Quantum Chaos, Measurement, and the Many Faces of Correspondence

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Abstract

Classical chaos is frequently claimed to pose a problem for quantum-classical correspondence. Recent work on quantum decoherence purportedly solves this problem. This essay attempts to rationally reconstruct these claims. When comparing quantum and classical distributions, it is argued that classical chaos poses no problem for quantum mechanics, even in the absence of quantum decoherence. By restricting our attention to relevant physical details — actions, timescales, and the limited measurement resolution of classical observables — we find that quantum distributions do not appreciably diverge from their classical, chaotic counterparts over relevant timescales, even without decoherence. This point has been obscured in the literature by an inattention to realistic physical parameters and measurement capacities. The remaining problem posed by chaos for quantum-classical correspondence is that chaos is a large natural channel for generating macroscopic quantum superpositions. Thus, the problem of quantum chaos is deeply tied up with the quantum measurement problem. The way in which this problem shows up, and the way in which decoherence "solves" it, is idiosyncratic to different interpretations of quantum mechanics. I illustrate this point using the Everett and Bohmian interpretations.

1 Introduction

Classical chaos is a phenomenon that can be readily observed. If we watch the chaotic trajectory of a double pendulum, we will see a system with a seemingly well defined state at every moment, but whose state moments from now will be effectively impossible to predict. The classical explanation of this is that given a small uncertainty in the "true" initial state of the pendulum, the dynamics of chaos stretches this uncertainty over a large range of possible future states, stifling long term prediction. The underlying trajectory remains sharp and well defined over time, it is our uncertainty of it that is being continuously pruned away by observations.

Despite its manifest appearance in observations, there is no widely accepted quantum explanation of this phenomenon. Given that only classical mechanics seems hospitable for chaos, many researchers have questioned whether chaos poses a problem for "the Correspondence Principle": the idea that quantum mechanics must reproduce classical mechanics within the macroscopic domain that classical physics remains accurate. While there are many ways of comparing the two theories, here are three where chaos is believed to present problems:[1]

(I) *von Neumann Correspondence:* correspondence between the behavior of classical states in phase space and quantum states in Hilbert space. Nearby classical states diverge exponentially in phase space under chaotic evolution, while the distance between quantum states in Hilbert space is constant under unitary Schrödinger evolution (Peres, 1984; Berry, 1989; Zurek and Paz, 1994; Penrose, 2011).

(II) *Ehrenfest Correspondence:* correspondence between a classical trajectory in phase space and an initially localized quantum wave-packet. The wave-packet behaves like the classical trajectory until the time where it has become highly delocalized (i.e. the Ehrenfest time). In chaotic systems, minimum uncertainty wave-packets become macroscopically delocalized on observationally relevant timescales, at which point we continue to observe seemingly definite classical trajectories rather than delocalized macroscopic quantum superpositions (Zurek, 1998; Berry, 2001).

(III) *Liouville Correspondence:* correspondence between quantum and classical distributions. Quantum distributions diverge from their classically chaotic counterparts in two respects.

a. *Quantum Suppression of Classical Chaos*: Under classical mixing, distributions develop arbitrarily fine grained phase space structure under time evolution, but quantum interference blocks these developments once structures appear on scales of the order of Planck's constant $\hbar$ (Berry et al., 1979; Korsch and Berry, 1981). This leads to a "clash of limits" because chaos is defined in the $t \to \infty$ limit, while quantum-classical correspondence is obtained by taking the $\hbar \to 0$ limit. In chaotic systems, these limits do not commute; chaos can only be recovered by taking the $\hbar \to 0$ limit first (Berry, 1995, 2001; O'Connor et al., 1992; Batterman, 1995, 2001; Bokulich, 2008).

[1] The terms "Ehrenfest" and "Liouville" correspondence are Emerson and Ballentine's (2001a; 2001b).

b. *Quantum Periodicity*: Quantum distributions are almost-periodic — meaning that expectation values of observables will exhibit recurrence behavior where they return arbitrarily close to their initial value — while classical chaos/mixing drives distributions asymptotically closer to the microcanonical distribution without the possibility of recurrence (Ford et al., 1991; Belot and Earman, 1997; Belot, 2000; Bishop, 2024).

Although conceptually distinct, these problems tend to be blurred together when discussing "the problem of quantum chaos" for the Correspondence Principle.

There is a growing consensus that quantum decoherence is essential for restoring correspondence for chaotic systems. The key insight is that macroscopic chaotic systems are not isolated, but are constantly having interference suppressed by environmental interactions. Decoherence addresses (I) because open quantum systems have non-unitary dynamics (Rosaler, 2016). The problem of (II) is answered by environmental monitoring choosing decohered quantum wave-packets as its preferred basis (Zurek, 2007; Wallace, 2012). The problem of (IIIa) is resolved because decoherence destroys fine structure in phase space, leading to agreement between classical and quantum distributions (Habib et al., 1998; Zurek, 2003; Franklin, 2024).

In this essay, I attempt to reconstruct this recent history, untangling various confusions along the way. I argue that the problems of (I), (IIIa), and (IIIb) are unfounded. The problem of (I) arises due to a mismatching of phase space and Hilbert space structure. Problems related to (IIIa) and (IIIb) are circumvented by focusing on the empirical grounds of classical chaos rather than purely formal correspondence. For realistic physical systems and measurement capabilities, the problem of (IIIa) is so small as to be undetectable, and the problem of (IIIb) only appears at astronomically large timescales. Drawing upon ideas from quantum statistical mechanics, I offer a reduction of classical mixing for few-body, isolated quantum systems. Thus, although these systems are undoubtedly undergoing rapid environmental decoherence, decoherence does not resolve these problems because they were never problems to begin with.

I argue that (II) is the only surviving problem posed by chaos to correspondence: isolated quantum chaotic systems can quickly go from small localized wave-packets to coherent macroscopic superpositions. Thus, quantum chaos leads to an inevitable clash between delocalized quantum states and the localized classical ones we observe. This is the quantum measurement problem. Here, decoherence has an important role to play, but that role is conceptually idiosyncratic to different interpretations of quantum mechanics. I demonstrate this using the Everett and Bohmian interpretations of quantum mechanics.

The paper begins in §2 with the correspondence principle. General descriptions of the correspondence principle fail to nail down what the correspondents are: what are the quantum and classical objects that are supposed to correspond to each other? In

§3 – 5, I describe three candidates and how correspondence fares for chaotic systems. In §3, I compare classical trajectories in phase space to quantum vectors in Hilbert space, and explain why the unitarity of quantum evolution does not bear on the question of whether quantum systems can be chaotic. I then compare classical states to quantum wave-packets in §4, where we find that quantum wave-packets cease to behave like classical trajectories on uncomfortably short timescales. Quantum and classical distributions are compared in §5. In §5.1, I introduce Weyl quantization as a tool for comparing distributions. I describe "the quantum suppression of classical chaos" in §5.2 and quantum periodicity in §5.3, arguing that neither poses a problem at the level of distributions. In §5.4, I propose a quantum reduction of mixing. In §6, I provide a minimal version of the correspondence principle at the level of distributions that survives for chaotic systems in the absence of decoherence, before arguing that it is insufficient to explain classical appearances. In §7, I discuss the role of decoherence. §7.1 is an overview of environmental decoherence. §7.2 and §7.3 describe how chaos appears in the Everett and Bohmian interpretations, respectively, aided by decoherence. In §7.4, I show how each interpretation of quantum mechanics faces an idiosyncratic problem posed by classical chaos, and how decoherence restores correspondence. I conclude in §8.

2 Formal vs. Empirical Correspondence Principles

A primary reason that the problem of quantum chaos is ill-defined is that the Correspondence Principle itself is ill-defined. Although its originator, Niels Bohr, intended a restricted use of the idea, its use in the literature has long since flown the coop of careful Bohr exegesis.[2] We will focus on two general uses that have become pervasive in the literature.

To delineate these uses, Rosaler (2015a) gives a helpful distinction between "formal" and "empirical" approaches to quantum-classical reduction:[3]

1. **Formal Reduction** is a two place relation between abstract theories. Classical mechanics (CM) is said to reduce to quantum mechanics (QM) when the former is shown to be a special or limiting case of the latter. This is an *a priori* investigation that compares the abstract mathematical structure of the theories.

2. **Empirical Reduction** is a three place relation between two theories and a shared domain of application (i.e. real physical systems and their observed behavior). CM is said to reduce to QM when quantum mechanics is able to recover the empirical successes of classical mechanics.

[2]For a discussion of how Bohr's original ideas relate to quantum chaos, see Batterman (1991).

[3]Here, we are using the philosopher's sense of reduction where a less fundamental theory reduces to a more fundamental theory (Nickles, 1973).

Both approaches are methodologically valuable, and discussions of the quantum-classical relation often integrate them. However, in distinguishing them, we will be able to trace much of the confusion surrounding "the problem of quantum chaos" back to formal investigations of reduction becoming detached from their empirical grounds.

Related to formal reduction, call the *Formal Correspondence Principle* the idea that the theoretical structure of QM *must* contain CM as a limiting case as $\hbar \to 0$. Since $\hbar$ is a physical constant, complete with dimensions, the interpretation of this limit is not transparent. Most commonly, the limit is interpreted to mean that $\hbar$ is a dimensionless parameter (such as $\hbar$ divided by the classical action) that goes to zero as the system increases in size (Landsman, 2007, 471).[4] I will adopt this common interpretation — which I denote as $\hbar_{\text{eff}} \to 0$ — because it is well suited to track how various dynamical quantities scale with system size. I will focus on few-body systems in the $\hbar_{\text{eff}} \to 0$ limit — as opposed to the many-body systems of statistical mechanics in the $N \to \infty$ limit — because these are the systems where problems related to correspondence are most acute.

On the other hand, the *Empirical Correspondence Principle* keeps close track of the empirical successes of CM, and only requires that QM reproduces these predictions to within the same error tolerances (Ford and Mantica, 1992; Belot and Earman, 1997). To understand this sense of correspondence, it is important to consider the fixed actions of relevant physical systems, the timescales over which real observations take place, and the level of accuracy that measurements are capable of producing. Questions over agreement between theories at unrealistic actions, timescales, or measurement resolutions are irrelevant for whether quantum mechanics can explain the empirical successes of classical physics.

