Borromean Rings: Chance & Complementarity Homology & Archetype, Analogy & Prototype

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Abstract. In this article we develop the Borromean rings metaphor for the complex entanglement of three threads with complementary pairs of concepts, like Chance & Complementarity, Homology & Archetype, and Analogy & Prototype. The context of application of this concepts includes biological theories of evolution, the means and methods used for studying divergence and convergence processes leading to life forms most beautiful, and their organization in a tree of life of phylogenetic history. Moreover, we suggest innovative definitions for the concepts of Analogy and Prototype, and motivate their use for acquiring valuable quantitative information leading to powerful methods of analysis that, in turn, can be helpful for reaching correct and clear conclusions while avoiding logical traps and pitfalls. Furthermore, we explain how the concepts of Analogy and Prototype can support teleological or functional interpretations. Finally, we consider philosophical consequences of these discussions concerning the use and meaning of language, and indicate some topics for further research.

Keywords: Complementarity; Probability & Inference; Discrete & Convex sets; Homology, Archetype & structure; Analogy, Prototype & function; Tree of Life.

> *It is evident that the attitudes termed mechanistic and finalistic do not present contradictory views on biological problems, but rather stress the mutually exhaustive observational conditions equally indispensable in our search for an ever richer description of life.* Niels Bohr (1957, p.100)

> *... also in other regions of human knowledge we meet apparent contradictions which might seem to be avoidable only from the point of view of complementarity.* Niels Bohr (1937, p.88)

> *Convexity is the essential property that warrants the emergence of eigen-solutions or equilibrium points and, by definition, convexity requires continuity. Moreover, in this scope of investigation, randomization can be geometrically interpreted as a "convexification / discretization operator".*
> This article, Sec.1.

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1. Complementarity in Borromean Rings

The two following quotations, by Charles Darwin (1809-1882) and Richard Owen (1804-1892), point to several perceived paradoxes in biological evolution and adaptation. First, the paradoxical role played by probability in its theoretical conception. On the one hand, randomness is seen as a source of noise or error; On the other hand, stochastic processes drive the evolution of well-adapted solutions. On the one hand, probability describes naturally unpredictable future events; On the other hand, statistical analysis is used to trace and confirm the paths already taken by nature in the development of life forms most beautiful. Second, the paradoxical aspects of functional adaptability vs. structural stability. On the one hand, we wonder at innovative features and designs perfected and optimized to best help a particular species to survive; On the other hand, we attest the inertia of persistent patterns or (ideal) archetypal forms that are inherited or shared by diverse genera of living creatures. This article examines how good methods of logic, probability and statistical inference can help to accommodate or solve these perceived paradoxes in a coherent theoretical framework. As fundamental conceptual tools, this article employs and further develops the notions of Homology, Analogy, Archetype, and Prototype. The last of these concepts, Prototype, is the focus of original contributions used to clarify some theoretical aspects and suggest effective application methods.

> *The view that each variation has been providentially arranged seems to me to make natural selection entirely superfluous, and indeed takes the whole case of appearance of new species out of the range of science. [FV] It seems to me that* VARIATIONS *in the wild and domestic conditions are due to unknown causes and are* WITHOUT PURPOSE *and insofar accidental; and that [LS] they* BECOME PURPOSEFUL *only when they are selected by man for his pleasure, or by what we call natural selection in the struggle for life under changing conditions. I do not wish to say that God did not foresee everything which would ensue; but here comes very nearly the same sort of wretched embroglio as between free-will and preordained necessity.*
>
> Charles Darwin letter to Ch. Lyell (1861)

> *Those physiologists who admit no other principle to have governed the construction of living beings than the exclusive and absolute adaptation of every part to its function, ... conceiving, quite gratuitously in my opinion, the idea of* CONFORMITY OF TYPE *to be* OPPOSED *to the idea of* DESIGN. *.. The teleologist.. finds the adaptation of the organ to its function square with his notions of the perfection of a machine constructed for such an end. .. The fallacy perhaps lies in judging of created organs by the analogy of made machines; but it is certain that in the instances where that analogy fails to explain the structure of an organ, such structure does not exist 'in vain' if its truer comprehension lead rational and responsible beings to a better conception of their own origin and Creator. .. For the Divine mind which planned the Archetype also foreknew all its modifications.*
>
> Richard Owen (1849, p.84-86) apud Cain (1989, p.9-10)

G - Chance & Complementarity. Discrete & Continuous random variables; Discretization & Convexification operations; Fortuitous variation & Probabilistic survival in evolution theories; Fortuitous branching & Probabilistic pruning in stochastic optimization; Multiple explanatory variables used in statistical models; Multiple *Abstract Belief Calculi* used in statistical modeling, etc.
R - Homology & Archetype. Structure & Divergence; Combinatorial concidences & Unlikely arrangement outcomes; Grammar & Communication; etc.
B - Analogy & Prototype. Function & Convergence; Fine tuning conditions & the Zero probability paradox (ZPP); Symbolic meaning & Understanding; etc.

TABLE 1. Borromean entanglement of three threads, color-coded RGB, each made of intertwined strands of conflating concepts.

In this article we critically examine three entangled threads, each of these made of intertwined strands of conflating concepts. These three threads are visually represented by Borromean rings, while conflating concepts in each thread are represented by white and colored strands in each ring, see Figure 1.l. In the sequel, we color-code these rings by the basic colors, RGB – Red, Green and Blue. Each thread in this Borromean metaphor is represented in the title of this article by a leading pair of conflating conceptual strands, see Table 1. These conflating concepts, doubly interlocked in such a complex configuration, are found in theories of evolution and their applications to the fields of biology, ethology, linguistics, etc.

This configuration doubly interlocks the aforementioned concepts by first intertwining single strands in one of the threads, and then entangling the three threads in the Borromean knot. This complex configuration is a mixed blessing: On the one hand, it allows intricate and subtle conceptual interactions that give modern evolution theories the structural complexity and the explanatory power needed to successfully dwell in some of the deepest mysteries of life and language, their nature, individual development, and shared evolution.

On the other hand, constant attention and careful scrutiny are required when using the same concepts, in order to avoid logical fallacies, inferential traps, and reasoning pitfalls resulting from careless or mistaken use of such conflating concepts deeply entangled in the Borromean rings. Fortunately, careful analyzes of these situations can help us to avoid such mistakes, as analyzed in the sequel, specially at Section 5. This article also advances novel ideas on how to use the concepts under scrutiny in applied sciences by, when convenient, modifying definitions of some key concepts or better adapting guidelines for their use. A distinctive and revolutionary aspect of Darwin's theory concerns the stochastic nature of biological evolution processes. Accordingly, the next subsection presents some basic concepts of probabilistic systems that will be useful in subsequent discussions.

1.1. Probability as a Discretization or Convexification Operator

Type conversion or *type casting* are operations that transform an argument or expression from one type to another, often avoiding a type mismatch fallacy. For example, in computer programming, the *round*, *floor* or *ceil* operators are standard alternatives

FIGURE 1. (l) Borromean rings. (c) Mathieu Lauweriks' (1895) – *The spirit of times*, showing diadic and triadic entanglements. (r) Random discretization / convexification operators acting by breaking a continuous symmetry - the spinning top, or by building one by asymptotic convergence - F. Galton's Quincunx machine.

used when "assigning" a floating point value to a variable of integer type. In general, such conversions are not arbitrary operations, for they correspond to real changes in the nature of the objects being represented, or to reinterpretations of the variables representing the properties of these objects. Random events, Stochastic processes, and other situations ruled by Probability theory are ubiquitous in modern scientific models. In some of these situations, a system described by continuous variables generates a discrete random variable, or a system described by discrete variables generates a continuous random variable, as discussed in the following examples:

(A) First, let us consider a prismatic spinning top, see Figure 1.tr; This is a *random number generator* or *randomization device* used for gambling. The top's rotation movement is characterized by continuous symmetries, namely, the radial symmetries of its inertia ellipsoid, see [86, ch.13]. Once spinning, the kinetic energy of the top slowly dissipates, until the movement's symmetries are suddenly broken when the top "falls upon" or "collapses into" one of its sides, hence generating a discrete state that, in turn, is associated to a Multivariate random variable whose possible values are engraved at the prismatic faces.

(B) Next, let us consider a classic example in Game Theory. In the Odds & Evens game (OEg), a.k.a. Matching-Pennies game, each of the players, Odd and Even, has to act by showing, simultaneously, a single bit, 0 or 1. If both bits agree, Odd wins, otherwise, Even wins. Hence, the set of possible actions for Odd and Even is given by the Discrete set $D = \{(0,0),(0,1),(1,0),(1,1)\}$. This is also the set of *Pure* or *Deterministic Strategies* in this game, that is, the ways players can choose how to act. Set D can be represented geometrically as the corners of the unit square in $I\!R^2$. *Mixed* or *Randomized Strategies* expand the ways players can choose how to act by allowing each player to chose probabilities of possible action. In the OEg, each player defines his strategy by a (continuous) parameter in the unit interval, $\lambda \in [0, 1]$, and then selects action 1 with probability λ, or action 0 with probability $(1 - \lambda)$. This (involuntary and independent) random selection can be accomplished by using a randomization device, like flipping a penny, hence the alternative name of this game. Mixed strategies for Odd and Even can be represented by a vector in the unit square, $[\lambda_O, \lambda_E] \in C = [0, 1]^2 \subseteq I\!R^2$. Notice that taking parameters at the extreme points of the continuous set C, i.e. $[\lambda_O, \lambda_E] \in D = \{0, 1\}^2$ recovers the deterministic strategies.

