Topology-induced structure and local-to-global symmetry

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Abstract

Grimmer and Read argue that certain flat but almost/asymmetric spatial topologies, especially the Hantzsche–Wendt manifold, affect two questions in the philosophy of spacetime: the equivalence of spacetime theories and the determinism of those theories. On fixed Hantzsche–Wendt product backgrounds, several familiar distinctions between global automorphism groups collapse, even though the corresponding structures remain locally distinct. This paper gives a sheaf-theoretic way of keeping these two facts separate. The first calculation is that, for any compact connected flat spatial manifold Σ, the Killing sheaf of the static Lorentzian product $(\mathbb{R} \times \Sigma, -\mathrm{d}t^2 + h)$ has global sections

$$H^0(\mathbb{R} \times \Sigma, \mathcal{K}) \cong \mathbb{R} \oplus H^0(\Sigma, \mathcal{K}_h).$$

Thus the Hantzsche–Wendt product has Poincaréan Killing-sheaf stalks but only the time-translation Killing field globally. The second calculation treats the rigging sector of the Newtonian/Leibnizian comparison as a torsor under vertical metric-preserving fields; in the static Hantzsche–Wendt case this torsor has a unique global section. Finally, the determinism definitions used by Grimmer and Read are written as extension and uniqueness conditions for restriction maps from full isometries to isometries of initial segments. The aim is not to replace their global automorphism-group analysis, but to record the local automorphism data that such an analysis suppresses.

Contents

1 Introduction

Grimmer and Read's *Equivalence and determinism in light of topologically-induced structure* introduces a class of flat spatial topologies into debates over equivalence and determinism in spacetime physics (Grimmer and Read, 2026). The purpose of the present paper is to spell out one mathematical feature of their examples: the distinction between local and global structure. In the examples at issue, the local Euclidean or Minkowskian geometry is unchanged. What changes is which local symmetries extend to global symmetries.

The central three-dimensional example is the Hantzsche–Wendt manifold Σ_{HW}. It is compact, orientable, flat, and locally Euclidean, but its isometry group has trivial identity component. Since the isometry group of a compact Riemannian manifold is a compact Lie group, it follows that the isometry group of any flat Hantzsche–Wendt metric is finite. The metric parameters affect the size of this finite group: in Grimmer and Read's terminology, the generic case has fewer isometries than the maximally symmetric case. Waldmüller's example goes further in dimension 141, giving a complete connected flat Riemannian manifold with trivial group of affinities and hence trivial isometry group (Waldmüller, 2003; Grimmer and Read, 2026). Thus these manifolds are locally homogeneous but globally inhomogeneous.

Grimmer and Read draw two consequences. First, on $M = \mathbb{R} \times \Sigma_{\mathrm{HW}}$, the usual global automorphism-group distinctions between Leibnizian, Maxwellian, Galilean, and Newtonian structures collapse. The same mechanism removes Lorentz boosts in the static Minkowskian product, so that the relativistic case also admits a recovered preferred temporal direction (Grimmer and Read, 2026, Secs. 4.1–4.2). Second, the same topology changes some determinism verdicts. Initial isometries, provided that they are defined on initial segments containing the whole spatial topology, are forced to extend uniquely in cases where the standard $\mathbb{R}^3$ topology permits time-dependent translations or rotations (Grimmer and Read, 2026, Sec. 5).

Their discussion also marks a limitation of the global comparison. The collapse is global, not local. A local detector of Newtonian structure would detect a rest field; a local detector of Galilean structure would not. Similarly, local relativistic dynamics and local Galilean dynamics remain different even if the corresponding global automorphism groups on $\mathbb{R} \times \Sigma_{\mathrm{HW}}$ are isomorphic (Grimmer and Read, 2026, Sec. 4.3).

The distinction can be put in sheaf-theoretic terms. A global automorphism group records automorphisms defined on all of M. A local automorphism sheaf records germs of automorphisms defined on open sets, together with their restriction and gluing behaviour. In the pseudo-Riemannian case, the infinitesimal version is the Killing sheaf. In the classical case, the recovered rest field can be described as a section of a torsor. In the determinism case, extension and uniqueness become properties of restriction maps from full isometries to initial isometries.

The plan for the paper is as follows. In §2, I fix conventions and recall the Hantzsche–Wendt facts used below. In §3, I define local automorphism pseudogroups and Killing sheaves, and prove the compact-flat product calculation

$$H^0(\mathbb{R} \times \Sigma, \mathcal{K}_{-\mathrm{d}t^2+h}) \cong \mathbb{R} \oplus H^0(\Sigma, \mathcal{K}_h).$$

In §4, I treat Newtonian riggings as a torsor under vertical metric-preserving fields and prove uniqueness of the Hantzsche–Wendt rigging in the static product case. In §5, I distinguish equivalence of global model categories from equivalence of local stacks. In §6, I express the relevant determinism notions as extension and uniqueness properties of local isometries.

2 Background and conventions

The following conventions will be used. A classical spacetime is presented using a smooth manifold M, a global time function $T : M \to \mathbb{R}$ when one is available, a temporal one-form $\tau = \mathrm{d}T$, and a spatial metric on the distribution $\ker \tau$ or, equivalently in the usual index notation, a degenerate contravariant spatial metric h^{ab} satisfying $h^{ab}\tau_b = 0$. This is the standard geometric presentation of classical spacetime structure used by Friedman (1983); Earman (1989); Malament (2012). Some index-theoretic details are suppressed when the product form $M = \mathbb{R} \times \Sigma$ makes the intended structure unambiguous. Unless explicitly stated otherwise, the convention is time-oriented, so that τ is preserved. If one instead follows a temporally unoriented convention in which the primitive temporal metric is $\tau_a\tau_b$, the vector-field uniqueness claims below should be read as claims about the corresponding timelike line field; time reflection then contributes an additional discrete automorphism but does not change the continuous-symmetry or H^0 calculations.