A crucial feature that these general approaches to the correspondence principle fail to nail down is what the classical and quantum relata are. There is no universally agreed upon template for how to relate the mathematical structure and empirical predictions of CM and QM. In the next three sections, I will describe three proposals for the classical and quantum relation — classical states to quantum states (von Neumann Correspondence), classical states to quantum wave-packets (Ehrenfest Correspondence), and classical distributions to quantum distributions (Liouville Correspondence). I will discuss the problems that chaos supposedly presents to each notion of correspondence, and why all but the second can be diffused without decoherence.

3 von Neumann Correspondence

Most frequently, the problem of quantum chaos is described in terms of a dynamical disparity between the behavior of classical states in phase space and quantum states in

[4] See Feintzeig (2020) for an alternative view.

Hilbert space. Roughly, classical states in chaotic systems separate exponentially under time evolution, while the separation between quantum states is a constant of motion. To understand this, we need to spell out some mathematical structure for both theories: the Hamiltonian formulation of CM and the von Neumann formulation of QM.

The state space of a Hamiltonian system is an even dimensional manifold Γ equipped with a symplectic form ω (a closed, non-degenerate 2-form). In canonical coordinates, a system of N particles in 3 dimensions has a phase space $\Gamma = \mathbb{R}^{6N}$ and states $x = (\mathbf{p}, \mathbf{q})$ where $\mathbf{q} = (q_1, ..., q_{3N})$ and $\mathbf{p} = (p_1, ..., p_{3N})$ are canonical positions and momenta. The symplectic form is $\omega = \sum_{i=1}^{3N} \mathrm{d}q_i \wedge \mathrm{d}p_i$.[5] The Hamiltonian energy function $H(\mathbf{p}, \mathbf{q})$ gives the dynamics via Hamilton's equations

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}. \tag{1}$$

The associated Hamiltonian flow generates integral curves $x(t) = (\mathbf{p}(t), \mathbf{q}(t))$ for initial conditions $x(0)$. The volume element obtained from the symplectic form is $d\mu = d^{3N}p\, d^{3N}q$.[6] Observables are given by smooth functions $f : \Gamma \to \mathbb{R}$. Probability densities are defined as smooth functions $\rho(x,t) : \Gamma \to [0, \infty)$ such that $\int_\Gamma \rho\, d\mu = 1$. The expectation value of observables are given as $\langle f \rangle_\rho = \int_\Gamma f \rho\, d\mu$. We can use Hamilton's equation to derive the Liouville equation

$$\frac{\partial \rho}{\partial t} = \sum_i \left(\frac{\partial H}{\partial q_i} \frac{\partial \rho}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial \rho}{\partial q_i} \right) \equiv \{H, \rho\}_{PB} \tag{2}$$

where $\{\cdot, \cdot\}_{PB}$ is the Poisson bracket. Liouville's equation describes the classical flow of probability densities and implies Liouville's theorem: phase-space volume is preserved under Hamiltonian flow, or, equivalently, ρ is constant along trajectories. As we will see, classical chaos manifests itself at the level of trajectories as well as densities.

The quantum state space is a complex Hilbert space $\mathcal{H}$ and (pure) quantum states are vectors $|\psi\rangle \in \mathcal{H}$.[7] This Hilbert space comes equipped with an inner product $\langle \cdot | \cdot \rangle$ and norm $\|\psi\|_{\mathcal{H}} = \sqrt{\langle \psi | \psi \rangle}$. Quantum states evolve according to the Schrödinger equation

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle . \tag{3}$$

where $\hat{H}$ is the Hamiltonian operator. Observables in QM are represented as

[5] Here, d is the exterior derivative and $\wedge$ the wedge (i.e. exterior) product.

[6] $d\mu = \frac{1}{(3N)!}\omega^{3N} = \frac{1}{(3N)!}\omega \wedge \omega \wedge ... \wedge \omega$ ($3N$ times) (Cannas da Silva, 2001, 191).

[7] More precisely, since multiplying each vector $|\psi\rangle$ by a pure phase $e^{i\theta}$ does not have any dynamical consequences, the unique states of QM are represented as rays in a projective Hilbert space.

self-adjoint operators $\hat{f}$ on $\mathcal{H}$. Mixtures of pure states are given by density operators $\hat{\rho}$; positive trace class operators on $\mathcal{H}$ such that $\mathrm{Tr}(\hat{\rho}) = 1$. Expectation values are given as $\langle \hat{f} \rangle_{\hat{\rho}} = \mathrm{Tr}(\hat{\rho}\hat{f})$, which for pure states becomes $\langle \hat{f} \rangle_{\psi} = \langle \psi | \hat{f} | \psi \rangle$. $\hat{\rho}$ evolves according to the von Neumann equation

$$i\hbar \frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = [\hat{H}, \hat{\rho}] \tag{4}$$

where $[\cdot, \cdot]$ is the commutator.

Some version of the following line of reasoning is often described as the problem of quantum chaos (Peres, 1984; Berry, 1989; Zurek and Paz, 1994; Penrose, 2011). Call $\|\cdot\|_{\Gamma}$ a reasonable norm on phase space that measures the distance between states (e.g. the Euclidean distance in canonical coordinates). In classically chaotic systems, two nearby initial states $x(0)$, $y(0) \in \Gamma$ diverge exponentially under Hamiltonian evolution:

$$\|x(t) - y(t)\|_{\Gamma} \approx \|x(0) - y(0)\|_{\Gamma}\, e^{\lambda t}. \tag{5}$$

Here, λ is called the system's maximal Lyapunov exponent. λ provides a measure for the rate that two states separate: when $\lambda = 0$ there is at most linear separation, and when $\lambda > 0$ there is exponential separation. Exponential separation occurs up until differences are on the order of the total accessible phase space.

Quantum states, however, do not "separate" at all under time evolution. This is because Schrödinger evolution is unitary, meaning that it preserves the inner product $\langle \psi(t) | \psi(t) \rangle = \langle \psi(0) | \psi(0) \rangle$ and thus the Hilbert space norm:

$$\|\psi_1(t) - \psi_2(t)\|_{\mathcal{H}} = \|\psi_1(0) - \psi_2(0)\|_{\mathcal{H}}. \tag{6}$$

Therefore, quantum states simply cannot behave in Hilbert space as classically chaotic trajectories do in phase space. Rosaler (2016, 70) suggests that decoherence can address this problem because only the entire system-environment pair is unitary, and the effective dynamics of the system alone can be non-unitary.

However, this setup is highly misleading. For one, these norms are encoding quite different notions of distance. To use an example from Belot and Earman (1997, 199), consider two free quantum wave packets whose centers start close together in configuration space but end up very far apart under Schrödinger evolution. There is a clear sense in which they have diverged, but $\|\cdot\|_{\mathcal{H}}$ is completely blind to it. It turns out that Hamiltonian mechanics produces the same result. Recall that Hilbert space comes equipped with $\|\cdot\|_{\mathcal{H}}$, while we added the classical $\|\cdot\|_{\Gamma}$ in by hand. What Hamiltonian systems come equipped with is a symplectic form, and this can be used to give a canonical L^1 norm on densities:

$$\|\rho_1(x,t) - \rho_2(x,t)\|_1 \equiv \int_{\Gamma} |\rho_1(x,t) - \rho_2(x,t)|\, d\mu. \tag{7}$$

The L^1 norm is a constant of motion under the Liouville equation:

$$\|\rho_1(x,t) - \rho_2(x,t)\|_1 = \|\rho_1(x,0) - \rho_2(x,0)\|_1. \tag{8}$$

Thus, probability densities preserve their $\|\cdot\|_1$ distance under Liouvillian evolution just as quantum states preserve their $\|\cdot\|_{\mathcal{H}}$ distance under Schrödinger evolution. Yet, this does not imply that classical systems cannot behave chaotically.[8]

An even more straightforward reason that quantum unitarity cannot be described as "the problem of quantum chaos" is that it extends to non-chaotic systems. Even integrable classical systems can separate at a linear rate in the $\|\cdot\|_\Gamma$ norm, which is also prohibited by unitary dynamics. Thus, if we are to find something distinctly problematic about chaos for quantum-classical correspondence, we will need to look beyond either formal or empirical *von Neumann Correspondence*.

4 Ehrenfest Correspondence

The textbook story of quantum-classical correspondence instead relates classical states to quantum wave-packets (Shankar, 1994, ch. 6, Sakurai 2017, 84-85). For a single particle in one dimension with potential $V(q)$, Ehrenfest's theorem tells us that

$$\frac{\mathrm{d}\langle q\rangle_\psi}{\mathrm{d}t} = \frac{\langle p\rangle_\psi}{m}, \quad \frac{\mathrm{d}\langle p\rangle_\psi}{\mathrm{d}t} = -\langle V'(q)\rangle_\psi \tag{9}$$

where $V'(q) = dV/dq$. This bears a striking resemblance to Hamilton's equations

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{p}{m}, \quad \frac{\mathrm{d}p}{\mathrm{d}t} = -V'(q). \tag{10}$$

Thus, the lesson of Ehrenfest's theorem, according to one popular QM text, is that "*expectation values obey the classical laws*" (Griffiths and Schroeter, 2018, 34).

In classical mechanics, the state is defined as a point in phase space (p, q). The closest initial quantum state $\psi(0)$ is a minimum uncertainty $\Delta p_0 \Delta q_0 = \hbar/2$ Gaussian wave-packet centered at (p, q). By identifying classical states with quantum wave-packets, Ehrenfest correspondence for trajectories can be stated as two conditions:

(i) $q(t) \approx \langle q\rangle_{\psi(t)}$, $p(t) \approx \langle p\rangle_{\psi(t)}$, and

(ii) $\Delta q(t)$ stays sufficiently small (i.e. the wave-packet remains localized).

[8]See Belot (2000, 459) for a similar point. Note that an analogous rationale can be used to explain why the linearity of the Schrödinger equation does not prohibit chaos. The Liouville equation is linear even when Hamilton's equations are not.

The first condition says that the "centroid" of the wave packet tracks the classical trajectory, and the second condition prohibits macroscopic superpositions.

To see when (i) holds, note that the difference between (9) and (10) is that, classically, the potential is evaluated at an exact position, whereas the quantum system's potential is evaluated over the width of the wave-packet. Expanding $V'(q)$ about the wave-packet's center and taking the expectation value gives

$$\langle V'(q)\rangle = V'(\langle q\rangle) + \frac{1}{2}V'''(\langle q\rangle)\langle(q-\langle q\rangle)^2\rangle + ... \tag{11}$$

Thus, the center of the wave packet will follow its classical trajectory when $\langle V'(q)\rangle \approx V'(\langle q\rangle)$. This occurs when the potential is at most quadratic (in which case the equality is exact) or approximately so over the size of the wave-packet.

