system.[37] The generalized probabilities $(\varphi_0, E_+\varphi_0) = |c_+|^2(\phi_+, P_+\phi_+) + |c_-|^2(\phi_-, P_+\phi_-)$ and $(\varphi_0, E_-\varphi_0) = |c_+|^2(\phi_+, P_-\phi_+) + |c_-|^2(\phi_-, P_-\phi_-)$ respectively of the effects E_+ and E_- in the initial spin state φ_0 are equal to the probabilities $(\Psi, (P_+ \otimes I)\Psi)$ and $(\Psi, (P_- \otimes I)\Psi)$ respectively of the position up P_+ and position down P_- projections in the evolved state Ψ.

If this is the correct interpretation of the claim then it is subject to the dilemma mentioned in Section 6.1. One can ask: What information is conveyed by the fact that a "generalized measurement" of the povm/gri $\{E_+, E_-\}$ has been made? And once again the answer is: So far, nothing that is not already conveyed by the information that as a result of the interaction between the object system and the ancillary system, the composite system is in state $\Psi = U(\psi \otimes \varphi_0) = c_+\phi_+ \otimes \varphi_+ + c_-\phi_- \otimes \varphi_-$.

But to continue the story, a sharp measurement of position up/down is

[37] The procedure here is contrary to the spirit of the construction outlined in Section 6.1. The idea was that a choice is first made of a "generalized observable" in the form of a povm on an object system, and then an ancillary system and a pvm are designed so that a suitable unitary transformation of the object + ancillary system state would produce the desired correlation between the povm and the pvm. Here, however, the povm for the object system is reverse engineered from the projective measurement to be performed on the ancillary system.

consummated by the location of the deposit on the upper/lower half of the screen. And it could be posited that a positive response for P_+ (respectively, P_-) "triggers" the unsharp result E_+ (respectively, E_-) for spin. If this is new information, and not just idle verbiage, then the state $\omega^{\Psi}_{P_{\pm\otimes I}}(\bullet)$, which is the update of Ψ on the result of the sharp position measurement, should be further updated on E_+ or on E_-, as the case may be, using an updating rule for effects. A clash of predictions between the von Neumanners and the povmers is in the offing.[38] Any bets on who will win?

8 Conclusion

The von Neumann mathematical framework for QM deserves reverence—it served to make clear that wave mechanics and matrix mechanics were not competing theories but different presentations of the same theory, and for many decades it successfully guided applications of the theory. But one can revere it while at the same time recognizing that, like everything in physics, it is subject to revision or replacement. The different strands of the new POVM/povm orthodoxy promote two distinct revisions. One proposes to expand on the von Neumann notion of observable to include observables that are represented by maximally symmetric but non-selfadjoint operators, and it claims that such an expansion is required if QM is to be empirically adequate, for example, in accounting for time-of-arrival experiments. We found no convincing argument for this claim. A weaker but nonetheless interesting claim is that maximally symmetric operators are useful in treating various quantum phenomena. We are sympathetic to this claim but remain agnostic.

The second revision retains the idea that observables are represented by selfadjoint operators but it rejects two companion assumptions of von Neumann orthodoxy: first, that quantum probabilities are assigned exclusively to elements of the projection lattice of a von Neumann algebra; and, second, that all quantum measurements are projective, i.e. measurements of individual projections or joint measurements of mutually orthogonal projections. The new orthodoxy proposes to generalize von Neumann quantum probability, construed as probability assignments to the lattice of projections, by

[38] A test of predictions would have to rely on a vesrion of the Stern-Gerlach experiment different from the one described here since this is a demolition measurement, ending with the deposit of the silver atoms on the screen.

making probability assignments to the algebra of effects and by extending the additivity axiom to non-orthogonal effects. We found no merit in this proposal; to the contrary, it involves awkwardnesses and implausibilities. It is perfectly proper to demand that the account of measurement in QM must accommodate measurements of fuzzy observables and noisy measurements, and it is also correct that proper effects, POVMs, and povms show up in such accounts. But these objects are not the starting point for analyses of measurements of fuzzy observables and noisy measurements; rather they show up as the result of applying the von Neumann apparatus plus auxiliary assumptions respectively about the inability of the measuring instrument to make arbitrarily accurate discriminations or the stochastic misbehavior of the recording device. Likewise we found no merit in the claim that "unsharp" measurements require the recognition of non-projective measurements. For example, in the alleged case of an unsharp spin measurement we rejected the claim that what is measured in an unsharp effect. The preferred description is that there is no spin measurement—sharp or unsharp; rather there is a projective measurement of another observable (position in the Stern-Gerlach case), and any legitimate post-measurement inferences about spin are to be drawn by Lüders updating on the result of the projective measurement. To claim otherwise is to flirt with empirical falsification.

To disguise my nearly superannuated status I would gladly put on the mantle of a rebel. But better to be a grumpy curmudgeon than a rebel without a good cause. I cannot find the good cause in the POVMs/povms challenge to von Neumann orthodoxy.

References

[1] Aharonov, Y., Bohm, D. (1961). "Time in the Quantum Theory and the Uncertainty Relation for Time and Energy," *Physical Review* 122: 1649-1658.

[2] Akhiezer, N. J. and Glazman, I. M. (1993). *Theory of Linear Operators in Hilbert Space* (Two volumes bound as one). New York: Dover Publications.

[3] Barnett, S. 2024. *Quantum Information.* Oxford: Oxford University Press.

[4] Beneduci, R. 2020. "Notes on Naimark's dilation theorem," *Journal of Physics: Conference Series* 1638: 012006.

