On the Structure of Charge in Classical and Quantum Physics

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Abstract

This paper takes up a suggestion that the structure associated with physical charge in quantum theories is best captured mathematically by the collection of charge representations of the quantum theory. In order to assess the evidence for this view, I investigate whether this charge structure has been preserved across theory change from classical to quantum physics. I show that one can analogously take charge structure in classical physics to be captured by the collection of charge representations of a classical theory. But one faces the potential problem that the collections of classical and quantum charge representations appear to *not* be isomorphic. I solve the problem by showing that if one attends carefully to more sophisticated category-theoretic tools for comparing structure, then one finds a precise and substantive sense in which charge structure is preserved from classical to quantum theories. I argue these results lead to greater understanding of inequivalent representations in quantum physics.

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1 Introduction: Charge and Preserved Structure

How radical are the shifts brought about when we accept new scientific theories and reject old ideas? Answering this question is essential for considering how much we should expect future theories to give up on what we currently believe. Existing and past scientific theories serve as a guide for setting our expectations about the future. However, the historical and philosophical literature leaves controversial and unclear just how significant past episodes of so-called scientific revolutions truly are. One finds *anti-realists* arguing that past instances of scientific revolutions do involve radical breaks, and that we ought to expect our current theories to be given up in the future in much the same way. On the other hand, *realists* find continuity across theory change in the very same historical episodes, which they take to support the view that some core part of what we currently believe will be held onto as science progresses.

The advent of quantum theory and its replacement of classical mechanics provides just one example of a supposed scientific revolution (Kuhn, 1984, 1987), and the one I will focus on in this paper. The construction of quantum mechanics of course brought about scientific change, in many ways drastically altering our conception of matter and its interactions. Yet, some also see significant preservation of our previous understanding of physics in the current quantum mechanical worldview (Hesse, 1952). My goal here is to choose just one aspect of classical physics to compare with its corresponding analog in quantum theory—an aspect that I believe has been given insufficient attention in the philosophical literature—and to attempt to assess whether this one aspect has been preserved across the classical–quantum divide.

I choose to focus on the understanding of *physical charge*. It is well known that charge plays a central role through its association with symmetry groups in both classical and quantum physics (Brading and Brown, 2003). I have two reasons to focus on charge. First, contemporary *structural realists* have prominently identified the structure of physical theories with aspects of these symmetry groups, sometimes arguing that the role of symmetry groups in physics even gives us reason to believe in key invariant structural features of our physical theories (Roberts, 2011; French, 2014). Second, my philosophical investigation of charge connects to existing foundational issues in quantum theory surrounding the appearance of inequivalent representations. The reason is that values of physical charge can be thought of as labels for *representations* of the symmetry groups that appear in a physical theory. In quantum theory, where observable quantities are represented by elements of a C*-algebra, the appearance of inequivalent group representations lines up precisely with inequivalent Hilbert space representations in a wide collection of physical systems. Since inequivalent Hilbert space representations have been a focus of philosophical discussions about quantum field theory and quantum statistical mechanics (Clifton and Halvorson, 2001; Ruetsche, 2011), understanding the structure of physical charge also connects to these controversial issues in the foundations of quantum theory more broadly.

In response to existing debates in the foundations of quantum theory concerning whether to understand distinct representations as competing theories, French (2012) puts forward the view that collections of representations themselves (and their relations) can be understood as capturing the structure of quantum theory. I take French's view as an interesting and hitherto unexplored hypothesis. There is of course existing work on understanding charge in the context of quantum field theory (Baker and Halvorson, 2010; Halvorson, 2007). But there has not been sufficient philosophical work on the comparison of charge in classical and quantum physics. French especially leaves unanswered the question of whether the structure captured by the charge representations of a symmetry group have been preserved from classical to quantum physics, which is of particular interest for structural realism. On the other hand, recent work by Manero (2019, 2022) tries to argue that structure is preserved across the classical–quantum divide, but I find these arguments wanting due in part to their emphasis on symmetry groups while ignoring the very structure of representations that connects symmetry groups to associated notions of charge.

In this paper, I will argue that one can do better to show that there is indeed a precise sense in which the structure associated with physical charge is preserved from classical to quantum physics. I will first motivate my project by establishing what I take to be the shortcomings of existing arguments for some kind of structural continuity. All existing discussions ignore what I take to be a fundamental problem in making the comparison: classical charge representations do *not* stand in one-to-one correspondence with quantum charge representations. I will present some background on what I mean by classical and quantum charge representations to substantiate this claim and the issues it raises. This will motivate my search for a new and precise sense in which charge structure has been preserved from classical to quantum physics.

I hope to show that there is a route to a better understanding of the preservation of charge structure. I will take inspiration from two distinct strands in the literature to arrive at my conclusion. First, a number of philosophers have argued that *quantization* procedures provide tools for comparing classical and quantum physics (Feintzeig, 2023).[1] In what follows, we will attempt to use quantization procedures to compare structure across the classical–quantum divide. Second, recent work on theoretical equivalence has proposed using *category-theoretic tools* to compare the structure of different formulations of the same theory (Weatherall, 2019a,b).[2] Moreover, Feintzeig (2019, 2025) and Steeger and Feintzeig (2021b) have argued that category-theoretic tools can be extended to structural comparisons of non-equivalent theories. Feintzeig (2025) even uses category-theoretic tools in combination with quantization to compare the structure of classical and quantum physics, but his results do not bear directly on our understanding of charge structure. The current paper thus builds upon existing work to investigate continuity of the structure for representing physical charge through quantization.

The paper proceeds as follows. In §2, I provide background on the role of symmetries in structural comparisons of theories, and I argue that the use of category-theoretic tools can improve upon existing claims of structural continuity across theory change. In §3, I present a framework for assessing continuity of the structure of physical charge between classical and quantum physics, with appropriate notions of charge representations and charge-structure preserving morphisms in each theoretical context. I then state the main result intended to vindicate the continuity of charge structure: a categorical equivalence between categories of models of classical and quantum physics motivated by the preceding discussion. In §4, I discuss the philosophical significance of the main result for the interpretation of quantum theories, and in particular for controversies surrounding inequivalent representations.

2 Symmetries and Structural Comparisons

Worrall (1989) proposed *structural realism* as an answer to the pessimistic induction—the challenge, to scientific realism often traced to Laudan (1981) based on the fact that many episodes of scientific change involve significant rejection of previous science. Anti-realists take these episodes as evidence that our current scientific theories will be rejected someday as well. Worrall attempts to defend realism by arguing that while historical examples show we often reject the *entities* proposed by past theories, we often maintain a *structural* resemblance to past science in our current theories. Worrall's position is sometimes called *epistemic structural realism* since it advocates that we only believe in the structure posited by our current best theories. Ladyman (1998) and French and Ladyman (2003) argue further for a position they call *ontic structural realism*, which claims that the structural relations provided by our best scientific theories are in some sense fundamental. French (2014), among others, has argued that these structural relations are captured by symmetry groups in physical theories. Both forms of structural realism rely for their justification on claims that these structures are preserved across episodes of theory change.[3] In this section, I present a challenge to existing work establishing these structural comparisons between classical and quantum physics. I do not myself subscribe to

[1] See also Thébault (2016); Yaghmaie (2020); Feintzeig (2020); Steeger and Feintzeig (2021b).

[2] See also Halvorson (2016); Weatherall (2016b, 2021); Barrett (2015, 2018); Bradley and Weatherall (2020); Bradley (2021).

[3] For a small sampling of the large literature on structural realism see, e.g., Bueno (1999), Psillos (2001), Cao (2003), Brading and Landry (2006), French and Saatsi (2006), and Frigg and Votsis (2011).

structural realism in this paper, yet I find the question of whether structure has been preserved across theory change to be sufficiently philosophically compelling to motivate the current investigation.

2.1 Insufficiency of Symmetry Groups

Recent work by Manero (2019, 2022), following the lead of French (2014), identifies the structure of physical theories with their corresponding symmetry groups. A symmetry group can be broadly understood as a collection of maps, defined on some space (of states, trajectories, observables, etc.) associated with the theory, that preserve the relevant structure of that space. Such maps are often called automorphisms, and they form a group under the standard operation of function composition. Manero uses a specific strategy to argue that structure is preserved across theory change: he searches for ways of identifying (parts of) symmetry groups in older theories with (parts of) the symmetry groups of new theories. In trying to assess whether the structure for representing *physical charge* is preserved across theory change, I will argue that Manero's strategy and results are not sufficient. The ultimate reason is that physical charge may correspond to the *same symmetry group* in classical and quantum physics, but the collection of charge representations in classical and quantum physics may look very different.

2.1.1 Classical Representation Theory

To proceed, we will need some background on representations of symmetry groups in classical and quantum physics. We now show how the action of a group of symmetries on the space of states of a classical Hamiltonian system gives rise naturally to the appearance of distinct classical representations. In what follows, we will only consider systems with finitely many degrees of freedom and finite-dimensional phase spaces, so we set aside complications with field theories in our examples.

In classical physics, the relevant space of states for our discussion is the phase space of possible instantaneous configurations and momenta of the system. The phase space carries the mathematical structure of a symplectic manifold S, with an anti-symmetric bilinear symplectic form ω. In the simplest case, for a configuration space given by a manifold M, the phase space consists in the cotangent bundle T^*M with the canonical symplectic form $\omega = \sum_i dp_i \wedge dq_i$ in canonical coordinates $(p_1, ..., p_n, q_1, ..., q_n)$. A physical charge is associated with a Lie group of symmetries G acting on M by diffeomorphisms and preserving the Hamiltonian, so that they may be understood as symmetries of the dynamics. Each element of the Lie algebra $X \in \mathfrak{g}$ corresponds to a vector field ξ^X on M giving the infinitesimal group action defined by

$$\xi^X_q(f) := \frac{d}{dt}_{|t=0}(f(e^{tX}q)) \tag{1}$$

for all $f \in C^\infty(M)$. In other words, ξ^X is the vector field whose integral curves trace out the orbits of the group G, obtained above by exponentiating elements of the Lie algebra $\mathfrak{g}$. Letting $\mathfrak{g}^*$ denote the dual vector space to the Lie algebra $\mathfrak{g}$, one can define a corresponding *momentum map* $j : T^*M \to \mathfrak{g}^*$ by

$$j(\eta_q)(X) := \xi^X_q(\eta_q), \tag{2}$$

for all $X \in \mathfrak{g}$ and $\eta_q \in T^*_qM$. For any $X \in \mathfrak{g}$, the function $j_X \in C^\infty(M)$ defined by

$$j_X(\eta_q) = j(\eta_q)(X) \tag{3}$$

for all $\eta_q \in T^*_qM$ is a Noether charge, which is conserved under the dynamics of any G-invariant Hamiltonian.

With a momentum map in hand, one can proceed to construct the Marsden-Weinstein reduction (Marsden and Weinstein, 1974) of the phase space T^*M for a point $\theta \in \mathfrak{g}^*$ as the space $P_\theta := j^{-1}(\theta)/G_\theta$, where G_θ is the isotropy group of θ, i.e., the set of group elements that leave θ invariant under the coadjoint action. If θ satisfies sufficient regularity conditions, then P_θ is itself a symplectic manifold that can be understood as the phase space of a Hamiltonian system. The reduction procedure essentially uses the constraint that the dynamics is invariant under the group action to infer the existence of conserved quantities of the motion. By fixing those conserved quantities as constant, one reduces the number of degrees of freedom needed to analyze the dynamical evolution of the system, thus yielding the lower-dimensional phase space P_θ.[4]

Generally a reduced phase space (arising from the quotient by a symmetry group) is not a symplectic manifold, but rather a *Poisson manifold* P with an antisymmetric bilinear Poisson bracket $\{\cdot,\cdot\}$ defined on $C^\infty(P)$. A symplectic form ω on P gives rise to a Poisson bracket

$$\{f, g\} := \omega(\xi_f, \xi_g) \tag{4}$$

for all $f, g \in C^\infty(P)$, where the Hamiltonian vector field ξ_f is defined by $\omega(\xi_f, \cdot) = df(\cdot)$, and likewise for g. On the other hand, not all Poisson manifolds are symplectic because a Poisson bracket can have singularities that prevent inverting it to obtain a symplectic form.