An *equilibrium-point* of a game, a.k.a. *fixed-point*, *invariant-solution* or *eigen-solution*, is a set of strategies that leaves each player at a local optimum, that is, each player, having full knowledge of the other player's strategy, can gain nothing by unilaterally changing his own. The OEg allowing only *Deterministic* strategies *does not* have any equilibrium point. In this situation, the OEg posees a dilemma - $\delta\iota\lambda\eta\mu\mu\alpha$ - a problem offering two possibilities, none of which is acceptable. Caught in this logical ambush, the players would be trapped, forever flip-flopping between the discrete options available. The OEg allowing *Mixed* strategies has a unique equilibrium point, namely, $[\lambda_O, \lambda_E] = [0.5, 0.5]$. Expanding the discrete space of deterministic strategies to the continuous space of mixed or randomized strategies allows the emergence of an equilibrium-point, hence offering a (eigen) solution to the unsolvable dilemma posed by the discrete case.

Geometrically, the operation of taking the Convex Hull transforms the aforementioned discrete space into the continuous space of strategies, while the operation of taking the Extreme Points goes the other way around. A *Convex Combination* of two points in an Euclinean space, x and y, can be interpreted either as a point z in the line segment joining them, or as a mixture or weighted average of these two points, namely, $z = (1 - \lambda)x + \lambda y$, $\lambda \in [0, 1]$. A *Convex Set* contains any convex combinations of its points. The *Convex Hull* of a set is the smallest convex set containing it. The *Extreme Points* of a set are those that cannot be obtained as convex combinations of other points in the set. Convexity is the essential property that warrants the emergence of eigen-solutions or equilibrium points and, by definition, convexity requires continuity. Moreover, in this scope of investigation, randomization can be geometrically interpreted as a "convexification / discretization operator". Nevertheless, notice that after a game has been played, the players' actions have been fully determined, and the continuum of future possibilities, once replete with probabilistic mixtures, has "collapsed into" the single discrete actions already taken.

(C) Finally, let us consider Francis Galton's Quincunx machine, see Figure 1.br. This machine uses a symmetrical arrangement of a *large number* of binary random events in order to generate (an approximation characterized by asymptotic convergence to) a continuous Gaussian (a.k.a. Normal) distribution. The Gaussian has some remarkable properties characterizing it as a *stable-distribution*; a.k.a. *eigen-*, *equilibrium-* or *invariant- / -form*, *-function* or *-solution* in the space of probability distributions. First, this distribution has a functional form that is invariant by a specific *coupling operation*, in this case, a linear combination. That is, if

X and Y are Gaussian random variables, then $Z = \alpha X + \beta Y$ is also a Gaussian random variable. Second, the same coupling operation can be used to aggregate a large number of random variables into a random variable that converges (in distribution) to this stable form. In this case, under appropriate regularity conditions, the (weighted sum) average of a large number of independent and identically distributed random variables converges to a Gaussian random variable. This cornerstone result of statistical theory is known as the *Central Limit Theorem*. Similar properties characterize the Bernoulli, Cauchy, Exponential, Gamma, Gumbel, Pareto, Poisson, and Weibull; distributions derived therefrom; and others of the most often used in Statistics. The other way around, one can argue that the characteristic properties of an eigen-solution explain why these distributions are so important to statistical theory, and also why they are so well-adapted to the practice of statistical modeling, that is, why these solutions keep "showing-up" in real applications; see [120, sec.5.5] and [103, 121, 148] for further comments and references. Finally, it is important to remark that the key concept of convergence, essential for our former considerations, only makes sense in continuous spaces.

The interpretations developed in the study of these simple examples will be used in the following sections to gain valuable insights about the evolution of complex systems. These examples also highlight the distinct nature and the different roles played by discrete and continuous variables, and the need to avoid type mismatch fallacies. For (honors) high school level introductions to Probability, Information, and Game Theory, see [3, 62, 85, 101, 119, 134, 137–139, 142, 147]. For basic introductions to Probability and Statistics at the level of college mathematics, see [29, 56–58, 89, 112]. For readable introductions to convex analysis pertinent to this context, see [81, 88, 149]; [93] is the fundamental work of modern Game Theory; [90, 120] give further interpretations for the role of randomization. For the historical development of calculus and the key concepts of convergence and continuity, see [6, 9, 16, 20, 31, 77].

2. Chance Coincidence, Random Rami & Probabilistic Pruning

In this section we examine three pairs of complementary concepts involving probability or improbability arguments, and the roles they play in the context of scientific theories of biological evolution; they are:

- "By Chance" explanations & Improbable coincidences in Biology;
- Fortuitous variations (FV) & Likely survival (LS) of well-fitted individuals;
- Random ramification (RR) & Probabilistic pruning (PP) in the tree of life.

The historical development of these concepts covers three millennia of human history, going from Aristotle (382-322ḂC), to Charles Darwin (1809-1882) and Alfred Wallace (1823-1913) classical theory of biological evolution, and from there to contemporary or extended versions of the theory of evolution used in biology, ethology, linguistics, computer science, etc. In this and the following sections we will present some key ideas in this development paths, organizing and focusing the presentation according to the specific objectives of this article.

2.1. "By Chance" Explanations & Improbable Coincidences in Biology

The role of chance, haphazardness, and similar concepts has been a point of controversy since the earliest writings in philosophy. Let us consider the following passages (condensed) of Aristotle (384-322 BCE):

> *Λεγεται δε και η τυχη και το αυτοματον των αιτιων...*
>
> *But chance [τυχη, tykhe; fortune, luck, haphazardness] and spontaneity [αυτοματον, automaton; self-acting, having no other reason] are reckoned among causes [αιτιων, aition; reasons, explanations]... Some people even question whether they are real or not. They say that nothing happens by chance, but that everything... has some definite cause and it is always possible... to find something which is the cause; but not chance, for if chance were real, it would seem strange indeed.*
>
> Aristotle, Physics (II, 4, 195-196)
>
> Το *γαρ μη τυχοντως αλλ' ενεκα τινος εν τοις της φυσεος εργοις εστι και μαλστα: ου δ' ενεκα συεστακεν η γεγονε τελους, την του καλου ζοραν ειληφεν.*
>
> *Absence of chance [τυχοντως, tykhontos; randomness, haphazardness] and conduciveness of everything for a purpose [ενεκα τινος, eneka tinos; for the sake of something] are to be found in nature's works in the highest degree, and the end [τελους, telus; completion, perfection] for which those works are put together and produced is a form of the beautiful [καλου, kalu; good, virtuous].*
>
> Aristotle, Parts of Animals (I, 5, 645)

These passages of Aristotle may suggest that chance has no proper place in explaining physical phenomena, while it may have some role to play in Biology. Nonetheless, a great breakthroughs of modern Physics is the incorporation of probability and statistics in the construction of theories leading to the formulation of exact laws that, in turn, render high precision models for science and technology. One of the most important of such theories is Statistical Mechanics, including the kinetic theory of gases, statistical thermodynamics, and the probabilistic interpretation of entropy, starting with the pioneering works of Daniel Bernoulli (1700-1782) and John Herapath (1790-1868), [10, 53]. For historical perspectives, see [19, 37, 83, 116]; for gentle introductory texts, see [3, 63, 91, 118, 139]. Nevertheless, despite its importance in Physics, the mathematical prerequisites of Statistical Mechanics limited its influence in the public mind. In contrast, Charles Darwin's (1859) Theory of Evolution, also relying on probabilistic reasoning and a focal point of this article, had a tremendous repercussion throughout science, philosophy, and the general public.

These passages of Aristotle also reveal a tension between two concepts: On the one hand, *τυχη*, *tykhe*, chance, indicating something that happens haphazardly, by coincidence; a concept closely related to the idea of *αυτοματον*, *automaton*, something that happens by itself, with no external cause or any other reason. On the other hand, *τελους*, *telus*, indicating a purpose, an end to be completed, a perfection to be accomplished. On the one hand, chance seems to be excluded from the admissible or legitimate explanations. On the other hand, the surprise and wonder in (the apparently unlikely) finding of a product fully put together, a task entirely finished, or a work already competed, seems to be, in an appropriate context, a valid reason to attest and praise the virtue of its existence. As we will see in following sections, the insights derived from this tension are of capital importance in modern theories of evolution, where they can be supported by quantitative arguments based on probability theory.

2.2. Fortuitous Variations & Likely Survival of Well-Fitted Individuals; and Random Ramification & Probabilistic Pruning in the Tree of Life

In this subsection we examine two distinct and complementary roles played by probability in modern theories of biological evolution, roles that have been clearly distinguished by Curtis Johnson in his book *Darwin's Dice*, [61] where he identifies these two dice by the names: (FV) *Fortuitous Variation*, and (LS) *Likely Survival*. Moreover, we contrast the complementary roles played by these dice at microscopic vs. macroscopic scales, concerning the survival of single individuals vs. the survival of collective species. The specific roles of these two dice are expressed in Charles Darwin letter to Charles Lyell of August 1, 1861, apud [61], whose core paragraph is the first opening quotation of this paper.

In biology, sex links single individuals and their species by providing mechanism for coding, preserving, transmitting, sharing and mixing of information concerning their characteristic capabilities and ways of life. For simpler life-forms, horizontal sexual exchange allows the communication of single genes – units of information coding relevant characteristics. For complex life-forms, sexual reproduction demands vertical genetic recombination at each reproductive cycle. Furthermore, at least in sexual reproduction, organism similarity at the level of biological species strongly limits the population in which genetic recombination is possible. Additional constraints, like bounded environments or isolated geographic areas, create conditions allowing for speciation - the evolutionary process in which species further evolve, drift apart, and eventually differentiate. See [59], and references therein, for relevant reinterpretations of these concepts in the context of biological and computational semantics, and related developments in genetic programming algorithms.

