Equivalence and determinism in light of topologically-induced structure

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Abstract

Contemporary philosophy of spacetime has thus far focused predominantly on the metrical properties of spacetime, with substantially less attention paid to its topological aspects. In this article, we go some way to rectifying this situation, by introducing philosophers to a class of '(almost) asymmetric' manifolds which possess flat geodesically complete Riemannian metrics yet admit only finite (or trivial) isometry groups. The resulting spacetimes exhibit a unique dual nature: they are locally homogeneous (indistinguishable from standard flat space) but globally inhomogeneous, such that every spatial point is individuated from its neighbours. We argue that this topology-induced structure has at least two significant philosophical upshots. First, we show that there are surprising upshots for existing popular notions of theoretical equivalence when one considers theories set on these manifolds: many theories which might appear *prima facie* inequivalent in fact become categorically equivalent on such topological settings. Second, we demonstrate that theories set on these manifolds satisfy what Manchak et al. (forthcoming) dub "de re* determinism" and "rigidity", thereby avoiding (e.g.) the indeterminism which some associate with Leibnizian spacetimes.

Contents

1 Introduction

The significance of topological structures for various debates in the foundations of spacetime physics has for a long time gone neglected. The purpose of this article is make some progress

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in rectifying that situation—in particular, by introducing to philosophers '(almost) asymmetric manifolds', which are manifolds that possess flat geodesically complete Riemannian metrics yet admit only finite (or trivial) isometry groups. The resulting spacetimes exhibit a unique dual nature: they are locally homogeneous (indistinguishable from standard flat space) but globally inhomogeneous, such that every spatial point is individuated.

We argue that this topology-induced structure has at least two significant philosophical upshots. First, we show that there are surprising upshots for existing popular notions of theoretical equivalence when one considers theories set on these manifolds: many theories which are inequivalent when set on 'standard' topologies (typically $\mathbb{R}^4$) become equivalent when set on (almost) asymmetric manifolds—throughout, we work with a notion of 'categorical equivalence' as presented by Weatherall (2016a,b). Second, we demonstrate that theories set on these manifolds satisfy what Manchak et al. (forthcoming) dub "de re* determinism" and "rigidity", thereby avoiding (e.g.) the indeterminism which some associate with Leibnizian spacetimes.

The plan for the article is as follows. In §2, by way of background, we recall a familiar hierarchy of spacetime structures. In §3, we present the technical background on (almost) asymmetric manifolds, with particular focus on the 'Hantzsche–Wendt manifold' and the 'Waldmüller manifold'. In §4, we consider the upshots of such manifolds for issues of theoretical equivalence, finding that the distinctions between (theories set on) many of the previously-introduced spacetimes collapse when one sets those theories on these topological structures, and canvassing various ways in which one might try to circumvent this collapse. In §5, we consider the upshots of such manifolds for issues of determinism, finding that many theories which were previously indeterministic (on various different formal definitions of determinism from Manchak et al. (forthcoming)) become deterministic when set on these topologies. We wrap up in §6.

2 The hierarchy of spacetime structures

We begin with the standard hierarchy of spacetimes as they exist on the topologically trivial manifold, $M \cong \mathbb{R}\times\mathbb{R}^3$. We follow the 'geometric' reconstruction of classical mechanics found in e.g. Earman (1989), Friedman (1983), and Malament (2012), which formalizes classical spacetimes using four-dimensional differential geometry comparable to general relativity. There is a standard 'hierarchy' of spacetime structures formulated in this way, ordered by the size of their symmetry group, or inversely by the amount of spacetime structure presumed—see Earman (1989, ch. 2).

2.1 Leibnizian spacetime

Leibnizian spacetime is the weakest structure in the classical hierarchy. It supports absolute temporal intervals and absolute spatial distances between simultaneous events, but possesses no definition of 'rest' or 'velocity' over time, and notably, no standard of rotation. To be specific, the structure (M, t_{ab}, h^{ab}) of a Leibnizian spacetime consists of:

1. A temporal metric, t_{ab}, which is a smooth covariant tensor field of signature $(1, 0, 0, 0)$. By assumption it can be written as $t_{ab} = \mathrm{d}T_a\mathrm{d}T_b$ for some cosmic time function $T : M \to \mathbb{R}$ which defines a unique foliation of spacetime into spacelike hypersurfaces of absolute simultaneity (with constant T).

2. A spatial metric, h^{ab}, which is a smooth contravariant tensor field of signature $(0, 1, 1, 1)$. It satisfies the orthogonality condition, $h^{ab}t_{bc} = 0$. This metric defines Euclidean distances on each of the simultaneity hypersurfaces.

We shall assume that each simultaneity hypersurface has the same spatial topology, $\Sigma_t \cong \Sigma$, and that h^{ab} is flat on each of them. In this section we take $\Sigma \cong \mathbb{R}^3$ to establish the usual hierarchy, but from §3 onwards we will explore a variety of different spatial topologies, e.g., the Hantzsche–Wendt manifold.

The symmetries of this Leibnizian structure are the diffeomorphisms $\Psi : M \to M$ that preserve both t_{ab} and h^{ab}. In standard coordinates $(t, \vec{x})$, these transformations take the form

$$\Psi(t, \vec{x}) = (\pm t + c, \mathbf{R}(t)\vec{x} + \vec{a}(t)). \tag{1}$$

Here, $\mathbf{R}(t) \in \mathrm{O}(3)$ is a time-dependent rotation matrix and $\vec{a}(t)$ is a time-dependent translation vector. The (smooth) time-dependence of $\mathbf{R}(t)$ and $\vec{a}(t)$ show that this Leibnizian spacetime is structurally unable to distinguish 'spinning' or 'accelerating' frames from inertial ones. There is no standard of rotation and no preferred affine connection to define 'straight' lines across time.

2.2 Maxwell spacetime

Maxwell spacetime adds to Leibnizian spacetime a standard of rotation, formalized as $\circlearrowright$ (Weatherall 2018). Its models take the form $(M, t_{ab}, h^{ab}, \circlearrowright)$. This structure allows one to distinguish rotating frames from non-rotating frames, but it does not distinguish (non-rotationally) accelerating frames from inertial ones.[1]

The symmetries of this spacetime must now preserve the standard of rotation $\circlearrowright$ in addition to the two metrics. This restricts the rotation matrix in the allowed coordinate transformations to be a constant,

$$\Psi(t, \vec{x}) = (\pm t + c, \mathbf{R}\vec{x} + \vec{a}(t)). \tag{2}$$

Here, $\mathbf{R} \in \mathrm{O}(3)$ is a fixed rigid rotation, but $\vec{a}(t)$ remains an arbitrary smooth function of time, allowing for non-rotational accelerations to remain symmetries.

2.3 Galilean spacetime

Galilean spacetime introduces (over and above Maxwell spacetime) a standard of absolute non-rotational acceleration, and as such introduces the structure of a full affine connection, denoted ∇_{gal}, which is compatible with h^{ab} and t_{ab}. Its models are hence of the form $(M, t_{ab}, h^{ab}, \nabla_{\mathrm{gal}})$. The connection ∇_{gal} is flat ($R^a{}_{bcd} = 0$), and defines straight lines across time (inertial trajectories) but not a preferred rest frame. This structure distinguishes inertial motion from all kinds of accelerated motion.

The symmetries of this spacetime must preserve the 'straightness' of inertial worldlines. This forces the time-dependent translation $\vec{a}(t)$ to be linear in t,

$$\Psi(t, \vec{x}) = (\pm t + c, \mathbf{R}\vec{x} + \vec{v}t + \vec{d}). \tag{3}$$

[1]For the formal definition of a standard of rotation $\circlearrowright$, as well as for what it means for said object to be compatible with h^{ab} and t_{ab}, see Weatherall (2018).

This group includes rigid rotations, $\mathbf{R} \in \mathrm{O}(3)$, constant velocity boosts, $\vec{v}$, and spacetime translations, $\vec{d}$. Acceleration is now absolute (distinguishable from inertial motion), but velocity remains relative. Importantly, this structure still provides no canonical way to identify spatial points across distinct time slices; that additional structure is exactly what Newtonian spacetime supplies.