Example 1. Let $M = G$ be a Lie group itself with action by left-multiplication (i.e., translation in G). In this case, the reduction of the phase space T^*G by the induced action of G is a Poisson manifold $P = T^*G/G$ that can be identified with $\mathfrak{g}^*$, carrying the Poisson bracket

$$\{f, g\}(\theta) := \theta\left([df_\theta, dg_\theta]\right) \tag{5}$$

for $f, g \in C^\infty(\mathfrak{g}^*)$ and $\theta \in \mathfrak{g}^*$. In this expression, we understand the covectors df_θ, dg_θ on the phase space $\mathfrak{g}^*$ as elements of $\mathfrak{g} \cong T^*_\theta\mathfrak{g}^*$ (since $T_\theta\mathfrak{g}^* \cong \mathfrak{g}^*$ and $\mathfrak{g}^{**} \cong \mathfrak{g}$). This Poisson bracket does not arise from a symplectic form. However, a Poisson manifold always foliates into a disjoint union of symplectic leaves $P = \bigcup_O S_O$, where the symplectic leaves are immersed submanifolds $S_O \hookrightarrow P$. In the example of the Poisson manifold $T^*G/G \cong \mathfrak{g}^*$, the symplectic leaves are *coadjoint orbits*.

Example 2. Suppose $P \to M$ is a principal fiber bundle with typical fiber G, whose physical interpretation is that curves on M represent possible trajectories of a particle interacting with a background Yang-Mills gauge field associated with the symmetry group G.[5] One can lift the natural action of G on the fibers by the pull-back to obtain an action of G on the cotangent bundle T^*P. The quotient space $(T^*P)/G$ is called the *universal phase space*, forming a Poisson manifold that again foliates into symplectic leaves S_O. Each leaf S_O is a fiber bundle with fiber given by a coadjoint orbit O in $\mathfrak{g}^*$ and base space T^*M, so that again there is a one-to-one correspondence between reduced phase spaces and coadjoint orbits. Each symplectic leaf represents the phase space of a particle with fixed charge, so that the coadjoint orbits are in one-to-one correspondence with possible values of charge associated with the interaction of the particle and the background gauge field.[6]

Example 3. Suppose a Lie group G acts on a manifold M. Consider the trivial bundle $\mathfrak{g}^* \times M$. This space can be given a canonical Poisson bracket making it into a Poisson manifold understood as the phase space of a system moving in M with symmetries G (Krishnaprasad and Marsden, 1987). Symplectic leaves in $\mathfrak{g}^* \times M$ are orbits under the diagonal action of G by the coadjoint action on $\mathfrak{g}^*$ and the given action on M.

[4] For philosophical literature on constraints and reduction, see Butterfield (2007); Bradley (2025a,b).
[5] See Gilton (2022) for background and discussion.
[6] For technical details, see Sternberg (1977); Weinstein (1978); Landsman (1993, 1995).

The structure we wish to extract from the foregoing is that the classical universal phase space of a system with associated dynamical symmetries is generally a Poisson manifold P, which nevertheless carries certain kinds of representations on symplectic manifolds. These classical representations, called *symplectic realizations* are given by smooth maps $j : S \to P$ from a symplectic manifold S to the Poisson manifold P, which preserve the Poisson structure determined by the symplectic form on S. Symplectic realizations include, but are not limited to, the symplectic leaves $S_O \hookrightarrow P$, which correspond with the possible values of a generalized physical charge of a classical system.

2.1.2 Quantum Representation Theory

Having a framework for representing classical charge representations in hand, we move on to the corresponding conception of charge representations in quantum theory. In quantum theory, it is perhaps even more familiar than in classical physics to treat charge as arising from *unitary representations* of a Lie group G on a Hilbert space of quantum states. Irreducible representations of G correspond to basic charge values, which serve as the building blocks of other composite representations. It is well known how Casimir invariants of the group G correspond to conserved quantities that label each irreducible representation by scalar values.[7] Irreducible representations can be thought of as analogous to the symplectic leaves from classical mechanics.

It is perhaps less well known (among philosophers) how to understand the structure of these group representations in a framework for representing states and observables of a quantum theory. As we will see later, quantization allows one to associate classical observables—functions on phase space with operators in a C*-algebra, which motivates what is ultimately an elegant theory of representations of C*-algebras coming from the action of symmetry groups. A *Hilbert space representation* of a C*-algebra $\mathfrak{A}$ is a *-homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$. How do these maps between spaces of operators, or physical observables, correspond to our previous notion of representation in classical physics? For a symplectic realization $j : S \to P$ understood as a map between spaces of classical states, one immediately generates by the pull-back a map between observables $j^* : C^\infty(P) \to C^\infty(S)$. It is this dual map that we take to be most closely analogous to a Hilbert space representation.[8] As before, we restrict attention to only systems with finitely many degrees of freedom here and set aside any complications with field theories. Notice that inequivalent Hilbert space representations appear in the following examples even without any field-theoretic considerations, or in other words without moving to infinitely many degrees of freedom.

Example 4. A Lie group G gives rise to a non-commutative C*-algebra of quantum observables associated with the symmetries and conserved quantities defined by G. The *(reduced) group C*-algebra*[9] $C^*(G)$ can be constructed from a dense subspace consisting of functions $f, g \in C_c^\infty(G)$ with multiplication acting like convolution

$$(f \cdot g)(x) := \int_G f(y) g(y^{-1}x) dy, \tag{6}$$

where dy is the invariant Haar measure on G.[10] One can define a C*-norm on $C_c^\infty(G)$ by considering its action on $L^2(G)$ in the *left regular representation* π^G given by

$$\left(\pi^G(f) \cdot \psi\right)(x) := \int_G f(y) \psi(y^{-1}x) dy \tag{7}$$

[7] These ideas on selection rules and sectors famously trace back to Wigner (1939, 1959).

[8] Likewise, the classical reduction procedure can be performed equally well on state spaces or observable algebras. Reduction by a symmetry group can be understood as either the construction of a new phase space as in §2.1.1, or equivalently as a restriction to the C*-algebra of invariant observables under the action of the symmetry group.

[9] In what follows, we drop the modifier "reduced" and refer to $C^*(G)$ as the *group C*algebra*, but generally this is only identical with the reduced group C*-algebra in case the group G is *amenable*.

[10] For our purposes, we restrict to the simple case of *unimodular groups* so that we need not distinguish left Haar measure from right Haar measure. The construction can be generalized.

for $\psi \in L^2(G)$. One then obtains the group C*-algebra as the completion $C^*(G) = \overline{C_c^\infty(G)}$ of the convolution algebra with respect to the operator norm. Hilbert space representations of the C*-algebra $C^*(G)$ of observables associated with the symmetry group G stand in one-to-one correspondence with continuous unitary representations of G itself. Hence, irreducible representations of the observable algebra $C^*(G)$ also carry labels corresponding to values of physical charge.[11]

Example 5. Consider now the quantum theory of a principal fiber bundle $P \to M$ with typical fiber G, representing a particle interacting with a background Yang-Mills gauge field. The action of G on P lifts to a unitary action on the Hilbert space $L^2(P)$. Taking the observables in the quantum theory to be G-invariant operators on $L^2(P)$, one can use the C*-algebra of all G-invariant compact operators $\mathcal{K}(L^2(P))^G$. It is known that this C*-algebra has a tensor product structure given by the isomorphism $\mathcal{K}(L^2(P))^G \cong \mathcal{K}(L^2(M)) \otimes C^*(G)$. Since there is a unique irreducible representation of the C*-algebra $\mathcal{K}(L^2(M))$ of compact operators on the spatial wavefunctions $L^2(M)$, the irreducible representations of the C*-algebra of observables $\mathcal{K}(L^2(P))^G$ correspond one-to-one with the irreducible representations of the group C*-algebra $C^*(G)$, which forms the other factor in the tensor product decomposition. Just as in the classical case, the irreducible representations of the algebra of quantum observables for a particle in a Yang-Mills field correspond also with the irreducible representations of the group itself. The very same possible values of physical charge determined by the group G label the distinct *charge sectors* of the quantum theory—the disjoint Hilbert spaces of wavefunctions with different values for the Casimir invariants.[12]

Example 6. Consider a configuration space given by a manifold M carrying the action of a Lie group of symmetries G. The *(reduced) transformation group C*-algebra*[13] $C^*(M, G)$ is generated from $C_c^\infty(M \times G)$ with multiplication defined by

$$(f \cdot g)(q, x) = \int_G f(q, y) \cdot g(y^{-1}q, y^{-1}x)\, dy \tag{8}$$

for $f, g \in C_c^\infty$, where dy is the Haar measure on G. Then $C^*(M, G) = \overline{C_c^\infty(M \times G)}$ is the completion in a suitable operator norm. Irreducible representations of $C^*(M, G)$ are labelled by points $q \in M$ and unitary representations of the isotropy group G_q leaving q invariant (Effros and Hahn, 1967).

As in the classical case, the upshot is that the entire state space of a C*-algebra of quantum observables, analogous to a classical phase space given by a Poisson manifold, decomposes in terms of state spaces given by irreducible Hilbert space representations. Those irreducible Hilbert spaces correspond to possible values of generalized physical charge, analogous to the symplectic leaves of a Poisson manifold in the classical setting.

2.1.3 Quantization of Charge

Despite the analogies between classical and quantum representations outlined in the previous sections, there is a significant difference between the corresponding theories. While the range of values of physical charge arising from the action of a Lie group is typically *continuous* in classical physics, in contrast it is typically *discrete* in quantum physics. This is the phenomenon known as the quantization of charge.

One can see this already in the example of the Lie group $G = U(1)$. This abelian group has Lie algebra $\mathfrak{g} \cong \mathbb{R}$, which one can see as the tangent space to a circle. Hence, the phase space is the Poisson manifold $\mathfrak{g}^* \cong \mathbb{R}$. It then follows from the commutativity of the Lie group action that coadjoint orbits are single points $q \in \mathfrak{g}^*$ corresponding to the conserved value of electric charge. The real-valued charge associated with $U(1)$ is thus continuous.

[11] See Landsman (1998) for background and technical details.
[12] See Landsman (1993) for more details.
[13] Our discussion below drops the modifier “reduced” just as for the group C*-algebra.

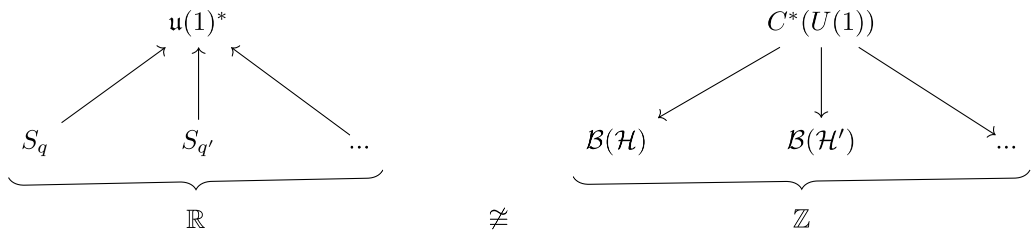

Figure 1: The collections of classical charge representations and quantum charge representations for the symmetry group $U(1)$ cannot be put in a one-to-one correspondence.

On the other hand, Hilbert space representations of $C^*(G)$ in the quantum theory correspond to unitary representations of G. Irreducible unitary representations of $G = U(1)$ are given by the action of $U(1)$ as unitary operators on a one-dimensional Hilbert space due to the commutativity of the group. Since the unitary operators on a one-dimensional vector space also form a copy of $U(1)$, each irreducible unitary representation of $U(1)$ corresponds to a group homomorphism $\rho : U(1) \to U(1)$. Denoting arbitrary group elements as $e^{i\theta} \in U(1)$, since $e^{2\pi i} = 1$, we must have $\rho(e^{2\pi i}) = \rho(1) = 1$, which constrains the homomorphism ρ to have the form $\rho(e^{i\theta}) = e^{ik\theta}$ for a fixed integer $k \in \mathbb{Z}$. Thus, irreducible continuous unitary representations of $U(1)$, and hence, irreducible representations of $C^*(U(1))$ are in one-to-one correspondence with the integers. The physical interpretation here is that the value of electric charge in quantum mechanics is constrained to be a discrete multiple of the fundamental electron charge.