A *Tree of Life* is, from a logical perspective, an acyclic graph mapping the distinctive branching process resulting from biological speciation, see Figure 2. The branching points in the tree represent two or more species diverging, in their phylogenetic path of evolution, from a single ancestor, or better said, from a single ancestral species. Pruning points in the tree, or (past) branch termination points, represent the extinction of a species, an event that may be caused by gradual or sudden changes in the environment, new predation or competition from better adapted species, or other adversities. Since, as explained in the previous paragraph, the phylogenetic history of a species is interlinked with the ontogenetic development and reproduction of its individuals, we mimic the aforementioned nomenclature of Darwin's Dice for the stochastic branching process forming the Tree of Life using the names: (RR) *Random Ramification* and (PP) *Probabilistic Pruning*.

Finally, is is interesting to contrast the historical role of mathematical probability and statistics in the aforementioned theories. On the one hand, quantitative

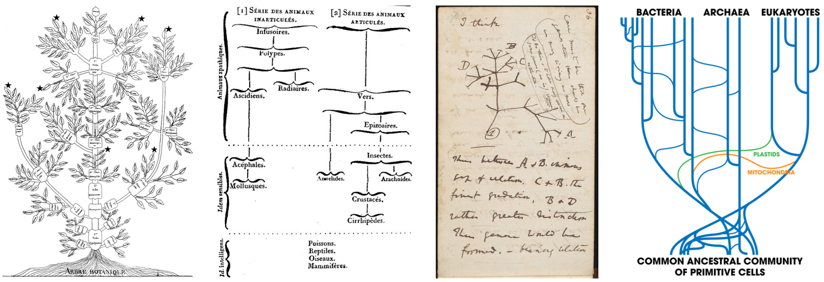

FIGURE 2. Family or Life Trees depicting proximity, trasformation, or evolution relations, by Augustin Augier (1809), Jean-Baptiste Lamarck (1815), and Charles Darwin, (1837); Contemporary version showing occasional horizontal genetic transfers.

probabilistic reasoning had an important role in Statistical Mechanics since its onset, see [37, 116]. On the other hand, the methods of mathematical analysis were incorporated in the Theory of Evolution at a much later time. One contributing factor for this delay is the equally late discovery of the mechanisms of genetic coding by Watson and Crick in 1953, [145]. Gregor Mendel (1822-1884) did extensive experiments with controlled hybridization of pea plants and developed a probability model for the underlying stochastic process, see [87]. Unfortunately, his work was almost forgotten until rediscovered by Hugo de Vries in the early XX century; see [126, sec.4.2], [25, 43], for historical comments. Furthermore DNA sequencing techniques only became available in the 1970s and, therefore, statistical models for random mutations, background noise timing along tree branches, measures of genetic proximity, etc. were only developed even latter in time. Nevertheless, biologists have been using much older probability arguments for reconstructing the tree of life, and this is the topic of the next section.

3. Structural Characterization of Diverging Similarities

Let us consider the vertical coordinate, t, in Figure 2.r or (3.1). The horizontal line at height $t = \tau$ represents a given instant in time; in Figure 3.r this is the nested oval line that starts at $t = \tau$ in the horizontal time coordinate. The region below this line, at $0 \leq t \leq \tau$, represents the past, and the tree fragment inside it represents the evolution of life as it has actually happen before $t = \tau$. An empty region above this line, at $t > \tau$, represents the future, replete of potential new innovations or possible evolutionary paths that are, however, contingent on random events yet to be realized.

The situation depicted in these Tree of Life diagrams is well described by the analogy in the following quotation from William Bragg. Adapting Bragg's analogy to the situation at hand, the "sieve passing through time" represents the horizontal line at $t = \tau$. Moreover, in this analogy, one could replace "in front of it" by "above it", referring to the region $t > \tau$. Furthermore, the expression "coagulates into" plays a role similar to "falls on" or "collapses into" in our former analysis of the

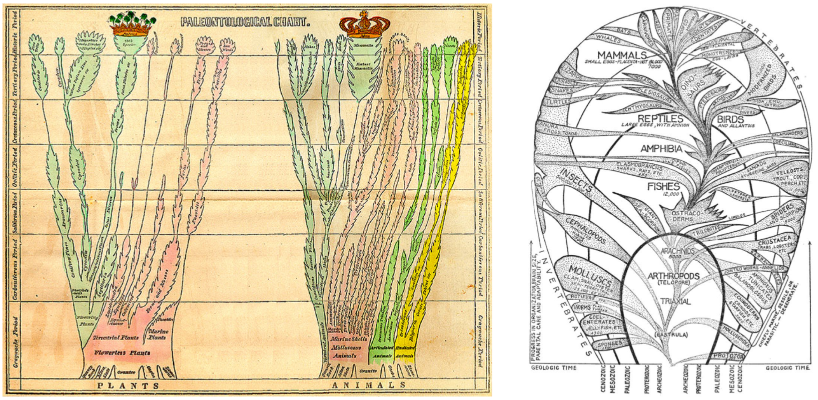

FIGURE 3. Trees of Life branching trough time and geological eras, by Edward Hitchcock (1840) and William Patten (1923).

spin top. Finally, the continuous (wavy, fluid, undetermined) vs. discrete (particular, fixed, determined) contrast in this analogy resembles a similar contrast in our analysis of the Odds & Evens game, an aspect to be further explored in the sequel.

> *The moment 'now' is like a sieve passing steadily through time. In front of it is a probability future, a* [continuous] *'wavy' future if you like, in which we can predict only how likely some result is to happen. As time streams through our sieve, it coagulates this wavy future into a* [discrete] *particle past, where the precise history of events is recorded.* William Lawrence Bragg (1968)

Living creatures can only be observed in the present; meanwhile paleontology studies life forms from the past using information obtained mainly from fossils. The order of geological strata provides good clews about the relative chronology of fossil findings, for example: Late / early times correspond to upper / lower deposit layers. There are reliable technologies for absolute timing of fossils, like radiometric dating based on proportional abundance of radioactive isotopes, but these were only developed late in the XX century. This is why, in Figures 2.r, 3.l and 3.r, the coordinate t refers to geological eras and not to standard time, as indicated in their vertical (horizontal) axis' labels, see [107] for further examples. Moreover, historically, paleontology and geological stratigraphy were developed in parallel with mutual and beneficial influence, as attested by the biological inspired names of many geological eras.

3.1. Richard Owen, Homology, and Archetype

Fossil records, together with relative spatial and temporal arrangements provided by geography and geology, were of fundamental importance in the construction and analysis of the first trees of life in the context of evolution theory. However, in this task, biologists needed additional help for further organizing and positioning these records in a tree diagram that was itself simultaneously under construction.

This help came in the form of rational and systematic methods for establishing and evaluating similarity relations between fossils, as studied in this subsection.

The (internal) skeleton has the hardest parts (bones) in the body of vertebrates, and these get fossilized, i.e. preserved by being incorporated into rock, much easier than softer parts, like muscle or internal organs. Similarly, the exoskeleton has the hardest parts in the body of arthropods, and these are also more easily and abundantly preserved in fossil records. Figure 4.tc show bones in forelimbs of some mammals and vein fields in wings of some insects, like they (usually incompletely) appear in fossils. Animals within these groups (of mammals and insects) share many similarities and, therefore, should (or are expected to) be closely connected in the tree of life. Fossils readily provide information concerning anatomical characteristics defined by the structure of organisms and their parts (and parts of their parts), and these are the basis for defining the similarity relations studied in the sequel.

In the Pick-up Sticks game, see Figure 4.b, a given throw is characterized by the way the fallen pieces are interconnected, that is, by how and where the sticks join or come in contact which each other. Likewise, Richard Owen (1804-1892) gave the following definitions (abbreviated) for the concepts of *Homology* and *Archetype* and, in consequence, a notion of identity (or sameness) between parts of animals. Subsequently, these key concepts proved to be essential in the study of biological evolution and in the construction of realistic tree of life diagrams.

> *Homologue: The* same *organ in different animals under every variety of form and function.* R. Owen (1843, p.379), apud [14]
> *These relationships (that is, homologies) are mainly, if not wholly, determined by the relative position and connection of the parts, and may exist independently of form, proportion, substance, function, and similarity of development.* R. Owen (1848, p.6), apud [14]
>
> *Archetype (Gr. αρχητυπος < αρχη, origin, τυπος, type): The original of that which is represented in a picture or statue. In the language of Plato, it means the world as it existed before creation in the mind of God.*
>
> *Archetype in Anatomy: Is that ideal original or fundamental pattern on which a natural group of animals or system of organs has been constructed, and to modifications of which the various forms of such animals or organs may be referred. The archetypal figure has been most clearly recognised in the study of the modifications of the skeleton of the vertebrate animals.*
> R. Owen in Brande (1866, p.146), apud [113]

The last sentence in Owen's definition of archetype in anatomy refers to its use, together with homology relations, to place animal species in the tree of life or, more extensively, to build a tree of life diagram capable of correctly organize all these relations between species. From the start, Owen used homological similarity relation as a tool for organizing organisms in tree of life diagrams, as in Figure 2.1,cl. One should be aware that Owen gave the aforementioned definitions of homology and archetype and started using them accordingly before 1859 – when Darwin and Wallace published their theories of evolution! Nevertheless, after some refitting and adjusting, biologists were able to repurpose these conceptual tools for the benefit of evolution theory, see [21].

