Cassirer and the Arithmetization of Physics*

Marco Giovanelli

Università di Torino
Department of Philosophy and Educational Sciences
Via S. Ottavio, 20 10124 - Torino, Italy

marco.giovanelli@unito.it

Drawing an explicit parallel with Felix Klein's account of the arithmetization of 19th-century mathematics, Cassirer maintains that 19th-century physics likewise underwent a process of arithmetization that culminated in the 20th century. This paper argues that this constitutes an original insight and a unifying thread running through Cassirer's philosophy of physics over several decades, one that has yet to receive adequate scholarly attention. In contrast to his fellow Marburg neo-Kantians, Cohen and Natorp, who regarded the continuum as central to modern scientific knowledge, Cassirer treated number-formation as the prototype of scientific concept-formation in general. The number-sequence combines the structuralist view that each individual number has no intrinsic identity apart from its reciprocal 'relation' to all other numbers with the constructivist view that such a relation is a 'productive relation', generating ordered sequences of all possible numbers. The paper concludes that, for this reason, Cassirer may be said to have defended a form of 'structural constructivism' rather than 'structural realism', as is often claimed.

Keywords: Ernst Cassirer • Arithmetization • Structuralism • Constructivism

Introduction

In a lecture delivered at the public session of the Royal Society of Sciences in Göttingen on November 2, 1895, Felix Klein famously argued that the 19th century witnessed a process of '*arithmetization of mathematics*.' Guided primarily by the demand for rigor, modern mathematics sought to progressively eliminate its dependence on geometrical intuition and to ground as much of mathematics as possible in arithmetic—that is, in constructions based on real and, ultimately, natural numbers (Jahnke and Otte 1981; Volkert 1986, ch. 5; Petri and Schappacher 2007). This paper aims to show that Cassirer sought to extend Klein's insight by claiming that the 19th century witnessed a parallel process of '*arithmetization of physics*,' involving a shift from intuitive geometrical models to abstract arithmetic formulas. Although Cassirer explicitly employed Klein's turn of phrase only in his later years (Cassirer 1940a, 113; 1944b, 219; see Biagioli 2016, sec. 5.4), this article—drawing together the threads of several earlier works (Giovanelli 2023a, 2023b, 2024, 2025)—argues that this theme runs through the fabric of Cassirer's philosophy of physics over three decades, despite having received little attention from interpreters.

*DOI: 10.1162/POSC.a.582

Section 1 argues that, in the 1910s, alongside his well-known claim that the mathematical science of nature progressively moved from substance-concepts to function-concepts, Cassirer introduced a further, less recognized layer: the shift from space-concepts to number-concepts. For Cassirer, scientific concept-formation in general is ultimately 'sequence-formation' (*Reihenbildung*),[1] of which the sequence (*Reihe*) of natural numbers is the ideal standard that the natural sciences strive to approximate. The other members of the so-called Marburg School—Hermann Cohen (1902), Paul Natorp (1910), and Dimitry Gawronsky (1910)—regarded the 'continuum' as the central concept of modern science; by contrast, Cassirer (1910b), already in his very early work, insisted on the centrality of the 'discrete' number as the epistemological model of scientific knowledge in general. Section 2 argues that, in the early 1920s, Cassirer (1921b) extended this insight to relativity theory, viewing it not as the beginning of a process of 'geometrization of physics' but as the culmination of a process of 'arithmetization of geometry,' in which space and time intuitions give way to a number manifold.[2] As examined in section 3, in the late 1920s Cassirer (1929) extended the motif of sequencing (*Reihung*) to non-scientific domains through the mediation of the concept of 'symbol.' The symbolic construction of the physical world occurs by replacing feeling-qualities, which serve an expressive function (*Ausdrucksfunktion*), with sensible qualities, the latter with 'schematic' geometric diagrams having an iconic role (*Darstellungsfunktion*), which ultimately give way to predominantly 'symbolic' arithmetic formulas with a purely syntactic-significative function (*Bedeutungsfunktion*). Indeed, number-symbols appear to Cassirer as the prototypical form of signs (*Zeichen*), whose meaning rests exclusively on their relations to one another and not on any resemblance to what they designate (*bezeichnen*).

The paper concludes that, in his writings from the exile years after 1933, Cassirer (1940a) explicitly portrayed the development of classical physics as a progressive *arithmetization* of nature, thereby challenging the dominant historiographical view that depicts it as a progressive *geometrization* (see, *e.g.*, Koyré 1935). Cassirer could then argue that, not only relativity theory, but also quantum mechanics could be seen as the latest stage of this trend. Quantum theory may have dismantled the 'determinism of things,' yet it enshrined the '*determinism of number*'—the postulate that nature must always be expressible through rule-based numerical sequences. Indeed, in his last published work, Cassirer (1944a) seems to present 'arithmetization' somewhat ambiguously as (1) a descriptive category: the history of physics has undergone a process of arithmetization comparable to that described by Klein in mathematics; (2) a regulative ideal: it is the goal of physics to eliminate all anthropomorphic elements, not only sensuous qualities but also intuitive spatial schemes, and to replace them with non-contentual numerical symbols. If (1) reveals an interesting yet largely unexplored aspect of Cassirer's historiography of science (Ferrari 2015), (2) highlights a central feature of his philosophy of science that has likewise remained insufficiently investigated.

For Cassirer 'arithmetization' does not mean that numbers constitutes the only

[1]In what follows, I have rendered *Reihe* and related expressions as 'sequence,' rather than 'series,' as is common in most translations. Cassirer mostly employs *Reihe* in the sense of an ordered succession of elements determined by a rule (what would today be called a *Folge*), and not in the modern mathematical sense of 'series,' understood as the sum of the terms of a sequence (a *Reihe* in today's parlance).

[2]It is worth noting that among Cassirer's contemporaries, Émile Meyerson (1925, §158) explicitly recognized this point: referring to the "panmathematicism of the relativists", Meyerson contrasted his own "pangeometrism" with Cassirer's "panalgebraism".

subject matter of mathematics or of the natural sciences, rather, that numbers embody, in their purest form, two distinct yet interrelated aspects of scientific knowledge in general, which, in modern terms, can be described as: (a) *structuralism*: each given individual number is defined not by intrinsic content but only by its *relation* to other numbers; (b) *constructivism*: this relation is a *productive relation*, namely the successor function, which generates the sequence of all possible numbers. In my view, recent literature tends to emphasize (a), presenting Cassirer as a precursor of a form of *structural realism* broadly understood—that is, the claim that only structures 'exist' in a Platonic or physical sense.[3] By contrast, (b) is often overlooked or outright dismissed as a remnant of the Kantian insistence on the "constitutive powers of the mind" (Coffa 1991, 1). However, in Cassirer's philosophy, (a) and (b) are two sides of the same coin. If one were to assign a label, it could be said that Cassirer, throughout his career, defended a form of *structural constructivism*.[4] The latter is neither an ontological description of reality nor a psychological account of the mind. Rather, Cassirer saw his work as belonging to the German-speaking tradition of the study of 'concept-formation' (*Begriffsbildung*), understood not merely as a 'descriptive' but as an inherently 'normative' enterprise. For Cassirer, scientific concept-formation is, in the final analysis, a process of sequence-formation (*Reihungsbildung*) (Koenig 2025, see). The number sequence exemplifies the latter in its paradigmatic form: what is stated about the number-sequence applies to any ordered sequence whatsoever. The concepts of natural science can grasp the *real* only insofar as they are able to assign it a place within the sequence of *possible* elements generated by a fixed principle.

1 The Energy Principle and Arithmetization of Physics

After completing his first monograph on Leibniz toward the end of 1901, Cassirer (1902) attempted to submit it as a *Habilitationsschrift*. Facing several rejections, in some cases motivated by blatant antisemitism, Cassirer (1906a) submitted the first volume of the *Das Erkenntnisproblem* as his thesis at the University of Berlin in 1906 (Cassirer Bondy 2003, 99–101). It was eventually accepted, and he immediately began working on the second volume (Cassirer 1907a). However, as was customary in the German university system, he was still required to deliver a *Habilitationsvortrag* in order to obtain the *venia legendi*. Among the topics proposed by Cassirer, the habilitation committee selected "Substanzbegriff und Funktionsbegriff." The lecture was delivered on July

[3] Cassirer's 'structuralism' has attracted considerable interest in recent decades (Gower 2000). Three main tendencies can be distinguished: (1) the philosophy of arithmetic literature highlights Cassirer's reception of Dedekind's treatment of the natural and real numbers, portraying him as anticipating '*ante rem* mathematical structuralism' —the view that mathematical objects are merely positions within a structure (Reck 2003, 336; 2020; Yap 2017; Biagioli 2025; Koenig 2024); (2) the philosophy of geometry literature focuses mostly on Cassirer's reception of projective geometry (Heis 2011) and Klein's Erlangen Program (Ihmig 1999), presenting him as one of the first to have perceived the 'structural turn' in nineteenth-century geometry (Schiemer 2018); (3) the philosophy of physics literature attributes to Cassirer a group-theoretical conception of the object, portraying him as a precursor of 'structural realism' (French 2014). These three strands have largely developed independently. This paper argues, however, that (2) and (3) are ultimately derivative of (1): for Cassirer, they are all examples of 'sequence formation,' modeled on number formation, which also underlies the formation of physical concepts.

[4] Ferreirós (2023) presents Cassirer as a proponent of a form of 'conceptual structuralism'; this form of structuralism, however, presupposes a reference to the 'agent-dependency' of mathematics that, in my view, is ultimately foreign to Cassirer's approach.

26, 1906, giving Cassirer (1906b) the opportunity to present, alongside his historical works, an overview of his systematic philosophy, which he would later develop into a comprehensive monograph bearing the same title (Cassirer 1910b).

1.1 Substance-Concepts vs. Function-Concepts. The Habilitation Lecture

Opening the lecture with a concise historical overview, Cassirer argues that the substance-concept (*Substanzbegriff*) had dominated ancient and medieval thought, where the distinction between substance and its properties served as the metaphysical counterpart to the logical relation between the subject and its predicates. Only in the early modern period was this paradigm challenged, as the function-concept (*Funktionsbegriff*) that emerged in modern mathematics began to take center stage within the mathematical science of nature as well (Cassirer 1906b, 5). Still, the dialectic between the substance-concept and the function-concept was not fully resolved even within the sciences. The substance-concept persisted in the 'concept of atom' (*Atombegriff*). Without the assumption of ultimate elements possessing absolute rigidity, impenetrability, and cohesion, it seemed impossible to ensure the identity of the subject of motion over time. According to Cassirer, it was only with the emergence of the 'concept of energy' (*Energiebegriff*) that the history of physics entered a fundamentally new logical dimension (Giovanelli 2023a). Already in its first appearance in the work of Robert Mayer, the principle of the conservation of energy received "a purely 'functional' interpretation" (Cassirer 1906b, 8).

As Cassirer had already pointed out in his previous work, the first step was taken by Leibniz, who elevated the concept of 'mechanical work' (*Arbeitsbegriff*) to the foundation of all physics (Cassirer 1902, 1907a). Starting from the axiom *causa aequat effectum*, Leibniz demonstrated that any mechanical change of state (the compression of a spring, the rotation of a water mill, etc.), regarded as the 'cause,' could be translated into a standard 'effect'—specifically, 'mechanical work' (the elevation of a given weight to a certain height). In the 19th century, after the discovery of the mechanical equivalent of heat, Mayer ventured to extend the *causa aequat effectum* principle to non-mechanical phenomena, including thermal and electrical ones. The question of the "elementary structure of matter" was thereby pushed into the background (Cassirer 1906b, 8). From Mayer's perspective, in order to grasp and describe natural processes mathematically, no mediating "*hypothetical schema*" concerning the structure of matter or the nature of heat was required (8). It was sufficient that the changes revealed by experience could be "directly related to one another and *measured* against each other", using mechanical work as the unit of measure (8).