Now we can see a challenge for understanding the continuity of charge structure from classical to quantum physics. Specifically, there is a challenge for anyone who thinks that the appearance of isomorphic symmetry groups in classical and quantum physics provides a sufficient condition for showing that charge structure is preserved. Classical charge values are indexed by coadjoint orbits, which form classical representations of a phase space. Quantum charge values are indexed by irreducible unitary representations, which form quantum representations of an algebra of observables. For even the Lie group $U(1)$, classical charge values and representations are indexed by continuous values in $\mathbb{R}$ while quantum charge values and representations are indexed by discrete values in $\mathbb{Z}$. Classical and quantum charge representations hence cannot be put in one-to-one correspondence with each other (See Fig. 1). It appears that what one might naively take to be the structures of classical and quantum charge—the collections of classical and quantum charge representations—are thus non-isomorphic. If isomorphism is our standard of sameness of structure (an assumption we will later question), then classical and quantum physics would appear to have different charge structure even when they carry the same symmetry groups. Hence, one should doubt whether sameness of symmetry group is sufficient for establishing preservation of charge structure.

One might be surprised by my claim that classical and quantum representations cannot be put in correspondence if one has some familiarity with representation theory. After all, the celebrated "orbit method" due to Kirillov (2004) gives a procedure to construct unitary representations of a group from coadjoint orbits in the dual of the Lie algebra. Why does this method not suffice to establish a correspondence between classical and quantum representations? As one can see in the example of $U(1)$, the orbit method requires some restrictions on the coadjoint orbits, and it is not possible to put all coadjoint orbits in $\mathfrak{u}(1)^*$ in bijective correspondence with unitary representations of $U(1)$. One can take only the integral coadjoint orbits, or one can construct many-one correspondences by, e.g., rounding each real number to its closest integer. But since $\mathbb{R}$ and $\mathbb{Z}$ have different cardinalities, no method can possibly establish a one-to-one correspondence between classical and quantum representations.

Why has this issue been overlooked? Existing literature (Manero, 2019, 2022; French, 2014) attempts to discern structural continuity by merely comparing symmetry groups in classical and quantum physics themselves. While this is perhaps helpful for some purposes, charge structure is associated with the collection of charge representations rather than the corresponding symmetry group. The foregoing shows that even when one employs the *same* symmetry group in classical and quantum physics, the nature of the classical and quantum representations theories may lead to non-isomorphic collections of charge representations.

French (2012) actually comes very close to this perspective, which we take in the remainder of this paper, in his discussion of inequivalent representations in quantum field theory, where he quotes approvingly from Halvorson, who proposes a view he calls *Representation Realism* and writes, "it is the structure of the category of representations that provides the really interesting theoretical content of QFT" (Halvorson, 2007, p. 783) (more on the notion of a *category* in the next section). Seemingly endorsing this view, French admits, "structural realism has perhaps been guilty of focussing too much on the group-theoretic characterisation of structure in the non-relativistic context and of downgrading or dismissing the ontological significance of the associated representations" (French, 2012, p. 133-134). This gives us reason to assess, in light of the foregoing challenges, whether there is any other sense in which the structure captured by a collection of charge representations is preserved between classical and quantum physics. Recall that so far we have only shown that if we take isomorphism of collections of charge representations as our standard for sameness of charge structure, then the appearance of isomorphic groups in classical and quantum theories is not sufficient to establish preserved charge structure. But we will see next that there is some reason to doubt specifically that isomorphism of collections of charge representations is a good necessary condition for sameness of structure. We will seek in the next section more permissive tools then to investigate when two theories share structure.

2.2 Category-theoretic Tools

Let us then take the view that models of classical and quantum physics should be associated with the structure of their classical and quantum charge representation theories, respectively. Even from this perspective, the fact that these representation theories are not isomorphic is insufficient to settle the question of whether classical and quantum charge share structure. The reason is that if one judges classical and quantum charge representation theories as equivalent only when they are isomorphic, one is appealing to what Barrett (2020, p. 1186) calls the *model isomorphism criterion* for structural equivalence. Barrett provides several counterexamples to illustrate that the model isomorphism criterion of structural equivalence is too strict. We take this to motivate using a more appropriate notion of structural equivalence to compare classical and quantum charge structure by treating classical and quantum theories as categories.

It has now become commonplace in the philosophy of science literature on theoretical equivalence to represent theories not merely as collections of models, but rather as *structured* collections of models called *categories* (Awodey, 2010; Halvorson, 2016; Weatherall, 2019a,b). A category $\mathbf{C}$ consists in a collection of *objects* and a collection of *arrows*, sometimes called *morphisms*, each of which has a *source* object and *target* object. We denote an arrow from source A to target B by $f : A \to B$. When the target of an arrow $f : A \to B$ is the same as the source of an arrow $g : B \to C$, then one can form the composition $g \circ f : A \to C$. Each object A possesses an identity arrow $1_A : A \to A$ whose composition with any other morphism leaves it unchanged. We denote the collection of morphisms between A and B by $Hom_{\mathbf{C}}(A, B)$. For our purposes, a scientific theory can be represented as a category in which the objects are models of the theory and the arrows are structure preserving maps (e.g., embeddings) between models.

Weatherall (2016a) proposes that we should compare theories by comparing the categories representing them via *functors*. A functor F between categories $\mathbf{C}$ and $\mathbf{D}$ consists in two maps: one between objects in $\mathbf{C}$ and objects in $\mathbf{D}$, and another between morphisms in $\mathbf{C}$ and morphisms in $\mathbf{D}$. We denote both maps by F and require that if $f : A \to B$ is an arrow in $\mathbf{C}$ with source A and target B, then $F(f)$ must be an arrow in $\mathbf{D}$ with source $F(A)$ and target $F(B)$, i.e., if $f \in Hom_{\mathbf{C}}(A, B)$, then $F(f) \in Hom_{\mathbf{D}}(F(A), F(B))$. Moreover, we require that F respects arrow composition in the sense that $F(f \circ g) = F(f) \circ F(g)$. More specifically, Weatherall proposes that two theories are equivalent when there is a functor between their associated categories that provides a *categorical equivalence*. A categorical equivalence is a functor $F : \mathbf{C} \to \mathbf{D}$ such that (i) for any two objects A and B in $\mathbf{C}$, the induced map $F : Hom_{\mathbf{C}}(A, B) \to Hom_{\mathbf{D}}(F(A), F(B))$ is bijective and (ii) for each object B in $\mathbf{D}$, there is an object A in $\mathbf{C}$ such that $F(A)$ is isomorphic to B in $\mathbf{D}$ (i.e., there are arrows $f : F(A) \to B$ and $g : B \to F(A)$ such that $f \circ g = 1_B$ and $g \circ f = 1_{F(A)}$). Equivalently, $F : \mathbf{C} \to \mathbf{D}$ is a categorical equivalence just in case there is an "almost-inverse" functor $G : \mathbf{D} \to \mathbf{C}$ such that $G \circ F$ and $F \circ G$ are naturally isomorphic to the identity functors on $\mathbf{C}$ and $\mathbf{D}$, respectively. Here, a *natural isomorphism* between two functors $F, F' : \mathbf{C} \to \mathbf{D}$ is an assignment to each object A in $\mathbf{C}$ of an arrow $\eta_A : F(A) \to F'(A)$ such that for any arrow $f : A \to B$ in $\mathbf{C}$, we have $\eta_B \circ F(f) = F'(f) \circ \eta_A$.

While Weatherall takes categorical equivalence to provide a standard for when two formulations of the *same* theory are equivalent, we note here that it serves equally well as a standard for when two *distinct* theories are structurally equivalent, where the relevant notion of structure is determined by the structure-preserving maps that serve as arrows in the categories. Categorical equivalence generally encodes sameness of structure because it establishes isomorphisms between all Hom sets of structure-preserving maps in the corresponding theories, and so satisfies a strong form of the criteria analyzed by Barrett (2018, 2020) for comparisons of structure. Indeed, Feintzeig (2025) also provides precedent for using categorical equivalence as a standard of structural equivalence between distinct physical theories.

Although Weatherall (2021) later retreats to the claim that categorical equivalence may be a merely necessary, and not sufficient, condition for structural equivalence, many of his concerns can be recast as worries about the appropriateness of the categories chosen to represent the theories we are judging to be equivalent. We hope to provide sufficient motivation for the categories we use to represent the charge structure of classical and quantum theories. We believe this wards off the worries discussed by Weatherall, so that we can take categorical equivalence to provide a standard of structural equivalence in what follows. We will return to the issue of the appropriateness of the categories we use later on, but first we define our categories and provide a categorical equivalence in the next section. We take the categorical equivalence discussed in the next section to demonstrate the preservation of charge structure between classical and quantum physics.

3 Classical and Quantum Induced Representations

So far, we have arrived at a framework for comparing charge structure in classical and quantum physics. We now wish to represent each of classical and quantum physics as a category of models. We will take each model to be associated with a collection of classical or quantum charge representations. Morphisms between models will then be structure-preserving maps between these collections of charge representations. Such a map provides a way of forming an *induced representation* of one model from a representation of another. Induced representations appear prominently in the work of Mackey (1968) on symmetries in quantum theories, which was lifted to a general C*-algebraic framework for quantum theory by Rieffel (1974a,b, 1978) and Green (1978, 1980)—see also Rosenberg (1994). Induced representations in classical physics are perhaps less well known, arising in the work of Xu (1991, 1992)—see also Landsman (1995, 1998).

We strive for some level of completeness in our discussion of the categorical equivalence provided below, although we omit some technical details and background due to constraints of space. Our main goal is to give a conceptual overview of the action of the quantization and classical limit functors for a wide class of models of classical and quantum physics, and to make clear the representation-theoretic structure that they preserve.

3.1 Induction Functors

We aim to define a category $\mathbf{Class}$ of models of classical physics and a category $\mathbf{Quant}$ of models of quantum physics. We take a model of classical physics to be the specification of a Poisson manifold P serving as its phase space. We associate P with its collection of symplectic realizations, which we denote $Rep_{\mathbf{Class}}(P)$. Likewise, we take a model of a quantum theory to be the specification of a C*-algebra $\mathfrak{A}$ of observable quantities. We associate with $\mathfrak{A}$ its collection of Hilbert space representations, which we denote $Rep_{\mathbf{Quant}}(\mathfrak{A})$. So objects of $\mathbf{Class}$ are of the form $Rep_{\mathbf{Class}}(P)$ and objects of $\mathbf{Quant}$ are of the form $Rep_{\mathbf{Quant}}(\mathfrak{A})$.

We note immediately that each of $Rep_{\mathbf{Class}}(P)$ and $Rep_{\mathbf{Quant}}(\mathfrak{A})$ has more structure than a mere set of representations. In fact, we can understand each of these objects themselves as a *category of representations*. Indeed, $Rep_{\mathbf{Class}}(P)$ is a category whose objects are symplectic realizations $j_S : S \to P$ and whose morphisms are smooth symplectic maps $u : S \to S'$ to another symplectic realization $j_{S'} : S' \to P$ satisfying $j_{S'} \circ u = j_S$. On the other hand, $Rep_{\mathbf{Quant}}(\mathfrak{A})$ is a category whose objects are Hilbert space representations $\pi_{\mathcal{H}} : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ and whose morphisms are linear maps $U : \mathcal{H} \to \mathcal{H}'$ to another Hilbert space representation $\pi_{\mathcal{H}'} : \mathfrak{A} \to \mathcal{B}(\mathcal{H}')$ satisfying $U\pi_{\mathcal{H}}(A) = \pi_{\mathcal{H}'}(A)U$. Thus, we take objects in each of $\mathbf{Class}$ and $\mathbf{Quant}$ to be categories of representations.