The familiar classical hierarchy can be summarized as follows. Leibnizian spacetime has temporal duration and spatial distance between simultaneous events but no standard of rotation or acceleration. Maxwellian spacetime adds a standard of rotation. Galilean spacetime adds a flat compatible connection and hence a standard of inertial motion. Newtonian spacetime adds a preferred rest frame, usually represented by a unit timelike vector field V^a whose integral curves identify the same spatial point through time. On the standard topology $\mathbb{R} \times \mathbb{R}^3$, the successive forgetful functors in this hierarchy are not equivalences because they discard structure that cannot be uniquely recovered (Earman, 1989; Malament, 2012; Weatherall, 2018; Grimmer and Read, 2026).

The Hantzsche–Wendt construction changes this verdict. A complete connected flat Riemannian manifold is a quotient $\mathbb{R}^n/\Gamma$ by a discrete group of Euclidean isometries acting freely and properly discontinuously; in the compact case Γ is a Bieberbach group (Charlap, 1986; Wolf, 1984). In dimension three, the compact flat cases are the ten platycosms described by Conway and Rossetti (2003). The Hantzsche–Wendt manifold is the orientable compact flat three-manifold with first Betti number zero; equivalently, its isometry group has trivial identity component. The original construction is due to Hantzsche and Wendt (1935); modern accounts are given by Charlap (1986) and Conway and Rossetti (2003).

Two facts about Σ_{HW} will be used repeatedly. First, every sufficiently small open set in Σ_{HW} is isometric to an open subset of Euclidean space. Hence any theory whose local geometry is Euclidean or Minkowskian will look locally standard on $\mathbb{R} \times \Sigma_{\mathrm{HW}}$. Second, the continuous spatial isometries of Σ_{HW} vanish. Since Σ_{HW} is compact, its isometry group is a compact Lie group; trivial identity component therefore implies that the full spatial isometry group is finite. The metric parameters can change the size of this finite group, as in Grimmer and Read's distinction between generic and more symmetric Hantzsche–Wendt metrics, but not the absence of continuous spatial isometries (Grimmer and Read, 2026, Sec. 3). The first fact is local homogeneity. The second is global inhomogeneity.

For sheaf theory, $\mathbf{Open}(M)$ denotes the poset category of open subsets of M. A sheaf of sets, groups, vector spaces, or groupoids is always understood on $\mathbf{Open}(M)$ with the usual open-cover topology. I write $H^i(M, \mathcal{F})$ for sheaf cohomology and use Čech descriptions when convenient. For the general background on sheaves and sheaf cohomology, see Bredon (1997) and Mac Lane and Moerdijk (1992). For torsors, nonabelian cohomology, and stacks, see Giraud (1971); only the elementary part of that machinery is needed here.

3 Automorphism sheaves and Killing sheaves

3.1 From global automorphisms to local automorphisms

Let S be a geometric spacetime structure on M. The global automorphism group $\mathrm{Aut}(M, S)$ records diffeomorphisms $M \to M$ preserving all the fields in S. The first local-to-global refinement is to remember not only global automorphisms, but also the pseudogroup of local automorphisms and its sheaf of germs.

Definition 3.1. Let (M, S) be a smooth manifold equipped with geometric structure S. The *local automorphism pseudogroup* $\mathcal{A}\sqcap\sqcup_S^{\mathrm{loc}}$ consists of all diffeomorphisms

$$\phi : U \longrightarrow V$$

between open subsets $U, V \subseteq M$ such that $\phi^*(S|_V) = S|_U$. Composition, inverse, and restriction are defined wherever they make sense. Equivalently, the germs of such maps form an ètale groupoid over M. Write $\mathcal{A}_S$ for this sheaf of germs, and write $\mathrm{Aut}(M, S) = \mathcal{A}\sqcap\sqcup_S^{\mathrm{loc}}(M, M)$ for its global bisections with source and target all of M.

The pseudogroup formulation is needed because the assignment $U \mapsto \{\phi : U \to U\}$ is not stable under ordinary restriction: a diffeomorphism of U need not map a smaller open $V \subseteq U$ into V. The sheaf-like object is the germ-valued local-map object. In Section 6, where maps between different models matter, the corresponding sheaf of local isometries is used.

When S is a pseudo-Riemannian metric, the infinitesimal object associated with $\mathcal{A}_S$ is the Killing sheaf.

Definition 3.2. Let (M, g) be a pseudo-Riemannian manifold. The *Killing sheaf* $\mathcal{K}_g$ is the sheaf of real vector spaces

$$\mathcal{K}_g(U) = \{X \in \Gamma(TU) : \mathcal{L}_X g|_U = 0\}.$$

Its global sections are the global Killing vector fields:

$$H^0(M, \mathcal{K}_g) = \Gamma(M, \mathcal{K}_g).$$

On constant-curvature backgrounds, this sheaf is part of the standard differential-geometric treatment of the Killing equation. Khavkine's work on the Calabi complex identifies the cohomology of that complex with the cohomology of the locally constant sheaf of Killing vectors (Khavkine, 2017). His later work on compatibility complexes for finite-type overdetermined PDEs gives a broader framework for the Killing equation and related systems (Khavkine, 2019).