Wherever we speak of energy as a being in its own right, we mean nothing other than this relation and this mathematical connection: "What is truly given to us, for example, in the conversion of heat into motion, are indeed only two qualitatively different processes, between which, however, we discover a constant quantitative relation of transition and thus a pure functional *dependence*" (9). Energy is not the mysterious 'carrier' we imagine as a permanent, quality-less substrate underlying the diverse manifestations of physical phenomena. The reality of energy thus resolves entirely into the understanding of relations of equivalence (*Aequivalenzbeziehungen*). The "substantiality" that Mayer ascribes to energy, Cassirer emphasizes, is "nothing other than a *constancy of* pure numerical relationships [*Zahlenverhältnisse*]" (12; emph. mine).

Cassirer could then conclude that in Mayer's energetics "the substance-concept [*Substanzbegriff*], as it is used here, has already passed through the function-concept

[*Funktionsbegriff*] and has thereby been elevated to a new logical level" (Cassirer 1906b, 12–13). It is true that the concept of energy always carries the risk of becoming an independent *substance*. Yet it was the merit of 19th-century positivism to insist that it signifies nothing but a *functional* relation among measurable quantities. However, according to Cassirer, positivism failed to recognize that such mathematical dependency is not derived from experience but rather constitutes a condition that makes experience possible in the first place. If no quantitative coordination (*Zuordnung*) between qualitatively different phenomena could be established, science itself, as Leibniz once put it, would become *quiddam vagum et absonum*,[5] something indeterminate and contradictory (Cassirer 1907a, 81). The assumption of the univocality of the coordination (*Eindeutigkeit der Zuordnung*) between equivalent numerical values (*Äquivalenzzahlen*), guaranteed by the principle of energy, was thus justified as a condition for the possibility of a mathematical science of nature.

Concluding the lecture, Cassirer pointed out that the centrality of the notion of 'coordination' dominated not only 19th-century physics but also 19th-century mathematics (Ryckman 1991). In particular, he briefly noted that "Dedekind undertook to ground the concept of number [...] on the operation of coordination [*Zuordnung*]" (Cassirer 1906b, 15). The entire number sequence can be generated from a fixed initial unit by repeatedly 'coordinating' each term with a unique successor, without bound. Cassirer elaborated on this point in his well-known 1907 article on Kant and modern mathematics for the *Kant-Studien*. Dedekind's work demonstrated that a complete foundation of arithmetic could be achieved by defining the natural numbers simply as a sequence of elements connected by a specific 'order,' determined by a transitive, asymmetric, and univocal relation—the 'generative relation' from which the sequence of numbers is constructed. On this view, finite numbers are not conceived as a 'plurality' of units but are instead identified solely by the 'position' they occupy within the overall sequence (Reck 2020). Cassirer (1907b, 12–13) shows in some detail how Dedekind was able to rigorously construct the real number continuum without any appeal to geometrical intuition: "In general, the lawfulness of pure number can be developed, and even extended to the positing of irrational numbers and thus to numerical 'continuity,' without conceiving of number as anything other than a 'position' [*Stelle*]" (Cassirer to Natorp, Jun. 28, 1906; Holzhey 1986, Vol. 2, Doc. 100).

1.2 Space-Concepts vs. Number-Concepts. The Hönigswald-Review

Cassirer's habilitation lecture was intended to show that the history of physics reveals a consistent progression from concrete, quasi-intuitive representations of things and their properties toward a framework increasingly dominated by purely functional relations between numerical values. An objection to Cassirer's position was raised by Richard Hönigswald in his 1906 booklet *Beitraege zur Erkenntnistheorie und Methodenlehre*. Drawing on Cassirer's account in *Das Erkenntnisproblem*,[6] Hönigswald concedes that Galileo may be regarded as the originator of a tradition that views physics as merely the *description* of the relations among phenomena (23). Yet, he argues, Descartes moves in the opposite direction, aiming instead at an *explanation* of phenomena through the construction of pictures (*Bilder*) or models (*Modelle*) (17–18). Cassirer (1909) felt compelled to respond to this objection and wrote an extensive review of Hönigswald's book for the *Kant-Studien*.

[5] Leibniz 1850, 3:208.
[6] Cassirer 1906a, 310.

Cassirer agreed that the question of whether, in the scientific representation of phenomena, one should proceed by means of abstract “hypotheses” about functional relations or search for “intuitive models [*Modelle*]” has always played a role in the history of physics (Cassirer 1909, 93). The difference between these two approaches is not that the first remains within the limits of experience while the other goes beyond them. Rather, it is a “differentiation of the *methods of empirical knowledge itself*” (93). The opposition between intuitive models and abstract functional relations is better understood as part of the more fundamental contrast between the *arithmetic* method and the *geometric* method. Here, for the first time, alongside the opposition between substance-concepts and function-concepts, Cassirer alludes to a further sub-opposition within the latter, between number-concepts and space-concepts. To clarify their role in physical practice, Cassirer, following Hönigswald, again contrasts Mayer’s energetics of the 19th century with the mechanical conception of nature.

According to the arithmetic approach, the goal of physics consists in demonstrating “*certain numerical ratios* [*Zahlenverhältnisse*]” (93). As he had already emphasized in his habilitation lecture, for Mayer, the ‘essence’ of the individual forms of energy is sufficiently understood when the fixed equivalence values that determine the numerical relations between the different domains can be univocally (*eindeutig*) established (93). The mechanical work done by a falling body in raising a load stands in a constant relation to the friction-induced increase in the water’s temperature: “In place of the peculiar ‘being’ of heat or motion, there arises merely the question of their arithmetic numerical ratio (*Verhältniszahl*) ” (93). Ultimately, these proportions remain simply juxtaposed. The inquiry ends with the assumption of distinct, mathematically definable forms of energy that cannot be further reduced to one another and whose differences must be accepted as a fact (93).

At this point, however, the question, for example, concerning the nature of heat, remains logically consistent and factually justified even after the establishment of the law of the conversion of work into heat: “It is this task that, in order to be fulfilled, repeatedly calls upon the geometric ‘model’ and schema, going beyond the abstract functional relationship” (93). A model is something distinct from the actual entity it represents; it omits the nonessential features and focuses exclusively on the ‘spatial structure.’ In particular, ‘mechanism’ attempts to explain all phenomena by reducing them to spatial displacements—for example, heat is nothing other than the collective motion of the particles of a body. Instead of being content with the quantitative comparability of different classes of phenomena, ‘mechanism’ seeks to reduce them to a single class, namely, changes of spatial relations. In this sense, the “requirement of geometric construction complements the determination of merely numerical relations” (94).

Both approaches are legitimate, Cassirer conceded. While the *arithmetic* approach is oriented toward ‘specification,’ the *geometric* approach reflects an interest in ‘homogeneity.’ Indeed, Cassirer continues, the search for geometrical ‘models’ does not originate merely from the interest in the intuitive illustration of abstract arithmetic relations but from the interest in developing a unified image of the world: “This connection is not guaranteed by mere number [*Zahl*] alone; rather, its establishment ultimately requires a reduction to space [*Raum*] as a unified and homogeneous basic schema” (94). When viewed in this way, Cassirer concluded his review, it becomes clear that both procedures are epistemologically justified; which one is more appropriate in a given case can be determined only by empirical investigation (94).

There is little doubt, however, that Cassirer was convinced that the history of physics had already embarked on the path of replacing the space-concept with the number-concept. By the end of 1909, Cassirer had likely already written most of his monograph *Substanzbegriff und Funktionsbegriff* (Cassirer 1910b), which was completed around July 1910. The book incorporates much of the material just discussed. Across nearly 500 pages, however, these brief reflections became part of Cassirer's broader line of argument, which began to take on more definite contours. In particular, in the Preface, Cassirer presented his research as a contribution to "a renewed analysis of the principles of concept-formation (*Begriffsbildung*) in general" (V/iii; tr. mod.),[7] beginning with mathematics and then extending to the exact sciences. Indeed, although the idea for the book originated in the philosophy of mathematics, Cassirer maintained that the problem of 'concept-formation' acquired broader significance only once it became clear that it could be extended to the whole of the exact sciences, and in particular to physics.

In the first chapter of the book, Cassirer famously contrasted the path of mathematical concept-formation (*Begriffsbildung*) with that of traditional logical concept-formation. Descriptive sciences such as botany and zoology rely on 'discursive concept-formation,' based on the *abstraction* of identical characteristics that an actual manifold of different elements have in common. The mathematizing sciences, by contrast, are engaged in a 'constructive concept-formation,' based on the *production* of a sequence (*Reihe*) of possibly different elements through the iterative application of an identical rule (196/147–148).[8] What gives unity to the sequence elements a, b, c (*Reihenglieder*) is not that they possess the *same property*, but rather that they are derived from one another by the repeated application of the *same sequence principle* (*Reihenprinzip*), the function F, so that $F(a, b)$ and $F(b, c)$ express the dependency between successive elements of the sequence (196/16).

In this way, Cassirer concluded, "a completely new field of investigation is opened for logic. In opposition to the logic of the genre-concept [*Gattungsbegriffs*], which, as we saw, represents the point of view and influence of the substance-concept [*Substanzbegriffs*], there now appears *the logic of the mathematical function-concept* [*Funktionsbegriffs*]" (27/21; tr. mod.). Cassirer, indeed, famously argued that the function-concept progressively acquired a central role in scientific thought to the detriment of the substance-concept (Heis 2014). However, it is worth noting that Cassirer seems to regard the 'rule-based' view of functions, as a 'mechanism' for producing a sequence of possible elements, as more fundamental than the 'relation-based' view, which treats functions as coordinations among a collection of actual elements. Indeed, Cassirer frequently likens the function of 'general concepts' in the mathematical sciences to that of the 'universal term' (*allgemeines Glied*) in a sequence, the formula that expresses a_n as a function of n. A single instance of such a concept is represented by the individual sequence member a_n. In this sense, the general concept defines the totality of individual cases. The latter, however, cannot be conceived as a 'collective' totality of actual elements; rather, it constitutes a 'distributive' totality of possible ones (Cassirer 1912, 91). What the concept achieves is a 'hypothetic generality.' Take the general term $a_n = n^2$: it

[7]The number following the slash indicates the corresponding page in the English translation given alongside the original edition in the bibliography.

[8]When a published English translation is available, the original page number is followed by a slash and the corresponding page number in the English edition.

ensures that *if* one is given any definite number, say $n = 5$, *then* one can be certain that $a_5 = 25$.

In this way, Cassirer sought to remain faithful to the spirit of Cohen's 'logical idealism,' that is, to the principle of 'origin' (*Ursprung*), the idea that scientific thought (conceived in a non-psychological sense) generates its own content. However, no privilege is accorded to the infinitesimal calculus, as Cohen (1902) and Gawronsky (1910) on the one hand, and Natorp (1910) on the other, had claimed (Giovanelli 2016). However, for Cassirer this is only one significant, though by no means unique, example of 'calculus,' alongside others such as barycentric calculus, vector calculus, and so on. In all these cases, the productive–constructive nature of scientific thinking "is fulfilled wherever the members of a manifold are deduced from a definite serial principle and exhaustively represented by it" (Cassirer 1910b, 131/99). The nature of a mathematical object is not fixed in itself prior to the principle of its generation but is determined only by the productive relation (*erzeugende Relation*) on which it rests. For Cassirer, the concept of number is the purest and most fundamental manifestation of this epistemological ideal.