What remains is to specify a definition of arrows in $\mathbf{Class}$ and $\mathbf{Quant}$. Taking seriously that objects in each category are categories of representations motivates treating arrows as functors between categories of representations, i.e., functors $F : Rep_{\mathbf{Class}}(P) \to Rep_{\mathbf{Class}}(P')$ in $\mathbf{Class}$ and $G : Rep_{\mathbf{Quant}}(\mathfrak{A}) \to Rep_{\mathbf{Quant}}(\mathfrak{A}')$ in $\mathbf{Quant}$ (See Fig. 2). A functor between categories of representations provides a means of constructing induced representations of one model from those of another, so we call these *induction functors*. In both the classical and quantum context, we must limit attention to a certain class of induction functors between categories of representations where the induced representations can be obtained by means of a tangible construction procedure.

In the classical case, induction functors can be determined by *symplectic dual pairs*. An $M - N$ symplectic dual pair for Poisson manifolds M and N is a symplectic manifold S with Poisson maps $j_M : S \to M$ and $j_N : S \to N$ such that for all $f \in C^\infty(M)$ and $g \in C^\infty(N)$, $\{j_M^* f, j_N^* g\} = 0$. Here, one gives N the negative Poisson bracket, or equivalently requires that j_N is an anti-Poisson map with respect to the original Poisson bracket. We depict this situation by a diagram

$$M \longleftarrow S \longrightarrow N, \tag{9}$$

and we often simply refer to the dual pair by its middle space S, leaving the remainder implicit when context is clear.

If we have such a dual pair in hand and we are given a symplectic realization $j : S_j \to N$ of the Poisson manifold N, we can construct an *induced symplectic realization* $j^S : S^j \to M$ of the Poisson manifold M as follows. First, consider the fibered product space

$$S \times_N S_j := \{(x, y) \in S \times S_j \mid j_N(x) = j(y)\}\,. \tag{10}$$

Restricting the product symplectic form to $S \times_N S_j$ gives rise in general to a non-trivial null space $\mathcal{N}_{S\times_N S_j}$, i.e., the collection of tangent vectors in $T(S \times_N S_j)$ that are symplectically orthogonal to all other tangent vectors in $T(S \times_N S_j)$. As long as the dual pair S satisfies some regularity constraints, the Marsden-Weinstein reduction procedure yields a symplectic manifold

$$S^j := (S \times_N S_j)/\mathcal{N}_{S\times_N S_j} \tag{11}$$

carrying a map $j^S : S^j \to M$ defined by

$$j^S\left([(x,y)]\right) := j_M(x) \tag{12}$$

for $[(x,y)] \in S^j$. It then follows that $j^S : S^j \to M$ is a symplectic realization of M. This association of representations of N with induced representations of M even gives rise to a functor $F_S : Rep_{\mathbf{Class}}(N) \to Rep_{\mathbf{Class}}(M)$, the induction functor determined by the symplectic dual pair S (Xu, 1991, 1992; Landsman, 1995).

We note, however, that the induction functor F_S generally does not depend on the full details of a symplectic dual pair S. The only information one needs to construct induced symplectic realizations from S is the image

$$(j_M, j_N)[S] = \{(j_M(x), j_N(x)) \mid x \in S\} \subseteq M \times N, \tag{13}$$

which is a *Lagrangian relation*, i.e., an isotropic and coisotropic submanifold of the product manifold, again with N carrying the negative Poisson bracket. If two symplectic dual pairs $M \leftarrow S \to N$ and $M \leftarrow S' \to N$ give rise to the same Lagrangian relation, then the corresponding induction functors agree, i.e., $F_S = F_{S'}$. For this reason, we will take arrows in $\mathbf{Class}$ to be equivalence classes of symplectic dual pairs giving rise to the same Lagrangian relation.

We must next proceed to define arrows in $\mathbf{Quant}$. In the quantum case, induction functors can be determined by *Hilbert bimodules* (Lance, 1995). An $\mathfrak{A}-\mathfrak{B}$ Hilbert bimodule for a pair of C*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ is a vector space $\mathcal{E}$ carrying a $\mathfrak{B}$-valued inner product $\langle\cdot,\cdot\rangle_{\mathcal{E}}$, a right action of $\mathfrak{B}$, and a left action of $\mathfrak{A}$ by adjointable operators.[14] The inner product is required to be compatible with the right action of $\mathfrak{B}$ by satisfying $\langle\psi,\varphi\cdot b\rangle_{\mathcal{E}} = \langle\psi,\varphi\rangle_{\mathcal{E}}\cdot b$ for all $\psi,\varphi \in \mathcal{E}$ and $b \in \mathfrak{B}$. The space $\mathcal{E}$ is required to be complete in the norm $||\varphi||^2 = ||\langle\varphi,\varphi\rangle_{\mathcal{E}}||$, where the right hand side uses the C*-norm in $\mathfrak{B}$. Finally, the left action of $\mathfrak{A}$ is required to be nondegenerate in the sense that $\mathfrak{A}\mathcal{E}$ is dense in $\mathcal{E}$. We depict this situation by a diagram

$$\mathfrak{A} \rightarrowtail \mathcal{E} \leftarrowtail \mathfrak{B}, \tag{14}$$

and we often simply refer to a bimodule by its middle space $\mathcal{E}$, leaving the remainder implicit when context is clear.

If we have such a Hilbert bimodule in hand and we are given a Hilbert space representation $\pi : \mathfrak{B} \to \mathcal{B}(\mathcal{H}_\pi)$ of the C*-algebra $\mathfrak{B}$, we can construct an *induced Hilbert space representation* $\pi^{\mathcal{E}} : \mathfrak{A} \to \mathcal{B}(\mathcal{H}^\pi)$ of the C*-algebra $\mathfrak{A}$ as follows. First, consider the tensor product space $\mathcal{E}\otimes\mathcal{H}_\pi$, with the inner product

$$\langle\psi\otimes v, \varphi\otimes w\rangle_0 := \langle\pi\left(\langle\varphi,\psi\rangle_{\mathcal{E}}\right)\cdot v, w\rangle_\pi \tag{15}$$

for $\psi,\varphi \in \mathcal{E}$, $v,w \in \mathcal{H}_\pi$ and $\langle\cdot,\cdot\rangle_\pi$ denoting the inner product in $\mathcal{H}_\pi$. This new inner product $\langle\cdot,\cdot\rangle_0$ on $\mathcal{E}\otimes\mathcal{H}_\pi$ in general has a non-trivial null space $N_{\mathcal{E}\otimes\mathcal{H}_\pi}$, so we take the quotient and define

$$\mathcal{H}^\pi := \overline{\mathcal{E}\otimes\mathcal{H}_\pi/N_{\mathcal{E}\otimes\mathcal{H}_\pi}}, \tag{16}$$

where the overline denotes completion relative to the inner product. This Hilbert space $\mathcal{H}^\pi$ then carries a representation $\pi^{\mathcal{E}} : \mathfrak{A} \to \mathcal{B}(\mathcal{H}^\pi)$ of $\mathfrak{A}$ defined by

$$\pi^{\mathcal{E}}(a)\left([\psi\otimes v]\right) := [(a\psi)\otimes v] \tag{17}$$

for $a \in \mathfrak{A}$, $\psi \in \mathcal{E}$, and $v \in \mathcal{H}_\pi$. This association of representations of $\mathfrak{B}$ with induced representations of $\mathfrak{A}$ even gives

[14] A linear operator a on $\mathcal{E}$ is called *adjointable* if there is a linear operator a^* on $\mathcal{E}$ such that $\langle\psi,a\varphi\rangle_{\mathcal{E}} = \langle a^*\psi,\varphi\rangle_{\mathcal{E}}$ for all $\psi,\varphi \in \mathcal{E}$.

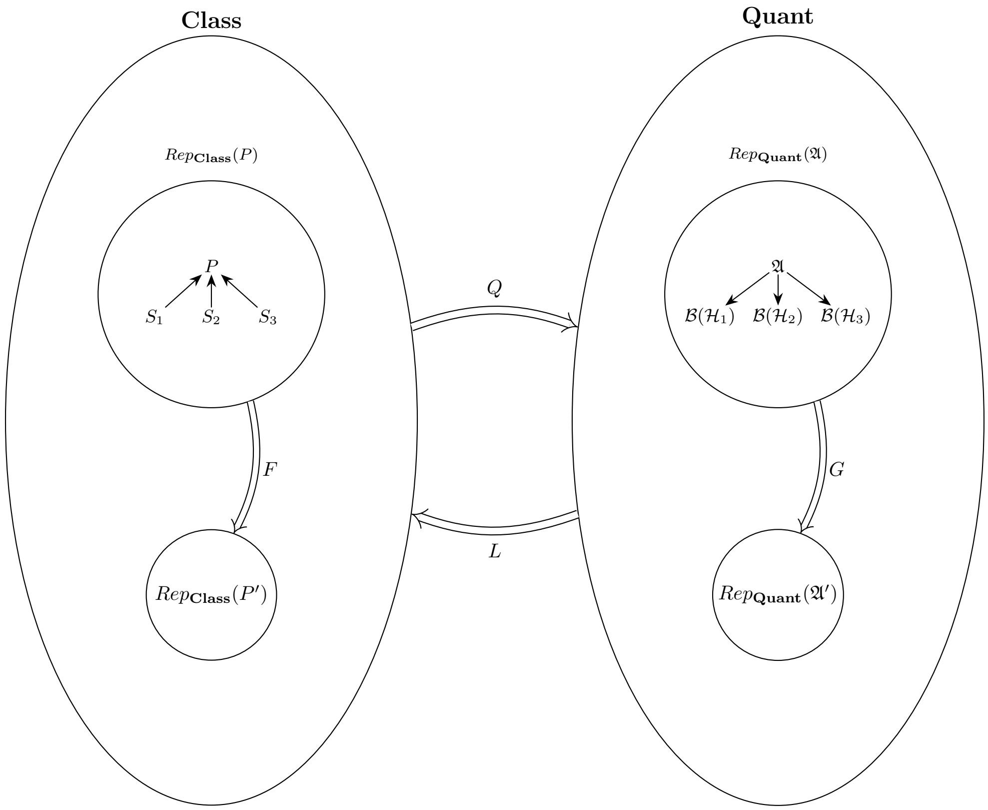

Figure 2: A depiction of the categories **Class** and **Quant**. Arrows in **Class** are induction functors F between categories of representations of Poisson manifolds, while arrows in **Quant** are induction functors G between categories of representations of C*-algebras. The quantization functor Q and classical limit functor L relate the two categories.

rise to a functor $G_{\mathcal{E}} : Rep_{\mathbf{Quant}}(\mathfrak{B}) \to Rep_{\mathbf{Quant}}(\mathfrak{A})$, the induction functor determined by the Hilbert bimodule $\mathcal{E}$ (Rieffel, 1974b; Landsman, 1995, 2001a).

We note, however, that the induction functor $G_{\mathcal{E}}$ generally does not depend on the full details of the Hilbert bimodule $\mathcal{E}$. Two Hilbert bimodules $\mathcal{E}$ and $\mathcal{E}'$ may give rise to corresponding induction functors that agree $G_{\mathcal{E}} = G_{\mathcal{E}'}$. One can find a necessary condition for agreement of induction functors by looking at the kernel of the tensor product representation $\chi^{\mathcal{E}} : \mathfrak{A} \otimes \mathfrak{B}^{op} \to \mathcal{B}(\mathcal{E})$ (where $\mathfrak{B}^{op}$ denotes the opposite algebra) defined by

$$\chi^{\mathcal{E}}(a \otimes b)\psi := a \cdot \psi \cdot b \tag{18}$$

for all $a \in \mathfrak{A}$, $b \in \mathfrak{B}$, and $\psi \in \mathcal{E}$. Note that the actions of $\mathfrak{A}$ and $\mathfrak{B}$ commute so we may leave the order unspecified. We call $\ker\left(\chi^{\mathcal{E}}\right)$ the *tensor product kernel* of $\mathcal{E}$. If the induction functors $G_{\mathcal{E}}$ of $\mathcal{E}$ and $G_{\mathcal{E}'}$ of $\mathcal{E}'$ agree, then it follows that their tensor product kernels are the same, i.e., $\ker\left(\chi^{\mathcal{E}}\right) = \ker\left(\chi^{\mathcal{E}'}\right)$. For this reason, we will take arrows in **Quant** to be equivalence classes of Hilbert bimodules that give rise to the same tensor product kernel.