3.2 The Killing sheaf of a flat quotient

The following proposition records the local-system description of the Killing sheaf on a flat space form.

Proposition 3.3. Let X be simply connected flat pseudo-Riemannian space, let $G = \mathrm{Iso}(X)$, and let $\mathfrak{g} = \mathrm{Lie}(G)$. Suppose $\Gamma \leq G$ acts freely and properly discontinuously on X, and put $M = X/\Gamma$. Then the Killing sheaf $\mathcal{K}_M$ is the locally constant sheaf associated with the representation

$$\Gamma \longrightarrow \mathrm{Aut}(\mathfrak{g}), \qquad \gamma \longmapsto \mathrm{Ad}_\gamma \,.$$

With the opposite deck-action convention this representation is written $\gamma \mapsto \mathrm{Ad}_{\gamma^{-1}}$; the H^0 invariant-subspace calculation below is unaffected, and the higher-cohomology statements should be read relative to the chosen convention. If X is contractible, then

$$H^i(M, \mathcal{K}_M) \cong H^i(\Gamma, \mathfrak{g}_{\mathrm{Ad}})$$

for group cohomology with coefficients in $\mathfrak{g}$ under this adjoint action.

Proof. Every germ of a Killing field on flat simply connected X extends uniquely to a global Killing field on X, because the Killing equation is a finite-type linear system and X is the model flat space; this is the flat-space instance of the Nomizu extension theorem for local Killing fields on simply connected analytic geometries (Nomizu, 1960). Hence the sheaf of Killing fields downstairs is obtained by descending the constant sheaf with fiber $\mathfrak{g}$ on X. Deck transformations act on lifted Killing fields by pushforward, i.e. by the adjoint representation. This is precisely the local system associated with $\mathfrak{g}_{\mathrm{Ad}}$. If X is contractible, M is a $K(\Gamma, 1)$ for the purposes of local-system cohomology, and the sheaf cohomology of the associated local system is canonically identified with group cohomology. This is the standard identification for locally constant sheaves; see Bredon (1997) and, in this specific Killing-sheaf setting, Khavkine (2017). □

The zeroth cohomology group in Proposition 3.3 is especially transparent:

$$H^0(M, \mathcal{K}_M) \cong \mathfrak{g}^{\Gamma}.$$

Thus a global Killing field is a local Killing field on the universal cover whose germ is invariant under all deck transformations. In the Hantzsche–Wendt case this invariant subspace is much smaller than the fiber $\mathfrak{g}$.

3.3 A computation for static compact flat products

The calculation needed for the Hantzsche–Wendt product is the following compact-flat product result.

Theorem 3.4. Let (Σ, h) be a compact connected flat Riemannian manifold, and let

$$(M, g) = (\mathbb{R} \times \Sigma, -\mathrm{d}t^2 + h)$$

be the corresponding static Lorentzian product. Then every global Killing field on (M, g) is uniquely of the form

$$X = a\,\partial_t + Y,$$

where $a \in \mathbb{R}$ is constant and Y is a time-independent Killing field on (Σ, h). Hence

$$H^0(M, \mathcal{K}_g) \cong \mathbb{R} \oplus H^0(\Sigma, \mathcal{K}_h).$$

For compact flat Σ, $\dim H^0(\Sigma, \mathcal{K}_h) = b_1(\Sigma)$, so

$$\dim H^0(M, \mathcal{K}_g) = 1 + b_1(\Sigma).$$

Proof. Let X be a global Killing field on M. Decompose it relative to the product splitting as

$$X = a(t, x)\partial_t + Y(t, x),$$

where $Y(t,\cdot)$ is tangent to Σ for each t. The Killing equations for $g = -\mathrm{d}t^2 + h$ are

$$\partial_t a = 0, \tag{1}$$

$$\partial_t Y = \operatorname{grad}_h a, \tag{2}$$

$$\mathcal{L}_{Y(t,\cdot)} h = 0. \tag{3}$$

Equation (3) says that, for each t, $Y(t,\cdot)$ is a Killing field on the compact flat manifold (Σ, h). A standard Bochner argument implies that every Killing field on a compact flat Riemannian manifold is parallel; equivalently, the identity component of the isometry group is a flat torus of dimension $b_1(\Sigma)$ (Charlap, 1986, Ch. 5, Sec. 6). Thus $\partial_t Y$ is a parallel spatial vector field. By (2), $\operatorname{grad}_h a$ is parallel. But a is independent of t by (1), and Σ is compact, so any smooth function on Σ with parallel gradient has zero gradient. Therefore a is constant and $\partial_t Y = 0$. The field Y is consequently a time-independent global Killing field on (Σ, h).

The converse is immediate: if a is constant and Y is a spatial Killing field independent of t, then $a\partial_t + Y$ is a Killing field of the product metric. This proves the displayed isomorphism. The dimension statement follows from the standard compact-flat fact just cited. □

Corollary 3.5. Let $M = \mathbb{R} \times \Sigma_{\mathrm{HW}}$ with $g = -\mathrm{d}t^2 + h$, where h is any flat Hantzsche–Wendt metric. Then

$$H^0(M, \mathcal{K}_g) \cong \mathbb{R},$$

generated by ∂_t. Nevertheless, every stalk of $\mathcal{K}_g$ is isomorphic to the ten-dimensional Poincaré algebra $\mathfrak{iso}(1,3)$.