After presenting the well-known dialectic between the substance-concept and the function-concept in the opening chapter on concept-formation, Cassirer develops it through a secondary dialectic in the next two chapters, focusing respectively on the number-concept (*Zahlbegriff*) and the number-concept (*Raumbegriff*). This latter dialectic has received much less attention. Yet, for Cassirer, only when the space-concept is resolved into the number-concept can the function-concept claim full victory over the substance-concept. Indeed, it is ultimately the logic of the number-concept, as articulated by Dedekind,[9] that represents the paradigmatic model of the functional–relational character of the scientific concept. The sequence of natural numbers, once a starting element is fixed, is generated by a process that, from each number already obtained, produces its unique successor, ensuring that no earlier number ever recurs in this progression. From this standpoint, to distinguish 2 from 1 and 3 from 2, nothing more is logically required than the capacity to distinguish each succeeding position in a sequence from the one that precedes it (Cassirer 1910a): "[The 'essence' of the numbers consists in nothing more than their positional value [*Stellenwert*]" (Cassirer 1910b, 51/39; tr. mod.) within a sequence of all possible numbers. The existence of the number e or π means nothing other than that, by means of the sequence used for its definition, one and only one position within the system of possible numbers is determined univocally (Cassirer 1912, 91). The number-concept is paradigmatic because the general features of the ordered 'progression,' what is here stated of number, apply to every progression whatsoever: it is therefore solely the form of the sequence itself (*Reihenform*), not the material that enters into it, that is defined here (see Koenig 2025, sec. 2.3).

As Cassirer wrote to Natorp, it is "the possibility of arranging elements in this way into a sequence unambiguously that precisely constitutes their numerical character [*Zahlcharakter*]" (Cassirer to Natorp, Oct. 30, 1909; Holzhey 1986, Vol. 2, Doc. 120). By contrast, Cassirer assigns no autonomous significance to the specific form of the space-concept (*Raumbegriff*), which still retains an irreducible intuitive–qualitative character that, for example, distinguishes points from instants. For Cassirer, the emergence of

[9] The Frege–Cantor approach conceives of *Zuordnung* as an operation that establishes a coordination among elements that are already given, that is, an equivalence relation that partitions a domain into classes. In contrast, Dedekind's conception of *Zuordnung* is *generative*: it defines the number sequence itself by assigning to each element a unique 'successor,' thus constituting an order relation. While equivalence relations generate classes or partitions, order relations generate sequences.

the abstract 'space-concept' of synthetic geometry was at most a kind of provisional stepping stone. In Descartes' 'analytic' geometry, the idea of a "completion of the space-concept through the number-concept" (Cassirer 1910b, 92/71) emerges: the intuitive 'shape' of the figure is replaced by the abstract *relation* between numerical values, opening the path to infinitesimal analysis (95–96/72–73). It is true that, in the 19th century, synthetic geometry regained prominence through the development of projective geometry (101–12/77–78). Still, the clarification of the exact relation between metric and projective geometry ultimately required a return to the analytic method. In Klein's (1872–1973) Erlangen Program, space is once again reduced to an n-dimensional number-manifold; geometric properties are those expressed by formulas that remain invariant under specific transformation groups acting on n-tuples of numbers (Cassirer 1910b, 116–119/88–91).

Cassirer regarded the history of 19th-century mathematics as a progressive "conversion of space-concepts into number-concepts" (93/71), that is, a transition from geometry to arithmetic.[10] Space can be made fully intelligible only by attributing to it the same logical nature that, until then, had belonged exclusively to number (114/87). Here, number is not seen merely as a technical tool for measurement; its deeper significance lies in the fact that it alone fully realizes the highest methodological postulate that each number is nothing but a position in a sequence (114–115/87). Concept-formation is sequence-formation; sequence-formation in its purest form is represented by the number-sequence, in which the identity of each element is determined solely by its position within the sequence (see Koenig 2025, sec. 2.4). The more mathematics advanced, the more it conformed to this regulative ideal. As one can infer from the Hönigswald review, Cassirer suggests that this same process of replacing space-concepts with number-concepts can also be observed in the development of 19th-century physics. Indeed, the same issue reappears in chapter IV of *Substanzbegriff und Funktionsbegriff*. The debate between mechanism and energetics, Cassirer argues once again, was, in essence, a debate between space-concepts and number-concepts.

Initially, it seems that the abstract number-schema could not be applied directly to reality: "In order to advance from numbers to material physical existence, it is necessary to have the mediation of the concept of space" (Cassirer 1910b, 206/156). Mechanism first replaces sensible qualities with abstract 'spatial' models, focusing on the measurable aspects of phenomena such as position and time, which are ultimately reducible to 'numbers.' However, these spatial models always ran the risk of being interpreted in a naive, realist manner as schematic copies of macroscopic systems. By contrast, Cassirer continues, energetics "contains a motive from the beginning, which protects it more than any other physical view from the danger of an immediate hypostatization of abstract principles. Its fundamental thought, from an epistemological point of view, does not *go back primarily to the concept of* space*, but to that of* number" (251/189; emph. mine).

For this reason, Cassirer presents the emergence of the energy principle as a kind of case study in which the concept-formation of physics can be analyzed in its purest form. Physical concepts "appear to be so many means of grasping the 'given' in sequences, and of assigning it a fixed place within these sequences" (196/148). Consider the case of a falling body. One starts with a list $a_1, a_2, a_3, \ldots, a_n$ representing the *actual* numerical values of the spatial coordinates of a falling body at a given time. The

[10] Gawronsky (1910, 16) complains about a process of 'arithmetrization of geometry,' which Cassirer, by contrast, embraces.

choice of these particular state variables is due to the fact that a rule can be found according to which the sequence of all *possible* values from a_1 to a_2, from a_2 to a_3, and so on, can be expressed as a function of time t as an independent variable (Cassirer 1910b, 343–344/259). The law of falling bodies does not concern the actual motion of particular bodies, but the possible motion they would have under idealized conditions. To approximate the actual motion of a real body, it is necessary to further determine "parameters" characterizing the particular system in question—such as air resistance, shape, mass distribution, distance from the Earth's center, and so on (199/150). The further physics strives to penetrate into the 'being' of its objects, Cassirer concluded, the more it encounters "new strata of numbers and numerical values" (199/150; tr. mod.).

Nevertheless, Cassirer continued, the embedding "of the sensuous manifold into sequences of purely mathematical structure remains inadequate as long as these sequences are separated from each other" (252/190). Scientific knowledge requires a principle that establishes a 'coordination' (*Zuordnung*) among the various numerical sequences, integrating them into a unified system. As mentioned above, the first step was the empirical discovery that the mechanical work performed by a falling body in lifting a load stands in a constant ratio to the increase in the water's temperature produced by friction. The mechanical equivalent of heat, "was soon extended beyond this starting point" (253/190). It was postulated that to each *quantum* of motion, heat, electricity, chemical affinity, and so on, there always corresponds a proportional amount of mechanical work:

> In this postulate, the essential content of the principle of conservation is already exhausted; for any quantity of work, which arose 'from nothing,' would violate the principle of the univocal coordination [*eindeutigen Zuordnung*] of all sequence. If we wish to represent the system schematically, we have here a number of sequences [*Reihen*] A, B, C, of which the members $a_1 a_2 a_3 \ldots a_n, b_1 b_2 b_3 \ldots b_n$, $c_1 c_2 c_3 \ldots c_n$, stand in a definite physical relation of exchangeability, such that any member of A can be replaced by a definite member of B or C without the capacity of work of the physical system in which this substitution is assumed being thereby changed. We briefly represent this relation of possible substitution not by always coordinating each individual member with the multitude of corresponding equivalents, but by ascribing to it once for all a certain value of energy, which draws all these coordinations [*Zuordnungen*] into a single pregnant expression. We do not compare the different systems with each other directly, but create for this purpose a 'common sequence,' to which they are all equally related. (254/191; tr. mod.)

For example, in the case of the mechanical equivalent of heat: equal increments in sequence A—such as lifting a body of known mass to a certain height—correspond to equal increments in sequence B, namely, the rise in temperature of a given mass of water. These, in turn, correspond to equal increments in sequence C, for instance, the work done by an electric current in producing the same amount of heat, and so forth. In this way, starting from a an arbitrary zero point, it is possible to "coordinate a certain *work-value*, a certain quantity of *energy*, to every individual member of the compared sequence" (262/197). The energy principle, the conservation of mechanical work, is the condition for the 'univocality of the coordination' between different sequences of changes, which yield the same numerical value when expressed in work units. If mechanical work were to arise from nothing, the coordination could not be maintained, that is, no symmetric, transitive relations could be defined between the different sequences (260–261/196–197)[11]

[11]I cannot elaborate on it here, yet it merits attention that, in sec. VIII, Cassirer extended this

The choice of mechanical work as the unit of measure is arbitrary and does not imply the reduction of all phenomena to mechanics, it depends only on the fact that mechanical effects are more easily measurable than other effects. Indeed, at first sight, it might seem that physics has simply accepted the existence of distinct classes of qualitatively different phenomena (mechanical, thermal, electrical, and so on) thereby forgoing further unification. However, Cassirer insists that the establishment of "equivalence values between the different sequences" provides a unification no less firm than the "reduction to a common mechanical model" (Cassirer 1910b, 269/202). The intellectual demand for unification is thus operative both in the energetic and in the mechanical conception of natural processes: "[T]he difference only consists in that, in the one case, its realization is based purely on the *concept of number*, while, in the other case, it also requires the *concept of space*" (260/202; emph. mine). However, the renunciation of spatial–mechanical models (269/202; fn.) in favor of purely numerical relations does not justify the positivistic rhetoric of a 'hypothesis-free' representation of natural phenomena: "for translation into the language of the abstract *number-concepts*, no less than translation into the language of *space-concepts*, involves a theoretical transformation of the empirical material of perception" (270/203).

2 Relativity Theory and the Arithmetization of Geometry

The reception of *Substanzbegriff und Funktionsbegriff* was allegedly tepid (Cassirer Bondy 2003, 128–129). It was, however, thanks to the success of *Das Erkenntnisproblem* that Cassirer emerged as the most prominent and widely read representative of the Marburg School—even more than Cohen or Natorp.[12] Yet all attempts to appoint Cassirer as Cohen's successor in Marburg after his retirement in 1912 failed. Cassirer remained a *Privatdozent* in Berlin for nearly two decades. During those years, Cassirer moved beyond epistemological questions (Cassirer 1914) toward the philosophy of culture, publishing *Freiheit und Form* in 1916. The book was intended as a historical investigation that emphasized the liberal–cosmopolitan aspects of German culture in order to counter its chauvinistic–nationalist appropriation during the First World War. However, Cassirer soon realized that such a 'philosophy of culture' still lacked an autonomous philosophical foundation (see Cassirer 1923a, 12–13). It was around that time that Cassirer began working on a program of 'Philosophy of the Symbolic,' as evidenced by a 32-page *Disposition* dated June 13, 1917 (Cassirer 1917).

It is interesting to note that in a diagram of this manuscript Cassirer indicates that the 'number-function' (*Zahlfunktion*) is the true center from which "the entire system of exact science emerges" (18/19). However, Cassirer also seems to present the space-function (*Raumfunktion*) as something autonomous. Nevertheless, in a marginal note Cassirer ultimately holds that the 'sequence-ordering function' (*Reihenordnungsfunktion*) is intended to represent a root from which number, space, time, and even set theory can be derived (18/19). Cassirer appears to suggest that this 'function' is the starting point for an 'epistemology of the symbolic' (*Erkenntnistheorie des Symbolischen*). Each individual element of the sequence is a 'symbol' of the entire sequence, since it is the result of the fixed rule by which the sequence is generated (20). At this point, Cassirer probably began to glimpse the possibility of using the symbol-concept as

'arithmetic' approach to chemistry as well (Cassirer 1910b, 270–291/220–233).