[5] Blank, J., Exner,P., and Havlíček, M. 1994. *Hilbert Space Operators in Quantum Physics.* New York: AIP Press.

[6] Brun, T. A. 2017. *Introduction to Quantum Information Processing.* https://viterbi-web.usc.edu/~tbrun/Course/index.ht

[7] Brunetti, R. and Fredenhagen, K. (2002). "Time of occurrence observable in quantum mechanics," *Physical Review A* 66: 044101.

[8] Busch, P. 2003. "Quantum states and generalized observables: A simple proof of Gleason's theorem," *Physical Review Letters* 91: 120403

[9] Busch, P., Grabowski, M., and Lahti, P. K. 1995. *Operational Quantum Physics.* Springer: Berlin.

[10] Cassinelli, G. and Zanghi, N. 1983. "Conditional Probabilities in Quantum Mechanics," *Nuovo Cimento B* 73: 237-245.

[11] Das, S. and Dürr, D. (2019). "Arrival Time Distributions of Spin-1/2 Particles," *Scientific Reports* 9: 2242; https://doi.org/10.1038/s41598-018-38261-4.

[12] Das, S. and Struyve, W. (2021). "Questioning the adequacy of certain quantum arrival time distributions," arXiv: [quant-ph] 2105.14744v1.

[13] de Groote, H. F. (2005). “On a canonical lattice structure on the effect algebra of a von Neumann algebra,” arXiv: math-ph 0410018v2.

[14] Earman, J. (2024). “Why $\sqrt{-1}$? The Role of Complex Structure in Quantum Physics,” https://philsci-archive.pitt.edu/id/eprint/23612.

[15] Earman, J. and Ruetsche, L. 2020. “Quantum Mechanics As a Theory of Observables and States (And, Thereby, As a Theory of Probability),” in M. Hemmo and O. Shenker (eds), *Quantum, Probability, Logic: The Work and Influence of Itamar Pitowsky* pp. 257-283. Springer Nature Switzerland.

[16] Fleming, G. (2000). “Operational Quantum Mechanics,” *Studies in History and Philosophy of Modern Physics* 31: 117-125.

[17] __________ (2013). “Response to Pashby: Time operators and POVM observables in quantum mechanics.” Presented at the Workshop on Cosmology and Time, April 16-17, 2013. Pennsylvania State University, University Park, PA. https://philsci-archive.pitt.edu/id/eprint/9674.

[18] Grabowski, M. 1989. “What is an Observable?” *Foundations of Physics* 19: 923-930.

[19] Hamhalter, J. 2003. *Quantum Measure Theory.* Dordrecht: Springer-Science + Business Media.

[20] Hegerfeldt, G. C. and Mayato, R. S. 2012. “Discrimination between the von Neumann and Lüders reduction rule,” *Physical Review A* 85:032116. arXiv: 1111.1088v2 [quant-ph]

[21] Kadison, R. V. and Ringrose, J. R. 1991. *Fundamentals of the Theory of Operator Algebras, Vol III: Elementary Theory—An Exercise Approach.* Providence, R. I.: American Mathematical Society.

[22] Kraus, K. (1983). *States, Effects and Operations: Fundamental Notions of Quantum Theory.* Berlin: Springer-Verlag.

[23] Kumar, C. S. , Sukla, A., and Mahesh, T. S. 2016. “Discrimination between the von Neumann and Lüders measuring devices: An NMR investigation,” *Physics Letters A* 380: 3612-3616. arXiv: 1607.05723v2 [quant-ph].

[24] Olson, M. P. 1971. "The selfadjoint operators of a von Neumann algebra form a conditionally complete lattice," *Proceedings of the American Mathematical Society* 28: 537-544.

[25] Pashby, T. 2014. *Time and the Foundations of Quantum Mechanics.* PhD dissertation, University of Pittsburgh.

[26] Reed, M. and Simon, B. (1980). *Methods of Modern Mathematical Physics*, Vol. 1. New York: Academic Press.

[27] Roberts, B. 2018. "Observables, Disassembled," *Studies in History and Philosophy of Modern Physics* 150-162. https://doi.org/10.1016/j.shpsb.2018.02.002

[28] Srinivas, M. D. and Vijayalakshmi, R. (1981). "The 'time of occurrence' in quantum mechanics," *Pramana* 16: 173-199.

[29] Twareque Ali, S. and Emch, G. (1974). "Fuzzy observables in quantum mechanics," *Journal of Mathematical Physics* 15: 176-182.

[30] Tumulka, R. (2022). "Distribution of the Time at Which an Ideal Detector Clicks," arXiv:1601.03715v2 [quant-ph]

[31] Uffink, J. 1994. "The Joint Measurement Problem," *International Journal of Theoretical Physics* 33: 199-212.

[32] von Neumann, J. 1932. *Mathematische Grundlagen der Quantenmechanik.* Springer-Verlag: Berlin.

[33] ______________ 1955. *Mathematical Foundations of Quantum Mechanics.* Princeton, NJ: Princeton University Press, 1955. (English translation of von Neumann (1932).

[34] Werner, R. (1986). "Screen observables in relativistic and nonrelativistic quantum mechanics," *Journal of Mathematical Physics* 27: 793-803.

[35] __________ (1987). "Arrival time observables in quantum mechanics," *Annales de l'Institut Henri Poincaré Physique théorique* 47: 429-449.

[36] Wightman, A. S. 1962. "On the localizability of quantum mechanical systems," *Reviews of Modern Physics* 34: 845-872.