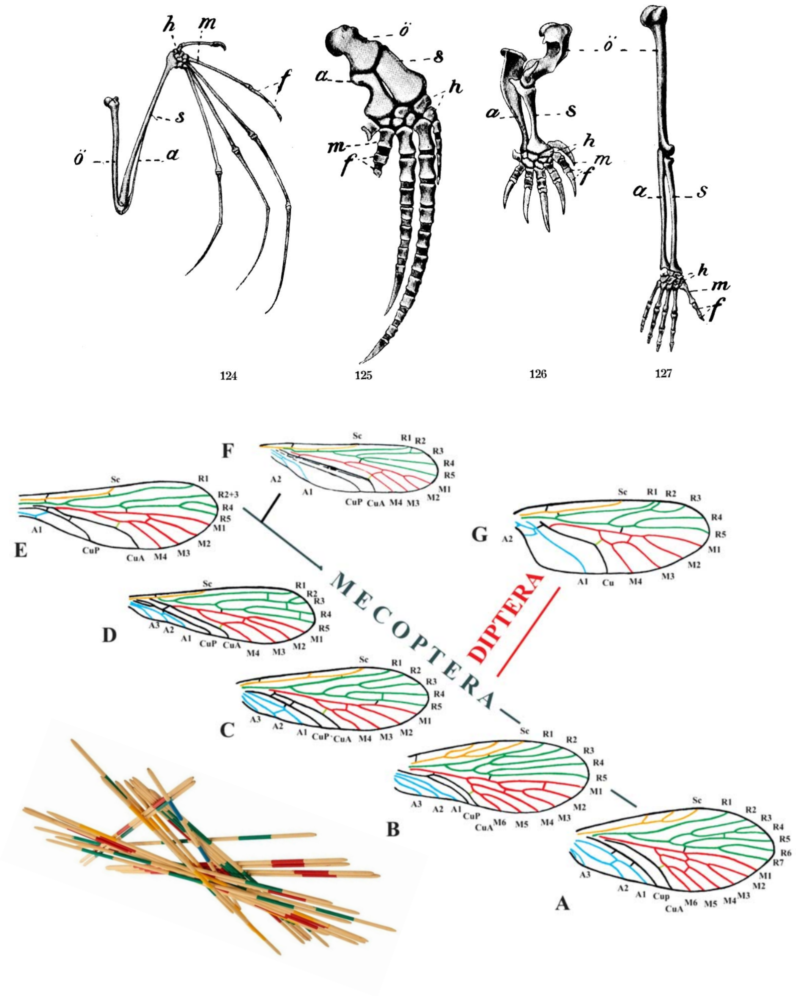

FIGURE 4. (t) Leche (1909): Homologous forelimb skeletons and functions: 124 Bat wing, flying; 125 Whale flipper, swimming; 126 Mole arm, digging; 127 Human arm, object manipulation. (c) Krzeminski (2003): phylogenetic relations between Triassic Mecoptera and Diptera. Veins fields: yellow, subcostal; green, radial; red, medial; black, cubital; blue, anal. (b) A throw in the Pick-up Sticks game showing how the fallen pieces interconnect.

Figure 5 displays Adolf Remane (1952) abstract representations of homology, related concepts, and how they can be used to establish correspondences between parts of structured systems: (5.tl) Proportional triangles in Euclidean geometry with same (homo) names (logos) for corresponding vertices and edges. (5.br) Continuous non-linear transformation of a rectangular coordinate grid, used for mapping

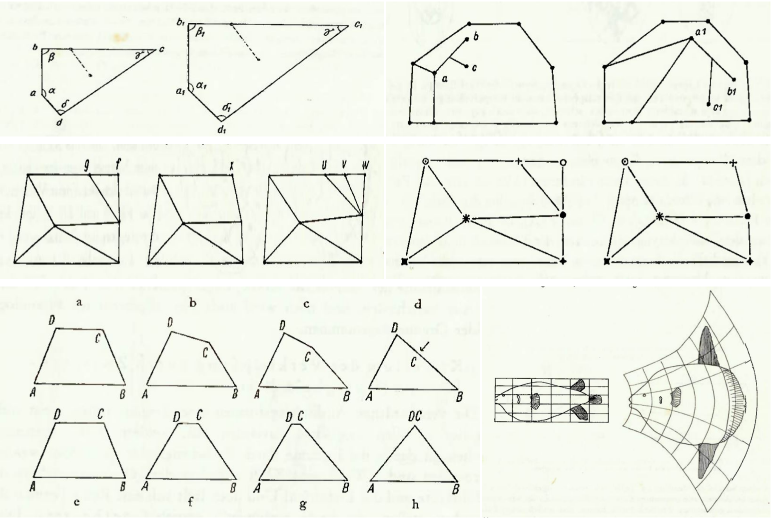

FIGURE 5. Remane (1952): Schemata for structural evolution and transformation, in/ex -clusion, spliting or fusion of elements.

corresponding anatomical features of fish. (5.tr) Homologous polygons with vertex correspondences established by topological relations, that is, by structural patterns of components' interconnection. (5.cl) "Nearly" or quasi homologous polygonal structures, where one extra vertex or edge has been either included or excluded. (5.bl) Intermediary steps in a gradual transformation of such a polygon, showing how splitting or merging of edges or vertices may occur. Homology, based on topological relations, is a fundamental tool used in biology to identify corresponding parts of structured systems and those, in turn, may be used to generate approximate coordinate transformation maps. Moreover, putative maps of this kind may, in turn, be a useful auxiliary technique in further or more detailed studies relating similar (homologous ou quasi-homologous) systems and their gradual transformation in time.

The definitions of homology and archetype in Table 2 have been updated and adapted having in mind the specific goals of this article. In the following sections, these definitions will be instrumental in preventing reasoning mistakes and logical fallacies often incurred when using these concepts.

3.2. Improbable Combinatorial Coincidences in Discrete Structures

Incipient Argument of Improbability by Richard Owen. Never developed to quantitative reasoning, but indicates veiled ideas already present. Owen's insight contains, in incipient form, ideas developed later in time in the form of quantitative argument.

[These structures] *...are too numerous and regular in their shape to allow for a moment the supposition of their being accidental...* Richard Owen (1832, p.28)

Improbability arguments to verify homology by qualitative (Gestalt perception) or crude combinatorial estimation. Konrad Zacharias Lorenz (1974) "Analogy as a Source of Knowledge" paper in *Science* celebrating his Nobel prize!

(H1) *(structural) Archetype:* Schematic plan, or Topological map characterizing the structure of a complex system (organism, organs, or parts thereof) by its construction template from simple components, or by the organization layout of its constituent elements, or by the interconnection pattern of its basic parts.
(H2) *Homologous:* Distinct organisms, organs, or parts thereof described by the same or similar structural archetypes.
(H3) Homologous systems may have different shapes, sizes, functions, constructions, geneses, material implementations, etc.
(H4) Homology may be used to establish an Identity (hence allowing the use of the same name) between corresponding parts in different systems.
(H5) In the context of biological evolution, homology may be a useful tool for investigating and tracing phylogenetic paths of development gradually diverging from a common ancestral species (at the root of a sub-tree).

TABLE 2. Definition, criteria and use of Homology and Archetype.

The improbability of coincidental similarity is proportional to the number of independent traits of similarity, and is, for n such characters, equal to $1/2^{n-1}$.
K. Lorenz (1974, p.230)

The probability of two forms of life evolving, by sheer coincidence, a certain number of identical characteristics, can be calculated. It is equal to $1/2^{n-1}$*,* n *being the number of similar or identical characteristics.* K. Lorenz (1981, p.88)

The natural computer of our Gestalt perception can take in and evaluate a much greater number of data than our rational computation can. Many truths become falsified and many obvious facts become invisible if one restricts his methods to quantification alone. Quantification, however, has the last word in verification, and all that our perception tell us becomes "science" only when we succeed in confirming it by rational verification . K. Lorenz (1981, p.95)

Lorenz calculation suggests an underlying model were the presence of any biological feature is represented by an equiprobable Boolean variable. For the Pick-Up Sticks game, a model of this kind could consider a topological characterization of a throw given by its connectivity pattern, in turn represented by a Boolean $(0/1)$ matrix B where, for $i, j \in \{1, ..n\}$, $i \neq j$, $B(i, j) = 1$ if and only if sticks i and j are connected (touch) in a given (partial) order (say, proximal-distal to a center). B has $d = (n^2 - n)$ off-diagonal elements and, therefore, at most $c \leq 2^d$ distinct possible configurations. Assuming a uniform distribution in this configuration space, the probability of any outcome is therefore $p = 1/c$. The models in this paragraph are naive, simplistic, and unrealistic, but they illustrate the discrete nature of the combinatorial mathematics underpinning models of this kind.