[12] Cohen to Natorp, May 10, 1911; Holzhey 1986, Vol. 2, Doc. 126, Cohen to Natorp, Jun. 10, 1912; Vol. 2, Doc. 131.

a tool to extend the sequence-concept beyond the domain of the exact sciences (see also Cassirer 1913, 19; fn. 15). A collection of over 240 numbered sheets and a manuscript on language (Cassirer 1919) shows that, in the ensuing years, Cassirer began to pursue the philosophy of the symbolic systematically (Schubbach 2016, 46–50). However, the project was temporarily put on hold following the confirmation of general relativity in November 1919. Critical philosophy could not ignore such a radical change in the 'fact of science,' and Cassirer was drawn back to epistemological work just after he obtained a professorship at the newly founded University of Hamburg.

Cassirer finished a booklet on relativity in August 1920, published in early 1921 (Cassirer 1921b), and gave thirteen lectures on the topic during the winter semester of 1920/1921 (Cassirer 1920–1921). However, his first published contribution to the relativity debate, a brief article for the *Neue Rundschau*, already stated his key, and somewhat surprising, but profound message: for Cassirer, relativity theory was not the beginning of a process of the 'geometrization of physics' but the culmination of a process of the 'arithmetization of geometry.' In this way, Cassirer returned more explicitly to the centrality of the concept of number over that of space. The ultimate goal of science, Cassirer writes in the article, is to construct "*a pure world of numbers* [...] out of the world of immediate intuition" (Cassirer 1920, 1354; emph. mine). Cassirer sees the hallmark of relativity in its application of this method to space and time themselves. Indeed, relativity theory became possible only after the intuitive nature of space and time had been replaced by an abstract four-dimensional 'number manifold': "In this relation to the fundamental motif of number [*Zahl*], space and time also appear unified in a new way" (1354).

Cassirer elaborated this point in detail, both in the published book (Cassirer 1921b) and in the lectures (Cassirer 1920–1921). In both cases, one can see that Cassirer sought to transpose what he had argued about the concept of energy in the 1910s to the concepts of space and time themselves. The energy principle is now presented as a typical example of the shift from the 'physics of models' (*Bilder*) to the 'physics of principles,' a distinction possibly taken from Poincaré (Giovanelli 2023b). It is true that Helmholtz attempted to express the energy principle in the language of classical mechanics, as a consequence of the mechanical view of nature based on the motion of point particles governed by central forces. However, with the advent of modern electrodynamics, this approach progressively lost its plausibility. By the end of the 19th century, energetics had adopted Mayer's approach, which presents the energy principle from the outset in a way not tied to any specific representation of the elementary processes as a mere "gaining of *equivalence-numbers*" (Cassirer 1921b, 17/359). Relativity theory deepened and developed this orientation.

In pre-relativistic physics, the difference between space and time persists as a qualitative distinction, even within the purely quantitative world-picture. Relativity theory, in its development, goes beyond this opposition: "the anthropomorphism of the natural sensuous picture of the world, the overcoming of which is the real task of physical knowledge, is here again forced one step further back" (45/381–382). Relativity managed to eliminate not only the differences in sensation but also those between spatial and temporal determinations. Space and time are famously merged into a four-dimensional manifold (62/397), where a point is nothing but a quadruple of numbers, its coordinates. In special relativity, these numbers have a direct meaning as the results of measurement; however, general relativity goes a step further, reducing them to meaningless parameters:

> The theory of relativity, however, pushes the dissolution of the form, of experiential quality significantly further, and it does not rest until it has succeeded in dissolving even the fundamental distinction between space and time *into one of mere number*. [...][Spacetime forms] a four-dimensional manifold: that is, each point within it (each individual 'event') is fully determined by specifying four numerical values x_1, x_2, x_3, x_4. These values have no intuitive or directly physical meaning; rather, as Einstein explicitly states, they serve only 'to assign numbers to the points of the continuum in a specific but arbitrary manner' And in this arbitrary numbering, the individual numbers used, x_1, x_2, x_3, x_4, exhibit no intrinsic differences among themselves. *They are nothing but numbers* [...] In accordance with this principle and the ideal of the general theory of relativity, it is no longer possible within it to distinguish the 'spatial coordinates' of an 'event' from its time coordinate; rather, all numerical values x_1, x_2, x_3, x_4 contribute equally and equivalently to determining the event as a physical occurrence. Even 'space' and 'time' belong to that content which, like the content of immediate sensation, vanishes in the final logical-mathematical formulation of physics. (Cassirer 1920–1921, 43)

That is precisely the meaning and the achievement of the theory of relativity: in order to preserve the constancy and universality of the laws, one must relinquish the demand for a unit of space and time as a kind of homogeneous 'form.' The requirement of the unity of space and time is reduced to the "univocality of the coordination [*Eindeutigkeit der Zuordnung*]" between the numerical values of the space and time coordinates (Cassirer 1921b, 82/415; tr. mod.).

However radical the departure of relativistic physics from classical physics might seem, Cassirer, as usual, seeks to demonstrate that, upon closer inspection, it merely brings to light a tendency already implicit in the latter: "In fact, it can be shown that the general doctrine of the invariability and univocality [*Eindeutigkeit*] of certain values, which is given first place by the theory of relativity, must recur in some form in *any* theory of nature, because it belongs to the logical and epistemological nature of such a theory" (45/384; tr. mod.). Once again, Cassirer draws on the concept of energy to make his point. Just as in climbing a mountain the change in height from base to summit is independent of the path, the change in energy has a unique value (*eindeutigen Wert*), expressed in work units, regardless of the process (mechanical, electrical work, heat, etc.) used to change the state of a system: "But that this univocality in fact exists, [...] is precisely what the 'principle of the conservation of energy' affirms" (46/385). However, the existence of a unique value, Cassirer conceded, is ultimately based on an empirical fact, the impossibility of a *perpetuum mobile* of the first kind.

Cassirer seems to suggest that both the energy principle and the relativity principle play the role of 'conditions' necessary to ensure the possible 'univocality of coordination' between the numerical values of certain parameters. However, Cassirer appears to have abandoned the assumption that each of these conditions is, even in some 'weak' sense, *a priori*. Writing to Schlick in October 1920, Cassirer famously conceded that he "would consider as '*a priori*' [...] only the 'uniqueness of coordination' [*Eindeutigkeit der Zuordnung*]. For me as well, however, how the latter is now specified into particular principles and presuppositions becomes clear only through the progress of scientific experience" (Cassirer to Schlick, Oct. 23, 1920; ECN, Vol. 18, Doc. 88). It was Kant's mistake not to have separated this general univocality from its various incarnations. What can now be required *a priori* is that it must be possible to achieve a progressive emancipation from the anthropomorphism of sensuous world-views, an ideal that Planck regarded as the true task of physics (Cassirer 1921b, 116/445). From this point of view, relativity is not a rupture in physics but a further step toward this goal: it represents an emancipation not only from sensuous qualities but also from the intuitive notions of space and time that still shaped classical physics: “The ideal with which scientific physics began with Pythagoras and the Pythagoreans finds here its conclusion; all qualities, including those of pure space and time, are translated into pure numerical values” (Cassirer 1921b, 120/448).

3 Arithmetization and the Symbolic Nature of Scientific Knowledge

The physics of the 19th and 20th centuries, in its progress, sought to cast off more and more the “hypotheses and images”, and did so in order to construct its entire framework all the more firmly and exclusively “in pure numerical determinations” (Cassirer 1921a, 57). Relativity theory is nothing but the extension of this process to space and time themselves (61). Concluding *Zur Einstein'schen Relativitätstheorie*, Cassirer famously insisted that the dissolution of qualities into numbers by physics was not the only way to access reality (Cassirer 1921b, 120/448). For example, in physics, time is fully reduced to the ‘continuum’ of real numbers. However, the fact that individual instants appear to be isolated from one another does not capture the flow immediately experienced in the ‘continuum’ of psychological time. Neither the physical nor the psychological time continuum, Cassirer concluded, should be considered as ‘real’ time: “The symbols that the mathematician and physicist take as the basis of their view of the outer world, and that the psychologist takes as the basis of his view of the inner world, must both be understood as *symbols*” (127/455). Between these two poles stand the manifold symbolic expressions of ‘time’—the historical, religious, and aesthetic conceptions of time. What time ‘truly’ is would be determined only if we succeeded in surveying “the *whole system* of symbolic forms” (199/447). However, this totality, according to the usual neo-Kantian rhetoric, is never ‘given’ (*gegeben*), but always ‘given as a task’ (*aufgegeben*).

3.1 Number-, Sequence-, Symbol-Concepts

Cassirer articulates this point further in his 1921 article, ‘Goethe und die mathematische Physik’. Comparing the conception of nature in modern physics with that of Goethe, Cassirer argues that natural phenomena can be considered from different perspectives, yet equally as ‘legitimate sequence principles’ (*Reihenprinzipien*). Modern physics derives “its sequence principle from the domain of number, the abstract fundamental form of sequence as such”, whereas Goethe derives his sequence principle from the “phenomenon of life” (51), requiring that each form contain within itself a continuous transformation from one form into another. Cassirer did not aim to determine whether Goethe's approach was justified. The case serves Cassirer to show that “a manifold of concepts of nature is possible” without the objectivity of one contradicting that of the other (73). What separates the scientific concept of nature from Goethe's concept of nature is the characteristic ‘principle of selection’ (*Auswahlprinzip*) operative within each. None of these concepts simply reflects the concrete totality of reality; rather, each highlights certain elements from it as the decisive and essential ones (73). Cassirer draws the general lesson of the paper by arguing that the real can be grasped from different perspectives, according to “different methodological selection principles” (75), corresponding to different symbolic forms.

The Goethe paper serves as a kind of hinge between Cassirer's earlier epistemological work and his new project of a philosophy of the symbolic (Sakata 2024, ch. 6.1). Indeed, the concept of ‘symbolic form,’ which is mentioned there somewhat in passing, became a main theme for the first time in a 1921/1922 talk given at a seminar of the Warburg Library (Cassirer 1923a), an institution which, as is well known, played an important role during Cassirer's Hamburg years (Ferrari 2003, ch. 8). All concept-formation, Cassirer argues, is characterized by the fact that it contains within itself a specific form of 'sequencing' (*Reihung*) (Cassirer 1922, 7). Just as in the natural sciences the 'number' serves as the most abstract model of sequence (ECN, 4:252), so myth and other cultural forms are governed by their own sequence principle (Ferrari 2002). Each actual sensible element, depending on the point of view, can stand 'symbolically' as a single member of a sequence of possible elements governed by a distinct sequence principle (Cassirer 1922, 7). Here, Cassirer seems to treat the symbol-concept (*Symbolbegriff*) as an extension of the sequence-concept (*Reihenbegriff*) beyond the exact sciences. Even in non-scientific domains, the individually given element does not simply remain what it is but becomes a symbol insofar as it is regarded as a member of a lawfully generated sequence of possible elements. Cassirer could then elevate 'symbolicity' to the unifying principle of the various cultural forms, as a 'doctrine of thought in general' (8).

Cassirer conceded that, at that time, he could only outline a phenomenology of linguistic form, within the broader framework of a 'general system of symbolic forms' (I). However, Cassirer argued that in each of these symbolic forms, despite their differences, one could identify: (a) a common *static* structure: in every case, the spirit manifests the capacity for the free creation of sensory 'signs' (*Zeichen*) and 'images' that refer to a designated (*Bezeichnet*) (Cassirer 1923a, 14); (b) a similar *dynamic* law governing their development of relation between sign and the designed. In every symbolic form, one can distinguish a process through whose stages the symbolic expression becomes increasingly complete. In particular, this includes a 'three-stage progression'—namely, the 'mimetic,' 'analogical,' and 'symbolic' stages in the process of linguistic symbol formation (18)—which corresponds to Goethe's model of the evolution of artistic creation: 'imitation,' 'manner,' and 'style' (22).