Both (i) and (ii) are met for non-chaotic systems. Take a free particle, whose centroid behaves exactly classically because $\langle V'(q)\rangle = V'(\langle q\rangle) = 0$. (ii) also holds over relevant timescales. The spread in position over time is going to be due to its initial spread in momentum. At long times, the spread will be

$$\Delta q(t) \sim \frac{\Delta p_0}{m}t = \frac{\hbar t}{2m\Delta q_0} \tag{12}$$

This becomes negligible as $m \to \infty$. For example, a 1 kg particle with initial spread $\Delta q_0 = 10^{-10}$ meters will take $\sim 6 \times 10^{16}$ years for $\Delta q(t)$ to reach 1 meter.

Chaotic systems obey a different scaling relation. Accounting for the Lyapunov exponent in (5) suggests

$$\Delta q(t) \sim \Delta q_0 e^{\lambda t}. \tag{13}$$

There is a characteristic length-scale L_{char} over which $V(q)$ varies. When $\Delta q(t) \approx L_{char}$, $\langle V'(q)\rangle \not\approx V'(\langle q\rangle)$. The time it takes $\Delta q(t)$ to reach this length-scale is called the **Ehrenfest time** t_E, and it can be obtained from (13) as

$$t_E \sim \frac{1}{\lambda}\ln\left(\frac{L_{char}}{\Delta q_0}\right) \sim \frac{1}{\lambda}\ln\left(\frac{S}{\hbar}\right) \tag{14}$$

where $S \sim \Delta p_0 L_{char}$ is the characteristic action of the system. For closed macroscopic systems, the Ehrenfest time is also the timescale of macroscopic superpositions, and thus the Ehrenfest time is when both (i) and (ii) break down. (Paz and Zurek, 2002, 126). Setting $\hbar/S = \hbar_{\text{eff}}$, it follows that for any fixed t_E we can recover classical trajectories by taking $\hbar_{\text{eff}} \to 0$. Thus, even chaotic systems have a limiting procedure for achieving *Formal Ehrenfest Correspondence*.

However, if we want to apply this to real world systems, we need to consider the details of (14). The logarithm in (14) means that it takes exponentially larger actions

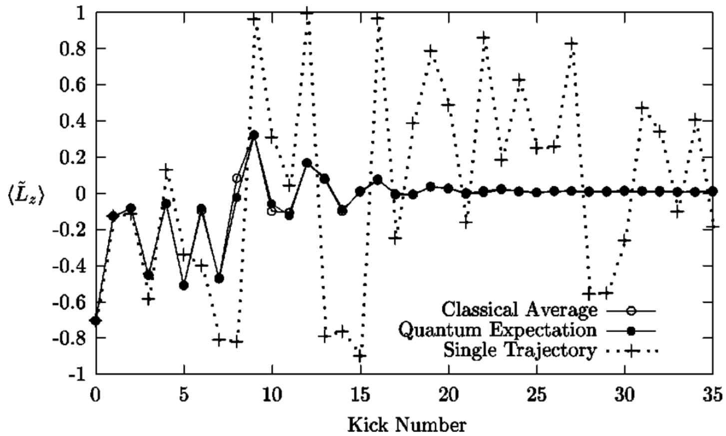

Figure 1: A chaotic system of two spins undergoing periodic interactions, taken from (Emerson and Ballentine, 2001a, 9), reprinted with permission. Given an initially localized distribution, the quantum and classical expectation values of angular momentum along the z axis $\langle L_z \rangle$ relax towards 0, and never stray far from one another in the process. Both of these expectation values begin to diverge from a single classical trajectory around the Ehrenfest time ($\sim$ 5 kicks). Observations for few-body macroscopic systems follow a single trajectory, not expectation values.

to obtain a linear extension of the Ehrenfest time. This has fittingly been termed the "logarithmic catastrophe" by Giorgio Mantica (2023). It is catastrophic because it yields unacceptably short time horizons of Ehrenfest Correspondence for realistically large systems. A well-known example is the chaotic rotation of Saturn's moon Hyperion (Zurek, 1998). Hyperion is shaped like a potato and not spherical, and so Saturn exerts a time-dependent torque on its different orientations. This leads to a chaotic evolution of the moon's orientation and rotational angular momentum. The Lyapunov time ($1/\lambda$) of this chaotic motion is 100 days, and its rotations have a characteristic action of $\sim \hbar \times 10^{58}$ (Berry, 2001, 46). This yields an Ehrenfest time on the order of decades, far smaller than the relevant timescales of astronomical observation. Under unitary dynamics we would expect Hyperion to rapidly evolve into a macroscopic superposition of orientations and cease to resemble the observed classical trajectory. For smaller chaotic systems, such as a 10^{-12} kg dust mote being tossed around in the atmosphere, this can occur in a matter of minutes rather than years (Wallace, 2012, 73). By examining realistic parameters, chaos effectively eliminates *Empirical Ehrenfest Correspondence* from contention.

It should be recognized that the problem of *Empirical Ehrenfest Correspondence* does not threaten most observables of interest for many-body chaotic systems in statistical mechanics. For these systems, observables settle down to their thermal values and stay there as long as we care to look. This thermal equilibration behavior is recovered at *both* the level of trajectories and expectation values of classical densities. Since the observable predictions of trajectories and expectation values are difficult to distinguish, an interpretive dilemma arises over which description is more fundamental.[9] For reasons discussed in §5.4, many-body quantum systems can recover the same equilibration behavior vis-à-vis expectation values.

However, for few-body, macroscopic systems like Hyperion, real-world observations are not tracked by quantum or classical expectation values beyond the Ehrenfest time. According to the quantum description of Hyperion, a *single quantum state* is coherently and macroscopically spread over orientations, making the classical state ill-defined. Additionally, the expectation value of, say, its angular momentum will settle into a single value and stay there, while the observed angular momentum will exhibit large fluctuations (similar to Figure 1). In this case, even if distributions line up (the topic of the next section), this does little to recover observations.

When we look at Hyperion we observe a single orientation, not a coherent macroscopic superposition. And yet, the former is what the unitary isolated quantum description predicts. This is the quantum measurement problem. Of course, macroscopic systems like Hyperion are rapidly entangling with their environment (or even unaccounted for internal degrees of freedom), and recovering the trajectory-level story will make essential use of decoherence.[10] Before we get to that, we have one more relation to consider.

5 Liouville Correspondence

5.1 Weyl Quantization

So far we have compared the behavior of classical and quantum states in their respective state spaces, and we have compared classical states with the minimum uncertainty quantum wave packets that most closely resemble them. In this section, we will try our luck at the level of classical and quantum probability distributions (i.e. Liouville correspondence). We will come to see that there are two ways in which quantum and classical distributions diverge for chaotic systems.

An increasingly prominent approach to Liouville correspondence is "deformation quantization" (either formal or strict), where a classical commutative algebra of

[9] The literature is largely centered on this interpretive dilemma, with Boltzmannian statistical mechanics favoring the trajectory interpretation and Gibbsian statistical mechanics favoring the density interpretation (Frigg and Werndl, 2024b).

[10] See Halliwell (2010) for a discussion of the distinction between external, environmentally-induced decoherence and internal, conservation-induced decoherence.

observables is deformed into a non-commutative quantum algebra.[11] One of the chief benefits of this approach is that we can formulate quantum theory on phase space. We will focus on the standard choice of the quantum chaos literature: formal Weyl quantization (Weyl, 1927; Zachos et al., 2005). Using Weyl quantization, we will walk through a common recipe for comparing observables of quantum and classical densities, discussing problems that chaos (supposedly) presents along the way.

First, we choose a "conventional" Hamiltonian $H(\mathbf{p}, \mathbf{q}) = \mathbf{p}^2/2m + V(\mathbf{q})$ of the standard kinetic plus potential form. Next, we choose an initial classical density $\rho(x, 0)$ and an appropriate corresponding (mixed or pure) quantum state $\hat{\rho}(0)$. Not every classical density corresponds to an allowable quantum state, so a common choice is a Gaussian wave packet with a width lower-bounded by the uncertainty principle.

To compare expectation values, quantum observables and states are cast onto phase space. Given a quantum operator $\hat{f}$, the Wigner transform produces a function on phase space:

$$f_W(\mathbf{p}, \mathbf{q}) = \int e^{-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{y}} \left\langle \mathbf{q} + \frac{\mathbf{y}}{2} \middle| \hat{f} \middle| \mathbf{q} - \frac{\mathbf{y}}{2} \right\rangle d^{3N}y. \tag{15}$$

where f_W is called the "Weyl symbol" of $\hat{f}$. Often, the transform yields the exact classical function f — such is the case for $\hat{p} \mapsto p$, $\hat{q} \mapsto q$, and conventional Hamiltonians $\hat{H} \mapsto H$ — but it can introduce $\hbar$ corrections when the operators feature ordering ambiguities (Zachos et al., 2005, 5). Given a mixed state $\hat{\rho}$ and its associated Weyl symbol ρ_W, we can obtain the Wigner function as

$$W(\mathbf{p}, \mathbf{q}) = \frac{1}{(2\pi\hbar)^{3N}} \rho_W(\mathbf{p}, \mathbf{q}) \tag{16}$$

whose prefactor allows us to derive the relation

$$\langle \hat{f} \rangle_{\hat{\rho}} \equiv \mathrm{Tr}\left(\hat{\rho}\hat{f}\right) = \int_\Gamma f_W \, W \, d\mu. \tag{17}$$

In this way, the Wigner function serves as a phase space weight for quantum expectation values, similar to the classical density: $\langle f \rangle_\rho = \int_\Gamma f \, \rho \, d\mu$. Thus, we can use it to help evaluate a Liouvillian correspondence condition at the level of expectation values:

$$\text{For some small } \varepsilon\text{:} \quad \frac{\left| \langle \hat{f} \rangle_{\hat{\rho}(t)} - \langle f \rangle_{\rho(t)} \right|}{f_{\text{char}}} < \varepsilon. \tag{18}$$

f_{char} is a characteristic value range over which f varies for the system. This allows us to cast the condition in dimensionless form and interpret "small" to mean $\varepsilon \ll 1$.