Proof. The Hantzsche–Wendt manifold has $b_1(\Sigma_{\mathrm{HW}}) = 0$ (Conway and Rossetti, 2003; Grimmer and Read, 2026). Theorem 3.4 therefore gives a one-dimensional space of global Killing fields. The stalk statement follows because every point of M has a neighborhood isometric to an open subset of Minkowski space. □

Corollary 3.5 separates the local and global facts. The local symmetry algebra is Poincaréan in every stalk, while the only global Killing fields are constant multiples of ∂_t:

$$\mathcal{K}_{g,p} \cong \mathfrak{iso}(1,3) \quad \text{for each } p \in M, \qquad H^0(M, \mathcal{K}_g) = \mathbb{R}\partial_t.$$

The global automorphism group records only the second fact.

Remark 3.6 (Temporal orientation)**.** Corollary 3.5 recovers a preferred timelike *line*. To recover a preferred future-directed vector field V^a, one also needs a temporal orientation, or else one must identify V^a and $-V^a$. The global isometry group still contains time reflection unless it is excluded by orientation or boundary conditions.

3.4 Higher cohomology

For $M = \mathbb{R} \times \Sigma$ with compact flat $\Sigma = \mathbb{R}^3/\Gamma$, the universal cover is $\mathbb{R}^{1,3}$. Proposition 3.3 therefore gives

$$H^i(M, \mathcal{K}_g) \cong H^i(\Gamma, \mathfrak{iso}(1,3)_{\mathrm{Ad}}).$$

The zeroth group records descended global Killing fields. The first group records the first gluing obstruction associated with local infinitesimal symmetries. In the language of the Calabi complex, these groups are the cohomological residue left by local exactness of the Killing equation on a globally nontrivial quotient (Khavkine, 2017). Only the H^0 calculation will be used below.

4 Riggings as torsors

The pseudo-Riemannian calculation is the cleanest place to begin, because the Killing equation is finite type and its sheaf is locally constant. The classical hierarchy is subtler. Leibnizian and Maxwellian structures have infinite-dimensional local automorphism pseudogroups on standard local patches: time-dependent translations, and in the Leibnizian case time-dependent rotations, are locally allowed (Earman, 1989; Malament, 2012; Weatherall, 2018; Grimmer and Read, 2026). A direct Killing-sheaf calculation is therefore not the best first formalism.

For the rigging part of the classical hierarchy, it is convenient to use torsors. The additional structures are local choices, and differences between such choices are acted on by a sheaf of relative symmetry data. This section applies that idea to Newtonian riggings.

4.1 The sheaf of local riggings

Let $L = (M, T, h)$ be a product Leibnizian background, with $M = \mathbb{R} \times \Sigma$, T the projection onto $\mathbb{R}$, $\tau = \mathrm{d}T$, and h a time-independent Riemannian metric on the leaves Σ_t. This product assumption is narrower than the most general Leibnizian model. It is the setting used for the Hantzsche–Wendt collapse in Grimmer and Read (2026), and it isolates the topological mechanism from additional curvature or time-dependence.

Definition 4.1. The *vertical local symmetry sheaf* $\mathcal{V}_L$ is the sheaf of real vector spaces whose sections over $U \subseteq M$ are vertical vector fields preserving the spatial metric:

$$\mathcal{V}_L(U) = \{Y \in \Gamma(TU) : \tau(Y) = 0,\ \mathcal{L}_Y h = 0\}.$$

Here $\mathcal{L}_Y h = 0$ is understood leafwise on the spatial metric.

Definition 4.2. The *local rigging sheaf* $\mathcal{R}_L$ is the sheaf of sets whose sections over $U \subseteq M$ are vector fields $V \in \Gamma(TU)$ satisfying

$$\tau(V) = 1, \qquad \mathcal{L}_V h = 0.$$

A section of $\mathcal{R}_L$ is an admissible local Newtonian rest field compatible with the Leibnizian background.

The preservation condition $\mathcal{L}_V h = 0$ is not intended to be the only possible definition of “admissible rigging” in every classical spacetime setting. It is the minimal condition needed here: the rigging must carry each spatial metric to the next without changing intrinsic spatial distances. Other choices, for example involving Maxwellian standards of rotation or Galilean connections, lead to analogous torsors under different coefficient sheaves.

Only the rigging sector relevant to Grimmer and Read’s recovery of a preferred rest field is being considered here. This is not a full torsorial reconstruction of Newtonian or Galilean spacetime. Including the compatible affine connection, the Maxwellian standard of rotation, or both would change the coefficient sheaves, while leaving the same sort of local-to-global question in place.

Proposition 4.3. If $\mathcal{R}_L$ is locally nonempty, then it is a torsor under $\mathcal{V}_L$.

Proof. If $V, V' \in \mathcal{R}_L(U)$, then $V' - V$ is vertical because $\tau(V' - V) = 0$, and

$$\mathcal{L}_{V'-V} h = \mathcal{L}_{V'} h - \mathcal{L}_V h = 0.$$

Thus $V' - V \in \mathcal{V}_L(U)$. Conversely, if $V \in \mathcal{R}_L(U)$ and $Y \in \mathcal{V}_L(U)$, then

$$\tau(V + Y) = 1, \qquad \mathcal{L}_{V+Y} h = 0,$$

so $V + Y \in \mathcal{R}_L(U)$. The action is free and transitive on each nonempty set of sections. The compatibility with restriction maps is immediate. $\square$

The usual torsor-cohomology theorem now applies.