Some sparse notes written for a 1922 course on the philosophy of language show how Cassirer was already attempting to apply this three-stage progression to the history of the mathematical sciences of nature. In the history of scientific knowledge, the first, mimetic stage is dominated by a 'physics of qualities'; in the second, analogical phase, sensible qualities are replaced by geometric models; in the final, purely symbolic stage, this 'physics of models' is superseded by the "physics of principles", in which any demand for similarity with reality is relinquished (ECN, 4:240). The tripartite division—mimetic, analogical, and (purely) symbolic expression—becomes central in the first volume on language (Cassirer 1923b, 26), and although not treated with the same degree of elaboration, Cassirer also employs this three-stage scheme in the volume on myth (Cassirer 1925a, 292–294), completed in 1924. However, by early 1925, there are indications that Cassirer (1925b) may already have begun to abandon the view that symbols were fundamentally linguistic (Cassirer to Goldstein, Jan. 5, 1925; ECN, Vol. 18, Doc. 54). Cassirer might have realized that myth was more fundamental than language, and that the volume on myth should, in fact, have inaugurated the sequence.[13] The result of this reorganization of the project of a philosophy of symbolic forms was first presented in June 1927 at the 3rd Congress for Aesthetics and General Art Science in Halle.

In the published version of that talk, Cassirer (1927) replaced the diachronic three-

[13] Krois 2009; for a more balanced stance on the question of the priority of myth over language, see Rudolph 1992, 2003; competing accounts include Endres 2023b, 2025.

stage scheme with a synchronic three-level classification of 'symbolic forms' according to their function: (1) the 'expressive function' (*Ausdrucksfunktion*), in which symbols directly express an internal state (desirable or hateful, comforting or threatening); here there is no distinction between sign and signified; (2) the 'representational function' (*Darstellungsfunktion*), in which the symbol points beyond itself but still retains a pictorial character, thereby introducing a distinction between sign and signified while preserving a similarity between them; (3) the purely symbolic 'significational function' (*Bedeutungsfunktion*), in which even the similarity between sign and designated is eliminated. At this level, the 'sign' (*Zeichen*) expresses nothing and represents nothing: its meaning is resolved into the abstract coordination (*Zuordnung*) among other signs (Cassirer 1927, 318). Only in the history of science did this final function come to full fruition. As examples of this general tendency, Cassirer cites Hilbert's conception of mathematics as a 'theory of signs' (*Lehre vom Zeichen*) and Hertz's notion of the 'image theory' (*Bildtheorie*) (318).

This three-function model became the blueprint for the third volume of the *Philosophie der symbolischen Formen,* which was completed in July 1927 but published in 1929.[14] As Cassirer explains in the *Preface*, the volume was intended to provide a 'phenomenology of knowledge,' where the term 'phenomenology' is used in its Hegelian sense (cf. Ferrari 2023; Endres 2023a). Consciousness moves from the sensible world of 'expression' (*Ausdruck*) to the intuitive world of 'representation' (*Darstellung*), and finally to a world of pure 'signification' (*Bedeutung*). Each level does not represent something absolutely alien to the preceding one but is rather the fulfillment of what was already implicit within it. For this reason, Cassirer does not abandon the diachronic three-phase model (see Endres 2020, 3.3; 2021). On the contrary, he seems to employ it to describe the historical evolution among the levels of the synchronic three-function model. In particular, as we shall see, the transition from the intuitive world of language to the scientific world of meaning is only gradual, passing through the mimetic, analogical, and symbolic phases (see below, 3.3).

3.2 Expressive, Presentational, and Significative Functions

It is impossible to do full justice to the 600-page third volume of the *Philosophie der symbolischen Formen* here. However, it is clear that Cassirer regards the concept of number as representing the final, crucial step in the development of the pure *Bedeutungsfunktion* of symbols that lack 'contentual' significance (Heis 2017; Koenig 2025, ch. 3): "All exact concept formation starts from the realm of number, from the determination and designation of the 'natural number series'" (Cassirer 1929, 396/341). Initially, the meaning of number-signs is tied to concrete things, (*Dingzeichen*)—the names for the hand, for the fingers and toes, etc., are used as names for specific numbers. Nevertheless, it is not the hand or the finger itself that is meant, but the fact that they always recur in the same 'order.' The sign, as it were, tears itself away from the sphere of things in order to become an order-sign (*Ordnungszeichen*), indicating an 'earlier' or 'later' step. The next stage consists in embedding the actual, individual number-signs within the sequence of all *possible* numbers by repeatedly associating each member with a unique successor (396–398/341–343). The number-signs thus

[14]Cassirer had initially hoped to publish the third volume together with a concluding section taking a stance toward contemporary philosophy (Cassirer 1929, IX/xvi). Yet this project stalled, prompting Cassirer to publish the third volume in its 1927 version. The preparatory materials intended for this concluding section were later published posthumously as the first volume of Cassirer's *Nachlass* (ECN).

become pure positional signs (*Stellenzeichen*), whose meaning rests entirely on their 'positional value' (*Stellenwert*) within the sequence, not on any relation to what they 'designate' (*bezeichnen*). In this way, after a detour through the most recent advances in the foundations of mathematics,[15] Cassirer ultimately gets back to defend a "purely ordinal derivation of the numerical concept" (Cassirer 1929, 449/387). The very core of number in general is pure 'sequencing' (*Reihung*), the sequential generation as such. The expansion of the concept of number—from whole numbers to fractions, rational, irrational, and imaginary numbers—is nothing but the manifestation of the same "universal *sequence-principle*" (470/404).

Indeed, for Cassirer, the "constitution of the *numerical domain*" is regarded "as the genuine prototype of an order that can be justified purely *constructively*" (483/415). However, Cassirer continues, once we enter the realm of physics, this process of sequence-formation seems to face an insurmountable obstacle. Mathematics offers us a constructive manifold, a sequence of *possible* elements that develop according to a defined 'sequence-principle'; in physics, by contrast, we are confronted with an irreducible multiplicity of *actual* elements given to us as such (472–493/424–447). The goal of physics is ultimately to bridge the gap between the 'constructive sequence-form' (*konstruktive Reihenform*) of mathematics and the 'empirical sequence-form' (*empirische Reihenform*). Physics can grasp an empirical sequence of actual elements a, b, c, d only if it can recognize them as part of a sequence of possible elements conceived as members of a constructive sequence $x_1, x_2, x_3, x_4, \ldots$ governed by a fixed rule. As Cassirer had put it earlier, to understand a phenomenon is to find something that plays the role of the general term of a sequence: "We attempt to order the elements $a, b, c, d \ldots$ in such a way that they can be thought of as members in a sequence $x_1, x_2, x_3, x_4 \ldots$ which is characterized by a determinate 'universal member' " (482/414). From the substitution of an arbitrary natural number n into this general member, one can calculate the individual member of the sequence that stands in the nth place. Once it is seen as the result of this calculation, the individual element appears as the 'symbolic' expression of the entire sequence.

As one can see, Cassirer seeks to give a symbolic 'twist' to the insight he had already articulated in the 1910s: that physical concept-formation is ultimately sequence-formation, a process of relating the empirical form of the sequence to the mathematical one, modeled on the number sequence. According to Cassirer, both dogmatic empiricism and dogmatic rationalism are unable to explain how these sequences can become processes of identification with one another. The 'philosophy of symbolic forms,' Cassirer argues, addresses the issue from a different perspective. The relation between the mathematical and the empirical sequence is not the result of a 'process of identification,' but

[15]The details cannot be discussed here. However, it is not surprising that Cassirer rejects Frege–Russell's logicism, according to which the concept of number is grounded in the existence of equinumerous sets (Cassirer 1929, 439/378). From the late 1920s onward, Cassirer shows greater sympathy for Brouwer's view that the concept of number rests on the mind's capacity to conceive the indefinite iteration of an act. Yet, in this way, Frege–Russell's realism seems at first to be replaced by a form of psychologism. For this reason, Cassirer explicitly prefers Weyl's formulation of intuitionism (430–431/360–361), based on the 'modal' distinction between the 'real' and the 'possible,' rather than on the psychological act of counting. Hilbert's 'formalism' sought to counter both tendencies by reducing mathematics to a system of 'signs.' However, in Cassirer's view, Hilbert takes 'signs' as intuitively given. By contrast, for Cassirer, mathematical signs are positional signs (*Stellenzeichen*), that is, placeholders for positions in an ordered sequence (449/386–387). Once again, the latter notion is taken as primitive.

of a ‘process of substitution’ (*Substitutionsprozeß*): ideal limit-constructs (*Grenzgebilde*) are placed in lieu of the empirical data of sensory perception (Cassirer 1929, 500/429). A concrete physical system is replaced by an idealized replica characterizing how the system would have behaved had the idealized conditions been met. The history of physics provides innumerable examples of such procedures, as can be seen from concepts like the ‘rigid body,’ the ‘ideal gas,’ the ‘incompressible fluid,’ or the ‘perfect thermodynamic cycle,’ among others. According to Cassirer, all these are nothing but applications of Klein’s requirement to replace “the limitation of the accuracy of intuition with unlimited accuracy” (Klein 1892, 355).

The most elementary ‘physiognomic’ experience of a concrete physical system, for instance, of a block sliding on an inclined plane, might at first possess an *Ausdrucksfunktion*—the block might appear, say, ‘recalcitrant.’ The world of sense perception emerges once these ‘feeling qualities’ are progressively separated from proper ‘perceptual qualities’ such as color, roughness, or warmth. However, the scientific treatment of the problem becomes possible only when this complex of qualities is replaced by a simplified ‘geometrical scheme.’ The block becomes a mass point without extension, the plane is rendered perfectly rigid and frictionless, the surrounding medium exerts no drag forces on the block, and so forth. The laws of physics “are no longer concerned with the real movements of given bodies, but relate to this limiting idea [*Grenz-Idee*], and only through it do they reach a concrete empirical content” (Cassirer 1929, 501/430), not with an *actual* inclined plane, but with a *possible* one: what an inclined plane *would be* in a frictionless environment, with the mass of the block concentrated at an extensionless point, etc. This ‘scheme,’ however abstract, still bears some resemblance to the real system; it still performs a *Darstellungsfunktion*. Yet it serves only as a stepping stone, a ‘mediator’ between the ‘world of approximation’ and the ‘universe of precision’.[16]

The ‘sensitivity’ of our sense organs has a threshold. It may be replaced by the ‘sensitivity’ of our physical instruments, which, however, also deliver only approximate ranges of values. Only in the ideal field of arithmetic does no such threshold exist. In Klein’s language, the ‘approximation mathematics,’ which deals with approximate values, is replaced by ‘precision mathematics,’ which calculates with real numbers: “It is transpositions of this sort that enable us to replace the pseudocontinuum, given to us in sense perceptions, with a genuine continuum. And it is through the relation to such a genuine continuum—ultimately to the fundamental sequence which analysis defines as the ‘continuum of all real numbers’—that perception is made ripe for mathematical-physical treatment and determination” (501/430). Once this substitution is accomplished, not only the ‘sensuous qualities’ are removed, but even abstract geometrical ‘schemes’ as tangible representations are abandoned and fully replaced by numerical ‘symbols’ that possess a pure *Bedeutungsfunktion*. The behavior of the sliding block is determined by a small number of parameters—say, space, time, and velocity—which bear no resemblance to the actual block.