To summarize: encoding the charge structure represented by classical and quantum physics in corresponding categories of models, we take objects to be categories of representations of Poisson manifolds and C*-algebras in the classical and quantum context, respectively. The motivation is that each representation corresponds to a possible value of charge so that a category of representations captures relevant relations among charges in the classical or quantum model. In this section, we have further motivated a choice of structure-preserving morphisms between these models of classical and quantum physics, certain functors between categories of representations. We take these as the relevant structure-preserving morphisms for our purposes of analyzing charge structure *precisely because they preserve the structure encoded in a category of representations*. We have restricted ourselves to a limited class of such functors, those corresponding to equivalence classes of symplectic dual pairs in the classical context and equivalence classes of Hilbert bimodules in the quantum context. Our justification for this choice is in part practical: induction functors determined by symplectic dual pairs or Hilbert bimodules follow a concrete pattern of construction that aids in their analysis, and especially in their comparison. As we shall see in the next section, working with symplectic dual pairs and Hilbert bimodules will allow us to construct functors comparing the categories $\mathbf{Class}$ and $\mathbf{Quant}$. These functors form a categorical equivalence, and thus establish a precise sense in which charge structure is preserved from classical to quantum physics.

3.2 A Categorical Equivalence

In this section, we will summarize a recent result (Feintzeig, 2026) establishing a categorical equivalence between categories whose objects are models of classical and quantum theories with morphisms understood as maps that preserve representation-theoretic structure. To do so, we provide two functors between the categories $\mathbf{Class}$ and $\mathbf{Quant}$ that are almost-inverse to each other (See Fig. 2). The functor from $\mathbf{Class}$ to $\mathbf{Quant}$ will be defined from a quantization procedure and denoted $Q : \mathbf{Class} \to \mathbf{Quant}$. In the other direction, the functor from $\mathbf{Quant}$ to $\mathbf{Class}$ will arise from the classical $\hbar \to 0$ limit and be denoted $L : \mathbf{Quant} \to \mathbf{Class}$.

To begin, we restrict the objects in $\mathbf{Class}$ to a collection of models that can all undergo a kind of generalized Weyl quantization procedure. These models arise from *Lie groupoids* (Mackenzie, 1987). A Lie groupoid is a generalization of a Lie group, where while all elements are required to have inverses, multiplication is only defined on certain pairs. More precisely, we can think of elements of a groupoid as arrows in a category, where multiplication is only defined (as composition) when the source of one arrow agrees with the target of another. A Lie groupoid is thus defined as a category with objects G_0 and arrows G both forming smooth manifolds. We require that every arrow has an inverse, and we denote the source and target projections by $s_G, t_G : G \to G_0$. In a Lie groupoid, we require that s_G and t_G are surjective submersions, the multiplication is a smooth map $G \times G \to G$, and the embedding $G_0 \hookrightarrow G$ of each object as its unique identity arrow is smooth. A groupoid is denoted by $G \rightrightarrows G_0$ to emphasize the two target and source projections, with multiplication always understood as composition.

Lie groupoids encompass a wide class of models of classical physics. For example, a Lie group (say, representing physical symmetries) is a Lie groupoid in which G_0 consists of just a single point. But Lie groupoids are also much more general, as illustrated by the following examples.

Example 7. Consider a manifold M representing the configuration space of a system. M can be associated with the *pair groupoid* $M \times M \rightrightarrows M$, with $s_{M\times M}(q_1, q_2) = q_2$, $t_{M\times M}(q_1, q_2) = q_1$, and multiplication defined by $(q_3, q_2) \cdot (q_2, q_1) = (q_3, q_1)$.

Example 8. Consider a principal bundle $\pi : P \to M$ with typical fiber G representing a particle interacting with a background Yang-Mills gauge field. The bundle π gives rise to the *gauge groupoid* $\mathcal{G} := (P \times P)/G \rightrightarrows M$ consisting of equivalence classes $[p_1, p_2]$ under the G-action $(p_1, p_2) \cdot x = (p_1 x, p_2 x)$ for $p_1, p_2 \in P$ and $x \in G$. The source and target projections are $s_{\mathcal{G}}([p_1, p_2]) = \pi(p_2)$ and $t_{\mathcal{G}}([p_1, p_2]) = \pi(p_1)$. To define multiplication, note that if $s_{\mathcal{G}}([p_1', p_2']) = t_{\mathcal{G}}([p_1, p_2])$, then $\pi(p_2') = \pi(p_1)$ so $p_1 = p_2' x$ for some $x \in G$. We then define multiplication by $[p_1', p_2'] \cdot [p_1, p_2] = [p_1', p_2'] \cdot [p_2', p_2 x^{-1}] = [p_1', p_2 x^{-1}]$.

Example 9. Consider a manifold M representing the configuration space of a system with a smooth action by a group of symmetries G. The group action gives rise to a *transformation groupoid* $M \rtimes G := M \times G \rightrightarrows M$ with source map $s_{M\rtimes G}(q,x) = x^{-1}q$ and target map $t_{M\rtimes G}(q,x) = q$ for $q \in M$ and $x \in G$. Multiplication is defined for elements (q,x) and (q',x') when $s_{M\rtimes G}(q,x) = t_{M\rtimes G}(q',x')$, so $q' = x^{-1}q$. We then define multiplication by $(q,x)\cdot(x^{-1}q,x') = (q,xx')$.

All these examples of Lie groupoids can serve as starting points for a generalized quantization procedure, which we will describe next (Landsman, 1998, 1999). Note that all Lie groupoids considered in this paper are finite-dimensional manifolds, so we restrict attention to theories with finitely many degrees of freedom and thus do not treat field theories.

The classical theory associated to a given Lie groupoid comes with a phase space determining the states and observables of the theory. The canonical phase space associated with a Lie groupoid G is the associated *dual* $\mathfrak{G}^*$ *of the Lie algebroid* $\mathfrak{G}$ (Mackenzie, 1987). Just like a Lie groupoid is a generalization of a Lie group, a Lie algebroid is a certain generalization of a Lie algebra, which can be thought of as the infinitesimal object associated with a Lie groupoid. While the Lie algebra of a Lie group can be thought of as the tangent space at the identity, since a Lie groupoid G has many distinct identity elements forming the manifold G_0, a Lie algebroid can be thought of as a subspace of the tangent space above each identity. More precisely, the Lie algebroid is the vector bundle $\mathfrak{G} \to G_0$ consisting of all vectors $\xi \in T_xG$ such that $(t_G)_*(\xi) = 0$. The vector bundle projection is given by $\tau_G(\xi) = t_G(x)$ when $\xi \in T_xG$. The dual $\mathfrak{G}^*$ of the Lie algebroid is the dual vector bundle to $\mathfrak{G}$, with vector bundle projection denoted $\tau^*_G : \mathfrak{G}^* \to G_0$. The Lie algebroid $\mathfrak{G}$ carries a further structure called an *anchor map*, which we omit here. What matters for our purposes is that with the structure of an anchor map, there is a canonical Poisson bracket $\{\cdot,\cdot\}$ defined on $C^\infty(\mathfrak{G}^*)$, allowing it to serve as the phase space for a Hamiltonian system (Landsman, 1998).

Example 10. Given a manifold M and the pair groupoid $M \times M$, the dual of its Lie algebroid is the phase space given by the cotangent bundle T^*M.

Example 11. Given a principal bundle $P \to M$ and the gauge groupoid $(P \times P)/G$, the dual of its Lie algebroid is the universal phase space $(T^*P)/G$.

Example 12. Given the smooth action of a group G on a manifold M and the transformation groupoid $M \rtimes G$, the dual of its Lie algebroid is the phase space given by the trivial bundle $\mathfrak{g}^* \times M$.

So thinking of phase spaces as duals of Lie algebroids allows us to encompass a wide range of examples, including all those discussed earlier in this paper. We will restrict the objects in our category **Class** to be phase spaces given by the dual of a Lie algebroid for systems associated with Lie groupoids.

The quantum system associated to a given Lie groupoid is the *(reduced) Lie groupoid C*-algebra* (Renault, 1980).[15] The Lie groupoid C*-algebra is a generalization of a Lie group C*-algebra. Whereas the Lie group C*-algebra was the completion of a convolution algebra defined by a Haar measure, now we must consider a family of measures, each on fibers of the Lie groupoid. A Lie groupoid G carries associated right/left Haar systems $(\nu^{s/t}_{x_0})_{x_0\in G}$ over each identity element $x_0 \in G$. Here, the right Haar measure $\nu^s_{x_0}$ is defined on $s_G^{-1}(x_0)$ while the left Haar measure $\nu^t_{x_0}$ is defined on $t_G^{-1}(x_0)$, and each is appropriately invariant with respect to the groupoid multiplication. Further, we consider a quasi-invariant measure ν_0 on the base space G_0 so that we can form the Hilbert space $L^2(G, \nu_0 \times \nu^s)$. Considering $C_c^\infty(G)$ with the convolution product, the left regular representation π^G of $C_c^\infty(G)$ on $L^2(G, \nu_0 \times \nu^s)$ is defined by

$$(\pi^G(f)\psi)(x) = \int_{s_G^{-1}(s_G(x))} d\nu^s_{s_G(x)}(x') f(xx'^{-1})\psi(x') \tag{19}$$

[15] Again, we omit the modifier "reduced" in this paper.

for all $f \in C_c^\infty(G)$ and $\psi \in L^2(G, \nu_0 \times \nu^s)$. The Lie groupoid C*-algebra $C^*(G)$ is the completion of $C_c^\infty(G)$ in the operator norm of the representation π^G. As for Lie groups, representations of a Lie groupoid C*-algebra are in one-to-one correspondence with suitably continuous representations of the Lie groupoid itself (Landsman, 1998, 2017). A Lie groupoid C*-algebra serves as the collection of observable quantities for a quantum system associated to a Lie groupoid, so we restrict the objects in our category $\mathbf{Quant}$ to be given by Lie groupoid C*-algebras.

Lie groupoid C*-algebras encompass all the examples of C*-algebras of observables for quantum systems presented in this paper. Clearly, the Lie groupoid C*-algebra of a Lie group is just the group C*-algebra.

Example 13. Given a manifold M and the pair groupoid $M \times M$, the Lie groupoid C*-algebra $C^*(M \times M)$ is isomorphic to the C*-algebra $\mathcal{K}(L^2(M))$ of compact operators on $L^2(M)$.

Example 14. Given a principal bundle $P \to M$ with gauge groupoid $\mathcal{G} = (P \times P)/G$, the Lie groupoid C*-algebra $C^*(\mathcal{G})$ is isomorphic to the C*-algebra $\mathcal{K}(L^2(M))^G$ of G-invariant compact operators in $L^2(M)$.

Example 15. Given a transformation groupoid $M \rtimes G$, the Lie groupoid C*-algebra $C^*(M \rtimes G)$ is isomorphic to the transformation group C*-algebra $C^*(M, G)$.

So Lie groupoid C*-algebras provide a way to unify many different C*-algebras for representing observables of quantum systems.

We further restrict attention to a special class of representation-theory preserving morphisms in our classical and quantum theories before going on to define functors between our categories. Since all of our classical and quantum systems are associated with Lie groupoids, we will restrict attention to generalized morphisms between Lie groupoids called *bibundles* (Moerdijk and Mrcun, 2003). These can be thought of as representation-theory preserving morphisms in the sense that they give rise to functors between representation theories of groupoids (with representations understood as actions of a groupoid on a vector bundle). A bibundle between Lie groupoids G and H is a manifold M with smooth maps $t_M : M \to G_0$ and $s_M : M \to H_0$, carrying a smooth left G-action and a smooth right H-action. For $x \in G$ and $q \in M$, the result of the left G-action xq is defined just in case $s_G(x) = t_M(q)$. For $y \in H$ and $q \in M$, the result of the right H-action qy is defined just in case $s_M(q) = t_H(y)$. Moreover, $t_M(qy) = t_M(q)$, $s_M(xq) = s_M(q)$, and $(xq)y = x(qy)$ for all $x \in G$, $q \in M$, and $y \in H$ on which the groupoid actions are defined. We denote this situation by a diagram $G \rightarrowtail M \leftarrowtail H$, and we sometimes refer to a bibundle only by its middle space M.