Far more realistic but complex and sophisticated models, are used for inferring, tracing and evaluating the plausibility of phylogenetic relations, including attested or unattested (hidden, latent) intermediate states. These models are used in many fields of application, including biology and (historical) linguistics. Such applications may include random mutation, transmission error or recombination mechanisms that can, in turn, be used to build clock models that can, latter on, be used to estimate the (time) length of branches in an evolution tree. The following references develop related models for applications like genetic programming, lineage tracing, and historical linguistics, [35, 59, 72, 133]

4. Functional Characterization of Converging Similarities

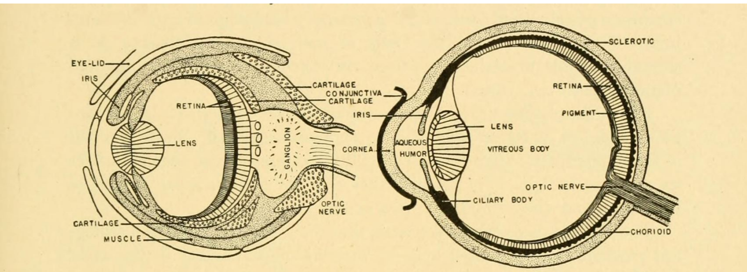

FIG. 479.—A diagram of median sections of the eyes of vertebrates and cuttlefish.

FIGURE 6. Neal & Rand (1936) Comparative anatomy diagrams showing median sections of eyes from cuttlefish and vertebrates.

Figure 6 introduces the main case study of this section, displaying schematic diagrams for eyes in cuttlefish (at class cephalopoda, phylum mollusca) and humans (at class mammalia, sub-phylum vertebrata, phylum chordata). Brief comparisons between the two diagrams reveal several overall morphological similarities, corresponding to common names used to identify basic components of these eyes like, for example, ocular globe, lens, ciliary muscles, iris, pupil, cornea, retina, optic nerve, etc. Notwithstanding such overall morphological similarities, the underlying construction templates for these two eyes, and the parts thereof, are quite different. Therefore, the similarity relations in this case study can *not* be appropriately handled using the concepts of Homology and Archetype developed in the previous section; making this clear is the first task of this section.

Consider the mission of finding out the structural characteristics of a complex building without having access to the corresponding architecture and engineering projects, construction plans, and other relevant documents. It is much easier to correctly conjecture the underlying structures following the building's construction step by step, rather than only looking at the final product as it stands. Figure 7.1 illustrates the ontogeny of each eye, that is, the unfolding of its structure during the life on an individual, from the early stages of an embryo to a fully formed adult. As depicted, the tissues forming the basic parts in each eye have distinct origins, allowing diverse interconnection patterns and construction templates, that are used to implement different working components, even if the final organs have similar functional characteristics or the same oveall purpose (eyes to see). The following examples illustrate such differences, see [41, 42, 47, 52, 128] and references therein.

As a first difference, one can observe that, in vertebrate eyes, the lens sits at a fixed position, while its shape can be controlled by muscles that squeeze it more or less if contracted or relaxed. In contrast, one can observe that, in cephalopod eyes,

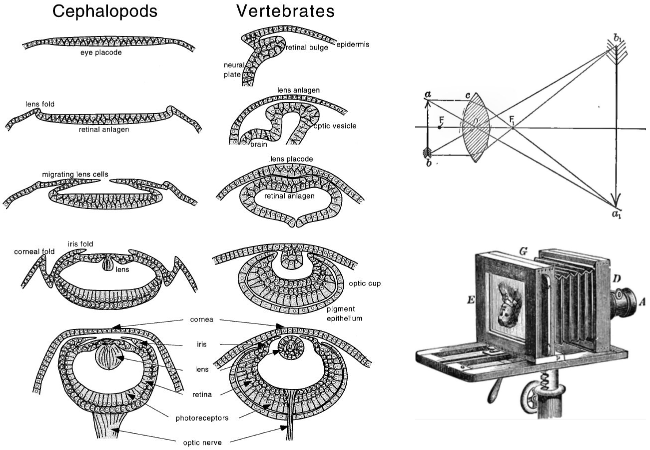

FIGURE 7. (l) W. Harris (1997) Comparative ontogeny diagrams showing eyes' developments. (r) Photographic camera optics.

the lens has a fixed shape, while its position can be controlled by muscles that move it closer of farther away from the retina at the back of the eye – a design much easier to implement from a material science perspective! Not surprisingly, the earliest optical instruments invented and build by humans, like cameras obscuras and telescopes, used a fixed-shape and variable-position lens design. These machines have fixed shape lenses made of rigid glass, that can be moved back and forth along the instrument main symmetry axis.

FIGURE 8. (l) Analogous pectoral fins; (r) Eye like a camera obscura.

As a second difference, one can observe that retinas in vertebrate eyes are built back-to-front, that is, before reaching the photo-receptors at the far-side of the retina, incoming light must go trough tissue layers responsible for support functions like blood supply, nerve connections and mechanical integrity. In contrast, retinas in cephalopod eyes are built with the right-way-out, that is, having the supporting layers at the rear side of the retina where they do not obstruct the path of incoming light – a much better design from an engineering perspective! Not surprisingly, the earliest photographic cameras invented and build by humans used the right-way-out design, see Figures 7.rb and 8.rb. These machines have, at the back of a camera

(A1) *(functional) Prototype:* A *Humanly invented machine, method or device*, *Proof-of-concept artifact*, or *Law-governed model*, demonstrating key functional aspects of a system conceived or built to achieve a certain aim or to serve a specific purpose.

(A1') *Fine-Tuning Condition:* A Prototype's configuration or set-up that is precisely adjusted or well-calibrated to optimize the system's operation or performance.

(A2) *Analogous:* Distinct organisms, organs, or parts thereof explained by the same or similar functional prototypes.

(A3) Analogous systems may have different structures, shapes, sizes, constructions, geneses or material implementations, ...

(A4) Analogy may be used to establish an Identity (hence allowing the use of the same name) between corresponding parts in different systems.

(A5) In the context of theories of biological evolution, analogy may be a useful tool for investigating and tracing distinct phylogenetic paths of development converging to similar solutions.

TABLE 3. Definition, criteria and use of Analogy and Prototype.

obscura, a glass plate that is coated, at the inner side, with a photo-sensitive emulsion. At a given moment, the plate is exposed to the light of an image that is then captured in latent form at the sensitive emulsion so that it can, latter on, be revealed (developed and fixed) by chemical post-processing.

In conclusion, notwithstanding the existence of several overall morphological similarities between the two eyes in case study at hand, from the comparative analyses in this section and the concepts already defined in Section 3, we understand that these similarities can *neither* be explained by shared original archetypes *nor* be properly characterized using homology relations.

We now proceed to the second task of this section, namely, to define the concepts of Analogy and Prototype that, in turn, lead to similarity relations appropriate for the case study at hand. Such concepts are based on functional characteristics of the organs under study, that is, their purpose and the way they work, see Table 3. The definitions in Tables 2 and 3 hope to be faithful to Owen's original ideas, although they deviate significantly from definitions found in classical textbooks, for they have been modified and adapted to serve specific goals of this article. Richard Owen defined Analogy as a concept parallel to that of Homology, that is – as Homology refers to structure and "conformity of type", Analogy refers to function and "teleological design"; see following quotations. In a similar way, we define Prototype as a concept parallel to that of Archetype, as suggested by the parallel structures of Tables 2 and 3. Our definition Prototype is also inspired by the work of Owen, see following quotations, although, to my knowledge, he never used this name and only used this notion *en passant* as a contrast to his concept of Archetype.

> *Analogue: A part or organ in one animal which has the same function as another part or organ in a different animal.* R. Owen (1843, p.374)

> *.. the teleologist.. finds the adaptation of the organ to its function square with his notions of the perfection of a machine constructed for such an end.*
> R. Owen (1849, p.84) apud [21]

4.1. Fine Tuning Condition in Continuous Prototypes

Figures 8.r,c depict a *camera obscura* as an explanation of the working principle of the Nautilus eye, as explained by R. Owen in the next quotation, see [94]. In a camera obscura, the size of the pinhole at the front poses a dilemma: the smaller the pinhole, the closer the entry point of any light ray is to its center, and the more sharply in focus the image formed at the back becomes. However, a smaller pinhole also reduces the quantity of light entering the camera, resulting in a dimmer image. Vertebrate and cephalopod eyes solve this dilemma replacing the pinhole by an optical lens that, in effect, enlarges the orifice's aperture while keeping the image in focus. The article [128] offers a self-contained explanation of lens geometry, presented in the context of the epistemological framework of Objective Cognitive Constructivism. For intuitive presentations of geometrical optics at honors high-school level, see [41, 42, 47, 128] and references therein.

> *In the Nautilus the eyes [present] the simplest condition of an organ of vision, consisting only of a darkened globular cavity or* camera obscura, *into which light was admitted by a single orifice, and a nerve expanded at the opposite side to receive the impression.* Owen (1836, p.551)

An essential characteristic of a convex lens is its focal length, f, which is determined by the curvature of its surfaces. In situations pertinent to this article, a smaller radius of curvature implies a shorter focal length. Finally, the condition for obtaining a sharp image is dictated by the *lens equation*, which relates the focal length, f, to the distances from the lens to the object, x, and to the image x', as shown in Figure 7.tr. When in focus, the system's magnification factor is $m = x/x'$. The aforementioned vertebrate and cephalopod eyes achieve the in-focus condition dictated by the lens equation by adjusting the lens' focal distance, f, and the image-to-lens distance, x, respectively.

$$\frac{1}{f} = \frac{1}{x} + \frac{1}{x'} \Rightarrow f = \left(\frac{1}{x} + \frac{1}{x'}\right)^{-1}, \; x = \left(\frac{1}{f} - \frac{1}{x'}\right)^{-1} .$$

In the case at study, the photographic camera is used as a *(functional) prototype*, that is, as a humanly invented device or law-governed model demonstrating key functional aspects of a system conceived or built to achieve a certain aim or to serve a specific purpose. Moreover, the Lens equation establishes a *fine-tuning condition*, that is, a prototype's configuration or set-up that is precisely adjusted or well-calibrated to optimize the system's operation or performance, see Table 3.