2.4 Newtonian spacetime

Newtonian spacetime posits a preferred rest frame which allows spatial points to persist through time. This is represented geometrically by a preferred unit timelike vector field, V^a, whose integral curves define the points of absolute space. The models of Newtonian spacetime are of the form (M, t_{ab}, h^{ab}, V^a). Importantly, this timelike vector field V^a fixes a unique Galilean connection—see Malament (2012, proposition 4.3.4)—such that one need not include ∇_{gal} in the models of Newtonian spacetime.

Any symmetry of this spacetime must preserve V^a. This forbids the Galilean boosts, $\vec{v}t$, as a boost would tilt the vector field V^a relative to the trajectory. As such, the coordinate representations of the symmetries of Newtonian spacetime are

$$\Psi(t, \vec{x}) = (\pm t + c, \mathbf{R}\vec{x} + \vec{d}). \tag{4}$$

The symmetry group is restricted to static rotations and translations. The spacetime possesses a preferred rest frame.

2.5 Minkowski spacetime

Whereas the previous four spacetimes were all comparable (each having sequentially more structure than the last) Minkowski space (M, η_{ab}) stands outside of that linear hierarchy (Barrett 2017). Its Lorentzian metric, η_{ab}, has signature $(-1, 1, 1, 1)$. The symmetry group of this spacetime (on $M = \mathbb{R} \times \mathbb{R}^3$) is the Poincaré group. In standard coordinates $\mathrm{x} = (t, \vec{x})$, these transformations take the form

$$\Psi(x) = \Lambda \mathrm{x} + \mathrm{d}, \tag{5}$$

where $\Lambda \in \mathrm{O}(3, 1)$ is a Lorentz transformation. This group includes Lorentz boosts which mix space and time. These are distinct from Galilean boosts which slide spatial slices relative to one another.

2.6 Categorical distinctions on $\mathbb{R}^3$

On the standard spatial topology, $\Sigma \cong \mathbb{R}^3$, these five spacetime structures are all categorically distinct. (For more on the framework of categorical equivalence with which we work in this article, see e.g. Weatherall (2016a,b). Since this framework is by now relatively standard in the literature, we will not explicate it from scratch here.) The concept of 'forgetful functors' elucidates this hierarchy. There is a forgetful functor from Newtonian to Galilean spacetime,

$$F : (M, t_{ab}, h^{ab}, V^a) \to (M, t_{ab}, h^{ab}, \nabla_{\text{gal}}). \tag{6}$$

Namely, one can use t_{ab}, h^{ab}, and V^a to construct a Galilean connection, ∇_{gal}, and then simply 'forget' which rest frame V^a one used to do this. Similarly one can forget the inertial structure encoded in ∇_{gal} to get $\circlearrowright$, and one can forget $\circlearrowright$ to get a Leibnizian spacetime.

Crucially, on $\Sigma \cong \mathbb{R}^3$, none of these functors are equivalences (they are not reversible). For instance, one cannot uniquely recover V^a from the Galilean structure because there are infinitely many inertial frames (related by boosts) that could serve as the 'rest' frame. These functors all 'lose' information that cannot be recovered. Thus, at least on $\Sigma \cong \mathbb{R}^3$, this categorical perspective aligns with our intuitions about how all of these spacetime structures are related to each other.

3 Asymmetric flat manifolds: Hantzsche–Wendt and Waldmüller

This section will introduce a series of spatial topologies, Σ, which admit a flat geodesically complete Riemannian metric, h^{ab}, yet have only a finite number of non-trivial isometries (almost asymmetric) or even none at all (asymmetric). Let us consider the flat Riemannian manifold (Σ, h), where h is the purely spatial metric h_{ab} defined in the previous section. By the classification of flat manifolds, any complete connected flat Riemannian d-manifold is isometric to a quotient of Euclidean space by some subgroup of Euclidean isometries, $\Gamma \subset \mathrm{E}(d)$, which act freely and properly discontinuously as[2]

$$\Sigma \cong \mathbb{R}^d/\Gamma. \tag{7}$$

If the resulting quotient space, Σ, is compact then the group, Γ, is called a 'Bieberbach group', i.e., a torsion-free crystallographic group.

In $d = 3$ dimensions one can recover several familiar cases including $\Gamma = \{1\}$ which gives $\Sigma \cong (\mathbb{R}^3, h)$ with isometries $\mathrm{E}(3)$, the Euclidean group. Alternatively, one might pick $\Gamma = \mathbb{Z}^3$ to be the group of translations in three orthogonal directions in which case the quotient then yields a 3-torus $\Sigma \cong \mathbb{T}^3$. If one assumes that all three of the translation generators in Γ have different lengths (i.e., L_x, L_y, L_z distinct) then the isometry group of the 3-torus consists of independent rotations/reflections in each of these directions, $\mathrm{Iso}(\mathbb{T}^3, h) \cong \mathrm{O}(2)\times\mathrm{O}(2)\times\mathrm{O}(2)$. By contrast, in the case of a cubic torus (with $L_x = L_y = L_z$) the size of the isometry group is expanded by a factor of $|S_3| = 6$, since one can now permute the three periodic axes.

Beyond these two familiar cases, however, there are several more possibilities. Indeed, there are exactly 18 distinct isomorphism classes of flat geodesically complete 3-manifolds:[3]

- *Non-compact cases (8):* There are 8 non-compact cases (4 orientable, 4 non-orientable). As Conway and Rossetti (2003, §6) discuss, these spaces are all products (or twisted fibrations) of some $\mathbb{R}^k$-factor over some compact base of dimension $3-k$, e.g., a point, a circle, a torus, or a Klein bottle. It follows from these spaces being non-compact that they each have a continuous isometry (i.e., a non-trivial Killing vector field).[4]

- *Compact cases (10):* There are 10 compact cases (6 orientable, 4 non-orientable) which Conway and Rossetti (2003) call "platycosms"—literally, flat universes. Of these only

[2] A textbook proof and discussion of this result can be found in Charlap (1986, ch. 2, §5, Corollary 5.1). Another standard reference is Wolf (1984, ch. 3.3).

[3] For the earliest works classifying the compact and non-compact cases see Hantzsche and Wendt (1935) and Nowacki (1935). Modern references for this classification are Wolf (1984, ch. 3.5) and Conway and Rossetti (2003).

[4] Concretely, because these spaces are non-compact, there exists some direction in the $\mathbb{R}^3$ covering space which is orthogonal to the lattice defined by the translation subgroup of Γ. Continuous translation in this direction commutes with the group action of Γ, resulting in a continuous isometry for $\Sigma \cong \mathbb{R}^d/\Gamma$.

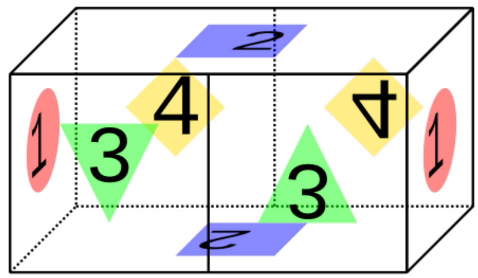

Figure 1: The glueing instructions for the Hantzsche–Wendt manifold. The symmetry group of this space includes three half-integer translations whose respective magnitudes determine the size of the space (L_x, L_y, and L_z) as well as its fundamental volume ($2L_xL_yL_z$). As each point in the space is individuated from its neighbours by its geometric place in the cosmos, neighbouring observer position might attribute a different shape to the universe's fundamental volume (Aurich and Lustig 2014). We are here showing the two-box construction from Conway and Rossetti (2003). Image from Wikipedia.

the Hantzsche–Wendt manifold (HW) has a discrete isometry group.[5] Figure 1 shows the gluing instructions to construct the HW space as a quotient of $\mathbb{R}^3$.