A bibundle $G \rightarrowtail M \leftarrowtail H$ gives rise to a symplectic dual pair. One can construct a symplectic dual pair between the corresponding duals of Lie algebroids $\mathfrak{G}^*$ and $\mathfrak{H}^*$ whose middle space is T^*M (with the negative Poisson bracket). To do so, define the Poisson morphism $j_G : T^*M \to \mathfrak{G}^*$ by

$$j_G(\eta_q)\left(\frac{d}{dt}_{|t-0}\gamma(t)\right) = -\eta_q\left(\frac{d}{dt}_{|t=0}\left(\gamma(t)^{-1} \cdot q\right)\right) \tag{20}$$

for $\eta_q \in T_q^*M$ and $\gamma : I \to G$ a curve with $\gamma(0) = t_M(q)$ and $\gamma(t) \in t_G^{-1}(t_M(q))$ for all $t \in I \subseteq \mathbb{R}$. Likewise, define the Poisson morphism $j_H : T^*M \to \mathfrak{H}^*$ by

$$j_H(\eta_q)\left(\frac{d}{dt}_{|t-0}\gamma(t)\right) = \eta_q\left(\frac{d}{dt}_{|t=0}\left(q \cdot \gamma(t)\right)\right) \tag{21}$$

for $\eta_q \in T_q^*M$ and $\gamma : I \to H$ a curve with $\gamma(0) = s_M(q)$ and $\gamma(t) \in t_H^{-1}(s_M(q))$ for all $t \in I \subseteq \mathbb{R}$. These momentum maps j_G and j_H are just generalizations of the construction in Eq. (2) to the context of a groupoid action. It follows that $\mathfrak{G}^* \leftarrow T^*M \to \mathfrak{H}^*$ is a symplectic dual pair (Landsman, 2001b). Further, suitable regularity properties of bibundles imply that corresponding symplectic dual pairs are also regular and hence composable as morphisms.[16]

[16]To some extent, the relation between general symplectic dual pairs and the more restrictive dual pairs determined by bibundles is analogous to

A bibundle $G \rightarrowtail M \leftarrowtail H$ also gives rise to a Hilbert bimodule. One can construct a Hilbert bimodule between the groupoid C*-algebras $C^*(G)$ and $C^*(H)$ whose middle space is a suitable completion of $C_c^\infty(M)$, which we denote $\mathcal{E}_M$. To do so, first note that the left Haar system $(\nu_{x_0}^t)_{x_0 \in G}$ for G induces a family of measures $(\mu_{y_0})_{y_0 \in H_0}$ on M, each with support contained in $s_M^{-1}(y_0)$. We further denote the left Haar system on H by $(\lambda_{y_0}^t)_{y_0 \in H}$. We can then define a $C^*(H)$-valued inner product by

$$\langle \varphi, \psi \rangle_{\mathcal{E}_M}(y) = \int_{s_M^{-1}(t_H(y))} d\mu_{t_H(y)}(q) \overline{\varphi(q)} \psi(qy) \tag{22}$$

for $\varphi, \psi \in C_c^\infty(M)$ and $y \in H$. The left $C^*(G)$-action and right $C^*(H)$-action are given by

$$(f \cdot \varphi)(q) = \int_{t_G^{-1}(t_M(q))} d\nu_{t_M(q)}^t(x) f(x) \varphi(x^{-1}q) \tag{23}$$

$$(\varphi \cdot g)(q) = \int_{t_H^{-1}(s_M(q))} d\lambda_{s_M(q)}^t(y) g(y^{-1}) \varphi(qy) \tag{24}$$

for all $\varphi \in C_c^\infty(M)$, $f \in C_c^\infty(G)$, $g \in C_c^\infty(H)$ and $q \in M$. Define $\mathcal{E}_M$ as the completion of $C_c^\infty(M)$ relative to the above inner product. The actions extend to actions of the groupoid C*-algebras on $\mathcal{E}_M$ by continuity. It follows that $C^*(G) \rightarrowtail \mathcal{E}_M \leftarrowtail C^*(H)$ is a Hilbert bimodule (Landsman, 2001b).

We are now in a position to codify the foregoing discussion in explicit definitions of our categories.

Definition 1. We denote by $\mathbf{Class}$ the category consisting in:

- *Objects:* An object is a dual $\mathfrak{G}^*$ to the Lie algebroid of a Lie groupoid G, understood as a Poisson manifold and subject to the condition that $\mathfrak{G}^*$ is integrable and G is source-simply connected.
- *Arrows:* An arrow is an equivalence class of symplectic dual pairs of the form $\mathfrak{G}^* \leftarrow T^*M \rightarrow \mathfrak{G}^*$ determined by a regular bibundle M between the Lie groupoids G and H as in Eqs. (20)-(21).

In the definition of objects, the requirement that $\mathfrak{G}^*$ be integrable and G be source simply-connected is required to ensure that each object has an identity arrow and that the identity arrow is determined by the identity bibundle from G to itself (Landsman, 2001b). In the definition of arrows, we understand equivalence classes of symplectic dual pairs as those that determine the same Lagrangian relation as in Eq. (13). Under these definitions, $\mathbf{Class}$ is indeed a category.

Definition 2. We denote by $\mathbf{Quant}$ the category consisting in:

- *Objects:* An object is a Lie groupoid C*-algebra $C^*(G)$, subject to the condition that $\mathfrak{G}^*$ is integrable and G is source-simply connected.
- *Arrows:* An arrow is an equivalence class of Hilbert bimodules of the form $C^*(G) \rightarrowtail \mathcal{E}_M \leftarrowtail C^*(H)$ determined by a regular bibundle M between the Lie groupoids G and H as in Eqs. (22)-(24).

In the definition of objects, once again the restriction that $\mathfrak{G}^*$ is integrable and G is source simply-connected ensures that each object has an identity arrow determined by the identity bibundle from G to itself. In the definition of arrows, we understand equivalence classes of Hilbert bimodules as those that determine the same tensor product kernel as in Eq. (18). Under these definitions, $\mathbf{Quant}$ is indeed a category.

the relation between general symplectomorphisms between phase spaces and the more restrictive point*-transformations determined by diffeomorphisms between configuration spaces, as discussed by Barrett (2019). Both bibundles and diffeomorphisms between configuration spaces can be lifted to the Poisson/symplectic manifolds associated with them, respectively, to define a morphism between phase spaces.

We now proceed to define functors $Q : \mathbf{Class} \to \mathbf{Quant}$ and $L : \mathbf{Quant} \to \mathbf{Class}$ that are "almost-inverse" to each other and hence provide a categorical equivalence. The functor Q corresponds to the operation of quantization, which constructs a quantum theory from the data of a given classical system. On the other hand, the functor L corresponds to the operation of the classical $\hbar \to 0$ limit, which constructs a classical theory as a large-scale approximation to a given quantum system. Both operations are physically significant because they give ways to construct mathematical models of *the same* physical system (understood on different scales) so that preservation of mathematical structure relative to these operations is the relevant issue for assessing structural preservation across theory change.

The quantization functor Q arises through strict deformation quantization following a generalized Weyl prescription. First, we describe how the quantization functor Q provides a map from objects of $\mathbf{Class}$ to objects of $\mathbf{Quant}$, i.e., from duals of Lie algebroids to Lie groupoid C*-algebras. This map is constructed by Landsman (1999), who provides a general formula for Weyl quantization on the dual of a Lie algebroid. The key is his construction of a map called the *Weyl exponential* $\exp^W : \mathfrak{G}^* \to G$, whose definition we omit. From this, one can define for each $\hbar > 0$ a quantization map $\mathcal{Q}_\hbar : C^\infty_{PW}(\mathfrak{G}^*) \to C^*(G)$ taking classical observables to quantum observables:

$$\mathcal{Q}_\hbar(f)(\exp^W(X)) = \hbar^{-n}\kappa(X)\grave{f}(X/\hbar) \tag{25}$$

for all $f \in C^\infty_{PW}(\mathfrak{G}^*)$ and $X \in \mathfrak{G}$, where $\grave{f} : \mathfrak{G} \to \mathbb{C}$ denotes the fiber-wise Fourier transform of $f : \mathfrak{G}^* \to \mathbb{C}$ and C^∞_{PW} denotes the class of smooth Paley-Wiener functions, i.e., those functions whose Fourier transform is smooth and compactly supported. Here, $\kappa(X)$ is a smooth cut-off function that ensures $\mathcal{Q}_\hbar(f)$ is well-defined as an integral operator in the left-regular representation of $C^*(G)$ acting on $L^2(G, \nu_0 \times \nu^s)$—see Landsman (1999) for further details. This family of maps serves as a quantization of the Poisson manifold $\mathfrak{G}^*$ in the sense that for each $f, g \in C^\infty_{PW}(\mathfrak{G}^*)$,

$$\lim_{\hbar \to 0} ||\mathcal{Q}_\hbar(f)\mathcal{Q}_\hbar(g) - \mathcal{Q}_\hbar(fg)|| = 0 \tag{26}$$

$$\lim_{\hbar \to 0} ||[\mathcal{Q}_\hbar(f), \mathcal{Q}_\hbar(g)] - \mathcal{Q}_\hbar(\{f, g\})|| = 0 \tag{27}$$

so that the product in the algebra of quantum observables approximates the pointwise product of classical observables and the commutator of quantum observables approximates the Poisson bracket of classical observables. These approximations form a *strict* quantization in the sense that the approximations arise in terms of the C*-norm $||\cdot||$. Moreover, it follows that for each $f \in C^\infty_{PW}(\mathfrak{G}^*)$, the map $\hbar \to ||\mathcal{Q}_\hbar(f)||$ is continuous, a fact we will use in what follows. The upshot is that Q associates the object $\mathfrak{G}^*$ in $\mathbf{Class}$ with the object $C^*(G)$ in $\mathbf{Quant}$.

Weyl quantization of Lie groupoids provides a general understanding of the quantization of a wide variety of classical systems, including all the examples discussed in this paper.

Example 16. Landsman (1999) shows that this general formula for Weyl quantization produces the standard quantization of a manifold M. If one starts with the pair groupoid $M \times M$, then Weyl quantization yields a map from the phase space given by the Poisson manifold T^*M to the C*-algebra of compact operators on $L^2(M)$.

Example 17. The Weyl quantization of a Lie group G understood as a Lie groupoid over a single point gives the quantization prescription presented by Rieffel (1990). Weyl quantization yields a map from the phase space given by the Poisson manifold $\mathfrak{g}^*$ to the group C*-algebra $C^*(G)$.

Example 18. The Weyl quantization of a principal bundle $P \to M$ with typical fiber G through its gauge groupoid $\mathcal{G} = (P \times P)/G$ gives the quantization prescription presented by Landsman (1993). Weyl quantization yields a map from the phase space given by the Poisson manifold $(T^*P)/G$ to the G-invariant compact operators on $L^2(P)$.

Example 19. The Weyl quantization of a manifold M with the action of a group G through its transformation groupoid

$M \rtimes G$ gives the quantization prescription generalized from Rieffel (1989, 1993) for the special case $G = \mathbb{R}^d$.[17] In this case, Weyl quantization is a map from the phase space given by the trivial bundle $\mathfrak{g}^* \times M$, understood as a Poisson manifold, to the transformation group C*-algebra $C^*(M, G)$.

So Landsman's formula for Weyl quantization provides a way to unify many different known quantization procedures under a single heading.

Moreover, Weyl quantization is a functor in the sense that Q also provides a map from arrows in $\mathbf{Class}$ to arrows in $\mathbf{Quant}$ (Landsman, 2001b, 2003). The quantization functor associates to each symplectic dual pair of the form $\mathfrak{G}^* \leftarrow T^*M \rightarrow \mathfrak{H}^*$ (constructed from a bibundle $G \rightarrowtail M \leftarrowtail H$ in Eqs. (20)-(21)) the corresponding Hilbert bimodule $C^*(G) \rightarrowtail \mathcal{E}_M \leftarrowtail C^*(H)$ (from Eqs. (22)-(24)). Landsman shows that this association of arrows with arrows respects composition of dual pairs in $\mathbf{Class}$ and bimodules in $\mathbf{Quant}$. Note also that dual pairs in the same equivalence class (i.e., with the same Lagrangian relation) will be mapped to Hilbert bimodules in the same equivalence class (i.e., with the same tensor product kernel) (Feintzeig, 2026). This completes the definition of Q as a functor.

