A Categorical Sketch for Viability: Formalizing the Structural Invariants of Self-Organizing Systems

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Abstract

We work in **Set** and define a P-system as a state object S with a self-production endomorphism $\varphi : S \to S$, a functional membrane $\partial S \hookrightarrow S$, and a flux action $\rho : R \times S \to S$. The endomorphism is evaluation-based: φ reads the currently installed metabolism h through $\mathrm{ev}(h, a) = h(a)$, so the active metabolism is part of the state. Endogeneity is stated as an E-relative non-factorisation condition through a class $K(S, E)$ of authorised mediations, with a separation lemma as the verification tool. The framework is worked out on an explicit finite Rosen-type model: 16 states, φ tabulated, membrane stability and permeability checked element by element, E-endogeneity verified by exhaustion, and a multi-level counterexample $T = S \times \{0, 1\}$ in which the same subsystem admits one admissible embedding and one not. The formal object is a *restricted generalised sketch* with an avoidance clause, not an ordinary finite-limit sketch. Section 5 places the formalisation in dialogue with recent BioSystems work on molecular autopoiesis, causation and codes, and empirical correlates of (M, R)-systems.

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1. Introduction

1.1. The problem

Self-organising living systems — from cells to ecological couplings — share recognisable structural features: they configure their own states, produce and maintain their own boundaries, and interact without mutual destruction. These features have been described in several formalisms. Maturana and Varela (Maturana and Varela, 1972) introduced autopoiesis, with Maturana's contribution central to the original biological and epistemological formulation; Varela (Varela, 1979) subsequently developed the concept of operational closure that we use for the membrane formalisation below. Rosen (Rosen, 1991) formalised metabolic closure through (M, R)-systems, following his earlier relational programme (Rosen, 1958a,b, 1959). Gánti (Gánti, 2003) proposed the chemoton model, coupling template replication, metabolism, and membrane as a minimal chemical living unit. Ehresmann and Vanbremeersch (Ehresmann and Vanbremeersch, 1987, 2007) provided a categorical framework for hierarchical self-organisation through Memory Evolutive Systems. Nicolis and Prigogine (Nicolis and Prigogine, 1977) formalised dissipative structures.

Cornish-Bowden and Cárdenas (Cornish-Bowden and Cárdenas, 2020) compared these theories and identified closure conditions shared by Rosen, Varela, and Gánti. The question we address is whether some of that shared content can be written down in a restricted formal grammar. Recent BioSys-

tems work on metabolic closure in real cells, on the replication of (M, R)-systems, and on empirical correlates of organisational closure provides the concrete backdrop (Rubin, 2023; Weckström, 2026; Hofmeyr, 2018; Igamberdiev and Kleczkowski, 2023; Louie, 2020); §5.3 returns to this literature.

1.2. Methodology: the Rosetta Stone approach

In the spirit of the Rosetta Stone approach of Baez and Stay (2011), we define an abstract object first, then verify that a domain-specific construction is a model of it. This avoids functorial maps between heterogeneous domains and any ontological commitment about the relation between, say, cells and social agents. The abstract object is fixed before any biological reading, so the verification is a check and not a definition in disguise.

1.3. Philosophical significance and the term 'sketch'

In the classical Ehresmann tradition, a sketch is a diagrammatic specification of a class of structured objects by specifying objects, arrows, cones, cocones, and commutative constraints. A model of such a sketch in a category $\mathcal{C}$ is an interpretation that preserves the stipulated positive conditions. Makkai's generalised-sketch framework (Makkai, 1997) extends this diagrammatic mode of specification to a broader logical setting.

Since our construction includes an avoidance condition—the non-factorisation of the self-production endomorphism through the environment—we refer to it as a restricted generalised sketch rather than as an ordinary finite-limit sketch. The avoidance clause is an explicit non-existence condition added to the positive sketch-data; it should not be read as claiming that the construction is a classical Ehresmann finite-limit sketch.

We refer to the sketch throughout as the *viability sketch* (or *P-sketch* where brevity is needed), since it encodes the structural conditions under which a P-system remains viable. This terminology replaces the opaque notation *ThP* used in earlier drafts.

Two consequences are worth noting. Organisational autonomy becomes a checkable property rather than a qualitative descriptor: a precise non-factorisation condition that can be verified against a given model. And the sketch supplies a portable vocabulary for comparing structural viability conditions across domains; each instantiation still depends on its own choice of state space, environment object, and authorised mediations. We make no claim that the viability sketch characterises the living as such. §5.3 places it in the formal-system tradition in the sense of Rubin (2023).

1.4. Overview of results

Definitions 1–10 and Criterion 1 give a restricted generalised sketch encoding three structural properties — self-configuration, self-limitation, recursive integrity — and a flux condition. An explicit finite Rosen-type model in **Set** verifies the conditions element by element. The multi-level criterion is presented as a diagnostic, with a finite counterexample showing that it has non-trivial content. No general theorem of viability is claimed.

§2 presents the sketch. §3 builds and verifies the finite model. §4 treats the multi-level diagnostic. §5 covers scope, related formalisms, recent BioSystems work, and extensions.

1.5. Relation to prior work

The paper sits in continuity with earlier categorical and relational work. Nomura (Nomura, 2001, 2007) gave formal descriptions of autopoiesis in category theory and distinguished it from (M, R)-systems. Letelier et al. (Letelier et al., 2003) studied organisational invariance in (M, R)-systems and gave a mathematical example of closure. Letelier, Cárdenas and Cornish-Bowden (Letelier et al., 2011) gave the canonical analysis on which our reading of Rosen rests. Louie (Louie, 2009, 2020) developed the relational-biological setting and connected it to computability theory. Rubin (Rubin, 2023) cartographed the formal systems of molecular autopoiesis, distinguishing the natural-system reading (Maturana) from the formal-system approach (Varela). The viability sketch belongs to the latter: it is a formal grammar, not a characterisation of the living as such.