Theorem 4.4. Let $\mathcal{V}$ be a sheaf of abelian groups on M, and let $\mathcal{R}$ be a $\mathcal{V}$-torsor. Then the isomorphism class of $\mathcal{R}$ determines a class

$$[\mathcal{R}] \in H^1(M, \mathcal{V})$$

with the following properties:

(a) $[\mathcal{R}] = 0$ iff $\mathcal{R}$ admits a global section;
(b) if $[\mathcal{R}] = 0$, then the set $\Gamma(M, \mathcal{R})$ of global sections is a torsor under $H^0(M, \mathcal{V})$;
(c) consequently, if $[\mathcal{R}] = 0$ and $H^0(M, \mathcal{V}) = 0$, then $\mathcal{R}$ has a unique global section.

Proof. Choose an open cover $\{U_i\}$ on which $\mathcal{R}$ has local sections V_i. On overlaps, $V_j - V_i$ is a section of $\mathcal{V}(U_i \cap U_j)$ by Proposition 4.3. The family $g_{ij} = V_j - V_i$ satisfies the Čech cocycle condition on triple overlaps. Changing the local sections changes g_{ij} by a Čech coboundary. Thus $\mathcal{R}$ determines a class in $\check{H}^1(M, \mathcal{V})$, and this class vanishes exactly when the local V_i can be adjusted to agree on overlaps, i.e. exactly when they glue to a global section. If V and V' are two global sections, then $V' - V \in H^0(M, \mathcal{V})$, and every global section of $\mathcal{V}$ acts in this way. This is the standard classification of torsors by first sheaf cohomology (Giraud, 1971; Bredon, 1997). $\square$

4.2 The Hantzsche–Wendt rigging theorem

The preceding torsor calculation gives the following version of the recovery of a rest frame on Hantzsche–Wendt backgrounds.

Theorem 4.5. Let $L = (\mathbb{R} \times \Sigma_{\mathrm{HW}}, T, h)$ be the static Leibnizian product with h a flat Hantzsche–Wendt metric on each spatial slice. Then the rigging sheaf $\mathcal{R}_L$ has exactly one global section, namely ∂_t relative to the product coordinate $t = T$.

Proof. The vector field ∂_t is a global section of $\mathcal{R}_L$, so $[\mathcal{R}_L] = 0$. It remains to show uniqueness. By Theorem 4.4, non-uniqueness is measured by $H^0(M, \mathcal{V}_L)$.

Let $Y \in H^0(M, \mathcal{V}_L)$. For each fixed t, the restriction Y_t is a global Killing field of $(\Sigma_{\mathrm{HW}}, h)$. Since $b_1(\Sigma_{\mathrm{HW}}) = 0$, the identity component of the isometry group of $(\Sigma_{\mathrm{HW}}, h)$ is trivial, and hence $Y_t = 0$ for every t (Charlap, 1986; Conway and Rossetti, 2003). Thus $Y = 0$. Therefore $H^0(M, \mathcal{V}_L) = 0$, and the global section ∂_t is unique. $\square$

In the static Hantzsche–Wendt case, the torsor obstruction vanishes because a global product rigging exists. The coefficient sheaf has no global sections because there are no continuous spatial isometries. Existence together with vanishing non-uniqueness gives a unique rest field.

By contrast, on $M = \mathbb{R} \times \mathbb{R}^3$ with the standard spatial metric, $\mathcal{V}_L$ has many global sections. In coordinates, $Y = f^i(t)\partial_i$ gives a vertical metric-preserving field for arbitrary smooth functions $f^i(t)$. The Hantzsche–Wendt topology does not change the local form of $\mathcal{V}_L$, but it eliminates these global sections.

4.3 Relation to the radiance obstruction

The torsor class $[\mathcal{R}_L] \in H^1(M, \mathcal{V}_L)$ is structurally analogous to the radiance obstruction of Goldman and Hirsch (1984). In the affine setting, an affine representation has a linear part and a translational part; the translational part defines a crossed homomorphism, and its cohomology class is the obstruction to the representation fixing a point. Equivalently, for a flat affine bundle, the vanishing of the radiance obstruction is equivalent to the existence of a global flat section (Goldman and Hirsch, 1984). The same pattern appears here: local riggings form an affine space over local relative-velocity data, and the obstruction to gluing them is a first cohomology class.

The analogy is only used here at the level of structure. Galilean and Newtonian structures are naturally affine-geometric: Galilean connections, inertial frames, and rest frames are choices of affine data compatible with the temporal and spatial metrics (Malament, 2012; Weatherall, 2018). The torsor class above plays the same formal role for riggings that the radiance class plays for fixed points of affine representations.

4.4 Categorical recovery from unique global sections

The torsor theorem also explains why the forgetful functor can become an equivalence on a fixed Hantzsche–Wendt topology.

Proposition 4.6. Let $\mathbf{L}(M)$ be a category of Leibnizian models on a fixed manifold M, and let $\mathbf{N}(M)$ be the corresponding category of Newtonian models obtained by adding an admissible rigging. Suppose that every object $L \in \mathbf{L}(M)$ has a unique admissible global rigging V_L, functorially in the following sense: for every Leibnizian isomorphism $\phi : L \to L'$, the pushforward $\phi_* V_L$ is an admissible rigging of L'. Then the forgetful functor

$$F : \mathbf{N}(M) \to \mathbf{L}(M)$$

is an equivalence of categories.