A few more remarks about the Wigner function are in order. The Wigner

[11] For more on strict deformation quantization see Landsman (2017) and Feintzeig (2022).

function is a real-valued quasiprobability distribution on phase space. For a pure state, integrating W over $\mathbf{p}$ or $\mathbf{q}$ recovers $|\psi(\mathbf{q})|^2$ and $|\psi(\mathbf{p})|^2$, respectively. The Wigner function is a "quasiprobability" distribution for two reasons. For one, it is not a positive definite function on Γ; it is negative in the presence of destructive quantum interference. Compare this with the Husimi Q function, which is a positive definite function on phase space that can be obtained by convolving the Wigner function with a Gaussian (Husimi, 1940). However, the negativity of the Wigner function is an asset in quantum chaos studies because it allows us to visualize and track the interference structure that emerges when the quantum state is behaving non-classically, structure which is obscured by the Q function. A second reason the Wigner function is a quasiprobability distribution is that even when it is positive definite — as it will be for Gaussians (Hudson, 1974) or convex mixtures thereof (Bröcker and Werner, 1995) — it cannot be said to represent a probability distribution of classical phase space. Points in this distribution do not represent possible states of an underlying quantum system that have their own independent dynamics (Wallace, 2021, 23).

Now we need to consider how these densities co-evolve. The literature has identified two worrisome features of quantum evolution. First, the *Quantum Suppression of Classical Chaos* refers to a time when the classical density develops fine-scale structure that the Wigner function cannot. Second, quantum states (pure or mixed) are generically periodic while classical densities are not. I will argue that neither of these turn out to be a problem for *Empirical Liouville Correspondence*.

5.2 Quantum Suppression of Classical Chaos

Classical evolution is given by Liouville's equation (2), $\frac{\partial \rho}{\partial t} = \{H, \rho\}_{PB}$, with $\{\cdot, \cdot\}_{PB}$ the Poisson bracket. In chaotic systems, Liouvillian evolution is mixing over surfaces of constant energy. μ_{mc} is the microcanonical measure: a uniform, normalized measure over the energy hypersurface at energy E: $\Gamma_E = \{x : H(x) = E\}$.[12] A system is mixing when for all $f, \rho \in L^2(\Gamma_E, \mu_{mc})$ such that $\int_\Gamma \rho \, d\mu_{mc} = 1$:

$$\lim_{t\to\infty} \langle f \rangle_{\rho(x,t)} = \lim_{t\to\infty} \int_\Gamma f(x)\rho(x,t) d\mu_{mc} = \int_\Gamma f \, d\mu_{mc} \equiv \langle f \rangle^E_{mc}. \quad (19)$$

This says the expectation values of observables will asymptotically approach their microcanonical values: $\langle f \rangle^E_{mc}$. Mixing can also be seen as the condition that there exists a unique stationary probability density towards which other densities $\rho \in L^2(\Gamma_E, \mu_{mc})$ relax (in a coarse-grained fashion). Geometrically, regions of initial conditions spread out in a way that will resemble this stationary density to greater and greater extent under time evolution. If we follow a density representing a patch of initial conditions

[12]The microcanonical measure is given by $d\mu_{mc} = \frac{\delta(H(\mathbf{p},\mathbf{q})-E)d\mu}{\int_\Gamma \delta(H(\mathbf{p},\mathbf{q})-E)d\mu}$.

under evolution, it will tend to be exponentially stretched along position and contracted in momentum, and then folded back onto itself. This creates a highly complex, filamentary structure of equal volume to the original patch (see the classical panes in Figure 2). From a coarse-grained perspective, this structure becomes indistinguishable from the stationary distribution.

In Weyl quantization, the Poisson bracket is replaced by the Moyal bracket $\{\cdot, \cdot\}_{MB}$, which is a deformation of the Poisson bracket with deformation parameter $\hbar$ (Moyal, 1949). The Moyal bracket produces the dynamics of Wigner functions in the same way that the Poisson bracket does for classical densities. For a single particle in one dimension:

$$\frac{\partial W}{\partial t} = \{H, W\}_{MB} = \{H, W\}_{PB} - \frac{\hbar^2}{24}\frac{\partial^3 V}{\partial q^3}\frac{\partial^3 W}{\partial p^3} + \mathcal{O}(\hbar^4). \tag{20}$$

Taking the classical limit here might correspond to considering only V and W that vary only on scales large compared to $\hbar$, in which case we obtain approximately classical Liouvillian flow. Unfortunately, in even macroscopic systems such as Hyperion, chaotic dynamics will tend to blow up the leading order correction in (20) until it becomes large enough to rival the Poisson bracket. Recall that chaotic evolution will tend to exponentially stretch the distribution in position and shrink it in momentum, and then fold these thinning filaments on top of each other. At some point, the classical filaments will develop $\hbar$ level structure that the Wigner distribution cannot reproduce; the quantum corrections due to $\frac{\partial^3 W}{\partial p^3}$ make the leading correction term in (20) as large as the Poisson bracket. At this point, the distribution no longer evolves classically and the Wigner function will develop negative troughs indicating destructive interference (Figure 2). This is also the point where semiclassical constructions of quantum wavefunctions break down (Berry, 1995; Batterman, 1995). Michael Berry has called this effect "the quantum suppression of classical chaos" (1987, 184).

To know whether this is a problem for empirical correspondence, we first need to estimate when the effect happens. Call the **Liouville break time**, t_L, the time when the Wigner function develops significant quantum interference structure. Given that the exponential shrinking in momentum is a compensation for the exponential stretching in position, this will be driven by roughly the same Lyapunov exponent λ responsible for sensitive dependence (5), leading to the same approximate scaling relation as the Ehrenfest time

$$t_L \sim \frac{1}{\lambda}\log\left(\frac{S}{\hbar}\right). \tag{21}$$

Since they scale in the same way, the Liouville break time is often not distinguished from the Ehrenfest time (Zurek, 1998). However, these times signify conceptually different points in chaotic evolution (Wiebe and Ballentine, 2005, 2). This can be seen

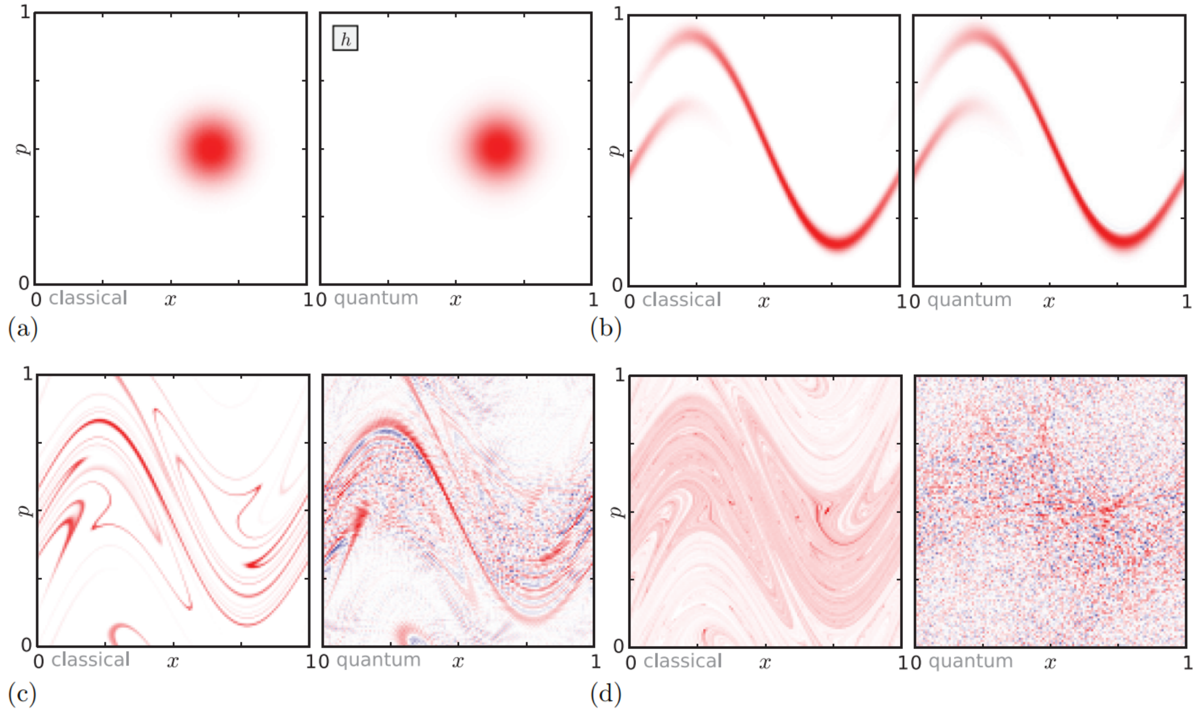

Figure 2: Phase space evolution of an initial Gaussian distribution by the kicked Harper map at (a) $n = 0$, (b) $n = 1$, (c) $n = 3$, (d) $n = 10$ kicks, taken from (Creagh et al., 2016, 7), reprinted with permission. A cell of size $h = 2\pi\hbar$ is given in (a). The Moyal flow of the Wigner function approximates Liouvillian flow until $\hbar$-scale filamentary structure creates quantum interference, which is displayed by the negative (blue) troughs.

in Figure 2, where the Ehrenfest time (the time that the wave packet has spread across scales where the potential varies) is shown in panel (b), and the Liouville break time (the time that significant interference shows up) occurs afterwards in panel (c). Note that only at the Liouville break time do the distributions start to become dissimilar. Because it also features logarithmic dependence on $\hbar_{\text{eff}} = \hbar/S$, the Liouville break time arrives quickly for even very large macroscopic chaotic systems such as Hyperion (decades rather than eons). Thus, as with the Ehrenfest time, the fact that we can push the Liouville break time out indefinitely in the $\hbar_{\text{eff}} \to 0$ limit is not sufficient for establishing empirical correspondence.

We can now address another common candidate for "the problem of quantum chaos." Often, the problem of quantum chaos is described in terms of the $\hbar_{\text{eff}} \to 0$ and $t \to \infty$ limits not commuting (Berry, 1995, 2001; O'Connor et al., 1992; Batterman, 1995, 2001; Bokulich, 2008). This is driven by work in semiclassical mechanics, where semiclassically constructed wave-functions become singular when phase space distributions develop $\hbar$ scale filamentary structure at the Liouville break time. If we take the $\hbar_{\text{eff}} \to 0$ limit first, we can keep these singularities at bay indefinitely. However, if we instead take the $t \to \infty$ limit first, then the filamentary structure becomes too finely packed in phase space to recover semiclassical correspondence. This is sometimes taken to show that chaos poses a problem for the reduction of CM to QM (Batterman, 2001, 109; Bokulich 2008, 16-17).