The contribution here is narrower: a restricted sketch-theoretic framework that places self-production, membrane, flux, and hierarchy in a single formal object, verified on an explicit finite Rosen-type model in **Set**.

2. The viability sketch

2.1. Informal pattern

The structural pattern P consists of three properties.

(P-a) Self-configuration. The system determines its own states without external instruction. There is no exogenous programmer. Self-configuration does not imply absence of environmental constraints, but absence of external determination of the system's organisational identity.

(P-b) Self-limitation. The system produces and maintains its own boundary (membrane). It distinguishes interior from exterior. The membrane is not a passive barrier but an active, selective, self-produced structure.

(P-c) Recursive integrity. Interaction between subsystems does not destroy their capacity for self-configuration. Interaction is coupling, not fusion or destruction.

Dynamic condition. The structural pattern alone does not guarantee maintenance of organisation. The system requires a flux of energy, information, or resources traversing the membrane. Maintenance of organisation requires both P and flux; we formalise their coupling below.

2.2. Ambient setting

We work throughout in the category **Set**. This choice is deliberate. It allows every object in the worked model (§3) to be written explicitly and checked element by element. It also forces a methodological constraint: certain notions that may later admit a more structural formulation — especially endogeneity — are here stated relative to a distinguished environment object E and an authorised class of mediations, not as interpretation-independent categorical properties.

Let E denote a distinguished environment object. In **Set**, no endomorphism $\varphi : S \to S$ can be "endogenous" in an interpretation-independent sense solely by virtue of the ambient category: whenever $|E| \geq |\mathrm{Im}(\varphi)|$, a factorisation through E can be manufactured. The relevant notion is therefore *E-relative endogeneity* (Definition 7), not structural endogeneity simpliciter.

2.3. P-system

Definition 1 (P-system)**.** A P-system in **Set**, relative to a distinguished environment object E and a resource object R, consists of the following data:

(i) a state object S in **Set**;

(ii) a subobject $\iota : \partial S \hookrightarrow S$ (the membrane);

(iii) an endomorphism $\varphi : S \to S$ (self-production);

(iv) a flux action $\rho : R \times S \to S$;

(v) when permeability is required, a restricted flux $\rho_2 : R \times \partial S \to \partial S$;

(vi) a specified class $K(S, E)$ of authorised mediations $k : S \to E$.

The system satisfies the viability-sketch conditions when permeability (Definition 4), stability (Definition 5), and E-relative endogeneity (Definition 7) hold relative to a chosen class $K(S, E)$ (Definition 6).

Throughout the paper, the symbol $\times$ denotes the cartesian product in **Set**. We use the cartesian product, not a tensor product: there is no monoidal-but-non-cartesian structure assumed in the ambient category, and making this commitment explicit avoids unwarranted appeal to monoidal machinery.

2.4. Evaluation-based self-production

We now make precise the form of the self-production endomorphism on the worked Rosen-type model. The same form is reused in §3.

Definition 2 (State object)**.** For the worked Rosen-type model, the state object is $S = A \times B \times H$, where A is the substrate set, B the product set, and $H = \mathrm{Hom}(A, B) = B^A$ the set of possible metabolisms.

Definition 3 (Self-production endomorphism)**.** The self-production endomorphism $\varphi : S \to S$ is defined by

$$\varphi(a, b, h) \;=\; \Big(a,\, \beta(\Phi(\mathrm{ev}(h, a))),\, \Phi(\mathrm{ev}(h, a))\Big),$$

where $\mathrm{ev} : H \times A \to B$ is the evaluation map $\mathrm{ev}(h, a) = h(a)$, $\Phi : B \to H$ is the repair map, and $\beta : H \to B$ is the closure map. Equivalently, the B-component satisfies $\pi_B \circ \varphi = \beta \circ \Phi \circ \mathrm{ev} \circ (\pi_H, \pi_A)$.

The active metabolism is not fixed externally: it is part of the state itself, and the endomorphism reads it through evaluation. With a fixed metabolism f, the endomorphism would depend only on the substrate coordinate and endogeneity would become trivial to lose. With evaluation through the installed metabolism h, the system's own organisational state becomes formally relevant.

2.5. Membrane permeability and stability

Definition 4 (Membrane permeability)**.** The flux action $\rho : R \times S \to S$ *traverses* the membrane if it factors through ∂S: there exists $\rho_2 : R \times \partial S \to \partial S$ such that the diagram in Figure 1 commutes, i.e. $\iota \circ \rho_2 = \rho \circ (\mathrm{id}_R \times \iota)$.

Definition 5 (Membrane stability)**.** The subobject ∂S is *stable* under φ if there exists $\varphi|_{\partial S} : \partial S \to \partial S$ such that $\iota \circ \varphi|_{\partial S} = \varphi \circ \iota$. Equivalently, $\varphi(\partial S) \subseteq \partial S$ in **Set**.

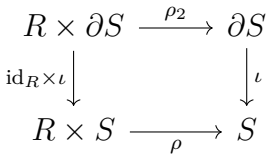

Figure 1: Permeability diagram. The product is cartesian.

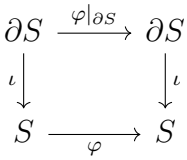

Figure 2: Stability diagram. The membrane is invariant under self-production.

The membrane is self-maintained: in the biological reading (§3.2), no external agent repairs or renews the boundary.

2.6. E-relative endogeneity

Definition 6 (Authorised mediations)**.** Let $K(S, E)$ be a class of *authorised mediations* $k : S \to E$. In the finite model of §3, these are the maps that factor through the substrate projection π_A:

$$K(S, E) \;=\; \{\, k : S \to E \;\mid\; k = \kappa \circ \pi_A \text{ for some } \kappa : A \to E \,\}.$$

The environment is allowed to encode substrate-side variation, but does not carry the internal organisational identity carried by the pair (b, h).

Thus $K(S, E)$ is not intended to be the class of all maps $S \to E$. It is part of the modelling choice: it specifies which environmental mediations are biologically admissible in the interpretation under consideration.