Before focusing on the Hantzsche–Wendt case in particular, it should be noted that all 18 of these possibilities (but especially the ten orientable cases including HW) are being actively investigated as candidate spatial topologies for our universe: see for instance Aurich and Lustig (2014). It should be stressed that all of these spatial topologies are perfectly consistent with the local physics of either Newtonian gravity or general relativity. Indeed, due to their global nature the only way to detect these topologies is to look at the universe on cosmological scales. In brief, researchers in the field of cosmic topology analyze the Cosmic Microwave Background (CMB) for 'circles-in-the-sky', i.e., matching patterns of temperature fluctuations that would indicate photons crossing the universe and returning from the other side.[6] The fact that the HW space uniquely lacks purely translational face identifications (instead relying entirely on half-turn corkscrew motions) means that it produces far fewer back-to-back matched circles on the CMB than any other cosmology. This implies that it is the hardest topology to detect by this (or really any such) method (Aurich and Lustig 2014). Thus, assuming that we continue to detect a null signal (no circles-in-the-sky), the HW space will always be the topology most compatible with our evidence; that is, of course, besides the null hypothesis $\mathbb{R}^3$.[7]

Let us now determine the size of the HW space's isometry group. Recall from our above discussion of the 3-torus that a space's isometry group, $\mathrm{Iso}(\Sigma, h)$, can depend not

[5]For any compact flat manifold (Σ, h), the identity component of its isometry group, $\mathrm{Iso}_0(\Sigma, h)$, is a torus whose dimension equals the space's first Betti number, $\beta_1(\Sigma)$ (Charlap 1986, ch. 5, §6, exercise 6.2). Conway and Rossetti (2003) tabulate the first Betti numbers of the 10 compact flat 3-manifolds in their Table 7 (column I). Note that only the Hantzsche–Wendt manifold (a.k.a. the didicosm, labeled 'c22') has $\beta_1(\Sigma) = 0$.

[6]For further philosophical discussion of testing topological properties of spacetimes, see Bielińska and Read (2026) and Read and Bielińska (2022).

[7]The science fiction author Greg Egan (2023) has recently used the HW space to dramatize the search for the spatial topology of the universe.

only upon its topology, Σ, but also upon its metric, h, through the length scales of its fundamental translations: L_x, L_y, and L_z. By contrast, the space's group of affinities $\text{Aff}(\Sigma, h)$ does not need to preserve these lengths but only the straightness of lines (the space's affine connection). Charlap (1986, ch. 5, §6, Example 6.1) computes this group for the Hantzsche–Wendt space, finding that $|\text{Aff}(\Sigma_{\text{HW}}, h)| = 96$. This group includes three half-integer translations (with magnitudes L_x, L_y, and L_z), a permutation symmetry between these translations as well as an extra reflection symmetry. In the most symmetric case (with $L_x = L_y = L_z$) all 96 of these affine equivalences are isometries. However, in the generic cases (with L_x, L_y, L_z distinct) the transformations which permute the three screw translations are no longer isometries; this reduces the size of the isometry group by a factor of $|S_3| = 6$ yielding $|\text{Iso}(\Sigma_{\text{HW}}, h)| = 16$.

The fact that the Hantzsche–Wendt space's isometry group is not trivial means that it is impossible for a flat geodesically complete Riemannian 3-manifold to be completely asymmetric. Indeed, the HW space is as good as one can do on this front. If, however, one relaxes the restriction to 3-dimensionality then one can achieve total asymmetry. Waldmüller (2003) has constructed a 141-dimensional example of a complete connected flat Riemannian manifold (denoted Σ_W) where the group of affine equivalences (hence the isometry group) is trivial: $\text{Iso}(\Sigma_W, h) = \{1\}$. To achieve this a Bieberbach group, $\Gamma \subset \text{E}(141)$, was identified with specific algebraic properties constructed from representations of one of the sporadic simple groups, M_{11}. As we shall discuss in §5.3, total asymmetry is necessary for what Manchak et al. (forthcoming) call de dicto* determinism.

These two examples establish that a flat geodesically complete Riemannian space can nonetheless be highly asymmetric. Indeed, both of these spaces exhibit a unique dual nature: they are locally homogeneous (every point looks locally like standard flat space) but globally inhomogeneous (every point is individuated from its neighbours by its geometric role in the manifold). Concretely, for a generic point in the HW case its geometric role is shared by at most 15 other points scattered across the manifold (the orbit of $\text{Iso}(\Sigma_{\text{HW}}, h)$). By contrast, in the Waldmüller case, every point is individuated *from every other point* by its geometric positioning in the manifold.

As we discuss in the following sections, these highly asymmetric spaces present profound challenges to standard accounts of spacetime equivalence and determinism. At each time-slice, each point can be individuated (at least from its immediate neighbours) by its global geometric context. This exact same identification procedure can then be performed at the next time-slice as well. Indeed, as we shall soon see, this gives us the resources to uniquely identify points in space across time. This, in turn, (a) collapses the theoretical distinction between the different elements of the hierarchy of spacetime structures introduced previously, and (b) renders more theories deterministic than one might have expected hitherto.

4 Asymmetric flat manifolds and equivalence

We now apply the HW and Waldmüller topologies to the spacetime hierarchies defined in §2. The ultimate result is the same in either case as it only depends upon the discreteness of the spatial isometry group. As such we will proceed focused on the HW case. We assume that the spacetime manifold is the product $M = \mathbb{R} \times \Sigma_{\text{HW}}$, where $\mathbb{R}$ represents time; in other words, each spatial surface is the HW manifold. The local metric structure remains flat. We investigate the global symmetries (isometries) of these spacetimes to see how its topology-induced asymmetry affects the allowable transformations, and the philosophical upshots of this for issues regarding theoretical equivalence.

In §4.1, we consider HW manifolds in the context of classical (i.e., non-relativistic) spacetime physics; in §4.2 we do the same for relativistic physics. In §4.3, we reflect on the broader ramifications of these results for philosophical discussions of categorical equivalence.

4.1 Application to classical spacetimes

Consider a Leibnizian spacetime $(\mathbb{R} \times \Sigma_{\text{HW}}, t_{ab}, h^{ab})$ where the spatial slices are now HW manifolds. Recall that the general Leibnizian symmetry is of the form

$$\Psi(t, x) = (\pm t + c, \psi_t(x)), \tag{8}$$

where for every time t, the map $\psi_t : \Sigma_{\text{HW}} \to \Sigma_{\text{HW}}$ must be a spatial isometry. The map $t \mapsto \psi_t$ represents the smooth evolution of the spatial frame over time. This map goes from the continuous timeline $\mathbb{R}$ into the isometry group of the spatial slice, $\text{Iso}(\Sigma_{\text{HW}}, h)$. As established above, this is a discrete (indeed, finite) group. A continuous map from a connected space ($\mathbb{R}$) to a discrete space ($\text{Iso}(\Sigma_{\text{HW}}, h)$) must be constant. Therefore, the spatial isometry cannot change over time. Hence,

$$\psi_t(x) = \psi(x) \quad \text{for all } t, \text{ where } \psi \in \text{Iso}(\Sigma_{\text{HW}}, h). \tag{9}$$

This is a significant restriction compared to what was previously permitted for Leibnizian spacetime with spatial leaves that are topologically $\mathbb{R}^3$. Recall that the symmetries of a Leibnizian spacetime on $\mathbb{R}^3$ were characterized by 'tumbling' frame changes: the rotation matrix, $\mathbf{R}(t)$, and the shift vector $a(t)$ could vary smoothly over time. These kinds of time-dependent transformations are no longer permitted on HW. Instead, the only symmetry transformations for a Leibnizian spacetime on HW are (a) time shifts, (b) time reflections, and (c) the few static spatial isometries allowed by HW (none in the Waldmüller case). Concretely, we have

$$\text{Iso}(\mathbb{R} \times \Sigma_{\text{HW}}, t_{ab}, h^{ab}) \cong (\mathbb{R} \rtimes \mathbb{Z}_2) \times \text{Iso}(\Sigma_{\text{HW}}, h). \tag{10}$$

This is structurally identical to the Newtonian symmetry group restricted to this topology. Indeed, as we shall now see, Leibnizian and Newtonian spacetimes are categorically equivalent when set on the HW manifold.