In the other direction, the classical limit functor L arises through the $\hbar \rightarrow 0$ approximation of a given quantum system. First, we describe how the classical limit functor L provides a map from objects of $\mathbf{Quant}$ to objects of $\mathbf{Class}$. This map is constructed by Steeger and Feintzeig (2021a,b) who provide a way of reconstructing the classical theory at $\hbar = 0$ from a strict deformation quantization yielding a quantum theory for values $\hbar > 0$. The key is to realize that the continuity condition for a strict deformation quantization allows one to generate a *(uniformly) continuous bundle of C*-algebras*. A uniformly continuous bundle of C*-algebras over the base space of values $\hbar \in (0, 1]$ consists in a family of C*-algebras $(\mathfrak{A}_\hbar)_{\hbar \in (0,1]}$ and a further C*-algebra $\mathfrak{A} \subseteq \prod_{\hbar \in (0,1]} \mathfrak{A}_\hbar$. The C*-algebra $\mathfrak{A}$ consists in *(uniformly) continuous sections*, understood as maps $a : \hbar \in (0, 1] \mapsto a(\hbar) \in \mathfrak{A}_\hbar$ with values in the fiber $\mathfrak{A}_\hbar$ above each base point $\hbar$. In the case of the quantization of the dual $\mathfrak{G}^*$ of a Lie algebroid, one has $\mathfrak{A}_\hbar = C^*(G)$ for all $\hbar \in (0, 1]$, and the algebra of sections $\mathfrak{A}$ is generated by tracing out the images of the quantization maps $\hbar \mapsto \mathcal{Q}_\hbar(f)$ ranging over all $f \in C^\infty_{PW}(\mathfrak{G}^*)$ and closing under algebraic operations.[18] Subject to certain continuity conditions, one can then reconstruct the unique algebra $\mathfrak{A}_0$ at $\hbar = 0$ as a quotient of $\mathfrak{A}$ defined by

$$\mathfrak{A}_0 = \mathfrak{A}/K_0^{\mathfrak{A}} \qquad\qquad K_0^{\mathfrak{A}} = \left\{ a \in \mathfrak{A} \mid \lim_{\hbar \rightarrow 0} ||a(\hbar)|| = 0 \right\}, \tag{28}$$

where one quotients out the ideal $K_0^{\mathfrak{A}}$ of sections whose norm vanishes continuously as $\hbar \rightarrow 0$. In the bundle of C*-algebras for the quantization of $\mathfrak{G}^*$, the construction yields the unique classical limit algebra $\mathfrak{A}_0 = C_0(\mathfrak{G}^*)$, which is the C*-algebra generated by the norm completion of the observables in $C^\infty_{PW}(\mathfrak{G}^*)$ with which we began the quantization procedure. Thus, the classical limit L yields the expected result by associating to each quantum theory $C^*(G)$ in $\mathbf{Quant}$ the classical theory $\mathfrak{G}^*$ in $\mathbf{Class}$.

The quantization maps $\mathcal{Q}_\hbar$ further allow one to extend L to a functor mapping arrows in $\mathbf{Quant}$ to arrows in $\mathbf{Class}$ (Feintzeig and Steeger, 2024). Given a Hilbert bimodule $C^*(G) \rightarrowtail \mathcal{E}_M \leftarrowtail C^*(H)$, one can use the quantization maps to continuously translate to all values of $\hbar \in (0, 1]$ and form a Hilbert bimodule $\mathfrak{A} \rightarrowtail \underline{\mathcal{E}}_M \leftarrowtail \mathfrak{B}$ between the algebras of sections $\mathfrak{A}$ and $\mathfrak{B}$ for the quantization of $\mathfrak{G}^*$ and $\mathfrak{H}^*$, respectively. Then one takes the classical $\hbar \rightarrow 0$ limit again yielding a unique extension to $\hbar = 0$ under certain continuity and nondegeneracy conditions. The classical limit is a bimodule of the form $\mathfrak{A}_0 \rightarrowtail (\underline{\mathcal{E}}_M)_0 \leftarrowtail \mathfrak{B}_0$, where $\mathfrak{A}_0 \cong C_0(\mathfrak{G}^*)$ and $\mathfrak{B}_0 \cong C_0(\mathfrak{H}^*)$ are the classical limits of objects as above in Eq. (28). The middle space is again a quotient $(\underline{\mathcal{E}}_M)_0 = \underline{\mathcal{E}}_M / \overline{\underline{\mathcal{E}}_M \cdot K_0^{\mathfrak{B}}}$ by the ideal of sections in $\mathfrak{B}$ vanishing in norm as $\hbar \rightarrow 0$. It turns out that the tensor product kernel of $(\underline{\mathcal{E}}_M)_0$ uniquely determines the Lagrangian relation of the dual pair $\mathfrak{G}^* \leftarrow T^*M \rightarrow \mathfrak{H}^*$ (Feintzeig, 2026). This implies that the classical limit L is well-defined

[17] See Bieliavsky and Gayral (2015) for significant generalization of Rieffel's quantization for group actions.

[18] This construction yields $\mathfrak{A}$ as the groupoid C*-algebra of what is called the *normal groupoid* of G (Landsman, 1999).

on arrows since bimodules in the same equivalence class in $\mathbf{Quant}$ are mapped to dual pairs in the same equivalence class in $\mathbf{Class}$. It further follows that L is a functor by virtue of respecting composition of arrows.

Finally, and most importantly, the fact that the classical limit of an arrow $\mathcal{E}_M$ in $\mathbf{Quant}$ yields the arrow T^*M in $\mathbf{Class}$ whose quantization was $\mathcal{E}_M$ implies that both $Q \circ L$ and $L \circ Q$ are naturally isomorphic to the identity functor. In other words, Q and L are "almost-inverse" functors. Thus, we can state the main result of Feintzeig (2026), establishing a sense in which representation-theoretic structure is preserved from classical to quantum physics.

Theorem (Feintzeig (2026))**.** *The functors $Q : \mathbf{Class} \rightleftarrows \mathbf{Quant} : L$ form a categorical equivalence.*

If structural equivalence of physical theories is analyzed as categorical equivalence between their corresponding categories of models, and if the relevant categories capturing charge structures are given by $\mathbf{Class}$ and $\mathbf{Quant}$, then this result shows a sense in which classical and quantum physics represent the same charge structure. In both categories, we understand charge structure as the structure of categories of charge representations, with structure-preserving maps given by induction functors between categories of charge representations. It is precisely this notion of charge structure that is preserved from classical to quantum physics.

Let us pause to consider the significance of the categorical equivalence just stated. One might at this point wonder what it could mean to say that continuous charge quantities in classical physics can be structurally equivalent to discrete charge quantities in quantum physics. How could such different conceptions of charge be equivalent?

Some context for the comparison of classical and quantum charge may make this result less surprising. It is analogous to the way we think of *energy* in classical and quantum physics as representing the same physical quantity, with a structurally equivalent role in both theories. The classical Hamiltonian can take on continuously many values, while for many systems the quantum Hamiltonian yields a discrete collection of energy levels. Nevertheless, the Hamiltonian plays the same role in both theories as the generator of dynamics, or time translations. The Hamiltonian in both theories represents a conserved quantity when the system is under the influence of no other forces. And further, we can think of Hamiltonians as having *relations* to other Hamiltonians. Many Hamiltonians decompose into a sum of a free part (kinetic energy) and a potential or interaction term, so that, e.g., free Hamiltonian systems can be embedded in interacting models. All these non-trivial relations are preserved in the transition from classical to quantum mechanics just as the structural relations for charge are.

In just the same way, the categorical equivalence stated here shows that *structural relations* among different possible charges in classical physics are preserved in quantum physics. Those structural relations are captured by embeddings and isomorphisms of models of charges in classical theories on the one hand, and embeddings and isomorphisms of models of charge in quantum theories on the other. I think this is exactly the kind of structure that structural realists are interested in, especially when they write about "modal structure" (Ladyman, 1998; French, 2012; Ladyman, 2020). The structure captured by a category of charge models (representation theories) consists in precisely these relationships among all the possible ways charge might have been according to the classical or quantum context. To say that charges in quantum physics share structure with charges in classical physics just is to say that the network of modal relations among the possible charges is preserved, even if the actual values of charge change from continuous to discrete. We have good evidence that these structural relations (embeddings and isomorphisms) suffice to implicitly define theoretical terms. Barrett (2018, 2020) has worked this out most explicitly for first-order theories in logic; my suggestion is that we should understand the same definability relations to hold in general mathematized theories.

To take one particularly clear example of the foregoing, the categorical equivalence stated here maps isomorphisms of models with the same charge representations in classical physics to isomorphisms of models with the same charge representations in quantum physics. We can see this concretely starting in the example of a classical system whose phase space is the dual of a Lie algebra $\mathfrak{g}^*$ for a Lie group G, which has classical representations labeled by coadjoint orbits. The charge representations here are isomorphic to those for a different classical system moving in the principal bundle P with structure group given by the same G and universal phase space $(T^*P)/G$, where classical representations are again labeled by coadjoint orbits in $\mathfrak{g}^*$. The two classical models have isomorphic charges in the sense that there is an equivalence between their categories of classical charge representations—this is a so-called *Morita equivalence of Poisson manifolds* given by a symplectic dual pair

$$(T^*P)/G \leftarrow T^*P \rightarrow \mathfrak{g}^*. \tag{29}$$

In the quantum theory, the model $\mathfrak{g}^*$ is quantized to the group C*-algebra $C^*(G)$, whose Hilbert space representations are indexed by unitary representations of G. And the universal phase space $(T^*P)/G$ is quantized the the G-invariant compact operators $K(L^2(P))^G$, whose Hilbert space representations are again indexed by unitary representations of G. The charge representations of the two models are likewise isomorphic in the sense that there is an equivalence between their categories of quantum charge representations—this is also a so-called *Morita equivalence of C*-algebras* given by a Hilbert bimodule

$$\mathcal{K}(L^2(P))^G \rightarrowtail L^2(P) \leftarrowtail C^*(G). \tag{30}$$

In both the classical and quantum contexts, we have many different models with the same notion of charge—constructed either by thinking of the degrees of freedom represented by G alone, or thinking of how the internal degrees of freedom captured by G apply on many different backgrounds given by principal bundles P. This notion of sameness of charge concept shared among different models in classical physics is preserved by the categorical equivalence and matches one-to-one with the notion of sameness of charge concept shared among different models of quantum physics.

Charge structure is physically significant in both classical and quantum theories for the role it plays in the dynamics, and it plays *structurally the same* role in both contexts. Take the case of a charged particle moving in a fixed background Yang-Mills field on a principal bundle P with structure group G. One can formulate a global classical dynamics on the universal phase space $(T^*P)/G$ with a single unified Hamiltonian. Moving to a classical charge representation, and hence fixing a particular value of charge, corresponds to constructing the reduced Hamiltonian in a particular charge sector, which ends up satisfying a generalization of the Lorentz force law for Yang-Mills theories (Gilton, 2022). Likewise, in the quantum context one can define a global dynamics through a single unified Hamiltonian operator. The representation of that operator on any charge sector introduces new terms that depend on the charge as a coupling (Landsman, 1993). Charge structure plays the same physically significant role in determining the dynamics of both classical and quantum theories.

Of course, the categorical equivalence stated here shows only that charge quantities in classical and quantum physics share structure *in some respects*. In particular, in the literature on theoretical equivalence many philosophers only consider categorical equivalence to establish that two theories are equivalent when the almost-inverse functors preserve empirical content. I hope it is manifest that the almost-inverse functors between classical and quantum charge structures do *not* preserve empirical content in that the classical and quantum theories make different predictions,[19] e.g., about the spectrum of the charge quantities in the two theories. The categorical equivalence considered here thus falls short of showing that classical and quantum physics are theoretically equivalent, and this is exactly what we should expect. Still, I believe the categorical equivalence presented here shows that some structure is preserved in the transition between these distinct theories, so that we can understand the structure of charge in quantum physics as corresponding with the structure of charge in classical physics in a substantive sense.