Definition 7 (E-relative endogeneity)**.** The endomorphism φ is *E-endogenous relative to* $K(S, E)$ if there is no factorisation $\varphi = g \circ k$ with $k \in K(S, E)$ and $g : E \to S$.

Lemma 1 (Separation)**.** *If for every $k \in K(S,E)$ there exist $x, y \in S$ with $k(x) = k(y)$ but $\varphi(x) \neq \varphi(y)$, then φ is E-endogenous relative to $K(S,E)$.*

Proof. Suppose $\varphi = g \circ k$ for some $k \in K(S,E)$ and $g : E \to S$. Then $k(x) = k(y)$ implies $g(k(x)) = g(k(y))$, hence $\varphi(x) = \varphi(y)$, contradicting the hypothesis. □

Remark 1. The E-relative formulation is honest about its interpretive character. A broader class of mediations would weaken or trivialise the endogeneity condition; a narrower one would strengthen it at the cost of generality. The present choice is calibrated to the Rosen-type instantiation of §3. The natural strengthening path moves toward Markov categories (Fritz, 2020) and causal kernels (Jacobs et al., 2019), where environmental mediation can be captured structurally. This paper does not achieve that step; it prepares it.

2.7. Admissible morphisms

Definition 8 (Admissible morphism)**.** A map $f : S_1 \to S_2$ between P-systems is *admissible* if:

(i) $f \circ \varphi_1 = \varphi_2 \circ f$ (compatibility with self-production);

(ii) $f(\partial S_1) \subseteq \partial S_2$ (membrane respect).

Proposition 1 (Category of P-systems)**.** *Under Definition 8, P-systems and admissible morphisms form a category* **P-Sys**.

Proof. Let $f : S_1 \to S_2$ and $g : S_2 \to S_3$ be admissible. Then

$$(g \circ f) \circ \varphi_1 = g \circ (f \circ \varphi_1) = g \circ (\varphi_2 \circ f) = (g \circ \varphi_2) \circ f = (\varphi_3 \circ g) \circ f = \varphi_3 \circ (g \circ f),$$

so condition (i) composes. Also, $(g \circ f)(\partial S_1) = g(f(\partial S_1)) \subseteq g(\partial S_2) \subseteq \partial S_3$, so condition (ii) composes. Identity maps satisfy both conditions trivially. □

Remark 2. A stronger endogeneity-preserving condition can be imposed as a supplementary property: that f does not create new factorisations of φ_2 through E via authorised mediations. We do not use this stronger condition in the finite verification below, and do not adopt it as definitional here, since its compositional stability in general remains open.

Remark 3. Flux compatibility is not included in the basic notion of admissibility. The category **P-Sys** used here is therefore the category of P-systems with self-production- and membrane-preserving morphisms. If resource maps are specified, one may impose an additional compatibility condition with ρ, yielding a stricter subcategory. We do not need this stronger notion for the finite verification below.

2.8. Predation: diagnostic vocabulary

The predation vocabulary introduced below is diagnostic terminology for non-admissible morphisms. It is not used as an established ecological category. The structural intuition behind the exit clause derives from Hirschman's analysis of exit and voice as organisational mechanisms (Hirschman, 1970); the heuristic is acknowledged as such and does no formal work.

Definition 9 (Predation)**.** Fix a specified class $\mathcal{I}$ of interaction maps between P-systems. A map $f : S_1 \to S_2$ in $\mathcal{I}$ is a *predation morphism* if it fails to be admissible, i.e. violates at least one of conditions (i)–(ii) of Definition 8.

A predation morphism $f : S_1 \to S_2$ is *distributed* if, relative to $\mathcal{I}$, there exists at least one competing interaction $g : S_3 \to S_2$ with $S_3 \neq S_1$ and at

least one admissible exit morphism $e : S_2 \to S_4$.

It is *monopolistic* if, relative to $\mathcal{I}$, no such competing interaction or no such admissible exit morphism is available.

We use the distributed/monopolistic distinction only as a diagnostic vocabulary. No theorem about ecological viability is claimed. The discussion stays narrow.

2.9. The viability sketch

Definition 10 (The viability sketch)**.** The *viability sketch* (*P-sketch*) is the restricted generalised sketch consisting of:

(a) an underlying graph G with vertices S, ∂S, E, R, 1 (terminal) and arrows $\iota : \partial S \to S$, $\varphi : S \to S$, $\rho : R \times S \to S$, $\rho_2 : R \times \partial S \to \partial S$, $\varphi|_{\partial S} : \partial S \to \partial S$;

(b) commutativity constraints expressing permeability ($\iota \circ \rho_2 = \rho \circ (\mathrm{id} \times \iota)$, Definition 4) and stability ($\iota \circ \varphi|_{\partial S} = \varphi \circ \iota$, Definition 5);

(c) an avoidance clause (Definition 7): there is no factorisation of φ through E via $K(S, E)$.

Parts (a)–(b) are ordinary positive sketch-data in the Ehresmann tradition. Part (c) is an additional non-existence condition, treated here in the broader generalized-sketch perspective of Makkai (1997). The avoidance clause is part of the present construction: it excludes interpretations in which self-production factors through the environment via an authorised mediation. The composite object is not a sketch in the strict classical sense; we call it a restricted generalised sketch with an avoidance clause.

3. Instantiation: an explicit finite Rosen-type model

3.1. Rosen's (M, R)-systems

Rosen (Rosen, 1991, 1958a,b, 1959) formalises the causal closure of living systems through (M, R)-systems (metabolism–repair). For the present Set-based instantiation, we represent the Rosen-type closure pattern by data $f : A \to B$, $\Phi : B \to \mathrm{Hom}(A, B)$, and $\beta : \mathrm{Hom}(A, B) \to B$. Here f is the metabolism (transformation of substrates A into products B), Φ is the repair map (products generate the capacity for production), and β is the closure map (the repair capacity itself produces components). Causal closure is achieved when every component of the system is produced by the system itself (Letelier et al., 2011).