Notice that the only continuous symmetry acting on the spacetime under consideration here is time translation $t \to t + c$. Using this fact, we can uniquely identify the 'same point in space' across time. Concretely, the orbit of the continuous isometry group, $\text{Iso}_0(\mathbb{R} \times \Sigma_{\text{HW}}, t_{ab}, h^{ab})$, provides a canonical identification of spatial points at different times. We can then define the absolute velocity field, V^a, as the generator of this unique continuous one-parameter subgroup of isometries. This defines an absolute rest frame.

In standard Leibnizian spacetime with spatial leaves that are topologically $\mathbb{R}^3$, this definition fails because there are many competing continuous subgroups (boosts, arbitrary accelerations) that traverse time. While many rest frames are definable, none are privileged. But when the spatial leaves are HW, the topological asymmetry of the spatial slices 'locks' them together in a rigid way. The global topology itself provides the 'rigging' usually given only in non-relativistic spacetimes at least as structured as the standard Newtonian spacetime. There is, however, an important distinction. For a Newtonian spacetime on $\mathbb{R}^3$ this rigging is unobservable. By contrast, when a Leibnizian (or even Newtonian) spacetime is set on HW this rigging is *in principle observable*—i.e., by a 'circles-in-the-sky' experiment.

As such, this rigging does not present the problems of theoretical underdetermination which so perturbed Leibniz in the *Correspondence* (Alexander 1956).[8]

In any case, coming back to issues of equivalence: we have constructed a map (properly speaking, a functor) from a Leibnizian spacetime, $(M, t_{ab}, h^{ab}) \mapsto (M, t_{ab}, h^{ab}, V^a)$, to a Newtonian spacetime. And, of course we still have the forgetful map, $(M, t_{ab}, h^{ab}, V^a) \mapsto (M, t_{ab}, h^{ab})$, from Newtonian spacetime to Leibnizian spacetime. However, because V^a is uniquely definable from the shared topological structure, the 'forgetful' functor is actually an equivalence. It forgets nothing that isn't already encoded in the topology. This demonstrates the categorical equivalence of these two kinds of spacetime when set on HW. As a corollary, the Maxwellian and Galilean spacetimes on HW are also categorically equivalent to Newtonian spacetime. This is because the addition of $\circlearrowright$ or ∇_{gal} does not affect the above construction of V^a.

4.2 Application to Lorentzian spacetimes

Perhaps surprisingly, Minkowski spacetime also becomes categorically equivalent to Newtonian spacetime when set on HW (so that we are dealing with a Lorentzian manifold of the form $(\mathbb{R} \times \Sigma_{\text{HW}}, \eta_{ab})$). We shall show this by demonstrating that (as above) the only possible isometries are: time reflections, time shifts, and static spatial isometries. This means that the only continuous isometries are time shifts, allowing for a definition of a rest frame V^a, thereby recovering once again a Newtonian spacetime.

We begin by showing that in standard coordinates $\mathrm{x} = (t, x)$ the only generator of continuous symmetries is $K = \partial_t$. Letting K' be an arbitrary Killing vector field on $(\mathbb{R} \times \Sigma_{\text{HW}}, \eta)$, we can decompose this into temporal and spatial components relative to the static slicing,

$$K' = A(t, x)\partial_t + X^i(t, x)\partial_i. \tag{11}$$

The Killing equation $\mathcal{L}_{K'}\eta = 0$ implies constraints on these components. Specifically, looking at the spatial components of the Lie derivative, we require,

$$(\mathcal{L}_{K'}\eta)_{ij} = \mathcal{L}_X h_{ij} + A\partial_t h_{ij} = 0. \tag{12}$$

Since the metric is static, $\partial_t h_{ij} = 0$, and the equation reduces to

$$\mathcal{L}_X h_{ij} = 0. \tag{13}$$

This condition implies that for any fixed time t, the spatial vector field $X^i(t, \cdot)$ must be a Killing vector field of the spatial geometry (Σ_{HW}, h). However, by our above discussion, (Σ_{HW}, h) possesses only a discrete isometry group. Consequently, we must have $X^i = 0$ everywhere. It follows that any global Killing vector must be of the form $K' = A(t, x)\partial_t$. Further analysis of the mixed Killing equations yield $\partial_t A = 0$ and $\partial_i A = 0$ confirms that A must be a constant. Thus, the Lie algebra of Killing fields is one-dimensional, generated by ∂_t.

It follows that every global isometry ψ must map this unique timelike Killing field to itself (up to sign), i.e., $\psi(K) = \pm K$. Consequently, ψ must map the integral curves of K (time) to integral curves, and the hypersurfaces orthogonal to K (the spatial slices Σ_t) to hypersurfaces. This precludes any mixing of space and time coordinates (boosts). The isometry group is therefore reducible to the product of temporal symmetries (translation

[8] For more on which see e.g. Pooley (2013).

and reflections) and static spatial isometries, exactly as in our previous discussion. Hence, the Minkowski isometry group on HW is

$$\mathrm{Iso}(\mathbb{R} \times \Sigma_{\mathrm{HW}}, \eta) \cong (\mathbb{R} \rtimes \mathbb{Z}_2) \times \mathrm{Iso}(\Sigma_{\mathrm{HW}}, h). \tag{14}$$

This is isomorphic to the collapsed Leibnizian/Galilean/Newtonian group derived in the previous subsection. The distinct features of the Poincaré group (the non-compact boosts which mix space and time) are eliminated by the asymmetry of the spatial topology.

Just as in the Leibnizian case, the fact that the only continuous symmetries of this spacetime are time translations allows us to define a preferred vector field $V^a = (\partial_t)^a$ purely from the symmetry group of the metric. We have thus constructed a map $(M, \eta) \mapsto (M, \eta, V^a)$. And, of course, we always have the forgetful map $(M, \eta, V^a) \mapsto (M, \eta)$. This demonstrates a first step of categorical equivalence. Using this preferred frame, V^a, we can split the relativistic metric η_{ab} into a temporal part $t_{ab} = -V_a V_b$ and a spatial part $h_{ab} = \eta_{ab} + V_a V_b$. We have thus constructed a map $(M, \eta, V^a) \mapsto (M, t_{ab}, h^{ab}, V^a)$. And, of course, we always undo these constructions to define a forgetful map $(M, t_{ab}, h^{ab}, V^a) \mapsto (M, \eta)$. This demonstrates a second step of categorical equivalence. Therefore, a Minkowski spacetime on HW is categorically equivalent to a Newtonian spacetime on HW.

4.3 Local categorical inequivalence

We have thus arrived at a striking result: when spatial leaves are topologically HW, models for Leibnizian spacetime, Maxwellian spacetime, Galilean spacetime, Newtonian spacetime, and even Minkowski spacetime all possess isomorphic automorphism groups. If we define theoretical equivalence via categorical equivalence (equivalence of the category of global models), then these spacetimes are all equivalent on this topology. So, in this topology, Newtonian gravity and special relativity are equivalent!

How might one escape having to bite this bullet? Here is one route that one might take. If one does not understand a theory as being set on one *particular* manifold topology (say due to considerations of 'naturality' from Weatherall and March (forthcoming)), then the inclusion of these other topologies will immediately break this categorical equivalence. And indeed, if one recalls the exhortation of Earman and Norton (1987, p. 518) to focus on 'local spacetime theories' without "unnecessary global assumptions", then one will take $M \cong \mathbb{R}^4$ (since all 4-manifolds are locally homeomorphic to $\mathbb{R}^4$ by definition), in which case this categorical equivalence is also broken.