Proof. Define $G : \mathbf{L}(M) \to \mathbf{N}(M)$ by sending L to (L, V_L). If $\phi : L \to L'$ is a Leibnizian isomorphism, functoriality implies that $\phi_* V_L$ is an admissible rigging of L'. By uniqueness, $\phi_* V_L = V_{L'}$, so ϕ is automatically a Newtonian isomorphism $G(L) \to G(L')$. Thus G is well-defined on morphisms. We have $F \circ G = \mathrm{id}_{\mathbf{L}(M)}$ strictly. Conversely, if (L, V) is Newtonian, then uniqueness gives $V = V_L$, so $G \circ F(L, V) = (L, V)$ up to equality. Hence F is full, faithful, and essentially surjective. ☐

Theorem 4.5 supplies the hypothesis of Proposition 4.6 for the static Hantzsche–Wendt Leibnizian models considered here. This is the sheaf-theoretic mechanism behind the global categorical collapse within the static product setting: once the Leibnizian data are restricted to this product class, the supposedly forgotten Newtonian rigging is represented by the unique global section of the associated rigging torsor.

5 Global categories versus stacks of local models

The preceding section explains how a forgetful functor can become an equivalence on a fixed topology. It does not imply that the corresponding theories are locally equivalent. The difference is expressed by distinguishing the category of global sections over M from the stack of local models over M.

5.1 Model stacks

Definition 5.1. Let T be a spacetime theory, understood kinematically as specifying a class of geometric structures and their isomorphisms. Its *prestack of local models* on M is the assignment

$$\mathfrak{Mod}_T : \mathbf{Open}(M)^{\mathrm{op}} \longrightarrow \mathbf{Gpd}$$

that sends an open set $U \subseteq M$ to the groupoid of T-models on U, and sends an inclusion $V \subseteq U$ to restriction of models from U to V. When descent for objects is imposed, or after stackification if needed, this becomes the stack of local T-models.

This definition is standard stack language applied to spacetime structures; see Giraud (1971) and Mac Lane and Moerdijk (1992). Its philosophical role is to separate two claims:

$$\begin{aligned} \text{global-model equivalence on } M: &\quad \mathfrak{Mod}_T(M) \simeq \mathfrak{Mod}_{T'}(M), \\ \text{local or stack-level equivalence on } M: &\quad \mathfrak{Mod}_T \simeq \mathfrak{Mod}_{T'} \text{ as stacks on } \mathbf{Open}(M). \end{aligned}$$

The first compares only global sections. The second compares all local sections and their restriction/gluing behavior.

5.2 The stack-level inequivalence theorem

The following result formulates the local inequivalence emphasized by Grimmer and Read. The claim concerns the natural forgetful comparison, not every possible abstract equivalence between underlying stacks.

Theorem 5.2. On $M = \mathbb{R} \times \Sigma_{\mathrm{HW}}$, the global category of static Leibnizian models considered in Theorem 4.5 is equivalent to the corresponding global category of Newtonian models. However, the natural forgetful morphism from the Newtonian stack of local models to the Leibnizian stack of local models is not an equivalence of stacks.

Proof. The global equivalence follows from Proposition 4.6. For the failure of local equivalence of the forgetful morphism, it suffices to look on any sufficiently small coordinate ball $U \subset M$. Such a U is indistinguishable from an open subset of standard Leibnizian spacetime on $\mathbb{R}^4$.

For this argument, an infinitesimal automorphism of a local model means a vector field whose local flow consists, wherever defined, of local isomorphisms of the relevant structure; equivalently, it is a section of the Lie algebra sheaf of the local automorphism pseudogroup. On U, the infinitesimal automorphisms of the Leibnizian structure include vector fields of the form

$$X = f(t)\partial_x$$

for arbitrary smooth germs f in the time coordinate; these generate local time-dependent spatial translations preserving temporal intervals and instantaneous spatial distances. By contrast, for a Newtonian model with rest field $V = \partial_t$, the additional preservation condition $\mathcal{L}_X V = [X, V] = 0$ forces $f'(t) = 0$. Thus the infinitesimal automorphism sheaf of the Leibnizian local model contains time-dependent shift germs that are absent from the Newtonian local model lying over it.

A stack equivalence is fully faithful, and full faithfulness preserves the automorphism-germ data of corresponding objects. The forgetful morphism sends the Newtonian object (U, τ, h, V) to the Leibnizian object (U, τ, h) but does not induce an isomorphism on these local automorphism germs, already at the infinitesimal level. Therefore this natural morphism is not an equivalence of stacks. $\square$

The same argument applies, with minor modifications, to the corresponding natural forgetful morphisms between other pairs in the hierarchy. Galilean local structure permits boost generators $t\partial_x$ that Newtonian local structure forbids. Maxwellian local structure still permits arbitrary time-dependent translations, though not time-dependent rotations. Minkowskian local structure has Lorentz boost generators such as $t\partial_x + x\partial_t$, while Newtonian local structure does not. Thus the local stacks recover the familiar distinctions even when global categories over $\mathbb{R} \times \Sigma_{\mathrm{HW}}$ collapse.

Corollary 5.3. The fixed-topology equivalence claims and the local-inequivalence claims are compatible. On $\mathbb{R} \times \Sigma_{\mathrm{HW}}$, the relevant theories can be equivalent on global sections while the natural forgetful comparisons remain inequivalent at the stack level.