3.2. Construction and biological reading

We now construct an explicit finite model. The state object is $S = A \times B \times H$ with $H = B^A$, and the self-production endomorphism is the evaluation-based map of Definition 3.

Biological reading. In a biological cell, the components of the state object admit the following interpretation. A represents the substrates available to the cell (nutrients, precursors). B represents the metabolic products (amino acids, lipids, nucleotides). $H = \mathrm{Hom}(A, B)$ schematically represents possible metabolic transformations, used here as a toy proxy for enzymatic configurations rather than as a biochemical state space. The evaluation map $\mathrm{ev}(h, a) = h(a)$ corresponds to the catalytic action of the currently installed enzymatic configuration h on a given substrate a.

The repair map $\Phi : B \to H$ captures the capacity of metabolic products to regenerate or select the active metabolism: products sustain the enzymatic machinery that produced them. The closure map $\beta : H \to B$ completes the cycle: the enzymatic configuration determines which products are generated, closing the loop of efficient causation in the sense of Rosen (Rosen, 1991). Hofmeyr (Hofmeyr, 2018) analysed this causal structure through the lens of causation, constructors, and codes, providing biochemical context for the closure conditions that φ formalises.

The membrane $\partial S = \{(a, b, h) \mid b = \beta(h)\}$ marks the states where the product component matches what the installed metabolism would produce. The criterion is functional, not spatial: organisational self-consistency rather than physical boundary.

3.3. Finite model

Base sets and Rosen data. Let $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. Let $H = B^A$, with four elements:

Name	$a_1 \mapsto$	$a_2 \mapsto$	Type
f_1	b_1	b_2	bijection
f_2	b_2	b_1	reversed bijection
f_3	b_1	b_1	constant b_1
f_4	b_2	b_2	constant b_2

Define the repair map $\Phi : B \to H$ by $\Phi(b_1) = f_1$, $\Phi(b_2) = f_2$. Define the closure map $\beta : H \to B$ by $\beta(f_1) = b_1$, $\beta(f_2) = b_2$, $\beta(f_3) = b_1$, $\beta(f_4) = b_2$. The state object is $S = A \times B \times H$ with $|S| = 16$.

The self-production endomorphism. The endomorphism is

$$\varphi(a, b, h) = \Big(a,\ \beta(\Phi(h(a))),\ \Phi(h(a))\Big).$$

Since $\Phi(b_1) = f_1$ and $\Phi(b_2) = f_2$, only f_1 and f_2 appear in the third component of $\mathrm{Im}(\varphi)$, so

$$\mathrm{Im}(\varphi) = \{(a_1, b_1, f_1),\ (a_1, b_2, f_2),\ (a_2, b_1, f_1),\ (a_2, b_2, f_2)\},$$

hence $|\mathrm{Im}(\varphi)| = 4$.

Table 1: Complete action of φ on all 16 states.

State	φ(State)	State	φ(State)
(a_1, b_1, f_1)	(a_1, b_1, f_1)	(a_2, b_1, f_1)	(a_2, b_2, f_2)
(a_1, b_1, f_2)	(a_1, b_2, f_2)	(a_2, b_1, f_2)	(a_2, b_1, f_1)
(a_1, b_1, f_3)	(a_1, b_1, f_1)	(a_2, b_1, f_3)	(a_2, b_1, f_1)
(a_1, b_1, f_4)	(a_1, b_2, f_2)	(a_2, b_1, f_4)	(a_2, b_2, f_2)
(a_1, b_2, f_1)	(a_1, b_1, f_1)	(a_2, b_2, f_1)	(a_2, b_2, f_2)
(a_1, b_2, f_2)	(a_1, b_2, f_2)	(a_2, b_2, f_2)	(a_2, b_1, f_1)
(a_1, b_2, f_3)	(a_1, b_1, f_1)	(a_2, b_2, f_3)	(a_2, b_1, f_1)
(a_1, b_2, f_4)	(a_1, b_2, f_2)	(a_2, b_2, f_4)	(a_2, b_2, f_2)

Observation 1. The image of φ on the a_1-branch consists of two fixed points, (a_1, b_1, f_1) and (a_1, b_2, f_2). The a_2-branch carries a 2-cycle between (a_2, b_1, f_1) and (a_2, b_2, f_2). States with $h \in \{f_3, f_4\}$ are mapped in one step into the closure-compatible image.

Membrane, stability, permeability. The membrane is $\partial S = \{(a, b, h) \in S \mid b = \beta(h)\}$, giving $|\partial S| = 8$. Explicitly,

$$\partial S = \{(a_1, b_1, f_1), (a_1, b_1, f_3), (a_1, b_2, f_2), (a_1, b_2, f_4),$$
$$(a_2, b_1, f_1), (a_2, b_1, f_3), (a_2, b_2, f_2), (a_2, b_2, f_4)\}.$$

Stability. For any $(a, b, h) \in \partial S$, we have $b = \beta(h)$. The output $\varphi(a, b, h)$ has second component $\beta(\Phi(h(a)))$ and third component $\Phi(h(a))$. Applying β to the third component yields $\beta(\Phi(h(a)))$, which equals the second component of the output. Hence the output is in ∂S. The restriction $\varphi|_{\partial S} : \partial S \to \partial S$ exists, and $\iota \circ \varphi|_{\partial S} = \varphi \circ \iota$ commutes.

Permeability. The resource object is $R = A$. The flux is $\rho(r, (a, b, h)) = (r, b, h)$. Since membrane membership depends only on (b, h), if $(a, b, h) \in \partial S$ then $(r, b, h) \in \partial S$ for every $r \in R$. The restriction $\rho_2 : R \times \partial S \to \partial S$ exists, and $\iota \circ \rho_2 = \rho \circ (\mathrm{id}_R \times \iota)$ commutes.

Endogeneity verified by exhaustion. Let $E = \{e_1, e_2, e_3\}$ with $|E| = 3$, and $K(S, E)$ as in Definition 6.