While the global structure/symmetries of these spacetimes are all the same, they are nonetheless very different locally. For the sake of illustration, suppose that spacetime structures are conceptually prior to the dynamics of matter.[9] Consider a super-powered physicist on a Newtonian HW spacetime, $(\mathbb{R} \times \Sigma_{\mathrm{HW}}, t_{ab}, h^{ab}, V^a)$, who can (magically) see all of a local region's spacetime structures; they would see the rest frame V^a. By contrast, consider this same super-powered physicist on a Galilean HW spacetime, $(\mathbb{R} \times \Sigma_{\mathrm{HW}}, t_{ab}, h^{ab}, \nabla_{\mathrm{gal}})$. Their magic local spacetime detector would reveal only the affine connection, ∇_{gal}, although (as already mentioned above) some other global 'circle-in-the-sky' experiment could reveal V^a. While these two theories have all of the same structures globally, they disagree about which of these structures are accessible locally.

The same point can be made without positing superpowers. Locally, a physicist in a Minkowski HW spacetime will observe the speed of light to be constant c and invariant;

[9] Of course, this will be anathema to the 'dynamical' approach of Brown (2005) and Brown and Pooley (2001, 2006), but we set this aside here.

from the local dynamical behaviour of matter they will conclude that the local structure of spacetime is Minkowskian. By contrast, a physicist in a Galilean HW spacetime will observe infinite signal speeds; from the local dynamical behaviour of matter they will conclude that the local structure of spacetime is Galilean. These two theories are different because their local dynamics are different.

While one might think that this is a problem for judgments that the various set on spatially HW manifolds are categorically equivalent (because, to repeat, the local physics is different), in our view a sensible resolution to this is to follow the above-mentioned line of *not* taking theories to be committed to some particular manifold structure.[10] Likewise (and again to repeat), if one instead follows the lead of Earman and Norton (1987) in just doing 'local spacetime physics', then one secures the right verdicts, because (of course) when $M \cong \mathbb{R}^4$, none of these spacetimes are categorically equivalent.

5 Asymmetric flat manifolds and determinism

So much for the implications for equivalence of these topological settings for spacetime theories. Our next order of business is to explore the upshots of these topological settings for determinism. In §5.1, we recall from Manchak et al. (forthcoming) various definitions of determinism, and present slight generalisations thereof. In §5.2, we consider the significance of our work for 'de re* determinism'; in §5.3, we do the same for 'de dicto* determinism'.

5.1 Definitions of determinism

In recent literature in the philosophy of physics, there has been an upturn of interest in formal definitions of determinism for physical theories. This work is summarised and codified by Manchak et al. (forthcoming), who offer six definitions of determinism; in this article, our interest lies with four of these.[11] Before proceeding, however: since we are interested in spacetimes which needn't be the Lorentzian manifolds of general relativity, all of these definitions must be generalised from the presentation of Manchak et al. (forthcoming).

To this end, consider a collection $\mathscr{C}$ of models of the form $(M, O_1, ..., O_n)$ where M is a differentiable manifold and the $O_1, ..., O_n$ are geometric object fields on M.[12] We say that $(M, O_1, \ldots, O_n)$, $(M', O'_1, \ldots, O'_n) \in \mathscr{C}$ are *isometric* just in case there is some diffeomorphism $d : M \mapsto M'$ such that $O'_i = d^* O_i$ for all O_i; in that case, we say that d *witnesses the isometry* of these models.[13] Following Manchak and Barrett (forthcoming), we restrict our attention to globally hyperbolic spacetimes, such that initial value problems can be posed. Moreover, we define an initial segment $U \subset M$ to be the timelike past of any Cauchy surface Σ in $(M, O_1, \ldots, O_n)$. For classical spacetimes these initial segments will always be of the form $(-\infty, t) \times \Sigma$. We can now present suitably generalised versions of the

[10] It is less obvious to us that the response of restricting to $M \cong \mathbb{R}^4$ globally is satisfactory, because the restriction to this particular manifold topology at the global level is *ad hoc*, and—even more significantly—ignores active empirical research in observational cosmology.

[11] The other two definitions (de re** determinism and de dicto** determinism) have to do with 'Heraclitus spacetimes'; i.e., spacetimes with no non-trivial local isometries. These are not our concern in this article, as we are only investigating flat spacetimes for which every pair of points are locally isometric.

[12] Here, let us limit attention to arbitrary tensor fields and derivative operators.

[13] For further discussion of the notion of isometry here, see Read and Manchak (2025). The term 'isometry' is apt here because our spacetime structures always include one or more metrics to be preserved as well as perhaps a metric compatible connection.

four definitions of determinism from Manchak et al. (forthcoming) as follows:[14]

Definition 1 (De dicto determinism)**.** *A collection $\mathscr{C}$ of globally hyperbolic spacetime models is* de dicto deterministic *iff, for any spacetimes $(M, O_1, \ldots, O_n)$, $(M', O'_1, \ldots, O'_n) \in \mathscr{C}$ and any initial segments $U \subset M$ and $U' \subset M'$, if there is an isometry $\varphi : U \to U'$, then there is an isometry $\psi : M \to M'$.*

In brief: de dicto determinism requires that the existence of an initial isometry implies the existence of a global isometry.

Definition 2 (De re determinism)**.** *A collection $\mathscr{C}$ of globally hyperbolic spacetime models is* de re deterministic *iff, for any spacetimes $(M, O_1, \ldots, O_n)$, $(M', O'_1, \ldots, O'_n) \in \mathscr{C}$ and any initial segments $U \subset M$ and $U' \subset M'$, if there is an isometry $\varphi : U \to U'$, then there is an isometry $\psi : M \to M'$ such that $\psi|_U = \varphi$.*

In brief: de re determinism requires that any initial isometry can be *extended* to a global isometry.

Definition 3 (De re* determinism)**.** *A collection $\mathscr{C}$ of globally hyperbolic spacetime models is* de re* deterministic *if, for any spacetimes $(M, O_1, \ldots, O_n)$, $(M', O'_1, \ldots, O'_n) \in \mathscr{C}$ and any initial segments $U \subset M$ and $U' \subset M'$, if there is an isometry $\varphi : U \to U'$, then there is a unique isometry $\psi : M \to M'$ such that $\psi|_U = \varphi$.*

In brief: de re* determinism requires that any initial isometry can be extended *uniquely* to a global isometry.

Definition 4 (De dicto* determinism)**.** *A collection $\mathscr{C}$ of globally hyperbolic spacetime models is* de dicto* deterministic *if, for any spacetimes $(M, O_1, \ldots, O_n)$, $(M', O'_1, \ldots, O'_n) \in \mathscr{C}$ and any initial segments $U \subset M$ and $U' \subset M'$, if there is an isometry $\varphi : U \to U'$, then there is a unique isometry $\psi : M \to M'$.*

In brief: de dicto* determinism requires that the existence of an initial isometry implies the existence of a *unique* global isometry.

Notice that the de re and de dicto versions of these definitions differ by whether or not the global isometry has to be an extension of the local isometry. Moreover, the difference between the starred versus unstarred versions of these definitions is the uniqueness clause on the latter. It should be noted that, with Manchak et al. (forthcoming), we adopt a pluralist approach to definitions of determinism, and do not take a stance on whether any one of these (or perhaps some other formal definition of determinism) should be regarded as the 'one true' definition thereof.

[14] The terminology of 'de dicto' and 're re' here is drawn from Dewar (2016, 2025), who writes that

> the two definitions are held to capture slightly different senses in which a theory might be deterministic—two senses that correspond, in fact, to the two species of possibility for which these definitions are named. Recall that de dicto possibility concerns how things could have been for the world as a whole, whereas de re possibility concerns how things could have been for the individuals within the world. Similarly, de dicto determinism asks: given the state of the world at a time, how many ways could things play out for the world as a whole? De re determinism asks: given the state of the world and the individuals within it, how many ways could things play out for the world *and for those individuals*? This makes de re determinism sensitive to variations that de dicto determinism ignores: whether the tower lands on *this* part of the disk or *that* one makes a difference to the parts of the disk (in that a tower lands on them, or doesn't); but it doesn't make a difference to the overall state of the world, which records merely that the tower landed on some part of the disk or other. (Dewar 2025, p. 15)

These definitions of determinism ought not to be confused with the following notions (see Manchak and Barrett (forthcoming)):

Definition 5 (Rigidity)**.** *A collection* $\mathscr{C}$ *of spacetime models is* rigid *if, for any spacetimes* $(M, O_1, \ldots, O_n)$, $(M', O'_1, \ldots, O'_n) \in \mathscr{C}$ *and any open subset* $V \subset M$*, if there are isometries* $\varphi_1, \varphi_2 : M \to M'$*, such that* $\varphi_1|_V = \varphi_2|_V$ *then* $\varphi_1 = \varphi_2$*.*

In brief: rigidity requires that if two global isometries agree upon an open subset then they are identical.