Figure 9 introduces the second case study of this section, displaying schematic diagrams for the geometry of honeycombs build by honeybees – *Apis mellifera*. Let us consider the benefit over cost ratio of a honeycomb's cell, evaluated by its volume over area ratio, i.e., its storage capacity over the amount of wax used for its construction. In order to maximize this ratio, the lozenges at the tip of a cell must obey fixed proportions between their small diameter, large diameter, and side, namely,

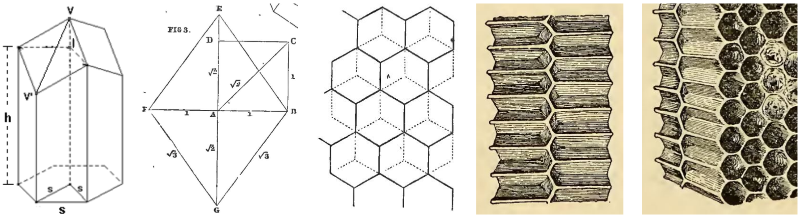

FIGURE 9. Honeycomb optimal geometry.

$1 : \sqrt{2} : \sqrt{3}$. This result can be readily obtained using basic calculus and vector geometry at the lever of honors high-school mathematics, see [8, prob.9.7.3,p.273-276].[1]Empirical observations attest the precision of this fine tuning condition in the bees' construction works. For further comments and historical references see [45, 82, 84, 108, 111, 141], [140, p.327-334], and [146, p.448-454]. The authors of these articles, some dating from the 16th century, analyze how bees build their honeycombs according to an optimal design and wonder about its philosophical consequences, including the capacity of these insects for teleological planning and intentional action, the nature of their psyche or soul, the origin of the underlying structural archetypes and functional prototypes (not with this terminology) and related topics.

From a Systems' Theory perspective, defining appropriate *constraints* and an *objective function*, *optimal control*, or *variational principle* that accurately describe a system's behavior fully enables intentional interpretations or teleological causal explanations when discussing how or why the system works the way it does (or, metaphorically, "wants to do"). This systemic perspective does not require an identifiable agent (distinct from the integral system) acting as the controller, wanting to achieve an objective, or driving an optimization process. For a discussion of these topics, including the possibility of reformulating systemic laws from efficient to final cause format, see [121, 128] and references therein. Accordingly, an appropriate functional Prototype, preferably reinforced by precise Fine Tuning conditions, fully enables the use of similarity relations of Analogy – as formally defined at Table 3.

4.2. Convergent Evolution to a Functional Solution

Vertebrate and Cephalopods are far away from each other in the tree of life, see Figure 3, situated in branches separated since a distant past when their ancestral forms had extremely simple eyes (if any). Moreover, arguments in this section demonstrate that Vertebrate and Cephalopods well-developed eyes at present times (as well as their constituent components) are based on different structural archetypes. Hence,

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[1] Although written for a different audience, namely, first year college students in biological sciences, this book overlaps almost perfectly with my high-school curriculum, and also with the mathematics' requirements of the admission exam at that time, 1977, at USP – the University of São Paulo, see also [36]. The end of high-school was celebrated by our teachers, at Colégio Bandeirantes, with special classes showing interesting things we could do with what we have learned. Solving the honeycomb's cell problem was one of these classes, that also discussed the use of teleological arguments. I was deeply impressed by this mathematical solution, and also by its possible consequences for causal thinking.

these eyes' respective designs evolved independently, in the sense that there was no direct exchange of information concerning their design along their separate phylogenetic histories. Notwithstanding their separate evolution histories, based on different structural archetypes, these eyes have converged to the same functional prototype (or similar variations thereof), that is, they are analogous to (i.e. they work like) a man-made machine, namely, a photographic camera equipped with a single lens that is shaped or positioned in order to achieve optimal focus conditions.

This situation begs the question: Why do the aforementioned eyes converged to the same functional prototype? A plausible answer is: Because the laws of optics are invariant and universal, so optical equipment must work in accordance to these same laws, no matter when or where it is, as suggested by the next quotation by R. Owen. Finally, it should be easy to understand that, like Hology in the case of divergence, see Table 2, Analogy may be a useful conceptual tool for investigating distinct phylogenetic paths converging to similar Prototypes, see Table 3.

> *The laws of light, as of gravitation, being the same in Jupiter as here, the eyes of such creatures as may disport in the soft reflected beams of its moons will probably be organized on the same dioptric principles as those of the animals of a like grade of organization on this earth.*
>
> R. Owen (1849, 83-84) apud [5]

5. Avoiding Logical and Inferential Fallacies concerning Evolution, Homology, Analogy, Archetype and Prototype

Richard Owen (1804-1892) rendered his definitions of homology and analogy in the 1840's, within the theoretical framework of pre-evolution biology, for pertinent historical remarks see [14, 15, 51, 102, 110, 113, 114]. In 1859, Charles Darwin and Alfred Russel Wallace published their versions of the theory of biological evolution. After some refurbishing, homology and analogy became key conceptual tools for tracing divergence and convergence in evolution processes, and are widely used in the field of biology and far beyond. Karl von Frisch (1886-1982) and Konrad Zacharias Lorenz (1903-1989) shared a Nobel prize, in 1973, for their discoveries concerning ethology and animal language, see [78]. In their work, homology and analogy are fundamental tools of investigation, see [38–40, 48–50, 75, 76, 78–80], and following discussions. More recently, these concepts found further applications in molecular biology, linguistics, genetic programming and other sciences, where homology and analogy became true *sources of knowledge* in the study of stochastic evolution processes.

Nevertheless, in spite (or because) of the ever expanding applications of these conceptual tools, their use has been plagued by pernicious and persistent reasoning fallacies. Some of these fallacies, listed in Table 4, are logical in nature, while others involve misconceptions in probability and statistics, or are generated by misunderstandings on how stochastic evolution works. This section pays special attention to such fallacies, including ways to avoid them.

Affirming a consequent fallacy. a.k.a. the *converse error*, in the form:
$(A \to B), B \therefore A$. That is: A implies B, B, therefore A.

Affirming a disjunct fallacy. a.k.a. the *mutual exclusion error*, in the form:
$(A \vee B), A \therefore \neg B$. That is: A or B, A, therefore *not* B.

Type mismatch fallacy. a.k.a. the *wrong type of argument error*. Such an error occurs when an agent, function, or operation uses an argument of inappropriate type. For example, in computer programming, such an error can be generated by assigning a continuous value to a discrete variable, or by defining a discrete variable to represent a continuous quantity in the pertinent application context.

TABLE 4. Logical fallacies associated to stochastic evolution.

5.1. Swapping Assumed Conditions and Inferred Consequences; or Assuming Similarity Relations to be Mutually Exclusive

This is our guiding-line for this subsection: Keep the definitions of key concepts clear and simple, precise and straight to the point, and use these concepts accordingly. In particular, avoid creating spurious associations, like *a priori* binding together concepts that can be defined independently or naturally separated. Such spurious associations between concepts unduly restricts their scope of application, fragments their space of possible logical relations, and constraints their range in formulation of admissible hypothesis. Therefore, the imposition of such spurious associations generates all sorts of artificial complexities, unexpected consequences, and unwanted side effects, see Stern et al. (2018, 2024, 2025) for related comments. Moreover, any *a posteriori* conclusion or synthetic empirical statement derived in such an adulterated theoretical framework may easily incur in vicious circularity and, therefore, is under suspicion of being either tautological or invalid.

Disregarding our guiding-line easily engenders logical errors of the first two kinds listed in Table 4. The following quotations give historical examples of (in)correct, (in)congruent or (in)consistent (according to our guiding-line) definitions or uses of homology, and equally relevant reactions found in the literature. The first two quotations, by Ch. Darwin, exemplify the good use of homology (as defined by Owen, see Section 3.1 and and Table 2) in the context of evolution theory.

> *The homological construction of the whole frame in the members of the same class is intelligible, if we admit their descent from a common progenitor, together with their subsequent adaptation to diversified conditions. (On any other view the similarity of pattern between the hand of a man or monkey, the foot of a horse, the flipper of a seal, the wing of a bat, etc, is utterly inexplicable.)* Ch. Darwin (1871,v.I, p.31)

> *By* unity of type *[common archetype] is meant that fundamental agreement in structure, which we see in organic beings of the same class, and which is quite independent of their habits of life. On my theory, unity of type is explained by unity of descent.* Ch. Darwin (1859, p.206)

In accordance to Darwin's ideas, homology is often used to investigate, trace and corroborate (statistical hypotheses concerning) common ancestry in evolutionary processes. Erroneously swapping the role of antecedent and consequent in logical or probabilistic implications (converse error in Table 4), some authors include common ancestry in the definition of homology. The next examples illustrate this kind of fallacy: The first, by E. Lankester, in 1870, [69], ironically expressing his support to Darwin's theory; The second, by K. Lorenz, from [78], celebrating his Nobel prize at *Science*. The following quotations, by [114], [60], give correct diagnostics for this kind of fallacy.