Of course, for most purposes, this ‘ideal’ system cannot be considered a good substitute for the real system. To arrive at the former, the values of further parameters that distinguish the behavior of ‘real’ inclined planes from that of ‘ideal’ ones must be fixed: the inclined plane’s specific dimensions, angle of inclination, cross-sectional area facing the airflow, drag coefficient, friction coefficient, and so on. If, for traditional empiricism, the concrete object was a ‘bundle of perceptions,’ Cassirer concludes,

[16] I borrow this opposition from Koyré (1948).

physics transforms it into a 'bundle of numbers' (Cassirer 1929, 508/436). At first sight, the multitude of 'sense qualities' is simply replaced by a colorful sequence of 'numerical values.' However, this transformation gives the notion of object a new dimension. Physics succeeds in establishing a 'hierarchy' among the values of material constants, progressively replacing them with universal constants that do not depend on the properties of any particular material: "In all these relations and connections, the intervention of the universal schematism of the number-concept [*des Zahlbegriffs*] is decisive. Number, one might say, functions as the abstract medium in which the various sensory spheres meet, and in which they relinquish their specific dissimilarity" (511/438; tr. mod.). A specific system of universal natural constants is posited—for example, the speed of light, the charge, or the mass of an electron, etc. Although these make no claim to similarity with the object, they can nevertheless be said to possess physical reality by virtue of their 'invariance,' the fact that they retain the same values across different theories (see also Cassirer 1938a, 112–120).

3.3 Analogical, Mimetic, and Symbolic Phases

According to Cassirer, just as the world of 'representation' emerges from mere 'expression' through language, it ultimately transcends itself and passes into a world of pure 'meaning' (Cassirer 1929, 386/448). It is only at this stage that scientific knowledge begins to take shape. However, as in the earlier shift from 'expression' to 'representation,' Cassirer argues that the transition from 'representation' to 'meaning' is not abrupt: "Rather, thought clings desperately [*mit klammernden Organen*] to the sphere of representation, even though driven beyond it by the inner law and necessary trend of its own unfolding" (386/448). To explain this development, in concluding the third volume of the *Philosophie der symbolischen Formen*, Cassirer returns to the diachronic three-stage model introduced in the first volume, now applying it to scientific knowledge. At this point, the earliest forms, the world of expression, have already been left behind. Yet in the shift from 'representation' to 'meaning,' science proceeds through a mimetic and then an analogical phase, before reaching its fully symbolic dimension, in which it "relinquishes all similarity to the empirical, sensuous world" (526/452).

Cassirer roughly identifies these phases with the names of Aristotle, Descartes, and Leibniz. Aristotelian physics is the most exemplary expression of the mimetic phase in the formation of scientific concepts. At this stage, the physiognomic qualities that dominate the mythical world have been replaced by sensuous qualities, which are ultimately objectified as properties of things. The qualities of 'heavy' and 'light,' 'cold' and 'warm,' 'moist' and 'dry,' etc., are not merely accidental properties of the basic elements; rather, they express their inner essence, their 'substantial form.' With early modern science, all qualities of sensation are banished from the objective view of nature and replaced by geometric 'schemes,' which, though abstract, still retain the ambition to depict the behavior of real, albeit unobservable, entities (atoms, ether waves, etc.). At this point, scientific thought enters an 'analogical phase.'

Cartesian physics, by aiming to reduce matter to space, represents the culmination of this phase: "Where spatial intuition forsakes us, where phenomena cannot be constructed *geometrically*, our insight into them ceases" (532/455). Whereas Descartes criticized the Aristotelian-scholastic failure to recognize the limitations of 'sensation,' Leibniz attacked Cartesian physics for failing to recognize the limitations of 'intuition.' The reality of the phenomenon, its objective nature, no longer rests on merely geometrical determinations: “For Leibniz came not from geometry but from *arithmetic*” (Cassirer 1929, 532/456; emph. mine). With Leibniz, physics enters the next, ‘symbolic phase.’ According to Cassirer, his physics aimed to free thought not only from sensory qualities but also from pictorial imagination.

However, Leibniz’s point of view did not initially prevail. Surprisingly, the ‘Newtonian’ Kant appeared to Cassirer as a kind of philosophical setback (420–421/362): Kant rejected Locke’s ‘sensification’ of knowledge, yet also resisted its ‘intellectualization’ by Leibniz: “the Leibnizian tendency toward intellectualization was followed by Kant’s concept of pure intuition” (534/458). In Cassirer’s view, from that point onward, the dialectic between ‘concept’ and ‘intuition,’ between ‘arithmetic’ and ‘geometry,’ came to define the history of physics well into the nineteenth century. The ‘physics of models’ is progressively replaced by a ‘physics of principles.’ This distinction, which made its first tentative appearance in the relativity book section 2, reemerges here with greater prominence. In nineteenth-century physics, Cassirer argues, “the primacy of *principles* over *models* is recognized and carried out in every particular” (540/463).

Whereas continental physicists still interpreted mechanical models realistically and considered them explanatory, British physicists had already begun to present such models as merely illustrative (ECN, 4:139–140). For this reason, Maxwell could employ various, even contradictory, models of the ether. Indeed, as Poincaré observed, for Maxwell the mechanical explanation of electricity and magnetism did not require the construction of actual mechanical models but only their ‘possibility.’ This possibility was linked to the principle of least action, from which the principle of energy conservation could be derived. However, it soon became clear that, although the principle of least action had been discovered within mechanics, it was more fundamental than mechanics itself. It proved to be “particularly fruitful in its application to ‘extramechanical physics’ ” (Cassirer 1929, 541/464), as demonstrated by the derivation of the equations of electrodynamics. The ‘physics of models’ thus gave way to the ‘physics of principles.’

According to Cassirer, the history of physics displays “a definite and unmistakable line runs from the principle of the conservation of energy to the general principle of relativity” (537/460). This historical progress (*Fortgang*) from models to principles is, in turn, grounded in the epistemological priority (*Vorrang*) of principles over models (538/463). Following Planck, Cassirer sees this process once again as a progressive de-anthropomorphization, in which not only are sensuous qualities overcome through intuitive-geometrical models, but these models themselves are transcended by means of abstract-algebraic requirements (Cassirer 1931, 125–126): “The schematism of images has given way to the symbolism of principles” (Cassirer 1929, 545/467). Unlike models, which retain a schematic–iconic dimension, principles possess a purely abstract–symbolic character, as they renounce any claim to similarity with their referents and are defined solely by their role within the conceptual framework of a physical theory. Yet through this renunciation the ‘objectivity’ of physics is not undermined: “For this objectivity is no problem of representation [*Darstellung*]; it is a pure problem of meaning [*Bedeutungsproblem*]. What we call the object is no longer a schematizable, intuitively realizable ‘something’ with definite spatial and temporal predicates; it is a point of unity to be apprehended in a purely intellectual way” (552/473).

Conclusion. The Determinism of Number

In concluding the third volume of *Philosophie der symbolischen Formen*, Cassirer conceded that the success of Bohr's planetary model of the atom might initially appear to contradict his account of the history of physics. Nevertheless, he warned his readers that the apparent 'intuitivity' (*Anschauulichkeit*) of Bohr's model was at most a provisional stepping stone toward a more abstract theory (Cassirer 1929, 554/474). Indeed, by the time the volume appeared in print in 1929, the new quantum theory was already established. Cassirer likely began planning to write on the topic in the early 1930s, but managed to complete his work only during his Swedish exile (Hansson and Nordin 2006). Already in one of his first lectures in Sweden, one can discern the contours of Cassirer's stance toward the new theory (Cassirer 1936b).[17] Whereas in classical physics intuitive models had become progressively *irrelevant*, with the advent of quantum mechanics the use of such models turned out to be *impossible*. Indeed, Cassirer interpreted Bohr's wave–particle complementarity as a sign that physics has had to give up the hope of exhaustively representing the entirety of natural phenomena with a fixed system of 'images' (144). Nevertheless, he concluded, "[t]he exact mathematical representability of natural phenomena is nowhere called into question by quantum mechanics" (146).

As Cassirer (1936a) put it in *Determinismus und Indeterminismus*, which was finished a few months later, quantum mechanics should not be understood as having engendered a 'crisis of causality' (*Krise der Kausalität*), at least if causality is properly understood not as predictability but as legality (Giovanelli 2025). Rather, Quantum mechanics represents a 'crisis of intuition' (*Krise der Anschauung*), a crisis of the possibility of a model-like grasp of phenomena (Ryckman 2021). However, for Cassirer this crisis was "not surprising, as it had been anticipated prior to the evolution of modern physics by developments in mathematics in the nineteenth century" (Cassirer 1936a, 204/166; tr. mod.).[18] Cassirer developed this parallel between the history of mathematics and the history of physics more extensively in the posthumous fourth volume of *Das Erkenntnisproblem* (Cassirer 1940a), which he began writing just after completing his quantum book (Cassirer to Bermann Fischer, Jun. 27, 1938; ECN, Vol. 18, Doc. 147). Here, Cassirer is explicit in emphasizing that in both nineteenth-century mathematics and nineteenth-century physics "the 'crisis of intuition' became openly apparent" (Cassirer 1936a, 204/166). Indeed, both disciplines underwent a parallel process of *arithmetization* (see also Biagioli 2016, sec. 5.4).

In chapter IV of the final volume of *Das Erkenntnisproblem*, devoted to the foundations of mathematics, Cassirer, alongside the discussion of non-Euclidean geometry, highlighted the emergence of 'pathological' cases (Weierstrass's continuous nowhere-differentiable curve and Peano's space-filling curve) that defy geometrical–spatial intuition but are nonetheless validated by arithmetical proof, presenting them as the motivation for the arithmetization program: "Felix Klein treated it as typical of modern mathematics, in that he recognized clearly and carried through completely this demand for the *arithmetizing of mathematics*" (Cassirer 1940a, 66–67/59–60). In its early phase, the nineteenth-century program of arithmetizing analysis, from Cauchy to Weierstrass, sought to ground analysis in arithmetic through the theory of real numbers. In its later phase, as developed by Cantor and Dedekind, the real numbers were themselves

[17] Later translated as Cassirer 1942a.

[18] Cassirer refers to Hahn (1933).

derived from the natural numbers, thereby advancing the reduction of analysis to arithmetic. This latter achievement became a focal point in the foundational debates on the very nature of number itself at the turn of the twentieth century. In contrast to the 'cardinal number' approach, grounded in the actual existence of classes of equinumerous sets, Cassirer expressed again a clear preference for the 'ordinal number' approach, which conceives numbers as positions in a sequence of possible elements generated by the successor function: "From the process of iteration, the possibility of an infinite progression within a sequence, can be derived the fundamental insights into the natural numbers on which the entire structure of pure mathematics is logically built" (Cassirer 1938a, 74; see also Cassirer 1936c; 1938b, 171–172).

In chapter V, dedicated to the foundations of physics, Cassirer argues that the same dialectic from *Raumbegriff* to *Zahlbegriff* also dominated the entire history of physics (see Cassirer 1938a, 112–120). Galileo famously claimed that the great book of nature is written in the language of mathematics, but this language was ultimately the language of geometry (Cassirer 1940b), whose characters are lines and angles, triangles, circles, and similar figures (Cassirer 1940a, 113/98). In modern philosophy, this geometrical ideal of knowledge found its most radical expression in Descartes' claim that physical objects must be entirely reducible to spatial determinations for exact knowledge of them to be possible (Cassirer 1939). However, starting from the nineteenth century, the tide turned: "Then, *however, an intellectual movement set in that may be compared in a way with the arithmetization of mathematics, which occurred at the same time*" (Cassirer 1940a, 113/98; emph. mine). The focus of theoretical knowledge of nature began to shift: number took the place that had previously been assigned to space. All secure and well-founded knowledge of natural processes was reduced to knowledge of specific numbers. Cassirer resorted to the usual examples (114, 111, 135/116–117), adding that with the advent of quantum theory this transition became irreversible (135/116–117). As Bohr recognized, without a "resignation" with respect to the demand for *Anschaulichkeit*, atomic physics could not be constructed. Heisenberg's matrix mechanics rejected from the outset the spatial representation of atoms, whereas Schrödinger's wave mechanics sought to preserve it but ultimately failed (135/116–117). In the end, the wave function serves as a computational tool for determining the probability of finding them in a particular place or state of motion.