Cardinality argument. Since every $k \in K(S, E)$ factors through $\pi_A : S \to A$, we have $|\mathrm{Im}(k)| \leq |A| = 2$. Hence $|\mathrm{Im}(g \circ k)| \leq 2$ for every $g : E \to S$. But $|\mathrm{Im}(\varphi)| = 4$. Therefore φ cannot equal $g \circ k$ for any $k \in K(S, E)$ and $g : E \to S$.

Separation argument. Every $k \in K(S, E)$ confounds states with the same substrate: $k(a_1, b_1, f_1) = k(a_1, b_1, f_2)$. But φ distinguishes them: $\varphi(a_1, b_1, f_1) = (a_1, b_1, f_1) \neq (a_1, b_2, f_2) = \varphi(a_1, b_1, f_2)$. By Lemma 1, φ is E-endogenous relative to $K(S, E)$.

3.4. Admissible and non-admissible morphisms

Admissible example. The morphism $r = \varphi : S \to S$ maps each state to its canonical image under self-production. We verify $r \circ \varphi = \varphi \circ r = \varphi^2$ on all 16 elements (see Appendix B). Since $\varphi(\partial S) \subseteq \partial S$, we have $r(\partial S) \subseteq \partial S$. Therefore r is admissible.

Non-admissible example. Define $\mu'(a, b, h) = (a, b_1, f_3)$ for all $(a, b, h) \in S$. This map forces the degenerate metabolism f_3 on every state — organisational capture. It preserves ∂S (since $\beta(f_3) = b_1$). But it violates condition (i):

$$\mu'(\varphi(a_1, b_1, f_1)) = \mu'(a_1, b_1, f_1) = (a_1, b_1, f_3),$$

while

$$\varphi(\mu'(a_1, b_1, f_1)) = \varphi(a_1, b_1, f_3) = (a_1, b_1, f_1).$$

Since $f_3 \neq f_1$, μ' does not commute with φ. In the diagnostic vocabulary of §2.8, μ' exemplifies a monopolising capture: a membrane-preserving interaction that destroys the internal dynamics.

3.5. Verification summary

Table 2: Verification of viability-sketch conditions on the finite Rosen-type model.

Axiom	Requirement	Realisation in finite model
P-a	φ E-endogenous	Cardinality + separation (§3.3)
P-b	∂S stable under φ	$\varphi(\partial S) \subseteq \partial S$ verified on all 8 states
Flux	ρ traverses ∂S	$\rho(R \times \partial S) \subseteq \partial S$ verified
P-Sys	Admissible morphisms compose	Proposition 1

4. Multi-level diagnostic

4.1. The problem of nested holons

Self-organising systems are typically nested: organelles in cells, cells in tissues, tissues in organs, organs in organisms. If level n is self-producing, what stops it from dissolving the self-production of level $n-1$? Ehresmann and Vanbremeersch (Ehresmann and Vanbremeersch, 1987, 2007) gave the categorical framework for this question through Memory Evolutive Systems, where a holon at level n is a colimit of its components at level $n-1$.

Koestler's *holarchy* (Koestler, 1967) is heuristic vocabulary for nested organisational levels. We use it descriptively; Criterion 1 and its counterexample do not depend on it.

4.2. Diagnostic criterion

Criterion 1 (Multi-level P-compatibility)**.** *We call a multi-level system* P-compatible *when every designated inter-level monomorphism (injective embedding in* **Set***)* $\iota_n : S_{n-1} \to S_n$ *is admissible in the sense of Definition 8.*

The status of the criterion is diagnostic, not theoremic: the construction below shows that it has non-trivial content, but the criterion itself is a definition of P-compatibility, not a theorem about it.

4.3. Finite counterexample

To demonstrate non-triviality, define $T = S \times \{0, 1\}$ with $|T| = 32$ and dynamics

$$\psi(s, 0) = (\varphi(s), 0), \qquad \psi(s, 1) = (\mu'(s), 1).$$

The membrane is $\partial T = \{(s, j) \in T \mid s \in \partial S\}$. Since both φ and μ' preserve ∂S, ψ preserves ∂T. The flux is $\rho_T(r, (s, j)) = (\rho(r, s), j)$ with $R = A$; since ρ preserves ∂S and the wing index j is unchanged, ρ_T preserves ∂T. These two checks establish the stability and permeability conditions for T. It remains to verify E-relative endogeneity.

Endogeneity of T. The image of ψ consists of $\mathrm{Im}(\varphi) \times \{0\}$, which has 4 elements, together with $\mathrm{Im}(\mu') \times \{1\}$, where $\mathrm{Im}(\mu') = \{(a_1, b_1, f_3), (a_2, b_1, f_3)\}$, which has 2 elements. Hence $|\mathrm{Im}(\psi)| = 6$. Let $K(T, E)$ consist of mediations factoring through the substrate projection $\pi_A : T \to A$, so $k(a, b, h, j) = \kappa(a)$ for some $\kappa : A \to E$. Then $|\mathrm{Im}(k)| \leq |A| = 2$ for every $k \in K(T, E)$. Since $|\mathrm{Im}(g \circ k)| \leq |\mathrm{Im}(k)| \leq 2 < 6 = |\mathrm{Im}(\psi)|$, ψ cannot factor through any $k \in K(T, E)$. Hence ψ is E-endogenous.

Together, stability, permeability, and E-relative endogeneity make T a P-system in the sense of Definition 1.

Two embeddings. Define $i_0(s) = (s, 0)$ and $i_1(s) = (s, 1)$.

- i_0 is admissible. For all $s \in S$: $i_0(\varphi(s)) = (\varphi(s), 0) = \psi(s, 0) = \psi(i_0(s))$. Verified on all 16 states.

- i_1 is not admissible. At $s = (a_1, b_1, f_1)$: $i_1(\varphi(s)) = (a_1, b_1, f_1, 1)$, but $\psi(i_1(s)) = (\mu'(a_1, b_1, f_1), 1) = (a_1, b_1, f_3, 1)$. Since $f_1 \neq f_3$, the diagram does not commute.