> *Structures which are genetically related, in so far as they have a single representative in a common ancestor, may be called homogenous.* [homologous]
> E. Lankester (1870, p.36), apud [18]

> *A homology can be defined as any resemblance between two species that can be explained by their common descent from an ancestor possessing the character in which they are similar to each other.*
> K. Lorenz (1974, p.230)

> *In due time, Darwinian evolutionary appropriation of the notion was so complete that homology not only became evidence of evolution, but came to be defined as a function of ancestry*
> N. Rupke (2009), apud [18]

> *Recently many writers have objected to this kind of definition of homology on the grounds that it is viciously circular, and the objection seems to be well-founded. By these definitions, in order to show that part x in one organism is homologous with part y in another organism, it must be shown that they are derived from a part z in a common ancestor. But homology itself is invoked in identifying parts x and y with part z.* Jardine (1967, p.125)

In some situations, homology and analogy offer valid alternative or complementary explanations for issues of interest. Nevertheless some authors erroneously assume homology and analogy to be mutually exclusive, see second item in Table 4. An example extracted from the influential work of Adolf Remane (1898-1976) is given in the next quotation. The following quotations by R. Owen, A. Boyden, and A. Panchen give correct diagnostics for this kind of fallacy.

> *Methodological criteria of analogy: For our purposes, we will proceed according to the general understanding of the following two indicators of analogy hold: (1) Its relation to function and way of life, (2) Its contrast to homology in the sense of either-or.* A. Remane (1971, p.83)

> *But homologous parts may be, and often are, also analogous parts in a fuller sense, viz. as performing the same functions; thus the fin or pectoral limb of a Porpoise is homologous with that of a Fish, it is inasmuch as composed of the same or answerable parts, and they are analogues of each other, inasmuch as they have the same relation of subserviency to swimming.*
> R. Owen, (1848, p.7), apud. Boyden(1943, p.230), Panchen (1994, p.44)

> *Owen distinguished two kinds of resemblance in the corresponding organs or parts of the bodies of different animals: (1) essential* structural agreements

R - Homology & Archetype. These concepts refer to similarities in structure that are primarily described by coincidences between discrete interconnection diagrams. Secondarily, homology arguments may include continuous maps of relative positions for the involved connections, given by their distances and angles, used for diachronic or synchronic comparative analyses; see Section 3.

B - Analogy & Prototype. These concepts refer to similarities in function or purpose that, in turn, are primarily described by continuous mathematical models. Such models often include fine tuning conditions or other precise hypotheses used to characterize teleology in systems' theory. Secondarily, analogy arguments may use structural correspondences, even if superficial, between discrete components of a system under study compared to another system whose function is already well-established and understood; see Section 4.

G - Probability & Statistics. Models used in probability and statistics are based on (scalar, vector, matrix, functional, etc.) random variables that are, essentially, either discrete or continuous. Moreover, the mathematical methods used to develop probability theory in discrete vs. continuous spaces (like combinatorial calculus and discrete set theory vs. differential and integral calculus and set topology), and corresponding techniques required in each case (like counting discrete events in the numerator and denominator for a betting quotient vs. evaluating the truth-value for a precise or zero-volume hypothesis), have essentially distinct characteristics.

TABLE 5. Boromean knot of three (RGB) threads, each made of intertwined discrete vs. continuous mathemathical variables.

> *relating particularly to relative position and connections; (2)* similarities in the function *or use to the organism. – These are really different qualities and they have no necessary dependency upon each other.*
>
> A. Boyden (1943, p.231)

> *There is an important and common misapprehension about the terms homologue and analogue. Ill-taught schoolchildren and students (and others who should know better) treat them as antonyms and thus mutually exclusive. Owen was quite clear in saying that this was not the case.*
>
> Panchen (1994, p.44)

5.2. Discrete vs. Continuous Type Mismatch

Each thread in the Borromean metaphor introduced in Section 1, visually represented in Figure 1, and concisely interpreted in Table 1, is reinterpreted in this section according to Table 5. Good mathematical models based on probabilistic or statistics are much needed and routinely used by scientists finding their way in this conceptual maze. However, careless reasoning in this complex entanglement of conflating concepts easily entails type mismatch fallacies, and they are ubiquitous in the scientific literature. The concept of Analogy, and its interpretation and use in evolution theory, is particularly prone to this kind or error, as examined in the sequel.

As carefully examined in this and previous articles, and summarized in Table 5, the concept of analogy is a similarity relation concerning function and purpose.

Moreover, such teleological notions can be abstracted and expressed in the form of exact laws, precise behavioral rules, or equational formulas used in science. Furthermore, such exact mathematical statements are naturally represented in statistical models by sharp hypotheses that, in turn, are formulated around continuous random variables; for further details see [121, 124–128, 131, 132] and references therein. Nevertheless, analogous systems may also exhibit secondary similarities (or superficial resemblances) concerning their structure that are, therefore, characterized by homology relations. Moreover, these structural similarities can be abstracted and expressed in the form of coincidences in combinatorial models. Furthermore, such coincidence statements are naturally represented by statistical hypotheses formulated around discrete random variables; for further details see [35, 51, 70–72, 105, 133] and references therein.

In the next quotation, Lorenz (1974, p.230) counts structural correspondences, characteristic of homology, as the most relevant evidence for functional similarity, characteristic of analogy. This kind of categorical misunderstandings lead to an impoverished and disempowered notion of analogy that is, therefore, perceived as a superficial resemblance or as a weak similarity relation, as states in [54, p.341]. The following quotation, by [14, p.230], reinforces the need to clearly distinguish prototype from archetype, analogy from homology, function from structure, continuous from discrete variables, etc. Under such conditions, that is, correctly understood and supported by appropriate means and methods, analogy is a powerful concept and a reliable tool that is capable of revealing deep systemic similarities and able to establish sound theoretical connections based on comparative function analysis. The final quotation, from Encyclopaedia Britannica (01/04/2026), repeatedly exemplifies reasoning mistakes and conceptual misunderstandings studied in this section, showing how pervasive they are in the literature.

> *...the concept of adaptation... forms the basis of the reasoning which the evolutionist applies to the phenomenon of* ***analogy****. ... Whenever we find, in two forms of life that are unrelated to each other, a similarity of form or of behavior patterns which relates to more than a few minor details, we assume it to be caused by parallel adaptation to the same life-preserving function. The* ***improbability*** *of coincidental similarity is proportional to the* ***number of independent traits of similarity****, and is, for n such characters, equal to 2^{n-1}. ... However, no such thing as a false analogy exists: An analogy can be more or less detailed and hence more or less informative.*
>
> Lorenz (1974, p.230)

> ANALOGY: *Superficial similarity that arises through functional convergence.*
>
> Hillis (1994,p.341)

> *Here is probably the germ of Owen's genius, to realize that structural and functional resemblances are different qualities, different kinds of agreement, and should be clearly recognized as such. This brings us to Owen's use of analogy, and the interrelation of the terms homology and analogy to each other.* Boyden (1943, p.230)

> ANALOGY, *in biology, similarity of function and superficial resemblance of structures that have different origins.*

> HOMOLOGY, *in biology, similarity of the structure, physiology, or development of different species of organisms based upon their descent from a common evolutionary ancestor. Homology is contrasted with analogy, which is a functional similarity of structure based not upon common evolutionary origins but upon mere similarity of use.*
>
> Encyclopaedia Britannica (01/04/2026)

6. Further Research

The two following subsections examine some paradoxes derived from misconceptions about the nature of evolution. These misconceptions, often compounded with logical fallacies, engender all sorts of erroneous conclusions including, as extreme cases, impossibility arguments against evolution theory. Moreover, careful examination of these paradoxes and related questions prompts interesting topics for further research.

6.1. Nothing Ever Begins When You Think It Does

Lillian Hellman (1969) celebrated aphorism, apud [117], refers to biology's habit of (re)deploying a well-developed organ ($O\rho\gamma\alpha\nu o\nu$, instrument, tool) into a new role, or (re)adapting a former working solution into a new function. Figure 4.t shows how the same basic 5-digit skeletal structure was readjusted in forelimbs used for flying, swimming, digging, and object manipulation. All these examples can be traced back to the forelimb of primitive fish adapted for walking in mud or close to water. Other examples of redeployment are far less obvious, for example: (a) Lungs, developed in fish as a swim blather, a gas-sac used for buoyancy regulation, and latter repurposed for breading air; (b) Feathers, developed for thermal insulation and regulation, and latter repurposed for gliding and flying. The next quotation by Charles Darwin's *Origin of Species*, 6th ed., Ch.VII, *Miscellaneous Objections to the Theory of Evolution*, apud [117], addresses this topic.

> *All Mr. [George] Mivart's objections will be, or have been, considered in the present volume. The one new point which appears to have struck many readers is, 'That natural selection is incompetent to account for the incipient stages of useful structures.' This subject is intimately connected with that of the gradation of the characters, often accompanied by a change of function, for instance, the conversion of a swim-bladder into lungs...*

• Old Archetypes, New Prototypes, and Organic Retuning (future research)

When an organ is repurposed for a new function, it can maintain many structural properties; in fact, this is how its evolution history can be traced. However, the description and explanation of how the organ accomplishes its new function requires, by definition, the postulation of a new prototype. In some cases, a sequence of increasingly complex and more sophisticated prototypes may be better suited to follow the readaptation process of an organ to its new function. This is a topic to be addressed in further research, focusing on the possible methodological contributions brought by the concept of prototype, as defined in this article.