There are reasons to doubt this presentation for both formal and empirical correspondence. On the formal side, as Steeger and Feintzeig (2021) point out, the fact that one mathematical construction becomes singular does not imply that others do as well.[13] In the Wigner representation, Moyal flow diverges from Liouvillian flow, but this does not amount to a mathematical singularity. On the empirical side, the onset of these problems is not one that is either encountered or avoided depending on how one arranges the limits. As (21) demonstrates, *these two parameters are intimately connected*. For a real system in question, we have a fixed value of $\hbar_{\text{eff}}$. This value then sets the timescale for the Liouville break time. So the question of empirical correspondence is not whether these limits commute, but whether observations diverge on empirically relevant timescales for real physical systems.

Let us now consider whether divergences at the Liouville break time amount to any empirical differences between distributions. To do so, we need to narrow down the class of observables under consideration. While the mathematical definition of classical observables is highly unconstrained — smooth $f : \Gamma \to \mathbb{R}$ — in practice the observables relevant for classical observations are those which vary on macroscopic scales. After all, it would hardly be a threat to empirical correspondence to find out that classical and quantum distributions disagree about sub-$\hbar$ level structure. This restriction still includes all the standard observables of classical systems such as position, momentum, kinetic and potential energy, angular momentum, etc. When probing these observables, I will argue that we should expect no differences at the level of distributions, both in terms of expectation values and overall distributional differences, even absent environmental decoherence.

In simulations, quantum and classical expectation values begin to exponentially diverge at the Liouville break time. These differences have been promoted as yet another place where decoherence is needed to establish correspondence (see Habib et al., 1998; Zurek, 2003, 728; Franklin, 2024, 295). This would be a problem for Liouville correspondence if the exponential growth lasted long enough to become macroscopic. However, these differences soon plateau, either level off or shrinking to some fixed value. In dimensionless form, an estimation for a pair of coupled rotors has these differences leveling out around $\mathcal{O}(\hbar_{\text{eff}}^{2/3})$ (Ballentine, 2004). The same scaling relation shows up for

[13] Steeger and Feintzeig work within the somewhat different setting of strict deformation quantization.

Hyperion, where Wiebe and Ballentine (2005, 11) estimate the maximum quantum-classical differences in $\langle J_z \rangle$ (dimensionless angular momentum) would be 5×10^{-37}, well below any classical observation. Thus, correspondence at the level of expectation values (18) appears to be met sans decoherence even after the Liouville break time.

Of course, two distributions can agree about expectation values while disagreeing in other respects. Past the Liouville break time, quantum and classical probability distributions over observables will diverge on their fine-grained structure. Summing over all of these small differences can yield a relatively large number. If instead we consider open system dynamics, environmental effects can bring quantum and classical distributions in closer correspondence by smoothing out fine structure (Habib et al., 1998; Hernández et al., 2025). However, coarse-grained measurement is also sufficient to make the distributions indistinguishable.[14] To illustrate, suppose we consider observables over bins of width Δ, limited by our measurement capabilities. If we let $\{s_n\}$ denote the set of discrete measurement outcomes, then the classical probability over each bin is

$$P_{cl}^{f}(s_n, t) = \int_{\{x: f(x) \in [s_n, s_n + \Delta]\}} \rho(x, t)\, d\mu \tag{22}$$

and the quantum probability is

$$P_{qm}^{\hat{f}}(s_n, t) = \mathrm{Tr}\Big(\hat{\rho}(t)\, \Pi_{[s_n, s_n+\Delta]}^{\hat{f}}\Big) \tag{23}$$

where $\Pi_{[s_n, s_n+\Delta]}^{\hat{f}}$ is the projector of $\hat{f}$ onto the interval $[s_n, s_n + \Delta]$. We can then define the difference measure

$$\|qm - cl\|_{\Delta}^{f} \equiv \sum_{s_n} \Big| P_{qm}^{\hat{f}}(s_n, t) - P_{cl}^{f}(s_n, t) \Big|. \tag{24}$$

How small does Δ need to be for (24) to detect fine structure differences? Using a similar coarse-graining procedure, Wiebe and Ballentine estimate that to noticeably distinguish the classical and quantum probability distributions of J_z for Hyperion after the Liouville break time, we would require a detector resolution of 10^{-60} rad/s (2005, 12).[15] They write that "At the macroscopic scale of Hyperion, the primary effect of decoherence is to destroy a fine structure that is anyhow much finer than could ever be resolved by measurement" (Wiebe and Ballentine, 2005, 13).

This can be explained with the aid of Figure 2. Figure 2 features an *extremely small phase space*, so small that cells of $\hbar$ scale are significant. The key point is that in the phase space of any macroscopic system, from dust motes to Hyperion, the $\hbar$ scale is

[14] See also Kofler and Brukner (2008).

[15] These authors convolve both distributions using a triangular filter (Wiebe and Ballentine, 2005, 8).

going to be imperceptibly small (unlike this Figure), and so differences between the distributions that show up in the $\hbar$ filamentary structure are going to be imperceptible as well. Although the quantum and classical distributions begin to look qualitatively different due to interference beginning at the Liouville break time in panel (c), coarse-grained observations are not going to be able to resolve fine-grained differences. Once the differences have saturated phase space in panel (d), both distributions are practically indistinguishable from the microcanonical distribution, and it does not matter that the Wigner function cannot follow the intricate classical developments because this structure is not detectable. Thus, the "quantum suppression of classical chaos" does not show up as a noticeable distributional difference, and there is at least no conceptual need to invoke decoherence to restore classicality.

5.3 Periodicity

There exists a timescale over which there will be large divergences. Mixing says that expectation values of classical densities asymptotically approach their microcanonical values. In phase space, classical densities become effectively indistinguishable from the microcanonical density and, unlike classical trajectories, never exhibit large fluctuations or recurrences. On the other hand, quantum expectation values for bound systems with discrete spectra *must* exhibit large scale recurrences. For these systems, the von Neumann equation (4) can be solved as:

$$\hat{\rho}(t) = \sum_{m,n} \rho_{mn}(0) e^{-\frac{i}{\hbar}(E_m - E_n)t} |E_m\rangle \langle E_n| \tag{25}$$

where E_n are the energy eigenvalues and $|E_n\rangle$ are the eigenvectors of $\hat{H}|E_n\rangle = E_n|E_n\rangle$, and $\rho_{mn}(0) = \langle E_m| \hat{\rho}(0) |E_n\rangle$. The expectation value of any observable $\hat{f}$ is

$$\langle \hat{f} \rangle_{\hat{\rho}(t)} = \mathrm{Tr}\Big(\hat{\rho}(t)\hat{f}\Big) = \sum_{m,n} \rho_{mn}(0) f_{nm} e^{-\frac{i}{\hbar}(E_m - E_n)t} \tag{26}$$

where $f_{nm} = \langle E_n| \hat{f} |E_m\rangle$. Thus, unlike classical mixing, quantum expectation values of a pure or mixed quantum states are periodic or almost-periodic in time.

Here is what this would look like in the Wigner representation. Our initial quantum and classical distributions will spread across phase space at the Ehrenfest time. They will diverge on fine structure at the Liouville break time, with these divergences evening out at the saturation time. At this point, both distributions are effectively indistinguishable from the microcanonical distribution. At some later time, the Wigner function will start to exhibit fluctuations, where it becomes noticeably different from the microcanonical distribution. Bizarrely, the Wigner function will even exhibit "wave-packet revivals" where the distribution relocalizes into a small wave-packet (Robinett, 2004). Here, the quantum and classical distributions will diverge significantly both at the level of $\langle f \rangle$ and $\|qm - cl\|^f_\Delta$.

Formally, the periodicity of (26) spoils any chance of a strict quantum version of mixing for bound, isolated quantum systems. This has led some researchers to question the correspondence principle for chaotic systems. Ford and collaborators (1991; 1992) showed that a quantized version of the Arnold cat map exhibits periodic behavior even though its classical counterpart is mixing, which is taken to pose a problem for the completeness of quantum mechanics (Ford and Mantica, 1992, 1093). Belot and Earman (1997, 170-173) struggle to find a defensible quantum version of mixing because of this periodic behavior, although they remain optimistic that some weaker approximation to mixing might be obtainable. However, Belot (2000) later argues that the absence of hallmark chaotic behaviors in quantum theory — including mixing — undermines certain notions of quantum fundamentalism.

Even though we should expect large distributional divergences to occur at some time, it is again important to consider details of the scaling behavior. Although classical trajectory recurrences for macroscopic few-body systems are short, quantum recurrences need not be. The timescale that a quantum system exhibits large recurrence behavior is called the **Heisenberg time**, t_H. This is the timescale that the system resolves the discreteness of its spectrum, undergoing dephasing and then rephasing. It is defined as

$$t_H = 2\pi\hbar\, \nu(E) \tag{27}$$

where $\nu(E) = dn/dE$ is the density of states of the initial state, which scales according to the relation $\nu(E) \sim \hbar_{\text{eff}}^{-d}$, where d is the number of degrees of freedom (Haake, 2010, 13). For even two degrees of freedom ($d \geq 2$ required for chaos), $\nu(E)$ will blow up for macroscopic systems. This means that the Heisenberg time is *much larger* than the other timescales considered, astronomically large for macroscopic systems well above their ground state. Back of the envelope calculations places $t_H \sim 10^{20}$ years for a 1 gram rigid rotor and $t_H \sim 10^{56}$ years for Hyperion, both many orders of magnitude greater than the age of the universe ($\sim 10^{10}$ years).[16] Thus, although there is a formal discrepancy between classical mixing and quantum periodicity, for macroscopic systems this does not amount to an empirical discrepancy over relevant timescales. Again, because this is a non-problem, decoherence is not necessary to address it.[17]

The history of statistical mechanics contains a parallel dialogue (Frigg and Werndl, 2024a, ch. 3).[18] Statistical mechanics was founded by Ludwig Boltzmann to

[16] Treating both objects as 3D rigid rotors, we have the semiclassical density of states $\nu(E) \approx 2I/\hbar^2$ with I the moment of inertia. Plugging this into (27) yields $t_H \sim I/\hbar$. For a 1 gram, 1 cm rigid rotor $I \sim mR^2 = 10^{-7}\,\text{kg}\,\text{m}^2$. For Hyperion, $I \sim 10^{28}\,\text{kg}\,\text{m}^2$ (using values taken from Thomas (2010)).

[17] That decoherence is needed here seems to be hinted at by Landsman (2007, 520).