Thus the global equivalence claim and the local inequivalence claim concern different comparison objects. The first concerns global sections; the second concerns the local stack and its automorphism germs.

6 Determinism as sheaf extension

The determinism results in Grimmer and Read are also local-to-global results. Their definitions, adapted from recent work by Manchak, Barrett, Halvorson, Weatherall, and others, concern whether isometries between initial segments imply, extend to, or uniquely determine isometries between full models (Manchak and Barrett, 2023; Manchak et al., 2025; Read and Manchak, 2026; Grimmer and Read, 2026). This is a restriction-map question. There is, however, a technical point to keep separate: a sheaf of local isometries has global sections that are local isometries defined on all of the source, whereas the determinism definitions require isometries from one full model *onto* another full model. Accordingly, local-isometry sheaves and global surjective isometries will be distinguished.

The theorem below is narrower than Grimmer and Read's full determinism discussion. It treats the static product cases responsible for the Hantzsche–Wendt-style de re* extension result. No attempt is made here to re-prove the de dicto* constructions involving boundary strategies, Waldmüller-type examples, Margulis spacetimes, or Cauchy-hyperbolic causal patches. In the present framework, those examples would be represented by the same restriction-map formalism, with de dicto* corresponding to singletonhood of the full-isometry set.

6.1 Local isometries and global isometries

Let $A = (M, O_i)$ and $B = (M', O'_i)$ be two spacetime models of the same type. Write

$$\mathrm{Isom}(A, B)$$

for the set of full isometries $A \to B$, i.e. diffeomorphisms $M \to M'$ preserving all of the fields. Also define a sheaf of local isometries as follows.

Definition 6.1. The local-isometry sheaf $\mathcal{I}_{A,B}$ assigns to each open set $U \subseteq M$ the set of local maps

$$\phi : U \longrightarrow M'$$

that are local diffeomorphisms and preserve the geometric fields locally: for every $p \in U$ there is a neighborhood $W \subseteq U$ of p such that $\phi|_W$ is a diffeomorphism onto its image and $(\phi|_W)^*(O'_i|_{\phi(W)}) = O_i|_W$ for every field in the model. Restrictions are given by restricting the domain of ϕ.

This is the usual sheaf of local maps, or equivalently the sheaf of sections of the ètale space of germs of local isometries. It is a genuine sheaf: compatible local isometries glue uniquely as smooth local maps, and preservation of geometric fields is local. The full-isometry set is generally a distinguished subset of $\Gamma(M, \mathcal{I}_{A,B})$, namely those global local isometries whose underlying map is a diffeomorphism from M onto M'.

For every open $U \subseteq M$ there is a restriction map

$$r_U : \mathrm{Isom}(A, B) \longrightarrow \Gamma(U, \mathcal{I}_{A,B}), \qquad \Psi \longmapsto \Psi|_U.$$

When an initial segment $U \subseteq M$ and a target initial segment $U' \subseteq M'$ are fixed, write

$$\mathcal{E}_{U,U'}(A,B) = \{\phi \in \Gamma(U, \mathcal{I}_{A,B}) : \phi \text{ is a diffeomorphism } U \to U'\}.$$

The relevant restriction map for de re determinism is then the map from full isometries $A \to B$ that send U to U' into $\mathcal{E}_{U,U'}(A,B)$.

Proposition 6.2. For a class $\mathcal{C}$ of globally hyperbolic spacetime models, the determinism and rigidity conditions used by Grimmer and Read can be expressed as follows.

(a) De re determinism says that, for every relevant pair $A, B \in \mathcal{C}$ and every pair of initial segments $U \subseteq M$ and $U' \subseteq M'$, every element of $\mathcal{E}_{U,U'}(A,B)$ lies in the image of restriction from a full isometry $A \to B$.

(b) De re* determinism says that this extension is unique; equivalently, the relevant restriction map from full isometries to initial isometries is bijective onto $\mathcal{E}_{U,U'}(A,B)$ whenever the latter is nonempty.

(c) Rigidity says that r_V is injective for every nonempty open set $V \subseteq M$.

(d) De dicto determinism says that if $\mathcal{E}_{U,U'}(A,B) \neq \varnothing$ for initial segments U, U', then $\mathrm{Isom}(A,B) \neq \varnothing$.

(e) De dicto* determinism says that if $\mathcal{E}_{U,U'}(A,B) \neq \varnothing$ for initial segments U, U', then $\mathrm{Isom}(A,B)$ is a singleton. Unlike de re*, it does not require the unique full isometry to restrict to the particular initial isometry with which one began.

(f) The giraffe condition says that $\mathrm{Isom}(A,B)$ has at most one element for every relevant pair A, B.

Proof. This is a direct unpacking of the definitions. A full isometry is an element of $\mathrm{Isom}(A,B)$. An initial isometry from U to U' is an element of $\mathcal{E}_{U,U'}(A,B)$. Requiring that a specified initial isometry extend is surjectivity of the corresponding restriction map; requiring a unique extension is bijectivity. Requiring two full isometries that agree on an arbitrary nonempty open set to be identical is injectivity of restriction to that open set. The de dicto clauses forget which initial isometry is extended and ask only for non-emptiness or singletonhood of the set of full isometries. ☐

This proposition has two advantages. First, it places all the determinism notions in one diagram. Second, it makes clear why the Hantzsche–Wendt case depends on initial segments rather than arbitrary small opens. The initial segment $(-\infty, t_0) \times \Sigma_{\mathrm{HW}}$ contains the whole spatial topology. A small coordinate ball does not.