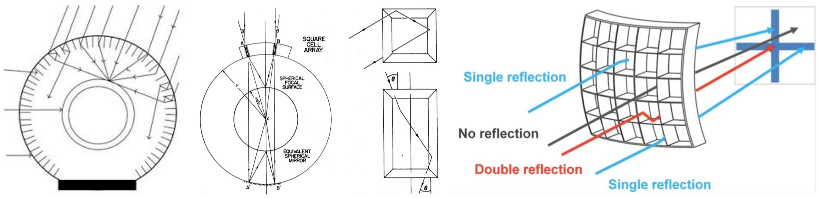

FIGURE 10. Lobster-eye with a square-cell-grid reflective lens.

A closely related topic for further research concerns the re-tuning process of an organ to its function, as defined by its new (assumed) prototype. Recent redeployment to a new function suggests an adaptation process in progress that is, therefore, far from optimal tuning and hence far from equilibrium conditions characteristic of convergence to a stable form. This non-equilibrium condition also suggests fast rates of change in its recent evolution history and, consequently, large variation of functional parameters in the population. In contrast, long employment in the same function suggests an already successful adaptation process that, therefore, is close to optimal tuning or close to equilibrium conditions characteristic of convergence to a stable form. This equilibrium condition also suggests slow rates of change in its recent evolution history and, consequently, small variation of functional parameters in the population. We believe that the concept of prototype and its accompanying fine-tuning conditions, as defined in this article, can be instrumental for an objective and quantitative and statistical assessments of the aforementioned convergence characteristics.

6.2. There Are More Ways To The Wood Than One, and He Which Knows Many Ways May Go The Nearest

Joseph Swetnam (1617) celebrated aphorism proclaims the advantage of having alternative ways to fight in a combat. In this paper, it refers to biology's talent for inventing different tools to do a task, or building alternative organs (instruments) to fulfilling a need. Just because a given functional prototype is precisely defined, it does not mean that it is the only possible way to solve a problem. For example, biologist were able to identify many independent reinventions of "eyes to see", converging not to one, but to a handful of viable (i.e. biologicaly implementable) but very different functional prototypes, see [68].

Figure 10 depicts diagrammatic schemata explaining the basic optics of a Lobster eye (order Decapoda, phylum Arthropoda). In the lobster eye we can recognize a few familiar structural components, most obviously, a "lens" and a retina. Nevertheless, the vertebrate and cephalopod eyes previously examined are equipped with a refractive lens used to project the captured image over a concave retina. In contrast, in the lobster eye, instead of the refractive lens, there is a spherical grid made of square reflexive cells, that is used as a focusing device for projecting the captured image over a convex spherical retina half-size of the lens, see Figure 10.

The lobster eye has fascinating histories concerning its phylogenetic evolution, its ontogenetic embriology, its anatomical description, and the (very late, in the

1970's) discovery of the prototype explaining how its actually works, see [1, 32–34, 44, 68, 115, 144]. Afterwards, this prototype was used to invent new technological devices in the field of x-ray optics, where no adequate materials are available for making a refractive lens, and even the best mirrors can only reflect light at very shallow (almost tangent) angles. This is an important example of Biomimetics – the art and science of reutilizing biological prototypes for creating or improving human technology, see [100, 140, 143].

- **Analogies Across Distinct Prototypes (future research)**

Although the prototypes explaining the optics of eyes based on refractive and reflective lenses, are quite different, both have some goals in common, namely, to focus the captured image into an appropriate (concave / convex) retina, allowing subsequent data acquisition and information processing, with the final purpose of giving a creature the capacity to see its environment. Moreover, the same quality parameters, like magnification, resolution, contrast, field depth, light sensitivity, etc., can be used to objectively evaluate and compare the performance of these eyes. The possibility of using well-established benchmarks to objectively relate the performance of different prototypes and their variant implementations has important consequences: First, it allows comparative analyses contrasting how and why each prototype is able to achieve its goals; Second, it allows the evaluation of how well each prototype effectively models the biological organ it is supposed to explain; Finally, this hierarchical setting allows the extension of analogical relations across different organs under scrutiny, even in the case where these organs are of different (proto-) types. The systematic development of such hierarchical *analogies across different prototypes* is a topic we intend to explore in future articles.

6.3. Archetypal Symbolic Language of the Honeybees (future research)

Karl von Frisch (1886-1982), Martin Lindauer (1918-2008) and John Haldane (1892-1964) pioneered the modern study of the honeybees' *waggle-dance* language. Bees use as dancing floor the flat surface of honeycombs hanging vertically inside their hive. The basic form (dance-move) of this language consist of a waggling run along a straight line, that is then embedded in an 8-shaped circuit, see Figure 11.1. The frequency and duration of waggling is interpreted by the bees as a fight distance, while the running line angle relative to the upward direction is interpreted as a flight direction angle relative to the sun. The language includes variations of the basic form, that are used according to strict rules concerning morphology (form), grammar (composition), and semantics (interpretation).

Although relatively simple, this language is surprisingly flexible and is used in different contexts to accomplish a variety of tasks. For example, bees use it for control and coordination of flight missions with the purpose of collecting resources like pollen or nectar. Moreover, bees make important collective decisions, like where to relocate the hive when necessary, by a voting process based on this language. Furthermore, this language has a fixed structure, or *archetype*, that is genetically transmitted and inherited, together with similarly pre-programmed behavioral patterns, giving each individual bee the inborn capacity for its appropriate use.

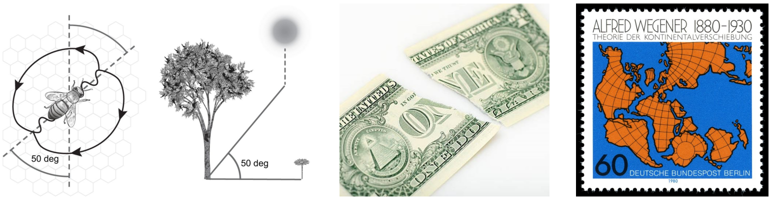

FIGURE 11. Symbolic (meaningful, reliable, precise) couplings: (l) Waggle-dance move and interpretation; (c) Matching tokens; (r) A.Wegener (1912) depiction of puzzle-like fitting continents.

The bee's interpretation of this language correctly and precisely integrates flight directions with information from multiple sensors (visual, vibration, olfactory, inertial, etc.) and several correction or modulation factors (diurnal dislocation of the sun's position in the sky, wind intensity and direction, terrain topography, etc). In this way, this language is an instrument ($o\rho\gamma\alpha\nu o\nu$) finely tuned to its function. Moreover, this language enables the sharing of meaningful, precise and reliable information and, in this sense, it is considered *symbolic* (from $\sigma\upsilon\nu$-$\beta\alpha\lambda\lambda\omega$ - syn-ballo = *I put together* $\sim$ well-fitting pieces or parts of a whole), see Figure 11.

The bee's do their work, and communicate (by using their language) the information they need to share in order to do it, without ever thinking about (or, even less, reflexively "talking" about) why, how, or what they are doing. In contrast, we humans try to understand all that by building models, i.e. *archetypes* and *prototypes*, that are based on theoretical assumptions and must be corroborated by empirical observations. While engaged in these studies, we may use our own language(s) to share our conclusions (with others humans). Moreover, in so doing, we may agree to some semantic shifts, or to introduce new words, or adopt new ways of articulation, or even to modify deeper features of our language. In contrast, for the bees' language, even the retuning of a single parameter (like adjusting the meters-to-waggle scale) can only be accomplished by genetic evolution over time spans of many generations; see [7, 22–24, 30, 65, 136].

In light of these contrasting characteristics, the relatively simple *archetypal symbolic language* of the honeybees should provide a good case-study for research on applying the concepts developed in this article to theory and philosophy of language. This case-study should also provide a stepping-stone for further applications of these ideas to philosophy of language and scientific ontologies.

7. Final Conclusions and Acknowledgments

This article develops parallel and comparable but separate and specific definitions for the concepts of Archetype and Prototype, that are considered in tandem with the derived concepts of Homology and Analogy. These novel definitions lead to implementable algorithms and methods of analysis that, in turn, facilitate their correct and productive application in science. Moreover, these definitions and methods preserve the possibility of offering simultaneous and complementary explanations, and allow for other useful and valid interactions between these concepts. Furthermore, the proposed definitions help to avoid some mistakes confounding structural and functional properties of biological organs (or technological instruments) and their evolution that, in turn, are linked to other mistakes confounding the use of discrete and continuous mathematical variables (either deterministic or random). In this context, a correct and clear understanding of the distinct roles played by structure vs. function and discrete vs. continuous mathematical variables offers helpful guidelines for choosing coherent probability models and consistent statistical methods in applications of interest. Finally, this article briefly considers possible consequences of these conceptual innovations and methodological developments in the areas of philosophy of language and science, and suggest some topics for further research.

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Figures: Position locators: t=top, b=bottom, l=left, c=center, r=right. Images in Public Domain or under Creative Commons license (commercial use allowed): 1.rt; 1.rb; 2.r; 4.b; 7.rt; 7.rb; 8.l,c,r; 11.c, 11.r. Images from books in public domain, more than 70 years past after dates of publication and author's death: 1.lb, from [2]; 1.c, from [73]; 2.l, from [4]; 2.cl, from [67]; 2.cr, from [26]; 3.l, from [55]; 3.r, from [104]; 4.t, from [74]; 6, [92]; 9.l,cl from [140]; 9.cr, from [146]; 9.r, from [106]; Images used with the kind permission given by the author, editor, or responsible for copyright: 1.lt, from [46], by the authors; 4.c, from [66], by the authors; 5, from [109], by the editor; 7.l, from [52], by the author; 10.l, from [44], by the editor; 10.cl,cr, from [1], by the author; 10.r, from [32], by the editor; 11.l, from [7], by the editor.

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