The same subsystem S admits two injective embeddings into the same ambient T: one admissible, one not. The diagnostic lives at the morphism level. Loss of self-production is not an internal defect of S; it depends on how the ambient system uses S.

5. Discussion

5.1. Scope and limitations

No general theorem of viability is claimed. The endogeneity condition is not interpretation-independent in arbitrary categories. The sketch does not generate biological content; it gives a grammar in which such content can be stated. These bounds are the scope of the contribution.

The conditions should be read as structural constraints, not as sufficient conditions for persistence. A P-system can still fail through resource depletion, environmental shock, or internal defects outside the structural pattern. The claim is only that systems violating P are structurally compromised in a way the framework makes explicit.

The endogeneity condition (Definition 7) is the most delicate axiom. The non-factorisation condition is well-defined in **Set**, but its verification depends on what $K(S, E)$ encodes: the environment, qua biological milieu, does not carry the organisational information that φ carries. The sketch makes this dependence explicit through the choice of $K(S, E)$ rather than hiding it in the ambient category. The strengthening path is Markov categories (Fritz,

2020) and causal kernels (Jacobs et al., 2019). The paper does not take that step; it prepares it.

5.2. Relation to existing formalisms

Rosen's (M, R)-systems give causal closure without an explicit membrane or flux. Varela's operational closure gives the membrane without the causal cycle. Memory Evolutive Systems give hierarchical organisation without an endogeneity condition. The viability sketch combines the three in a single restricted generalised sketch, with the explicit caveat that the resulting object is not an ordinary finite-limit sketch.

Table 3: Comparison with neighbouring categorical and relational formalisms — structural axioms.

Formalism	Membrane	Endogeneity
Nomura (Nomura, 2001, 2007)	addressed via autopoietic formalisation, not isolated as ∂S	not formulated as E-relative non-factorisation
Letelier et al. (Letelier et al., 2003, 2011)	not foregrounded	treated through organisational closure
Louie (Louie, 2009, 2020)	relational	relational
Ehresmann & Vanbremeersch (Ehresmann and Vanbremeersch, 2007)	not isolated as ∂S	not formulated as E-relative non-factorisation
Present work	explicit (∂S)	E-relative + Lemma 1

Table 4: Comparison with neighbouring categorical and relational formalisms — multi-level structure and finite instantiation.

Formalism	Multi-level	Finite model
Nomura (Nomura, 2001, 2007)	not the focus	not the focus
Letelier et al. (Letelier et al., 2003, 2011)	not foregrounded	yes (specific example)
Louie (Louie, 2009, 2020)	treated	relational rather than finite-element verification
Ehresmann & Vanbremeersch (Ehresmann and Vanbremeersch, 2007)	central (colimits)	not the focus
Present work	Criterion 1	yes (16 elements)

5.3. Recent BioSystems work and biological anchoring

Recent work in this journal covers the empirical and biochemical dimensions the sketch does not.

Rubin (Rubin, 2023) cartographed the formal systems associated with molecular autopoiesis, distinguishing the natural-system reading of Maturana from the formal-system approach of Varela and subsequent modellers. The viability sketch is in the formal-system tradition. It does not characterise the living as such; it isolates a structural pattern that can be instantiated and checked.

Hofmeyr (Hofmeyr, 2018) analysed the causal structure of metabolic organisation through causation, constructors, and codes. The closure maps Φ and β can be read alongside that analysis, since they formalise how products

participate in the regeneration of the conditions of their own production. Igamberdiev and Kleczkowski (Igamberdiev and Kleczkowski, 2023) examined the role of nucleotides in metabolically closed autopoietic organisation — a concrete biochemical system in which closure conditions of this shape can be examined. Louie (Louie, 2020) extended relational biology in connection with Church's thesis, touching the computability-theoretic dimension that the avoidance clause (non-factorisation of φ) brushes against.

Weckström (Weckström, 2026) examined empirical correlates for a general theory of living organisation, revisiting the (M, R)-system and its replication. Which empirical observables correspond to the structural conditions of the viability sketch — what measurable features distinguish an E-endogenous system from a heteronomous one — is open, and a natural direction for future work.

The biological reading of the finite model in §3.2 is one interpretation among others. The sketch does not generate biological content on its own.

5.4. Extensions and limits of transfer beyond biology

Three candidates for further instantiation. Each requires its own substantial development.

Dissipative structures. The dissipative structures of Nicolis and Prigogine (Nicolis and Prigogine, 1977) are a candidate for a second instantiation: spontaneous emergence as self-configuration, gradient maintenance as the membrane, energy dissipation as the flux. The obstruction is temporal: dissipative structures have intrinsic dynamics that static category theory does not capture. Time-indexed categories (Ehresmann and Vanbremeersch, 2007) may

be needed.

Compositional game theory. The compositional game theory of Ghani et al. (Ghani et al., 2018) is a categorical framework for interacting agents. The open problem for any transfer is what endogeneity should mean here — what it would mean for a game to "produce its own rules".

Social systems (Luhmann). Luhmann (Luhmann, 1984) applied autopoiesis to social systems. Whether that application is substantive or merely terminological remains debated. We mention it only as a possible future domain. The biological argument and formal results of this paper make no claim about social systems.

6. Conclusion

What the paper contributes is a restricted formal core, deliberately checkable. The five elements are: (i) an evaluation-based self-production endomorphism; (ii) an E-relative endogeneity condition supported by a separation lemma; (iii) a two-condition admissibility under which P-systems and admissible morphisms form a category; (iv) an explicit finite Rosen-type model with element-by-element verification; (v) a multi-level diagnostic criterion shown to be non-trivial by a counterexample.

Possible next steps: richer examples, stronger endogeneity formalisms (Markov categories, causal kernels), more general categorical settings, and empirical correlates of E-endogeneity.