[19]One should not go as far as to think that the almost-inverse functors presented here have been chosen arbitrarily without regard for empirical content. In the classical $\hbar \to 0$ limit, the empirical predictions of the quantum theory come to approximate those of the classical theory arbitrarily well (Feintzeig, 2020). So the almost-inverse functors respect an empirical correspondence that holds approximately and in appropriate regimes.

4 Interpreting Inequivalent Representations

The main result of the previous section is that there is a sense in which the models of quantum physics—understood as C*-algebras of observables— share structure with the models of classical physics—understood as phase spaces represented by Poisson manifolds. The structure preserved across the classical–quantum divide is *representation-theoretic* structure, where both the classical notion of representations as symplectic realizations and the quantum notion of representations as Hilbert space representations capture possible values of physical charge associated with a symmetry group. We showed that when the classical and quantum models are understood as elements of categories $\mathbf{Class}$ and $\mathbf{Quant}$ with structure-preserving morphisms understood as induction functors between their associated categories of representations, then the classical and quantum categories are categorically equivalent.

There are some technical limitations to the results that bear mentioning. First, we restricted attention to models of classical and quantum physics associated to Lie groupoids. As we discussed, this class of models is still broad enough to encompass a wide range of systems. Second, we restricted attention to induction functors between categories of representations that were constructed concretely from symplectic dual pairs and Hilbert bimodules. This might not be the only class of representation-theoretic structure-preserving morphisms of interest, although we note that it is still a wide class of morphisms that can be understood as encompassing all the ones generated from generalized morphisms (bibundles) between Lie groupoids we might begin with. Finally, recall that in each of the classical and quantum contexts we took morphisms to be *equivalence classes* of symplectic dual pairs and Hilbert bimodules, respectively. In $\mathbf{Class}$, equivalent symplectic dual pairs that give rise to the same Lagrangian relation are precisely those that give rise to the same induction functor. In contrast, in $\mathbf{Quant}$, one only has one direction of the analogous implication: if two Hilbert bimodules give rise to the same induction functor, then they are equivalent in the sense of having the same tensor product kernel. Having the same tensor product kernel is merely a necessary condition, but not a sufficient condition, for two Hilbert bimodules to give rise to the same induction functor. Because of this, one might take the structure encoded by the morphisms in $\mathbf{Quant}$ to be slightly different than representation-theoretic structure captured by induction functors between categories of representations. We believe this deserves further investigation to better understand the representation-theoretic structure of quantum theories, which we now know at least partially lines up with the representation-theoretic structure of classical theories.

We conclude now with a discussion of further interpretive issues that the arguments of this paper bear upon in the foundations of quantum theory. Specifically, Ruetsche (2011) poses a *problem of inequivalent representations* as a central issue in the interpretation of quantum theories. While the ordinary quantum mechanics of finitely many particles moving in the phase space $\mathbb{R}^{2n}$ takes place in a unique (in a sense) Hilbert space representation on $L^2(\mathbb{R}^n)$, many more complicated quantum systems have algebras of observables with inequivalent representations. Ruetsche focuses on inequivalent representations that appear when a system has infinitely many degrees of freedom, as in quantum field theory or quantum statistical mechanics in the thermodynamic limit. Ruetsche, among many others, sees these inequivalent Hilbert space representations as potentially competing quantum theories and outlines distinct philosophical attitudes one might take. One can either follow the *Hilbert Space Conservative* and choose a single Hilbert space out of the many competing ones in which to formulate the theory, or one can follow the *Algebraic Imperialist* in committing only to the common C*-algebraic structure that all representations have in common. Ruetsche's ultimate philosophical conclusion is that none of these interpretive positions suffices for all the explanatory aims of quantum theory, forcing one to give up a univocal picture of what the theory is, a prospect explored further by Jacobs (2021).

The significance of inequivalent representations in Ruetsche's arguments has been questioned by others, such as Wallace (2006) and Fraser (2023). Wallace argues that the starting point of algebraic quantum field theory is not apt for philosophical investigation. His reason is that the modern viewpoint of effective field theories teaches us that quantum field theories do not describe the world at arbitrarily small length scales, and so he argues that we should not treat those theories as fundamental. Doing so might seem to give philosophers permission to ignore mathematical complications of the algebraic approach, including inequivalent representations. But Wallace (2006, §4) actually goes on to provide an interpretation of the significance of some inequivalent representations that we take to be largely in line with the viewpoint taken in this paper, in which disjoint representations correspond to distinct sectors of a theory, often associated with different values of charge. We merely add here that viewing representations as corresponding to distinct sectors does not rely on any assumptions about small scales in quantum field theory, as inequivalent charge representations appear already in all the systems with finitely many degrees of freedom considered in this paper. Fraser provides further arguments for disregarding inequivalent representations that arise from infinitely many degrees of freedom from a similar perspective on effective field theories. However, Fraser (2023, §5.2) also recognizes that inequivalent representations can arise already in systems with finitely many degrees of freedom having a topologically or geometrically more complicated phase space than $\mathbb{R}^{2n}$, and in those systems he also recognizes inequivalent representations as corresponding to values of a global charge. The current paper serves to emphasize the significance of inequivalent representations for making sense of charge structure in quantum theories even with finitely many degrees of freedom, as in all models considered in this paper.

I have further argued that taking the structure of inequivalent representations seriously is essential for the philosophical goal of understanding how structure is preserved from classical to quantum physics. I believe that seeing how the structure of inequivalent representations in quantum systems corresponds to the structure of distinct symplectic realizations in classical physics lends some insight into the interpretation of the often more perplexing quantum theories. I suggest even that seeing preserved charge structure from classical to quantum physics gives some reason for conceptualizing charge representations in quantum physics along the same lines as in classical physics. Namely, since distinct charge representations in classical physics are clearly *not* competing theories that we need to choose between, neither are inequivalent charge representations in quantum physics. We should think of charge representations in quantum physics as different sectors of a theory *because* the analogous charge representations in classical physics are likewise different sectors (e.g., symplectic leaves) of a Hamiltonian theory formulated in a universal phase space.

Accepting the significance of inequivalent representations for understanding the structure of charge in quantum mechanics immediately rules out Hilbert Space Conservatism as a viable interpretive stance since it will not suffice to focus on only a single value of charge when other values are physically realizable. But Halvorson (2007, p. 844) has argued that the significance of distinct charge representations should not lead one to the position of Algebraic Imperialism because one requires the structure of the category of representations of an algebra to understand physical charge, rather than merely the structure of the abstract C*-algebra itself:

> The naive transcription of Algebraic Imperialism to the current context would say: the representations [...] are surplus structure; the physical content of the theory is in $\mathfrak{A}$, the abstract algebra of observables. (Halvorson, 2007, p. 844)

Halvorson calls the position in which one takes the structure of the category of representations as part of the content of the theory *Representation Realism*. But we now have the resources to see that Algebraic Imperialism also has access to the category of representations of an algebra, so Halvorson's criticism of Algebraic Imperialism is inapt.

Indeed, there is a sense in which the category of representations of a C*-algebra is determined by the C*-algebra itself, which we can make precise in the framework of this paper. Consider the category $\mathbf{C^*Alg}$ whose objects are C*-algebras and arrows are *-homomorphisms, and the category $\mathbf{Reps}$ whose objects are categories of representations of C*-algebras and whose arrows are functors between categories of representations (See Fig. 3). There is a functor $F : \mathbf{C^*Alg} \to \mathbf{Reps}$ that takes each C*-algebra to its category of representations. A *-homomorphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ between C*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ determines a Hilbert bimodule $\mathfrak{A} \rightarrowtail \mathfrak{B} \leftarrowtail \mathfrak{B}$ whose middle space is $\mathfrak{B}$ itself with the $\mathfrak{B}$-valued inner product $\langle b, b' \rangle_{\mathfrak{B}} = b^* b'$, the left action of $\mathfrak{A}$ by $a \cdot b = \varphi(a)b$ and right action of $\mathfrak{B}$ by $b \cdot b' = bb'$

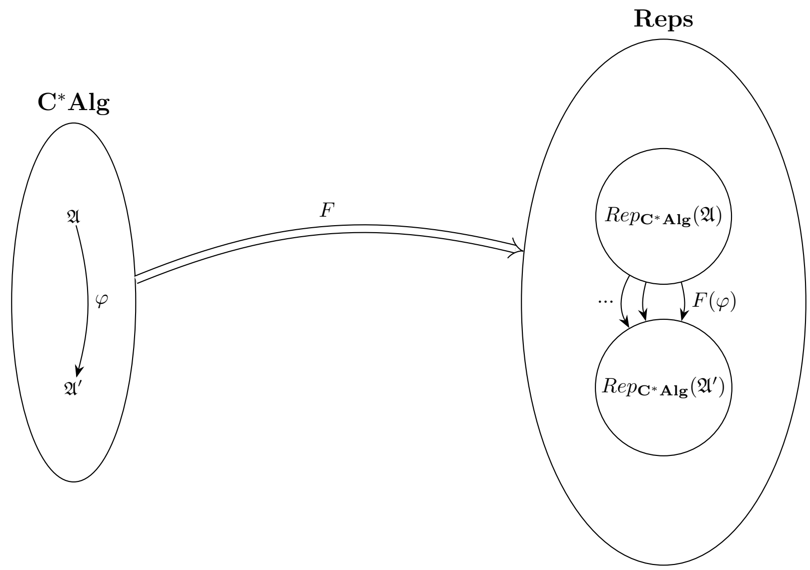

Figure 3: The forgetful functor from C*-algebras to their categories of representations.

for all $a \in \mathfrak{A}$ and $b, b' \in \mathfrak{B}$. Since this Hilbert bimodule $\mathfrak{A} \rightarrowtail \mathfrak{B} \leftarrowtail \mathfrak{B}$ determined by the *-homomorphism φ gives rise to an induction functor $Rep_{\mathbf{C^*Alg}}(\mathfrak{A}) \to Rep_{\mathbf{C^*Alg}}(\mathfrak{B})$, we can define F as associating each *-homomorphism φ in $\mathbf{C^*Alg}$ with its corresponding induction functor $F(\varphi)$ in $\mathbf{Reps}$. The induction functor $F(\varphi) : Rep_{\mathbf{C^*Alg}}(\mathfrak{A}) \to Rep_{\mathbf{C^*Alg}}(\mathfrak{B})$ has a simple form, taking any representation $\pi_{\mathcal{H}} : \mathfrak{B} \to \mathcal{B}(\mathcal{H})$ of $\mathfrak{B}$ and constructing the induced representation $\pi^{\mathcal{H}} := \pi_{\mathcal{H}} \circ \varphi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ of $\mathfrak{A}$. The key point I wish to make is that it follows that the functor F *forgets structure* because there are generally more induction functors than those constructed from *-homomorphisms. So according to the standard category-theoretic tools for comparing structure, the category of representations of a C*-algebra has *less structure* (more morphisms) than the abstract C*-algebra itself. We take this to be a sense in which an abstract C*-algebra determines its category of representations, and it follows that Algebraic Imperialism has access to all of the interpretive tools of Representation Realism.

We conclude by returning briefly to the suggestion of French (2012), which motivated the overarching investigation of this paper. Recall, French suggests that in the context of quantum field theory, a structural realist should take the collection of all representations of an abstract algebra as part of the structure of the theory. In this paper, we have shown that this viewpoint arises already for systems with finitely many degrees of freedom without any of the complications of quantum field theory. Moreover, we have established a precise sense in which the structure of these collections of representations is preserved from classical to quantum physics, thus connecting to a central issue for structural realism and providing a *justification* for treating the representation-theoretic structures of the two theories with the same interpretation. We take this to give an interpreter reason to view the structure of a quantum theory as consisting in (at least) the structure of its category of representations. Ultimately, we hope to have shown that the understanding of charge structure in quantum theories is a rich and underexplored philosophical area. We can only hope to encourage future work in the area, and especially its extension to quantum field theory following the work of Halvorson (2007).

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