6.2 Initial segments on Hantzsche–Wendt products

The following theorem is the sheaf-extension form of the de re* determinism result for the classical Hantzsche–Wendt models. It is stated with the time-orientation appropriate to past initial segments. If temporal orientation is not part of the structure, the full spacetime may admit a time-reflection isometry, but such a reflection sends a past half-space to a future half-space and therefore is not an isometry between two past initial segments of the form used in the determinism definitions.

Theorem 6.3. Let A and B be static Leibnizian or Maxwellian product models of the form $\mathbb{R} \times \Sigma$ and $\mathbb{R} \times \Sigma'$, where Σ and Σ' are compact connected flat Riemannian manifolds and the set of spatial isometries $\mathrm{Iso}(\Sigma, \Sigma')$ is discrete. Let

$$U = (-\infty, t_0) \times \Sigma, \qquad U' = (-\infty, t_0') \times \Sigma'$$

be past initial segments. Then every initial isometry $\phi : U \to U'$ extends to a unique full isometry $\Psi : A \to B$. In particular, the Hantzsche–Wendt models considered above are de re* deterministic.

Proof. An isometry of initial segments must preserve the temporal metric and the spatial metrics on simultaneity slices. Thus its time component has the form $t \mapsto \epsilon t + c$ with $\epsilon \in \{+1, -1\}$, and its spatial component is a family of spatial isometries $\psi_t : \Sigma \to \Sigma'$. Since ϕ maps the past interval $(-\infty, t_0)$ onto the past interval $(-\infty, t'_0)$, the time sign must be $\epsilon = +1$; a sign -1 would map the past half-line to a future half-line. Hence $c = t'_0 - t_0$.

The source time interval is connected and $\mathrm{Iso}(\Sigma, \Sigma')$ is discrete, so the map $t \mapsto \psi_t$ is constant. Therefore every initial isometry has the static form

$$\phi(t, x) = (t + c, \psi(x))$$

for a fixed spatial isometry $\psi : \Sigma \to \Sigma'$. The same formula defines a full isometry

$$\Psi(t, x) = (t + c, \psi(x))$$

from $\mathbb{R} \times \Sigma$ to $\mathbb{R} \times \Sigma'$. This proves existence of an extension. If two full isometries restrict to the same initial isometry, their constants c and their spatial isometry ψ agree on U, so they agree everywhere. This proves uniqueness. □

The proof uses the same discreteness argument as Grimmer and Read's determinism section, but presents it as a statement about a restriction map. The common point is that a connected time parameter cannot move continuously through a discrete spatial isometry set.

6.3 The cohomology of extension obstructions

Sheaf language also identifies where extension can fail, although the general statement is nonabelian and is best handled as a cocycle construction rather than as an ordinary linear cohomology group. Suppose an initial section $s_U \in \mathcal{E}_{U,U'}(A, B)$ is to be extended to M. Choose an open cover $\{W_i\}$ of M such that local extensions $s_i \in \Gamma(W_i, \mathcal{I}_{A,B})$ exist and agree with s_U where appropriate. On overlaps $W_i \cap W_j$, the local extensions differ by local automorphisms of B:

$$g_{ij} = s_j \circ s_i^{-1}.$$

When the domains and images are tracked carefully, the g_{ij} can be regarded as defining, in favorable cases or after formulating the automorphism germs as a nonabelian sheaf/groupoid, a nonabelian Čech 1-cocycle. The associated cocycle class is then the obstruction to gluing the local extensions to a single global section. If the class vanishes, extension exists. If the relevant zeroth automorphism data are trivial, or if the restriction map is separated in the sense of rigidity, the extension is unique.

This observation is not needed to prove Theorem 6.3; there the discreteness argument is enough. It is included only to indicate how determinism, like rigging recovery, fits a first-cocycle obstruction and zeroth-order uniqueness pattern once the relevant nonabelian coefficient object has been specified. The exact coefficient object depends on the theory and on the pair of models.

7 Conclusion

In this paper, I have considered the local/global structure behind Grimmer and Read's Hantzsche–Wendt examples. The relevant spacetimes remain locally Euclidean or Minkowskian, but the global topology restricts which local automorphisms, riggings, and isometries extend over the whole model.

This was made explicit in three ways. First, for static compact flat products,

$$H^0(\mathbb{R} \times \Sigma, \mathcal{K}_{-\mathrm{d}t^2+h}) \cong \mathbb{R} \oplus H^0(\Sigma, \mathcal{K}_h),$$

so $\mathbb{R} \times \Sigma_{\mathrm{HW}}$ has Poincaréan Killing-sheaf stalks but only the time-translation Killing field globally. Second, in the rigging sector, local Newtonian rest fields form a torsor under vertical metric-preserving fields; in the static Hantzsche–Wendt case this torsor has a unique global section. Third, the determinism notions used by Grimmer and Read can be written as extension and uniqueness conditions for restriction maps from full isometries to initial isometries.

The resulting picture leaves the Grimmer–Read global analysis intact while adding the corresponding local data. The global model category records global sections. The local stack, local automorphism pseudogroup, and Killing sheaf record the local structure and its gluing behaviour. This is the sense in which the Hantzsche–Wendt examples both collapse certain global distinctions and preserve the local distinctions emphasized in their discussion.

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