[18] Thanks to an anonymous referee for suggesting this comparison.

provide a microphysical foundation for thermodynamics, including the irreversible approach to equilibrium described by the second law. As Loschmidt (1876) and Zermelo (1896) showed, the ergodicity of classical trajectories in statistical mechanical systems implies that out of equilibrium systems will equilibrate, but it also implies large fluctuations and recurrences *away from equilibrium*. The widely accepted resolution is that the timescales for recurrences and large fluctuations in many-body statistical mechanical systems are astronomically large, leading thermodynamic and statistical mechanical predictions to coincide for relevant timescales.

In the present case, classical densities play the role of the exceptionless second law and quantum densities behave like classical statistical mechanical trajectories. Mixing allows classical densities to exhibit irreversibility at the level of expectation values. The more fundamental quantum description empirically mirrors this behavior for a very long time, but not indefinitely. Like statistical mechanics, differences at unobservable timescales do not threaten empirical correspondence, even if they might threaten some forms of formal correspondence.

5.4 A Nagelian Reduction of Mixing

Apparently, for macroscopic systems, quantum divergences from Liouvillian mixing occur only at imperceptible measurement resolutions or astronomically large timescales. This suggests that Belot and Earman's (1997, 173) hope for an approximate reduction of classical mixing is attainable. This reduction would follow the standard Nagel-Schaffner model of reduction: the higher level behavior — i.e. irreversible equilibration of expectation values — is not exactly realized from the lower-level theory, but a corrected version is (Nagel, 1961; Schaffner, 1967). Something of this sort is already on offer for the $N \to \infty$ limit, and recent work indicates that it can be extended to the $\hbar_{\text{eff}} \to 0$ limit as well. Here, I offer only a schematic proposal.

In quantum statistical mechanics, the corrected quantum version of mixing is called the Eigenstate Thermalization Hypothesis (ETH). A system obeys ETH when at high energies the matrix elements $f_{nm} = \langle E_n | \hat{f} | E_m \rangle$ of a suitable class of observables $\hat{f}$ satisfy the ansatz (Srednicki, 1999; D'Alessio et al., 2016; Wang and Wang, 2025)

$$f_{nm} = \mathcal{F}(\bar{E})\delta_{nm} + \nu(\bar{E})^{-1/2}\mathcal{G}(\bar{E},\omega)r_{nm} \tag{28}$$

where

- δ_{nm} is the Kronecker delta, $\bar{E} = (E_n + E_m)/2$, and $\omega = E_m - E_n$.
- $\mathcal{F}(\bar{E})$ is the microcanonical expectation value of f at energy $\bar{E}$: $\langle f \rangle_{mc}^{\bar{E}}$. $\mathcal{F}(\bar{E})$ is assumed to be a slowly varying function of energy, which rules out observables that can probe the fine structure of the spectrum (e.g. projectors of single energy

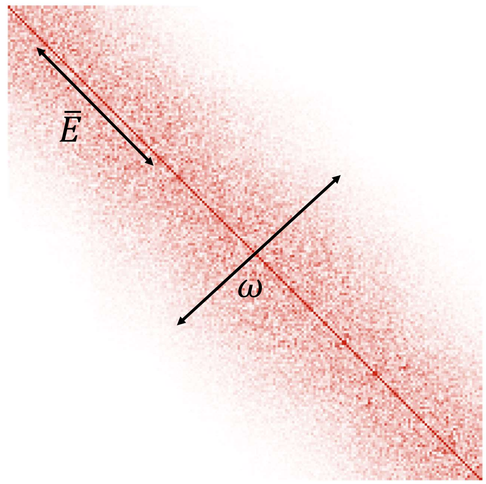

Figure 3: ETH observable $\hat{f}$ in the energy eigenbasis $\{|E_n\rangle\}$. Color saturation indicates the size of each matrix entry. Diagonals vary smoothly with $\bar{E}$, and agree with classical microcanonical values up to small fluctuations. The small off-diagonals connect nearby energies and fall off for large ω.

eigenstates). Slow variation also guarantees that these terms are effectively constant over narrow energy windows.

- $\nu(\bar{E})$ is again the density of states.[19] For macroscopic systems in either the $N \to \infty$ or $\hbar_{\text{eff}} \to 0$ limit, $\nu(\bar{E})$ is enormous, and so $\nu(\bar{E})^{-1/2}$ is very small.
- $\mathcal{G}(\bar{E}, \omega)$ is an envelope function that controls the size of the off-diagonals. It is a slowly varying function that decays at large ω (observables are local in energy).
- $r_{nm} = r^*_{mn}$ is a pseudo-random complex number with zero mean and unit variance.

Figure 3 shows an example matrix that satisfies the ETH ansatz.

The ETH ansatz is intimately connected with various results from quantum chaos. For the diagonal elements, (28) implies that f_{nn} approximates its microcanonical value, up to small random fluctuations. This follows from a central result from quantum chaos: in the limit of large energies, probability distributions of energy eigenstates for non-integrable quantum systems converge to the classical microcanonical distribution (Shnirelman, 1974; Zelditch, 1987). For the off-diagonal elements, (28) implies that they are small, local in energy, and random. The smallness is due to the large $\nu(\bar{E})$ for

[19] $\nu(\bar{E}) \sim e^{S(\bar{E})}$ for many body systems with thermodynamic entropy $S(\bar{E})$, in which case the ansatz features the more standard term $e^{-S(\bar{E})/2}$ (D'Alessio et al., 2016, 33). (28) is a generalized ansatz that also applies to few-body systems in the $\hbar_{\text{eff}} \to 0$ limit (Wang and Wang, 2025; Wang et al., 2025).

macroscopic systems, the energy locality is connected to the rate of classical correlation decay, and the randomness follows from the fact that the spectral statistics of chaotic systems are well described by Random Matrix Theory (D'Alessio et al., 2016).

For a macroscopic system with a relatively small energy uncertainty, an initial state in the energy eigenbasis will be supported along a narrow band of energies $E_n \in [E, E + \Delta E]$, with an enormous number of distinct E_n in this band. By separating the diagonal and off diagonal elements of (26), we can now state ETH as the following two conditions:

$$\langle \hat{f} \rangle_{\hat{\rho}(t)} = \underbrace{\sum_n \rho_{nn}(0) f_{nn}}_{\approx \langle f \rangle^E_{mc}} + \underbrace{\sum_{n \neq m} \rho_{mn}(0) f_{nm} e^{-\frac{i}{\hbar}(E_m - E_n)t}}_{\approx 0 \text{ for almost all } t}. \tag{29}$$

The first condition says that the diagonal sum will closely match the classical microcanonical value $\langle f \rangle^E_{mc}$. This follows from (28) because the f_{nn} within a narrow $[E, E + \Delta E]$ window will be effectively constant, and each entry will individually approximate $\langle f \rangle^E_{mc}$. Thus, the specific $\rho_{nn}(0)$ weights are irrelevant.[20] Given the diagonal sum is close to the microcanonical value, deviations from this value must come from the off-diagonal sum. The second condition says that the off-diagonal sum is near zero for almost all times. This follows from the off-diagonals being small and random. Large contributions from this sum involve delicate phase alignments of many terms, and time evolution will dephase and cancel these contributions.

We can now compare the classical mixing condition — $\lim_{t\to\infty} \langle f \rangle_{\rho(t)} = \langle f \rangle^E_{mc}$ for **all** $f, \rho \in L^2(\Gamma_E, \mu_{mc})$ with $\int_\Gamma \rho \, d\mu_{mc} = 1$ — to its quantum analog given by ETH. This says that for a **restricted** set of observables $\hat{f}$ and initial states $\hat{\rho}(0)$:

$$\text{For some } \varepsilon \ll 1 \text{ and almost all times } t, \quad \frac{\left| \langle \hat{f} \rangle_{\hat{\rho}(t)} - \langle f \rangle^E_{mc} \right|}{f_{\text{char}}} < \varepsilon. \tag{30}$$

f_{char} is again a characteristic value range of f. This comparison highlights the two differences between classical and quantum chaos previously discussed. First, correlation decay does not last forever. The asymptotic decay of classical mixing is replaced by a dephasing of the off diagonals. Once this dephasing is accomplished, remaining differences between $\langle \hat{f} \rangle_{\hat{\rho}(t)}$ and $\langle f \rangle^E_{mc}$ will be due to the stationary diagonal terms and thus will not continue to shrink. However, all that is needed empirically is that ε is well

[20] If each $f_{nn} \approx \langle f \rangle^E_{mc}$ in this window, then

$$\sum_n \rho_{nn}(0) f_{nn} \approx \langle f \rangle^E_{mc} \sum_n \rho_{nn}(0) = \langle f \rangle^E_{mc}$$

below observational thresholds. Second, the limiting behavior of mixing now gives way to the periodic behavior of (29), with large dynamical fluctuations being suppressed by the system's size. Thus, ETH explains why macroscopic systems exhibit ersatz mixing for certain observables over relevant timescales even though quantum systems are both periodic and limited in phase space resolution.

For this account to be satisfactory, more needs to be said about the types of observables and states that ETH applies to, particularly in the context of few-body chaotic systems such as Hyperion. ETH is only expected to hold for simple observables, often described as observables that are "few-body" or "local" for many-body systems. In statistical mechanics, this is believed to cover standard thermodynamic observables. There are obvious difficulties applying this to systems that are themselves few-body. Here, less is known about the appropriate restriction of observables.[21] From our discussion of Hyperion, one natural suggestion is to consider only coarse-grained observables whose Weyl symbols are smooth over phase space areas $\gg \hbar$. I leave this issue for future investigation.

6 A Defensible Correspondence Principle?

In surveying candidates for correspondence, we have seen that formally recovering classical behavior in the $\hbar_{\text{eff}} \to 0$ limit is not sufficient for explaining classical observations. The Ehrenfest time marks the departure point from classical trajectory dynamics, and the Liouville break time is the departure point from classical distributional dynamics. Although both times go to infinity in the $\hbar_{\text{eff}} \to 0$ limit, we receive little solace. Since they scale as $\ln\left(\hbar_{\text{eff}}^{-1}\right)$, both occur on observationally relevant timescales for real physical systems.

However, in two other places we do get encouraging scaling as $\hbar_{\text{eff}} \to 0$. First, after the Liouville break time, the maximum size of expectation value differences are undetectable because they scale as some small power of $\hbar_{\text{eff}}$. Second, the Heisenberg time — the timescale that a system will display large fluctuations and therefore conspicuous violations of mixing — scales as $\hbar_{\text{eff}}^{-d}$ (with d the degrees of freedom), making it astronomically long for macroscopic systems.