Declaration of competing interests

The author declares that there are no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Declaration of generative AI and AI-assisted technologies in the writing process

During the preparation of this work, the author used AI-assisted tools (ChatGPT, OpenAI; Claude, Anthropic; Gemini, Google) for language editing, organisation of revision materials, and checking the clarity of explanatory passages. After using these tools, the author reviewed, edited, and verified all content, mathematical claims, references, and final wording. The author takes full responsibility for the content of the article.

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Appendix A. Rosen ↔ viability sketch dictionary

Table A.5 records the vocabulary correspondence between Rosen's (M, R)-systems and the viability sketch.

Table A.5: Vocabulary correspondence.

Rosen (M, R)-systems	Viability sketch (this paper)	Notes
Substrates A	First component of $S = A \times B \times H$	Input to metabolism
Products B	Second component of S	Output of metabolism
Metabolism $f : A \to B$	Element $h \in H = \mathrm{Hom}(A, B)$, read via ev	Not fixed; part of state
Repair $\Phi : B \to \mathrm{Hom}(A, B)$	$\Phi : B \to H$	Products generate metabolic capacity
Closure $\beta : \mathrm{Hom}(A, B) \to B$	$\beta : H \to B$	Closes the causal cycle
Causal closure $\beta \circ \Phi \circ f$	$\varphi(a, b, h) = (a, \beta(\Phi(\mathrm{ev}(h, a))), \Phi(\mathrm{ev}(h, a)))$	Evaluation-based, not fixed-f
Closure to efficient causation	E-relative endogeneity (Def. 7, Lemma 1)	Relative to $K(S, E)$
Operational closure (Varela)	Membrane $\partial S = \{s \mid b = \beta(h)\}$	Functional, not spatial
Structural coupling	Admissible morphism (Def. 8)	Commutes with φ, respects ∂S
Pathological interaction	Non-admissible morphism (e.g. μ')	Destroys organisational identity

Appendix B. Finite model calculations

B.1 Rosen data

$$A = \{a_1, a_2\}, \quad B = \{b_1, b_2\}, \quad H = B^A = \{f_1, f_2, f_3, f_4\}$$

$$\Phi : B \to H,\ \Phi(b_1) = f_1,\ \Phi(b_2) = f_2.$$

$$\beta : H \to B,\ \beta(f_1) = b_1,\ \beta(f_2) = b_2,\ \beta(f_3) = b_1,\ \beta(f_4) = b_2.$$

$$S = A \times B \times H, \quad |S| = 16, \quad \varphi(a, b, h) = (a, \beta(\Phi(h(a))), \Phi(h(a))).$$

B.2 Computation of φ — worked examples

$$\varphi(a_1, b_1, f_1): \quad h(a) = f_1(a_1) = b_1,\ \Phi(b_1) = f_1,\ \beta(f_1) = b_1 \Rightarrow \varphi = (a_1, b_1, f_1).$$

$$\varphi(a_1, b_1, f_2): \quad h(a) = f_2(a_1) = b_2,\ \Phi(b_2) = f_2,\ \beta(f_2) = b_2 \Rightarrow \varphi = (a_1, b_2, f_2).$$

$$\varphi(a_2, b_1, f_1): \quad h(a) = f_1(a_2) = b_2,\ \Phi(b_2) = f_2,\ \beta(f_2) = b_2 \Rightarrow \varphi = (a_2, b_2, f_2).$$

$$\varphi(a_1, b_2, f_3): \quad h(a) = f_3(a_1) = b_1,\ \Phi(b_1) = f_1,\ \beta(f_1) = b_1 \Rightarrow \varphi = (a_1, b_1, f_1).$$

The remaining 12 values follow the same procedure and are recorded in Table 1 (§3.3).

B.3 Membrane verification

$|\partial S| = 8$.

Stability check. For each of the 8 membrane states, φ outputs a state in ∂S:

B.4 Endogeneity — exhaustive verification

$E = \{e_1, e_2, e_3\}$, $|E| = 3$, $K(S,E) = \{k = \kappa \circ \pi_A \mid \kappa : A \to E\}$. There are $|E|^{|A|} = 9$ possible maps $\kappa : A \to E$, each yielding a $k \in K(S,E)$ that maps the 16 states to at most $|A| = 2$ values, since k factors through the substrate projection π_A.

Cardinality argument. Since every $k \in K(S,E)$ factors through $\pi_A : S \to A$, we have $|\mathrm{Im}(k)| \leq |A| = 2$. Hence $|\mathrm{Im}(g \circ k)| \leq 2$ for every $g : E \to S$. But $|\mathrm{Im}(\varphi)| = 4$. Therefore φ cannot equal $g \circ k$ for any $k \in K(S,E)$ and $g : E \to S$.

Separation argument (independent verification). Take $\kappa(a_1) = e_1$, $\kappa(a_2) = e_2$. Then $k(a_1, b_1, f_1) = e_1 = k(a_1, b_1, f_2)$, but $\varphi(a_1, b_1, f_1) = (a_1, b_1, f_1) \neq (a_1, b_2, f_2) = \varphi(a_1, b_1, f_2)$. By Lemma 1, φ is E-endogenous. The same separation pair works for all 9 maps in $K(S,E)$, since every $k \in K(S,E)$ confounds states differing only in (b, h).

B.5 Admissibility checks

$r = \varphi$ *is admissible.* Condition (i): $r \circ \varphi = \varphi \circ r = \varphi^2$. The table below records φ^2 explicitly only to make the induced dynamics checkable.

For the a_1-branch, φ^2 reaches a fixed point in one step. For the a_2-branch, φ^2 cycles between (a_2, b_1, f_1) and (a_2, b_2, f_2), reflecting the 2-cycle in $\mathrm{Im}(\varphi)$.

Condition (ii): $\varphi(\partial S) \subseteq \partial S$ verified in B.3. Therefore $r = \varphi$ is admissible.

μ' *is not admissible.* $\mu'(a, b, h) = (a, b_1, f_3)$.

- Condition (ii) holds: $\beta(f_3) = b_1$, so $\mu'(s) \in \partial S$ for all s.