Definition 6 (Giraffe)**.** *A collection* $\mathscr{C}$ *of spacetime models is* giraffe *if, for any spacetimes* $(M, O_1, \ldots, O_n)$, $(M', O'_1, \ldots, O'_n) \in \mathscr{C}$ *and any isometries* $\varphi_1, \varphi_2 : M \to M'$*, then* $\varphi_1 = \varphi_2$*.*

In brief: the giraffe condition requires that if two global isometries exist then they are identical.

Notice that, unlike in the above four definitions of determinism, V does not have to be an initial segment in Definition 5. Indeed, the spacetime does not even need to be globally hyperbolic. Rigidity is important because when combined with de re determinism it implies de re* determinism. Similarly, the giraffe property is important because when combined with de dicto determinism it implies de dicto* determinism.

5.2 Topology and de re* determinism

We focus for the time being on de re* determinism and rigidity. A line in the recent philosophical literature, taking the lead from Earman (1977, p. 96), has it that non-relativistic theories set in Leibnizian spacetimes are not deterministic in this sense, because Leibnizian spacetime is not rigid (see Manchak et al. (forthcoming, p. 6) for discussion). To see this, consider the isometries of a Leibnizian spacetime to itself, noting that $\mathbf{R}(t)$ and $\vec{a}(t)$ can be any smooth functions. Fixing how one of these isometries acts in an open region (how V maps to V') will not uniquely fix the isometry globally. For Stein (1977), such symmetries do not implicate the theory in indeterminism, because they merely represent a change of reference frame (a gauge transformation) rather than a distinct physical possibility. But for Earman (1977), if one takes the mathematical structures literally and enriches the spacetime with material content (e.g., particle worldlines, γ), these transformations do threaten determinism. A non-trivial, time-dependent map φ from this Leibnizian spacetime to itself can leave the past identical while altering the future trajectories ($\varphi^*[\gamma] \neq \gamma$), thereby appearing to violate determinism; as such, past does not fix the future.[15]

One might question whether it is sensible to invoke the 'Earmanian' notion of determinism in order to maintain that theories set on Leibnizian spacetime are indeterministic. Here, however, we simply adopt this perspective; the question which we explore is how topologically-induced structure might change verdicts on determinism. To this end, note that when we spoke in §4.1 about the HW topological structure on spatial slices in Leibnizian spacetimes as locking slices together in a 'rigid' way, our choice of terminology was deliberate: in fact, this spatial topological structure *does* suffice to render Leibnizian HW spacetimes rigid. We have the following result:

Proposition 1. *Let* (M, t, h) *or* $(M, t, h, \circlearrowright)$ *be a Leibnizian or Maxwellian spacetime where* $M \cong \mathbb{R} \times \Sigma$ *with* Σ *being a connected Riemannian manifold where* $\mathrm{Iso}(\Sigma, h)$ *is discrete (e.g,. HW or Waldmüller). In either case, the collection of such spacetimes is rigid.*

[15] The distinction between 'Steinian' versus 'Earmanian' approaches to determinism was presented by Weatherall (2020, p. 83). See also the discussion by Read and Manchak (2025).

Proof. Let $\varphi_1, \varphi_2 : M \to M'$ be two global isometries which agree upon some open subset V such that $\varphi_1|_V = \varphi_2|_V$. The above results show that the only possible global isometries of such spacetimes are: time reflections, time shifts, and static spatial isometries. Hence, both maps are of the following form:

$$\Psi(t, x) = (\pm t + c, \psi(x)),$$

for some $\psi \in \mathrm{Iso}(\Sigma, h)$. Notice that any isometry will be uniquely specified once we know its sign, $\pm$, as well as its offset in time c and its constant offset in space, $\psi \in \mathrm{Iso}(\Sigma)$. We shall now prove that the local action of φ_1 and φ_2 at V are sufficient to fix these parameters.

The action on the time coordinate in V uniquely determines the sign (direction of time) and the shift c. The action on the spatial coordinates in V uniquely identifies which discrete spatial isometry ψ is acting.[16] Since c, the sign $\pm$, and the spatial isometry, ψ, are uniquely fixed by the action at V and φ_1 and φ_2 agree at V, they must be identical. □

We also have the following result regarding de re* determinism:

Proposition 2. *Let (M, t, h) or $(M, t, h, \circlearrowright)$ be a Leibnizian or Maxwellian spacetime where $M \cong \mathbb{R} \times \Sigma$ with Σ being a connected Riemannian manifold where $\mathrm{Iso}(\Sigma, h)$ is discrete. The collection of such spacetimes is de re* deterministic (and hence de re deterministic).*

Proof. Let (M, O_i) and (M', O_i') be two such spacetimes, and let $U \subset M$ and $U' \subset M'$ be initial segments. For these globally hyperbolic classical spacetimes, the initial segments take the form $U = (-\infty, t_0) \times \Sigma$ and $U' = (-\infty, t_0') \times \Sigma'$. Suppose there exists an initial isometry $\varphi : U \to U'$. Note that time-dependent continuous isometries (e.g., rotations, boosts, accelerations) are all forbidden because U contains the entire spatial slice Σ and $\mathrm{Iso}(\Sigma, h)$ is discrete. Thus, φ must take the static form

$$\varphi(t, x) = (\pm t + c, \psi(x))$$

for all $(t, x) \in U$, where $\pm$ determines the time direction, $c \in \mathbb{R}$ is a constant time shift, and $\psi : \Sigma \to \Sigma'$ is a fixed spatial isometry. We construct a global extension $\Psi : M \to M'$ by applying the exact same functional form over the entire spacetime manifold,

$$\Psi(t, x) = (\pm t + c, \psi(x))$$

for all $(t, x) \in M$. Thus, any initial isometry can be extended to a global isometry, proving the collection is de re deterministic.

As Manchak et al. (forthcoming) note, any collection of spacetime models that is both rigid and de re deterministic is de re* deterministic. To see this, note that de re determinism supplies the extension and rigidity supplies its uniqueness. □

It is important that we are here being asked to extend an *initial isometry* (over an initial segment, U) rather than a local isometry (over an open set, V). Rigidity tells us that *if* a local isometry can be extended then its global extension is unique. But not all local isometries can be extended to global isometries in this Leibnizian–HW spacetime. The local-to-global extension is only guarateed to work if the patch is already large enough to cover the spatial topology (as in an initial segment). Otherwise there is no reason for a local isometry to play by the global rulebook.

[16] Let at $p \in V$ be some point and note that our two φ_1 and φ_2 each restrict to spatial isometries on $V \cap \Sigma_p$, and moreover they agree there. In particular, they agree upon $\varphi_1(p) = \varphi_2(p)$ as well as the differentials $d\varphi_1|_p = d\varphi_2|_p$. From this it follows that φ_1 and φ_2 must agree upon all spatial geodesics passing through p. For a connected manifold, this implies that φ_1 and φ_2 must agree everywhere on Σ_p. That is, they implement the same spatial isometry, ψ.

5.3 De dicto* determinism with and without boundaries

In order to achieve de dicto* determinism, we require a spacetime with a strictly trivial isometry group (i.e., possessing neither continuous nor discrete symmetries).[17] Note that if we construct a spacetime as a simple topological product $M = \mathbb{R} \times \Sigma$ with a static metric, this will always preserve at least the continuous time-translation isometry, $t \mapsto t + c$. Thus, spatial asymmetry alone is insufficient to secure de dicto* determinism; we must also break the temporal symmetry.