This holds a methodological lesson for correspondence and reduction. Formal convergence results in the $\hbar_{\text{eff}} \to 0$ limit give us invaluable insights into the theoretical relationship between classical and quantum theory. However, we must also keep track of whether this convergence happens fast enough to recover appropriately classical behavior for real physical systems with fixed $\hbar_{\text{eff}}$. In sorting out which of these are genuine problems for correspondence, we have made essential use of specific scaling behavior and limited observational capacities. Many of the problems attributed to

[21] For some recent work, see (Villaseñor et al., 2023; Wang and Wang, 2025; Wang et al., 2025).

chaos — such as those related to distributional differences or periodic behavior — are formal problems that can be circumvented or diffused by paying close attention to observations. To successfully navigate the many faces of the quantum-classical relation, formal approaches to correspondence needs to be tightly integrated with their empirical grounds.

Throughout this investigation, we have seen that one empirical form of correspondence does survive for macroscopic chaotic systems:

> **Empirical Liouvillian Correspondence Principle (ELCP):** for all macroscopic systems that are classically chaotic, single-time differences between quantum and classical distributions must remain negligible for coarse-grained observables of interest for longer timescales than we have ever confirmed.

Thus far in the literature, chaos has not posed any problems for **(ELCP)**. Given our discussions of Weyl quantization and Eigenstate Thermalization, there are good theoretical reasons to think that **(ELCP)** is safe. Therefore, **(ELCP)** was not saved by decoherence because it was never under threat. Under certain non-representational interpretations of quantum mechanics, **(ELCP)** might be all we were looking for in the first place, and so there never was a problem of quantum chaos. This is the motivating force behind Ballentine's simulations; to critique of the necessity of decoherence. Ballentine (1970) is an advocate of the ensemble interpretation, whereby quantum states represent ensembles of systems rather than being directly representational. For him, **(ELCP)** is sufficient.

However neither **(ELCP)** nor the ensemble interpretation are capable of explaining classical observations. As Schlosshauer points out (2008, 800), the ensemble interpretation is really just a hidden-variable theory that stops short of describing the hidden variables. Therefore, it is not a serious candidate to recover classical trajectory-level observations. Past the Liouville break time, the ensemble interpretation does not rule out the idea that the individual systems are moving around in highly non-classical ways even though the distributions stay aligned (as we will see, this is what occurs in Bohmian mechanics without decoherence).[22]

To recover classical observations, we need to be able to locate something resembling classical trajectories inside the quantum formalism. Chaos provides a large channel for nature to create macroscopic superpositions. However, when we observe classical systems, we do not observe them in superpositions but singular states, following classical trajectories. Going from quantum mechanics to Hamiltonian

[22]See (Drummond and Reid, 2020; Friederich, 2024) for a recent ensemble interpretation using the Husimi Q function that features different dynamics than Bohmian mechanics. An interesting question is how the problem of quantum chaos appears in this interpretation compared to Bohmian mechanics.

trajectories cannot be simply naive Ehrenfest correspondence, where classical states are put into correspondence with pure state quantum wave-packets. Something else must be added. In the following section, I will describe how classical chaos emerges within two existing interpretations of quantum mechanics, and the role decoherence plays in each.

7 Decoherence and Chaos within Interpretations of QM

In this section I will describe how chaos is understood in two of the leading interpretations of quantum mechanics: Everettian many-worlds QM and Bohmian pilot-wave QM. I will also describe the role of decoherence in reestablishing a link to Hamiltonian mechanics. As we will see, decoherence only provides part of the solution to the problem of quantum chaos, the rest must be provided by the interpretation itself. While the mathematics of decoherence is essentially the same for each interpretation (Rosaler, 2016), it plays a remarkably different conceptual role in restoring classicality for each. This points to a final reason why the problem of quantum chaos has no canonical form; the problem varies across different interpretations of quantum mechanics.

7.1 Environmentally Induced Decoherence

Here I will present a brief overview of environmentally induced decoherence with a focus on chaos. For further details see Zurek (1998, 2003, 2007) and Schlosshauer (2019).

Decoherence involves the smoothing of fine structure and the suppression of quantum interference via the system weakly interacting with its environment. The joint system-environment state space is $\mathcal{H}_S \otimes \mathcal{H}_E$. Given a pure state $|\Psi(t)\rangle \in \mathcal{H}_S \otimes \mathcal{H}_E$ tracing over the environmental degrees of freedom yields the system's density operator

$$\rho_S(t) = \mathrm{Tr}_E[|\Psi(t)\rangle \langle\Psi(t)|] \tag{31}$$

Interaction with the environment drives $\rho_S(t)$ towards approximate diagonalization in the "pointer basis." The states in this basis, called the "pointer states," are the states that are most stable (i.e. least entangled) under this interaction. For a broad class of system-environment interactions, the *approximate* pointer states will be Gaussian wave-packets. Since the environment is often effectively monitoring the position of the system, the exact pointer states would be position eigenstates. However, these feature infinite uncertainty in momentum, leading to large dispersions in position that would rapidly entangle the system. Instead, the quantum states that remain as localized as possible under environmental interaction are Gaussian wave-packets, and these are the states that are selected by environmental interaction.

Off-diagonal coherences in this basis decay as

$$\rho_{nm}(t) \sim e^{-\Lambda(q_n - q_m)^2 t} \tag{32}$$

with Λ the decoherence strength and q_n, q_m the centers of the Gaussians.[23] Therefore, the off-diagonal interference terms are suppressed on a timescale

$$t_D \sim [\Lambda(q_1 - q_2)^2]^{-1}. \tag{33}$$

This is utterly negligible for macroscopic superpositions like Schrödinger's cat ($\sim 10^{-35}$ seconds) or Hyperion ($\sim 10^{-53}$ seconds).[24]

Here is how decoherence plays out in the Wigner representation for chaotic systems. Recall that unitary evolution of the Wigner function is given by the Moyal bracket (20), which is the Poisson bracket plus higher order corrections with a leading order term $-\frac{\hbar^2}{24}\frac{\partial^3 V}{\partial q^3}\frac{\partial^3 W}{\partial p^3}$. In chaotic systems, regions of phase space are stretched primarily along position at a rate given by the Lyapunov exponent λ, and Liouvillian evolution then requires contraction along conjugate momentum by the exponential rate $-\lambda$. This means that an initial momentum spread of width Δp_0 shrinks as $\Delta p(t) \sim \Delta p_0 e^{-\lambda t}$. Chaotic evolution will fold these thinning strips on top of each other, until eventually the quantum corrections from $\frac{\partial^3 W}{\partial p^3}$ begin to rival the Poisson contributions.

For a wide class of models, the master equation for the system under environmental decoherence will include a diffusion term:[25]

$$\frac{\partial W}{\partial t} = \{H, W\}_{MB} + \hbar^2 \Lambda \frac{\partial^2 W}{\partial p^2} \tag{34}$$

where Λ is again the decoherence strength. One effect of this diffusion term is to quickly smooth out any fine-grained oscillatory fringes corresponding to quantum interference. Additionally, diffusion will prevent Δp from shrinking below a minimum threshold. This is the threshold where the classical Lyapunov shrinking and the diffusion term offset, given by the relation (Zurek, 2003, 745)

$$\Delta p_{min} \sim \sqrt{\frac{\hbar^2 \Lambda}{\lambda}}. \tag{35}$$

[23] For a scattering environment, such as the system subjected to an incoming stream of photons or air molecules, the decoherence strength is given as $\Lambda = k^2 \Phi \sigma$ where k is the wave number of incoming particles, Φ is the flux, and σ is the interaction cross section (Kiefer and Joos, 1999, 111).

[24] See (Wallace, 2012, 80-81) for the cat estimate and (Berry, 2001, 48) for Hyperion.

[25] There is also a friction term which has been omitted, and would be negligible for a case like Hyperion.

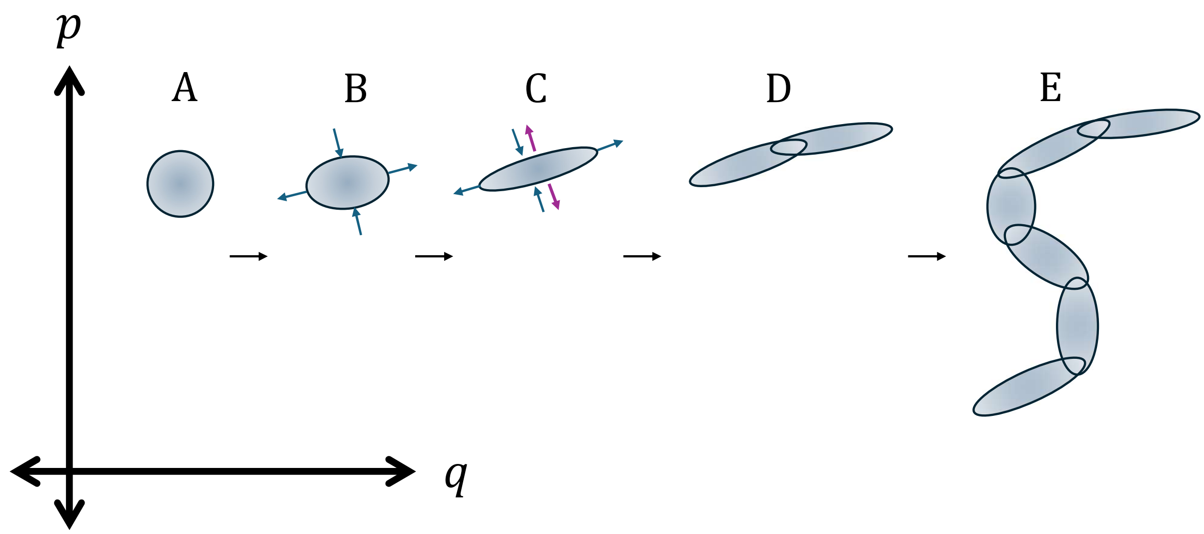

Figure 4: Evolution of the Wigner distribution under decoherence, adapted from (Hernández et al., 2025, 13). The Gaussian in (A) will initially follow Liouvillian evolution (B) where it is stretched along position and squeezed in momentum (blue arrows). When the distribution hits Δp_{min} at (C), the Liouvillian squeezing and diffusion due to decoherence (purple arrows) offset, and the distribution begins to grow in phase space support in a way well approximated as a mixture of Gaussian states (D) and (E).

(35) sets the most stable momentum for wave-packets for the system. For moderate decoherence strengths, Δp_{min} will be large enough that the quantum corrections of the Moyal bracket remain negligible, and interference will be quickly suppressed.