- Condition (i) fails: $\mu'(\varphi(a_1, b_1, f_1)) = (a_1, b_1, f_3)$, while $\varphi(\mu'(a_1, b_1, f_1)) = \varphi(a_1, b_1, f_3) = (a_1, b_1, f_1)$.

μ' is a membrane-preserving interaction that destroys organisational identity: a monopolising capture in the diagnostic vocabulary of §2.8.

B.6 Multi-level counterexample verification

$T = S \times \{0, 1\}$, $|T| = 32$, $\psi(s, 0) = (\varphi(s), 0)$, $\psi(s, 1) = (\mu'(s), 1)$, $\partial T = \{(s, j) \mid s \in \partial S\}$.

Stability and permeability. $\psi(\partial T) \subseteq \partial T$ because both φ and μ' preserve ∂S. It remains to verify E-relative endogeneity.

Endogeneity. $\mathrm{Im}(\psi) = \mathrm{Im}(\varphi) \times \{0\} \cup \mathrm{Im}(\mu') \times \{1\}$; $|\mathrm{Im}(\varphi)| = 4$; $\mathrm{Im}(\mu') = \{(a_1, b_1, f_3), (a_2, b_1, f_3)\}$, so $|\mathrm{Im}(\mu')| = 2$; hence $|\mathrm{Im}(\psi)| = 6$. $K(T, E)$ factors through π_A, so $|\mathrm{Im}(k)| \leq 2 < 6$ for every $k \in K(T, E)$. Therefore ψ is E-endogenous.

Together, stability, permeability, and E-relative endogeneity make T a P-system in the sense of Definition 1.

Embeddings. For $i_0(s) = (s, 0)$,

$$i_0(\varphi(a_1, b_1, f_1)) = (a_1, b_1, f_1, 0),$$
$$\psi(i_0(a_1, b_1, f_1)) = (\varphi(a_1, b_1, f_1), 0) = (a_1, b_1, f_1, 0).$$

Verified on all 16 states.

For $i_1(s) = (s, 1)$,

$$i_1(\varphi(a_1, b_1, f_1)) = (a_1, b_1, f_1, 1),$$
$$\psi(i_1(a_1, b_1, f_1)) = (\mu'(a_1, b_1, f_1), 1) = (a_1, b_1, f_3, 1).$$

i_0 is admissible. i_1 is not.

Table B.6: Membrane test: $(a, b, h) \in \partial S$ iff $b = \beta(h)$.

State	$\beta(h)$	$b = \beta(h)$?	$\in \partial S$?
(a_1, b_1, f_1)	b_1	yes	yes
(a_1, b_1, f_2)	b_2	no	no
(a_1, b_1, f_3)	b_1	yes	yes
(a_1, b_1, f_4)	b_2	no	no
(a_1, b_2, f_1)	b_1	no	no
(a_1, b_2, f_2)	b_2	yes	yes
(a_1, b_2, f_3)	b_1	no	no
(a_1, b_2, f_4)	b_2	yes	yes
(a_2, b_1, f_1)	b_1	yes	yes
(a_2, b_1, f_2)	b_2	no	no
(a_2, b_1, f_3)	b_1	yes	yes
(a_2, b_1, f_4)	b_2	no	no
(a_2, b_2, f_1)	b_1	no	no
(a_2, b_2, f_2)	b_2	yes	yes
(a_2, b_2, f_3)	b_1	no	no
(a_2, b_2, f_4)	b_2	yes	yes

Table B.7: Stability: $\varphi(\partial S) \subseteq \partial S$.

∂S state	φ(state)	φ(state) $\in \partial S$?
(a_1, b_1, f_1)	(a_1, b_1, f_1)	yes
(a_1, b_1, f_3)	(a_1, b_1, f_1)	yes
(a_1, b_2, f_2)	(a_1, b_2, f_2)	yes
(a_1, b_2, f_4)	(a_1, b_2, f_2)	yes
(a_2, b_1, f_1)	(a_2, b_2, f_2)	yes
(a_2, b_1, f_3)	(a_2, b_1, f_1)	yes
(a_2, b_2, f_2)	(a_2, b_1, f_1)	yes
(a_2, b_2, f_4)	(a_2, b_2, f_2)	yes

Table B.8: φ and φ^2 on the full state space.

s	$\varphi(s)$	$\varphi^2(s)$
(a_1, b_1, f_1)	(a_1, b_1, f_1)	(a_1, b_1, f_1)
(a_1, b_1, f_2)	(a_1, b_2, f_2)	(a_1, b_2, f_2)
(a_1, b_1, f_3)	(a_1, b_1, f_1)	(a_1, b_1, f_1)
(a_1, b_1, f_4)	(a_1, b_2, f_2)	(a_1, b_2, f_2)
(a_1, b_2, f_1)	(a_1, b_1, f_1)	(a_1, b_1, f_1)
(a_1, b_2, f_2)	(a_1, b_2, f_2)	(a_1, b_2, f_2)
(a_1, b_2, f_3)	(a_1, b_1, f_1)	(a_1, b_1, f_1)
(a_1, b_2, f_4)	(a_1, b_2, f_2)	(a_1, b_2, f_2)
(a_2, b_1, f_1)	(a_2, b_2, f_2)	(a_2, b_1, f_1)
(a_2, b_1, f_2)	(a_2, b_1, f_1)	(a_2, b_2, f_2)
(a_2, b_1, f_3)	(a_2, b_1, f_1)	(a_2, b_2, f_2)
(a_2, b_1, f_4)	(a_2, b_2, f_2)	(a_2, b_1, f_1)
(a_2, b_2, f_1)	(a_2, b_2, f_2)	(a_2, b_1, f_1)
(a_2, b_2, f_2)	(a_2, b_1, f_1)	(a_2, b_2, f_2)
(a_2, b_2, f_3)	(a_2, b_1, f_1)	(a_2, b_2, f_2)
(a_2, b_2, f_4)	(a_2, b_2, f_2)	(a_2, b_1, f_1)