The simplest way to eliminate this temporal symmetry is to introduce a boundary of some kind—e.g., an initial 'starting point' to the universe. This strategy will be pursued in §5.3.1. After this, §5.3.2 will pursue a boundary-less approach which succeeds but at the cost of introducing some causal pathologies. Finally, §5.3.3 will fix these causal pathologies by reintroducing a past boundary condition.

5.3.1 The boundary strategy: Riemannian manifolds and the giraffe

The simplest method for eliminating temporal symmetry in a product manifold is to introduce a past boundary condition—effectively, positing an initial 'starting point' to the universe. If we restrict our Riemannian spacetime constructions to the half-open interval, $M = (0, \infty) \times \Sigma$, time translation symmetry is broken, and any isometry of this spacetime is immediately reduced to a purely spatial isometry of the underlying manifold Σ.

Applying this strategy to the Riemannian examples discussed in §3 yields mixed results. For the Hantzsche–Wendt space, introducing a past boundary successfully kills all time-translation symmetries, but the spacetime inherently retains the 16 discrete spatial symmetries of the HW manifold. It therefore falls short of de dicto* determinism. In the Waldmüller case, however, the spatial isometry group is already strictly trivial. The resulting spacetime $M = (0, \infty) \times \Sigma_{\mathrm{W}}$ possesses absolutely no isometries, satisfying the criteria for de dicto* determinism perfectly. The obvious drawback of the Waldmüller construction, however, is its exorbitant dimensionality ($141 + 1$ dimensions), rendering it physically unrealistic.

To achieve de dicto* determinism in a standard $(3+1)$-dimensional setting, we might be tempted to turn to inherently Lorentzian structures with asymmetrical boundaries. Consider the so-called 'giraffe' construction presented by Manchak and Barrett (forthcoming, p. 7):

> Malament has suggested an elegant way to construct a giraffe spacetime: take Minkowski spacetime and excise a compact region 'shaped like a giraffe' (Barrett et al. 2023). The shape of a sufficiently asymmetric giraffe ensures that there are no global symmetries.

It is instructive to view this construction from both a local and a global perspective. Locally, every point in this spacetime is identical to every other point (each is seated within a flat, featureless Minkowskian neighbourhood). Globally, however, every point in this spacetime is individuated by its precise geometrical relationship to the giraffe-shaped boundary. This flat $\mathbb{R}^4$ giraffe construction achieves a minimum of global isometries alongside a maximum of local isometries.

Importantly, however, the asymmetry achieved by the giraffe construction is not a manifestation of the spacetime's intrinsic topological structure. Because the points are individuated solely by their relationship to the excised boundary, the boundary itself is only

[17] De dicto determinism follows from de re determinism. Hence, de dicto* determinism becomes equivalent to the giraffe condition.

asymmetric *relative* to the particular background metric. (One could, for instance, topologically deform the giraffe-shaped hole into a sphere and then place a new, conformally tailored, spherically symmetric metric around it). We are thus left with a critical question: can de dicto* determinism be achieved via natural topological structures, without resorting to arbitrary excised boundaries, and in low dimensions?

5.3.2 The geodesically complete strategy: Margulis spacetimes

To secure a boundary-less asymmetric spacetime, we abandon static product manifolds ($\mathbb{R} \times \Sigma$) and return to the quotient-based methods of §3. As in the Riemannian case, any geodesically complete connected flat Lorentzian $(d+1)$-manifold is isometric to a quotient of Minkowski space by some discrete subgroup of Poincaré group, $\Gamma \subset \mathrm{ISO}(d,1)$, acting freely and properly discontinuously as[18]

$$M \cong \mathbb{R}^{d,1}/\Gamma. \tag{15}$$

It is a standard result that the full isometry group of such a quotient spacetime is isomorphic to $N(\Gamma)/\Gamma$, where $N(\Gamma)$ is the normalizer of Γ within the Poincaré group (Thurston and Levy 1997, §3.4). Thus, we are seeking a subgroup $\Gamma \subset \mathrm{ISO}(d,1)$ of the kind described above which has no normalizers outside of itself. As we shall now discuss, we can find what we are looking for in the realm of 2+1D *Margulis spacetimes*.

Working to disprove a conjecture of Milnor (1977) regarding affine manifolds, Margulis (1983, 1987) demonstrated that such a quotient can yield a non-compact manifold where Γ is a non-abelian free group. Furthermore, Fried and Goldman (1983) proved that any 3-dimensional counterexample to Milnor's conjecture must be non-compact and inherently Lorentzian. These resulting spaces have become known in the physics literature as 'Margulis spacetimes'. Interestingly, these spacetimes arise naturally as the zero cosmological constant limit (the flat limit) of complete anti-de Sitter 3-manifolds (Danciger et al. 2016). For our purposes, however, they are crucial because they they have absolutely no continuous isometries (and can often be chosen to have no discrete isometries either).

Before proving this, however, let us introduce a bit of notation. Let us write elements of the Poincaré group as pairs $\gamma = (B, w)$ acting by $x \mapsto Bx + w$, with $B \in \mathrm{O}(2,1)$ the *linear part* and $w \in \mathbb{R}^{2,1}$ the *translational part*. Let $L(\gamma) = B$ extract the linear part, and let $L(\Gamma)$ denote the image of Γ under L.

Proposition 3 (Margulis spacetimes have no continuous isometries)**.** *Let $M \cong \mathbb{R}^{2,1}/\Gamma$ be a geodesically complete, connected, flat $2+1$-dimensional Lorentzian manifold (i.e., $\Gamma \subset \mathrm{Isom}(\mathbb{R}^{2,1})$ is discrete and acts freely and properly discontinuously). If M is a time-oriented Margulis spacetime (Γ is a non-abelian free group with $L(\Gamma) \in \mathrm{SO}_0(2,1)$), then M has no continuous isometries.*[19]

Proof. Recall that the isometry group of the quotient spacetime is isomorphic to $N(\Gamma)/\Gamma$, where $N(\Gamma)$ is the normalizer of Γ within the Poincaré group (Thurston and Levy 1997, §3.4). That is, any isometry must conjugate the elements of Γ onto each other. Now consider any continuous isometry (generated by some Killing vector field, K) and consider how each element $\gamma \in \Gamma$ would flow within Γ according conjugation by K. It follows from Γ

[18] See Wolf (1984, pg. 128).

[19] One can also prove the converse claim: if such an M has no continuous isometries then it must be a time-oriented Margulis spacetime. We omit the proof since it requires technical details which are not relevant to this article.

being discrete that the path of each γ must be constant; conjugation by K must map each element of Γ onto itself. That is, K must *commute* with each element of Γ.

It follows that its linear part, $L(K)$, must commute with every element of $L(\Gamma)$. By construction, Γ is a non-abelian free group and thus is not virtually solvable (Margulis 1987). If $L(\Gamma)$ were a so-called elementary group, Beardon's classification (Beardon 1983, Ch. 5) would restrict it to geometric types that are all virtually solvable in $2+1$ dimensions. Hence, $L(\Gamma)$ is a non-elementary group. It follows from this that $L(\Gamma)$ must be Zariski dense (i.e., algebraically dense) within $\mathrm{SO}_0(2,1)$, see Raghunathan (1972, Sec 2.1). Because commutator relations are polynomial, any transformation commuting with a Zariski-dense subgroup (e.g., as our $L(K)$ does) must commute with the entirety of $\mathrm{SO}_0(2,1)$. This forces $L(K) = I$.