When Cassirer left Sweden in May 1941, he left behind the 400-page manuscript of the fourth volume of *Das Erkenntnisproblem*, which remained unpublished during his lifetime (Hansson and Nordin 2006). However, he clearly brought with him its central message (Cassirer 1942b, 83–84; 1944b). Indeed, in the *An Essay on Man* (Cassirer 1944a)—his last book published during his lifetime, written in American exile—he insists on this point more strongly than perhaps in any of his other writings: "In a historical survey of the development of mathematical thought during the nineteenth century", Cassirer writes again, "Felix Klein declared that one of the most characteristic features of this development is the *progressive 'arithmetization' of mathematics. Also in the history of modern physics we can follow this process of arithmetization*" (219; emph. mine). From Hamilton's quaternions to tensor calculus to Heisenberg's matrices, progressively more abstract schemes were introduced. However, "the general form of number is preserved in all these subsequent schemes" (219). Calculations made on the basis of these schemes involve numbers; they are adopted since they serve to assist in the manipulation of numbers: "In all its single branches physics tended to one and the same point; it attempted to bring the whole world of natural phenomena under the control of number" (Cassirer 1944a, 214).

If we consider the history of physics from this point of view, Cassirer concluded, the tension between classical and modern quantum theory is only apparent (214). To be sure, with the advent of quantum mechanics the metaphysical "*determinism of things*" has to be abandoned, but the methodological determinism, the "*determinism of number*", endures (214; emph. mine). The Laplacean predictability of phenomena could not be maintained; however, by introducing a richer and more flexible mathematical language it was still possible to reduce phenomena "to strict laws and precise numerical rules" (220; emph. mine). Scientists recognize that vast domains of phenomena still resist this reduction, it is not always possible to compress a list of data into rule-governed sequences. Nevertheless, they remain faithful to a 'Pythagorean creed,' that "in number, and in number alone, we find an *intelligible* universe" (211). Of course, the scientist does not give us a logical or empirical proof of this fundamental assumption: "The only proof that he gives us is his work. He accepts the principle of numerical determinism as a guiding maxim, a regulative idea that gives his work its logical coherence and its systematic unity" (219). This vaguely defined 'numerical determinism' is the last faint trace of regulative *a priori* that remains in Cassirer's work (Giovanelli 2022). While claiming to capture a tendency manifested in past physics, it at the same time expresses the minimal assumption without which future physics would not be worth pursuing (Cassirer 1944a, 191).

Cassirer's position can appear disappointing, almost to the point of banality. Yet, it becomes more understandable if one keeps in mind that it is not grounded in a neo-Pythagorean conviction of the 'ontological' preeminence of numbers as the ultimate constituents of reality (77, 211–212), but in the neo-Kantian assumption of their 'epistemological' preeminence. For Cassirer, the sequence of natural numbers encapsulates the two central aspects of concept-formation in general: (a) *structuralism*: each actual individual number is defined not by intrinsic content but by its relation to all other numbers; (b) *constructivism*: this relation is a productive one, namely the successor function, which generates the sequence of all possible numbers. Recent literature tends to emphasize (a) the structuralist–positional dimension (positional identity) of Cassirer's philosophy, attributing to him a form of '*structural realism*'—the view that only structures possess mind-independent (physical or Platonic) reality. By contrast, the constructive–productive aspect (b) of Cassirer's stance (sequential generation) is often dismissed, as it appears to lead to 'psychologism'—as if what were at stake were the mental act of generating a sequence.

However, as this paper has sought to show, both (a) and (b) are essential elements of Cassirer's philosophy of the natural sciences. In this sense, Cassirer might be thought to have defended a form of *structural constructivism.* The latter is neither a metaphysical description of the constituents of reality nor a psychological account of mental activity. Rather, it aims to be a *normative* theory of scientific concept-formation in general: the concepts of natural science can grasp the 'real' only by assigning it a fixed place within the sequence of 'possible' elements generated by a single principle. Instead of, say, classifying *actual* perceptible colors, physics introduces the concept of monochromatic light waves, defined by their position within the continuum of *possible* wavelength values. Ultimately, even the notion of ether vibrations becomes superfluous; what remains is only the functional relation between the numerical components of the electromagnetic field and the numerical values of the space and time coordinates.[19]

[19]This example is borrowed from Weyl (1932), whose position is, in my view, possibly the closest to

Abbreviations

ECN Cassirer, Ernst. 1995–2022. *Nachgelassene Manuskripte und Texte.* Edited by John Michael Krois. 19 vols. Hamburg: Meiner.

ECW Cassirer, Ernst. 1998–2009. *Gesammelte Werke: Hamburger Ausgabe.* Edited by Birgit Recki. 26 vols. Hamburg: Meiner.

References

Biagioli, Francesca. 2016. *Space, Number, and Geometry from Helmholtz to Cassirer.* Berlin: Springer.

———. 2025. "Cassirer on the Concept of Number: A Neo-Kantian Perspective on Dedekindian Abstraction." In Pulte et al. 2025, 38–57.

Cassirer, Ernst. 1902. *Leibniz' System in seinen wissenschaftlichen Grundlagen.* Marburg: Elwert. Repr. in ECW, Vol. 1.

———. 1906a. *Das Erkenntnisproblem in der Philosophie und Wissenschaft der neueren Zeit.* Vol. 1. Berlin: B. Cassirer. Repr. in ECW, Vol. 2.

———. 1906b. "Substanzbegriff und Funktionsbegriff." Probevorlesung an der Königlichen Friedrich-Wilhelms-Universität in Berlin (26. Juli 1906). Repr. in ECN, Vol. 8, 3–16.

———. 1907a. *Das Erkenntnisproblem in der Philosophie und Wissenschaft der neueren Zeit.* Vol. 2. Berlin: B. Cassirer. Repr. in ECW, Vol. 3.

———. 1907b. "Kant und die moderne Mathematik: (Mit Bezug auf Bertrand Russells und Louis Couturats Werke über die Prinzipien der Mathematik)." *Kant-Studien* 12:1–49. Repr. in ECW, Vol. 9, 37–82.

———. 1909. "Rev. of Hönigswald 1906." *Kant-Studien* 14:91–98. Repr. in ECW, Vol. 9, 447-459.

———. 1910a. "Rev. of Cohn 1908." *Deutsche Literaturzeitung* 31:Sp. 2437–2445. Repr. in ECW, 9460–467.

———. 1910b. *Substanzbegriff und Funktionsbegriff: Untersuchungen über die Grundfragen der Erkenntniskritik.* Berlin: B. Cassirer. Eng. trans in Cassirer 1923b.

———. 1912. "Das Problem des Unendlichen und Renouviers 'Gesetz der Zahl'." In *Philosophische Abhandlungen. Hermann Cohen zum 70sten Geburtstag dargebracht (4. Juli 1912),* 85–98. Berlin: Bruno Cassirer.

———. 1913. "Erkenntnistheorie nebst den Grenzfragen der Logik." *Jahrbücher der Philosophie* 1:1–59. Repr. in ECW, Vol. 9, 139–200.

———. 1914. "Die Grundprobleme der Kantischen Methodik und ihr Verhältnis zur Nachkantischen Spekulation." *Die Geisteswissenschaften* 1:784–787 and 812–815.

———. 1916. *Freiheit und Form: Studien zur Deutschen Geistesgeschichte.* Berlin: B. Cassirer. Repr. in ECW, Vol. 7.

———. 1917. "Philosophie des Symbolischen: allg. Disposition." Unpublished manuscript. Ernst Cassirer Papers, GEN MSS 98, Box 24, Folders 440 and 441; pub. in Schubbach 2016, Beinecke Rare Book/Manuscript Library: New Haven, CT.

———. 1919. "Untitled manuscript from 1919." Unpublished manuscript. Ernst Cassirer Papers, GEN MSS 98, Box 25, Folders 476–480, Beinecke Rare Book/Manuscript Library: New Haven, CT.

———. 1920–1921. "[Die philosophischen Probleme der Relativitätstheorie]." 13 lectures at the University of Hamburg October 13, 1920–January 26, 1921. Publ. in (ECN, Vol. 8, 29–116).

———. 1920. "Philosophische Probleme der Relativitätstheorie." *Die Neue Rundschau* 31:1337–1357. Repr. in ECW, Vol. 9, 217–239.

———. 1921a. *Idee und Gestalt: Goethe/Schiller/Hölderlin/Kleist. Fünf Aufsätze.* Berlin: B. Cassirer. Repr. in ECW, Vol. 9, 241–435.

———. 1921b. *Zur Einstein'schen Relativitätstheorie: Erkenntnistheoretische Betrachtungen.* Berlin: B. Cassirer. Eng. trans. in Cassirer 1923b.

———. 1922. "Die Begriffsform im mythischen Denken." In *Studien der Bibliothek Warburg,* vol. 1. Leipzig: Teubner. Repr. in ECW, Vol. 16, 2–73.

———. 1923a. "Der Begriff der symbolischen Form im Aufbau der Geisteswissenschaften." In *Vorträge der Bibliothek Warburg,* edited by Fritz Saxl, I:11–39.

———. 1923b. *Substance and Function, and Einstein's Theory of Relativity.* Edited by Marie Taylor Swabey. Chicago, London: Open Court.

that of Cassirer.

Cassirer, Ernst. 1925a. *Philosophie der symbolischen Formen.* Zweiter Teil: Das mythische Denken. Berlin: B. Cassirer. Repr. in ECW, 12.

———. 1925b. *Sprache und Mythos: Ein Beitrag zum Problem der Götternamen.* Leipzig/Berlin.

———. 1927. "Das Symbolproblem und seine Stellung im System der Philosophie." *Zeitschrift für Ästhetik und allgemeine Kunstwissenschaft* 21:295–322. Repr. in ECW, Vol. 17, 253–282.

———. 1929. *Philosophie der symbolischen Formen.* Dritter Teil: Phänomenologie der Erkenntnis. Berlin: B. Cassirer. Eng. trans. in Cassirer 1957.

———. 1931. "Die Einheit der Wissenschaft." Two lectures given on October 21 and October 28, 1931 for the program *Hochsculfunk* aired by the radio station *Deutsche Welle.* Pub. in (ECN, Vol. 8, 117–134).

———. 1936a. "Determinismus und Indeterminismus in der modernen Physik: Historische und systematische Studien zum Kausalproblem." *Göteborgs Högskolas Årsskrift* 42. Eng. trans. in Cassirer 1956.

———. 1936b. "Die Funktion der Sprache im Aufbau der naturwissenschaftlichen Erkenntnis." Publ. in ECN, Vol. 4, 107–149.

———. 1936c. "Inhalt und Umfang des Begriffs: Bemerkungen zu Konrad Marc-Wogau: Inhalt und Umfang des Begriffs." *Theoria,* 207–232. Repr. in ECW, Vol. 22, 28–31.

———. 1938a. "Ziele und Wege der Wirklichkeitserkenntnis." Pub. in ECN, Vol. 2.

———. 1938b. "Zur Logik des Symbolbegriffs." *Theoria* 4:145–175.

———. 1939. "Die Philosophie im XVII. und XVIII. Jahrhundert: Philosophie. Chronique annuelle, Bd. 5 / Actualités scientifiques et industrielles, Bd. 841." Paris. Repr. in ECW, Vol. 22, 193–283.

———. 1940a. *Das Erkenntnisproblem in der Philosophie und Wissenschaft der neueren Zeit.* Vol. 4: Von Hegels Tod bis zur Gegenwart(1832–1932). Berlin: B. Cassirer. Eng. trans in Cassirer 1950.

———. 1940b. "Mathematische Mystik und mathematische Naturwissenschaft: Betrachtungen zur Entstehungsgeschichte der exakten Wissenschaft." *Lychnos,* 248–265. Repr. in ECW, Vol. 22, 284–303.

———. 1942a. "The Influence of Language upon the Development of Scientific Thought." *The Journal of Philosophy* 39:309–327. Repr. in ECW, Vol. 13, 115–134.