At this point, it is clear that this diffusion process does *not* resemble long-term classical Liouvillian flow, except in a coarse grained sense. Unlike a classical region under Liouvillian evolution, a decohering Wigner function will increase in phase-space support (Figure 4). In a chaotic system, an initial wave-packet will be stretched in position and shrunk in momentum until its momentum has reached Δp_{min}, and then it will continue to be stretched in position while remaining fixed in momentum. At this point, the distribution will be well-described as multiple non-interfering wave packets, with this approximation geting better as they continue to separate. According to Zurek, in this regime "the number of pure states needed to represent the resulting mixture increases exponentially," as does phase space support (Zurek, 2003, 746-749). Thus, the Wigner function features a branching structure where wave packets are being split and decohered, and these descendents are in turn split and decohered, etc. That looks suspiciously like...

7.2 Everett/Ehrenfest Correspondence

... the branching multiverse of Everettian quantum mechanics. We have now come full circle back to Ehrenfest correspondence, relating classical states to quantum wave-packets. However, each wave-packet is being continuously split into numerous, similarly sized wave-packets. If we loosely associate wave-packets with classical states, then we have a proliferation of classical states. When we observe a classical system with a well-defined trajectory, we are following a "thin red line" through this branching structure (Roberts, 2022, 35). The other branches still exist, but we have become cut off from them due to decoherence.[26]

As proponents of Everett point out, accepting this conclusion does not require tinkering with the machinery of unitary quantum mechanics (Wallace, 2012, 81). If you were prepared to accept Ehrenfest correspondence for integrable systems, this is its analog for chaotic systems. Decoherence alone does not solve the measurement problem by turning interfering superpositions into an improper mixture of quasi-classical states. Due to the proliferation of quasi-classical states, we cannot treat this as a single ignorance distribution that is being evolved forward by some dynamics, as with Liouvillian dynamics. A single wave-packet, corresponding to a quasi-classical state, will branch into many wave-packets corresponding to distinct quasi-classical states, and nothing singles one of these future states as unique or "real." Therefore, says the Everettian, they all are. If this conclusion is unpalatable, then something must be added or subtracted from the above presentation.

7.3 Bohmian Correspondence

Another popular interpretation of quantum mechanics seeks to solve the measurement problem by adding particle position to the ontology of QM, and a guiding equation to determine the change in position. This interpretation is called Bohmian mechanics, named after David Bohm (1952). I will describe the basics of Bohmian mechanics before applying it to chaos.

For a system of N particles, Bohmian mechanics will feature a wave function $\psi(\mathbf{q},t) = \psi(\mathbf{q}_1, ..., \mathbf{q}_N, t)$ and a configuration of those particles $\mathbf{Q}(t) = \mathbf{Q}_1(t), ..., \mathbf{Q}_N(t)$. The wave function evolves according to Schrödinger's equation (3). Particle positions evolve according to a "guiding equation"

$$\frac{\mathrm{d}\mathbf{Q}_k}{\mathrm{d}t} = \frac{\hbar}{m_k} \operatorname{Im}\left\{ \frac{\psi^*(\mathbf{q})\nabla_k \psi(\mathbf{q})}{\psi^*(\mathbf{q})\psi(\mathbf{q})} \right\}\Bigg|_{\mathbf{q}=\mathbf{Q}} \tag{36}$$

[26] See Dawid and Thébault (2025) for a recent discussion of the emergence of probability via decoherence from the Wigner formalism.

which yields a velocity field for the particle configurations. Bohmian predictions will coincide with quantum predictions so long as we consider ensembles of the systems that are initially distributed by $\rho(\mathbf{q},0) = |\psi(\mathbf{q},0)|^2$, as this Born rule distribution is equivariant under time evolution in Bohmian mechanics (Dürr et al., 1992).

Under Bohmian mechanics, quantum-classical correspondence takes on an appealingly simple form: quantum particle positions correspond to classical particle positions. This gives us a clear target for how Bohmian mechanics needs to reproduce chaos in macroscopic systems: Bohmian trajectories need to behave as their classically chaotic counterparts. In doing so, it is not sufficient that Bohmian trajectories are sensitively dependent on their initial conditions. They must follow approximately classical chaotic trajectories. For example, a massive Bohmian particle being violently tossed around by quantum interference will exhibit sensitive dependence, but its trajectory will not be remotely classical.

To investigate this question, it is common (Allori et al., 2002, 484) to follow Bohm's (1952) original formulation of the theory, where the wave function is expressed in polar form

$$\psi(\mathbf{q},t) = R(\mathbf{q},t)e^{iS(\mathbf{q},t)/\hbar} \tag{37}$$

with $R, S \in \mathbb{R}$ and $R \geq 0$. We can then obtain a quantum version of Newton's equations

$$m_k\,\ddot{\mathbf{Q}}_k(t) = -\nabla_k\Big[\,V(\mathbf{q}) + V_{quantum}(\mathbf{q},t)\,\Big]\Big|_{\mathbf{q}=\mathbf{Q}(t)}. \tag{38}$$

$V(\mathbf{q})$ is the classical potential and

$$V_{quantum}(\mathbf{q},t) \equiv -\sum_{j=1}^{N}\frac{\hbar^2}{2m_j}\frac{\nabla_j^2 R}{R} \tag{39}$$

is the quantum potential. (38) is classical when the quantum potential is negligible. In general, $V_{quantum}$ will blow up to rival the classical potential near "nodal points" where R (the amplitude of the wave-function) goes to zero (Contopoulos and Tzemos, 2020). Thus, when the wave-function features destructive interference, the quantum potential will become large, leading to highly non-classical trajectories.

The problem of chaos for Bohmian mechanics can be illustrated with reference to a more general problem related to bounded regions (Romano, 2016). Suppose we have a particle that is confined to some bounded spatial region — e.g. a 2 dimensional billiard table — and our initial $\psi(\mathbf{q},0)$ is a narrow wave packet. As the wave packet bounces off the boundaries, it will tend to delocalize. Delocalized segments of the wavefront will then collide with one another and interfere. Given that the guiding equation for velocity (36) is a single valued function of $\psi(\mathbf{q},t)$ at every $\mathbf{Q}$ in configuration space, the ensemble trajectories associated with the same $\psi(\mathbf{q},t)$ cannot cross in configuration space. When this collisional interference exists, trajectories that would cross classically are instead repelled by the quantum potential $V_{quantum}$. We could appeal to the $\hbar_{\text{eff}} \to 0$ limit to ensure that our initially localized wave-packet will stay localized long enough for this not to happen. However, if the boundaries are chosen to make the system chaotic — a stadium shaped billiard table would do the trick — then this is to no avail. Even highly localized initial wave-functions will spread out quickly under reflection, and we would soon expect our macroscopic Bohmian particle to be exhibiting large deviations from its expected Newtonian trajectory.

Decoherence restores the classicality of trajectories by removing the problematic interference. The Bohmian account makes use of essentially the same off-the-shelf decoherence formalism as other interpretations.[27] However, the conceptual role decoherence is playing is very different than in Everettian QM. Although each of the branching wave-packets still exists, the actual particle position selects only one of them. Therefore, the combination of decoherence and the particle's position allows an "effective" collapse of the wave function, where only a shrinking portion of the wave-function remains dynamically relevant to the particle's evolution: the wave-packet that actually holds the particle (Romano, 2023). Since Bohmians take quantum particle positions, and not wave-packets, to correspond to classical particle positions, these dynamically inert wave-packets are also ontologically inert in that they do not correspond to independent classical states.

7.4 The _ Problem of Quantum Chaos

Insofar as chaos posed any empirical problem for correspondence, it is at the level of Hamiltonian trajectories and not Liouvillian densities. Thus, "the problem of quantum chaos" is ensnared by the quantum measurement problem: how do we observe only one classical outcome when the quantum description generates superpositions?

From here, the nature of the problem varies depending on one's preferred interpretation of quantum mechanics. For certain non-representational interpretations of quantum mechanics — such as QBism (Fuchs, 2023) or the ensemble interpretation (Ballentine, 1970) — Liouville correspondence alone might be correspondence enough, meaning that there never was a problem of quantum chaos to begin with that needed to be solved by decoherence. For the two representational interpretations of quantum mechanics surveyed, quantum chaos poses distinct problems:

> **The Everettian Problem of Quantum Chaos**: chaos continuously generates macroscopic, coherent superpositions which lack the appropriate classical-state structure (i.e. localized quantum wave-packets).

[27] However, see Rosaler (2015b, 1185) for why Bohmian QM requires a stronger decoherence condition.

> **The Bohmian Problem of Quantum Chaos**: chaos generates interference that leads Bohmian trajectories to deviate from their classical counterparts.

For Bohmians, chaos poses a problem of fit; trajectories of isolated chaotic systems will display highly non-classical behavior. For Everettians, chaos poses a fundamental intelligibility problem; it removes the very candidates for quantum-classical correspondence (wave-packets). Thus, across these interpretations chaos poses two distinct problems.

Insofar as decoherence solves "the" problem of quantum chaos, it does so within an interpretation of QM. For Everettian QM, decoherence restores an emergent classical-state structure by selecting wave-packets as the pointer states of environmental interactions. For Bohmian QM, decoherence restores classical-looking trajectories. Saying that "decoherence solved *the* problem of quantum chaos" is akin to saying that "the medicine cured the patient's disease" despite it being unclear if the patient was even sick or the nature of their affliction. It may be true, but to evaluate it we need more detail.

8 Conclusion

Failure to recognize the measurement problem's central role in this topic makes the dialectic difficult to reconstruct. There is no canonical statement of the form "chaos once posed problem X for classical-quantum correspondence, but decoherence has solved problem X." To give such a statement, we must be precise about what corresponds to what in each theory. While it is agreed that chaos invalidates the 1-1 correspondence between pure state wave-packets and classical trajectories (Ehrenfest correspondence), decoherence ain't bringing that back. However, if correspondence is defined in terms of distributions (Liouville correspondence), then we have good reasons to believe that quantum and classical distributions will not appreciably disagree for macroscopic systems over relevant timescales even in the absence of decoherence. On the other hand, to recover the classical trajectories over states that appear in observation, we need something more than Liouville correspondence. A representational interpretation of quantum mechanics proposes a quantum analogue of the classical state. In Everettian QM, classical states loosely correspond to mixed state wave-packets, and in Bohmian QM, classical particle positions correspond to quantum particle positions. Only after these targets have been set can we recognize the full nature of the problem of quantum chaos, and evaluate whether and how decoherence solves it.[28]

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