The only remaining possibility for continuous isometries are pure translations (generated by a fixed spacetime direction, v). For this to generate an isometry, v would have to be fixed by every element of $L(\Gamma)$ such that $L(\gamma)v = v$. However, given the Zariski density of $L(\Gamma)$ and that the standard representation is irreducible, this is only possible if $v = 0$. Consequently, every Margulis spacetime has no continuous isometries. □

One can further eliminate discrete isometries from these Margulis spacetimes by ensuring that their fundamental domain is constructed asymmetrically. To see how this is possible, we must first introduce a way of visualizing these spacetimes. The fundamental domain of a Margulis spacetime has a non-convex structure, quite unlike Euclidean fundamental domains. Being a free non-abelian group, we can view Γ as being constructed from some number, $n \geq 2$, of generators as $\Gamma = \langle h_1, h_2, \ldots h_n \rangle$ where each $h_i = (B_i, v_i)$ is an element of the Poincaré group. Drumm (1992) showed that each generator $h_i \in \Gamma$ contributes a pair of 'wedges' $W_i^+,\ W_i^- \subset \mathbb{R}^{2,1}$ to the fundamental domain. Each wedge is an open region in $\mathbb{R}^{2,1}$ bounded by two null half-planes (the 'wings') meeting at a spacelike quarter-plane (the 'stem'). This 'crooked plane' construction can be seen in Figs. 1–3 of Drumm (1992). Each such pair of wedges is related by the generating element, h_i, such that ∂W_i^- is mapped to $\partial(W_i^+ + v_i)$. As is usual in affine quotient spaces, a geodesic that 'exits' the spacetime through ∂W_i^- then 're-enters' through $\partial(W_i^+ + v_i)$.

Under the 'allowable translation' conditions of Theorem 3.5 of Drumm, the $2n$ wedges are pairwise disjoint. Drumm (1992, Corollary 3.8) proves that resulting quotient manifold is homeomorphic to a solid handlebody of genus n (the number of generators in Γ). In general, one can arrange this collection of wedges in such a way that they have no further symmetry beyond the transformations $\Gamma = \langle h_1, h_2, \ldots h_n \rangle$ which generated the wedges in the first place. In this case, the resulting Margulis spacetime has no isometries (continuous or discrete).

Unfortunately, despite their topological elegance, Margulis spacetimes suffer from two fatal roadblocks that prevent them from serving as viable models for de dicto* determinism:

1. *Causal pathologies:* As demonstrated by Charette et al. (2003), Margulis spacetimes are inherently not globally hyperbolic and inevitably contain closed timelike curves (smooth, but not geodesic). Without global hyperbolicity, the initial value problem is ill-posed, arguably rendering the spacetimes conceptually problematic in discussions of determinism.

2. *The dimensional limitation:* The construction cannot be scaled straightforwardly to $(3+1)$ dimensions. Consider $\mathrm{SO}(p,q)$ with $p \geq q$. A theorem by Abels et al. (2011) states that for any $|p-q| \geq 2$ signature, $L(\Gamma)$ will not be Zariski dense in $\mathrm{SO}(p,q)$ if it

acts properly discontinuously on *the entirety of* $\mathbb{R}^{p,q}$. Recall from above that Zariski density played a key role in establishing the asymmetry of Margulis spacetimes.

Fortunately, however, both of these issues can be overcome by switching back to our earlier boundary-laden strategy, although as we shall now see, it is shaped more like a Big Bang than a giraffe.

5.3.3 The causal patch strategy: Cauchy-hyperbolic spacetimes

To bypass both the causal and dimensional pathologies of Margulis spacetimes, we synthesize the boundary approach with the affine quotient approach. Indeed, all of these bad causal properties of a Margulis spacetime can be removed by simply introducing a past boundary condition; there exists a convex, future-complete domain, $\Omega^+ \subset R^{2,1}$ invariant under Γ such that the quotient Ω^+/Γ is globally hyperbolic and Cauchy-complete (Danciger et al. 2016). The resulting quotient space Ω^+/Γ is, quite literally, the causally well-behaved future patch of a Margulis spacetime, excised from its pathological past. These kinds of constructions are the *Cauchy-hyperbolic* spacetimes considered by Barbot (2005).

The past boundary $\partial\Omega^+$ in the universal cover is a jagged, irregular surface composed of intersecting null hyperplanes. Within the quotient $M = \Omega^+/\Gamma$, Barbot defines a 'cosmological time' T, measured as the proper time elapsed from this boundary. The level sets of T form complete, spacelike Cauchy surfaces, guaranteeing that the spacetime is globally hyperbolic. As one traces time backward ($T \to 0$), the physical distance between points identified by the group action scales with T. At $T = 0$, the spatial volume of the universe crushes to zero. This represents a flat 'Big Bang'—a crushing initial singularity arising entirely from the global topology pinching off, with no local Riemann curvature whatsoever. This topological Big Bang cleanly and naturally breaks time-translation symmetry.

Importantly, because we are no longer quotienting the entirety of Minkowski space, the non-elementary discrete subgroup Γ which we are quotienting by does not need to act properly discontinuously on all of Minkowski space but only $\Omega^+ \subset R^{2,1}$. This is what allows us to circumvent the above-discussed dimensional limitation (Abels et al. 2011) and extend these spacetimes to $3+1$D and beyond. Concretely, because the group is no longer required to act properly discontinuously on the *entire* Minkowski space $\mathbb{R}^{n,1}$, it becomes possible for the linear part of the group, $L(\Gamma)$ to be Zariski dense in $\mathrm{SO}_0(n,1)$. Given this key property, one can prove that these spacetimes lack continuous isometries by the exact same methods as used above.

Proposition 4 (No continuous isometries in Cauchy-hyperbolic spacetimes)**.** *Let* $M = \Omega^+/\Gamma$ *be a Cauchy-hyperbolic spacetime, where* Γ *is a discrete Poincaré group acting freely and properly disconintuously on* Ω^+ *with* $L(\Gamma)$ *being a non-elementary subgroup of* $\mathrm{SO}_0(n,1)$ *Then* M *admits no continuous isometries.*

Proof. The full isometry group of M is isomorphic to $N(\Gamma)/\Gamma$, where $N(\Gamma)$ is the normalizer of Γ in the Poincaré group. Because Γ is discrete, any continuous isometry (generated by some Killing vector field, K) must commute with every element of Γ. From this it follows that $L(K)$ must commute with every element of $L(\Gamma)$. Because $L(\Gamma)$ is non-elementary it is Zariski dense in $\mathrm{SO}_0(1,n)$. Because commutator relations are polynomial, any transformation commuting with a Zariski-dense subgroup (e.g., as our $L(K)$ does) must commute with the entirety of $\mathrm{SO}_0(n,1)$. This forces $L(K) = I$.

The only remaining possibility for continuous isometries are pure translations (generated by a fixed spacetime direction, v). For this to generate an isometry, v would have to be fixed by every element of $L(\Gamma)$ such that $L(\gamma)v = v$. However, given the Zariski density of $L(\Gamma)$ and that the standard representation is irreducible, this is only possible if $v = 0$. Consequently, every such spacetime has no continuous isometries. □

Just as with the crooked planes in $(2+1)$ dimensions, one can arrange the fundamental domain of these spacetimes to ensure that Ω^+ possesses no discrete isometries either. Such a spacetime would have no isometries whatsoever (continuous or discrete).

In sum, by trading past geodesic completeness for global hyperbolicity we can excise the pathological past with a topological Big Bang thereby producing a flat, $(3+1)$-dimensional spacetime with a well-posed initial value problem and a strictly trivial isometry group. This finally achieves a rigorous, physically plausible, purely topological realization of *de dicto** determinism.

6 Conclusion

In this article, we have considered the philosophical upshots of (almost) asymmetric manifolds for two longstanding and central foundational issues in the philosophy of spacetime—namely, (i) issues of the equivalence of spacetime theories, and (ii) issues of determinism. *Ad* (i), we have found that setting theories on such manifolds collapses differences between them, though there are various ways in which that inequivalence can be restored. *Ad* (ii), we have seen that setting theories on these manifolds renders them de re* deterministic when they otherwise would not have been; we have also seen that there is a viable topological implementation of de dicto* determinism via Margulis spacetimes.

This work on spacetime topology is broadly 'metaphysical' in nature; as such, it is a complement to recent 'epistemological' work on spacetime topology due to e.g. Bielińska and Read (2026) and Read and Bielińska (2022). But taken together, we hope that all this work whets philosophers' appetites for topological fare more exotic and adventurous than the familiar staple of $M \cong \mathbb{R} \times \mathbb{R}^3$.

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