———. 1942b. "Zur Logik der Kulturwissenschaften: Fünf Studien." *Göteborgs högskolas årsskrift* 48.

———. 1944a. *An Essay on Man: An Introduction to a Philosophy of Human Culture.* New Haven: Yale University Press. Repr. in ECW, Vol. 23.

———. 1944b. "The Concept of Group and the Theory of Perception." *Philosophy and Phenomenological Research* 5:1–36.

———. 1950. *The Problem of Knowledge Philosophy, Science, and History since Hegel.* New Haven: Yale University Press.

———. 1956. *Determinism and Indeterminism in Modern physics: Historical and Systematic Studies of the Problem of Causality.* Translated by Theodor Benfey. With a foreword by Henry Margenau. New Haven: Yale University Press.

———. 1957. *The Philosophy of Symbolic Forms: Volume Three: The Phenomenology of Knowledge.* With an introduction by Charles W. Hendel. New York: Yale University Press.

———. 1995–2022. *Nachgelassene Manuskripte und Texte.* Edited by John Michael Krois. 19 vols. Hamburg: Meiner.

Cassirer Bondy, Toni. 2003. *Mein Leben mit Ernst Cassirer.* Hamburg: Meiner (1948).

Coffa, Alberto J. 1991. *The Semantic Tradition from Kant to Carnap: To the Vienna Station.* Edited by L. Wessels. Cambridge: Cambridge University Press.

Cohen, Hermann. 1902. *Logik der reinen Erkenntnis.* Berlin: B. Cassirer. Repr. in Cohen 1977–, Vol. 6.

———. 1977–. *Werke.* Edited by Helmut Holzhey. 19 vols. Hildesheim: Olms.

Cohn, Jonas. 1908. *Voraussetzungen und Ziele des Erkennens: Untersuchungen über die Grundfragen der Logik.* Leipzig: L. Engelmann.

Endres, Tobias. 2020. *Ernst Cassirers Phänomenologie der Wahrnehmung.* Hamburg: Meiner.

———. 2021. "Phenomenological Idealism as Method: The Hidden Completeness of Cassirer's Matrix of the Symbolic." In *The Method of Culture,* edited by Andreas Pollok and Luca Filieri, 121–148. Pisa: Edizioni ETS.

———. 2023a. "Negativity in Cassirer? On the Scope and Limits of a Hegelian Reading of The Philosophy of the Symbolic Forms." In *The Being of Negation in Post-Kantian Philosophy,* edited by Gregory S. Moss, 487–502. Cham: Springer.

———. 2023b. "On the Reality and Value of Technology." *Argumentos de Razón Técnica* 26:51–83.

Endres, Tobias. 2025. “Formvarianz durch Invarianz: Zum Zusammenhang von Metapher, Reihe und Gruppenbegriff.” In *Metapher und Erkenntnis: Beiträge zur Epistemologie der Metapher,* edited by Stefan Feger et al., 103–120. Berlin/Boston: De Gruyter.

Ferrari, Massimo. 2002. “Begriffslehre und mythisches Denken bei Ernst Cassirer: Festschrift für Ernst Wolfgang Orth zum 65. Geburtstag.” In *Die Stellung des Menschen in der Kultur,* edited by Christian Bermes et al., 197–212. Würzburg.

———. 2003. *Ernst Cassirer: Stationen einer philosophischen Biographie: Von der Marburger Schule zur Kulturphilosophie.* Hamburg: Meiner (1996).

———. 2015. “Cassirer and the History of Science.” In Friedman and Luft 2015, 261–284.

———. 2023. “On the Eve of the ‘Philosophy of Symbolic Forms’: Cassirer and Hegel.” *European Journal of Philosophy* 31:1125–1134.

Ferreirós, José. 2023. “Conceptual Structuralism.” *Journal for General Philosophy of Science* 54:125–148.

French, Steven. 2014. *The Structure of the World: Metaphysics and Representation.* Oxford: Oxford University Press.

Friedman, J. Tyler, and Sebastian Luft, eds. 2015. *The Philosophy of Ernst Cassirer: A Novel Assessment.* Berlin/Boston: De Gruyter.

Gawronsky, Dimitri. 1910. *Das Urteil der Realität und seine mathematischen Voraussetzungen.* Marburg: Dissertation.

Giovanelli, Marco. 2016. “Hermann Cohen’s *Das Princip der Infinitesimal-Methode*: The History of an Unsuccessful Book.” *Studies in History and Philosophy of Science* 58:9–23.

———. 2022. “Motivational Kantianism: Cassirer’s Late Shift Towards a Regulative Conception of the a Priori.” *Studies in History and Philosophy of Science* 95:118–125.

———. 2023a. “Cassirer and Energetics: An Investigation of Cassirer’s Early Philosophy of Physics.” *British Journal for the History of Philosophy.*

———. 2023b. “Relativity Theory as a Theory of Principles: A Reading of Cassirer’s *Zur Einstein’schen Relativitätstheorie*.” *HOPOS: The Journal of the International Society for the History of Philosophy of Science* 13:261–296.

———. 2024. “The Philosophical Coming of Age of Science: Euler’s Role in Cassirer’s Early Philosophy of Space and Time.” *Studies in History and Philosophy of Science.*

———. 2025. “Parallel Convergences: Cassirer and Vienna Indeterminism.” *European Journal for Philosophy of Science* 30.

Gower, Barry. 2000. “Cassirer, Schlick and ‘Structural’ Realism: The Philosophy of the Exact Sciences in the Background to Early Logical Empiricism.” *British Journal for the History of Philosophy* 8:71–106.

Hahn, Hans. 1933. “Die Krise der Anschauung: Fünf Wiener Vorträge.” In *Krise und Neuaufbau in den exakten Wissenschaften.* Leipzig/Wien: Franz Deuticke.

Hansson, Jonas, and Svante Nordin. 2006. *Ernst Cassirer: The Swedish Years.* Bern: Lang.

Heis, Jeremy. 2011. “Ernst Cassirer’s Neo-Kantian Philosophy of Geometry.” *British Journal for the History of Philosophy* 19:759–794.

———. 2014. “Ernst Cassirer’s *Substanzbegriff und Funktionsbegriff*: A Detailed Analysis.” *HOPOS* 4:241–270.

———. 2017. “Arithmetic and Number in the Philosophy of Symbolic Forms.” In Friedman and Luft 2015, 123–140.

Holzhey, Helmut. 1986. *Cohen und Natorp.* Vol. 2. Basel: Schwabe.

Hönigswald, Richard. 1906. *Beitraege zur Erkenntnistheorie und Methodenlehre.* Leipzig: Fock.

Ihmig, Karol-Nobert. 1999. “Ernst Cassirer and the Structural Conception of Objects in Modern Science: The Importance of the “Erlanger Programm”.” *Science in Context* 12:513–529.

Jahnke, Hans Niels, and Michael Otte. 1981. “Origins of the Program of “Arithmetization of Mathematics”.” In *Social History of Nineteenth-Century Mathematics,* edited by Herbert Mehrtens et al., 21–49. Boston; Basel; Stuttgart: Birkhäuser.

Klein, Felix. 1872–1973. *Vergleichende Betrachtungen über neuere geometrische Forschungen (Das Erlanger Programm) (1872).* In *Gesammelte mathematische Abhandlungen,* 1:460–497. Berlin: Springer.

———. 1892. *Nicht-Euklidische Geometrie: I. Vorlesung, gehalten während des Wintersemesters 1889–90. – II. Vorlesung, gehalten während des Sommersemesters 1890.* Göttingen.

———. 1895. “Über Arithmetisirung der Mathematik.” *Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen* (Göttingen), 82–91.

Koenig, Daniel. 2024. “From Magnitudes to Real Numbers: Cantor’s and Dedekind’s Number Extensions and Their Reception in the Marburg School of Neo–Kantianism.” In Pulte et al. 2025, 58–76.

———. 2025. *Mathematik als Erkenntnisform: Ernst Cassirers Philosophie der Mathematik und ihre paradigmatische Bedeutung für die Philosophie der symbolischen Formen.* Hamburg: Meiner.

Koyré, Alexandre. 1935. “Études sur Galilée.” *Annuaire 1934-1935. École Pratique des Hautes Études. Section des sciences religieuses,* 53–54.

———. 1948. “Du monde de l’à peu près à l’univers de la précision.” *Critique* 4-28:806–823.

Krois, John Michael. 2009. “The priority of “symbolism” over language in Cassirer’s philosophy.” *Synthese* 179:9–20.

Meyerson, Émile. 1925. *La déduction relativiste.* Paris: Payot.

Natorp, Paul. 1910. *Die logischen Grundlagen der exakten Wissenschaften.* Leipzig Berlin: Teubner.

Petri, Birgit, and Norbert Schappacher. 2007. “On Arithmetization: Origins of the programme and its mathematical meaning.” In *The Shaping of Arithmetic after C. F. Gauss’s Disquisitiones Arithmeticae,* edited by Catherine Goldstein et al., 343–374. Berlin, Heidelberg: Springer.

Pulte, Helmut, et al., eds. 2025. *New Perspectives on Neo-Kantianism and the Sciences.* New York/ Abingdon: Routledge.

Reck, Erich H. 2003. “Dedekind’s Structuralism: An Interpretation and Partial Defense.” *Synthese* 137:369–419.

———. 2020. “Cassirer’s Reception of Dedekind and the Structuralist Transformation of Mathematics: Structuralist Transformation of Mathematics.” In *The Prehistory of Mathematical Structuralism,* edited by Erich H. Reck and Georg Schiemer, 329–351. New York: Oxford University Press.

Rudolph, Enno. 1992. “Sprache zwischen Mythos und Erkenntnis: Zu Cassirers Diagnose der Tragik sprachlichen Fortschritts.” In *Sagen, was die Zeit ist: Analysen zur Zeitlichkeit der Sprache,* edited by Enno Rudolph and Heinz Wismann, 79–92. Stuttgart: J. B. Metzler.

———. 2003. *Ernst Cassirer im Kontext: Kulturphilosophie zwischen Metaphysik und Historismus.* Tübingen: Mohr Siebeck.

Ryckman, Thomas. 1991. “Conditio Sine Qua Non? *Zuordnung* in the Early Epistemologies of Cassirer and Schlick.” *Synthese* 88:57–95.

———. 2021. “Quantum Mechanics As the Ultimate Mode of Symbol Formation.” In *Interpreting Cassirer: Critical Essays,* edited by Simon Truwant, 89–108. Cambridge: Cambridge University Press.

Sakata, Tomoki. 2024. *Die Spannung zwischen Wissenschaft und Mythos: Die Entwicklung des symbolischen Geistes.* Hamburg: Felix Meiner Verlag.

Schiemer, Georg. 2018. “Cassirer and the Structural Turn in Modern Geometry.” *Journal for the History of Analytical Philosophy* 6:182–212.

Schubbach, Arno. 2016. *Die Genese des Symbolischen: Zu den Anfängen von Ernst Cassirers Kulturphilosophie.* Hamburg: Felix Meiner Verlag.

Volkert, Klaus Thomas. 1986. *Die Krise der Anschauung: Eine Studie zu formalen und heuristischen Verfahren in der Mathematik seit 1850.* Göttingen: Vandenhoeck & Ruprecht.

Weyl, Hermann. 1932. *The Open World: Three Lectures on the Metaphysical Implications of Science.* New Haven: Yale University Press. Repr. in Weyl 2009, Ch. 4.

———. 2009. *Mind and Nature: Selected Writings on Philosophy, Mathematics, and Physics.* Edited by Peter Pesic. Princeton: Princeton Univerity Press.

Yap, Audrey. 2017. “Dedekind and Cassirer on Mathematical Concept Formation.” *Philosophia Mathematica* 25:369–389.