Countable Dutch Book Arguments

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Abstract

I show that a standard argument against the possibility of a valid Dutch book argument for countable additivity is too quick because it fails to distinguish different notions of convergence for random variables. Making this distinction allows for the precise formulation of such a Dutch book argument.

Keywords— Dutch book; countable additivity; Bayesian epistemology; probability

1 Introduction

Dutch book arguments are among the most valuable tools in the Bayesian toolbox. There are Dutch book arguments for finitely-additive probability (de Finetti 1974, Ramsey 1926), for countable additivity (Adams 1962, Williamson 1999), for Conditionalization (Lewis 1999, first reported in Teller 1973; see also Rescorla 2018, Rescorla 2022), for probability kinematics (Skyrms 1987), for Reflection (Van Fraassen 1984, Goldstein 1983, Huttegger 2013), and for the Principal Principle (Pettigrew 2020), among others. As with any good piece of philosophy there is considerable debate over their validity (McGee 1999, Arntzenius, Elga, and Hawthorne 2004).

Quite briefly, the structure of a Dutch book argument is as follows. We fix some set Ω to be our sample space—in the philosophical literature it is commonplace to gloss the elements of Ω as possible worlds, though no mathematical consequences hinge on that interpretation. We suppose that you have *credences* regarding some family $\mathcal{F}$ of events; these events are subsets $A_i \subseteq \Omega$. That is, for some (perhaps all) events $A_i \in \mathcal{F}$, you have a credence $P(A_i) = p_i$ representing your *degree of belief* in A_i. We call the function P your *credence function*. When Ω is a finite set, we generally assume that $\mathcal{F}$ is the powerset $\mathcal{P}(\Omega)$, and when Ω is infinite, we generally assume that $\mathcal{F}$ is some σ-algera of subsets.

A Dutch book argument then attempts to establish that some *requirement* on your credence function is rationally required. They do so by showing that, if your credences fail to satisfy that requirement, then you are exploitable by a *Dutch book*, a collection of bets, all of which look favorable to you, but on whose conjunction you are guaranteed not to win.[1] By a *bet* we mean a measurable function $\phi : \mathcal{F} \to \mathbb{R}$ of the form

$$\phi(\omega) = \sum_{i \in I} c_i [1_{A_i}(\omega) - \mu(A_i)], \tag{1}$$

where $\omega \in \Omega$, 1_{A_i} is the indicator function (also called the characteristic function) of the measurable set $A_i \in \mathcal{F}$, $\mu(A_i)$ is the agent's credence in A_i, and c_i is the stakes of the bet. The index set I can be finite or countably infinite.

Dutch book arguments for countable additivity have generated a particularly active controversy. Here 'countable additivity' refers to the assumption that, if $A_1, A_2, \ldots$ is a countably infinite sequence of disjoint sets, then your credence function P satisfies

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

These arguments attempt to establish that countable additivity is a normative

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[1] Or, sometimes, on whose conjunction you are guaranteed to lose. This distinction is discussed later.

requirement for Bayesian agents by showing that, if your credences are not countably additive, then you are exploitable by a Dutch book.

The recent book (Pettigrew 2020) provides an excellent, careful discussion and evaluation of many of these Dutch book arguments. In particular, Pettigrew argues *against* the possibility of a valid Dutch book argument for countable additivity. This section of the book is rather brief, consisting of only three pages. In it he presents what he claims are two distinct styles of objection to countable Dutch book arguments (hereafter 'CDBAs'): one style he attributes to de Finetti (de Finetti 1974), with an expanded form by Vann McGee (McGee 1999), while the other he attributes to Arntzenius et al. (Arntzenius, Elga, and Hawthorne 2004). Both McGee and Arntzenius et al. construct 'defeaters' for the possibility of a CDBA: countably infinite collections of bets which appear to Dutch book even a countably additive probability measure. Thus, even countably additive measures cannot avoid exploitation by countable Dutch books. If countable Dutch books are unavoidable then it cannot be *irrational* to be exploitable by them. Pettigrew concludes that these objections are decisive and, thus, there cannot be a valid CDBA.

I argue that this dismissal is too quick, and that there is a valid CDBA, but its statement requires some mathematical care. To do so I first review McGee's countable bet construction in §2. In §3 I isolate the mathematical property responsible for its Dutch-book-like behavior by comparing it to a countable Dutch book theorem due to Freedman (Freedman 2003). With this understanding in place I argue that there is a natural class of bets which permit the statement of a valid CDBA.

2 McGee's Defeater

In order to understand the force of McGee's construction we must be clear on its dialectical role in a CDBA. Pettigrew helpfully lays out an argument schema that generalizes the various Dutch book arguments scattered across the literature. Each argument begins with the same first premiss:

(DB1) If you have credence p in proposition A, then

- (i) you are rationally permitted to pay £x for a £S bet on A for any $x \leq pS$,
- (ii) you are rationally required to pay £x for a £S bet on A for any $x < pS$.

The second premiss in the argument schema is a theorem schema. Here one inserts the relevant theorem which shows that if your credences fail to have some property then they are, in some sense to be made precise, exploitable. For example, the second premiss of Pettigrew's CDBA is as follows:

(DB2CA) If your credences violate countable additivity then they are countably strongly fully exploitable.

Your credences are *countably* exploitable if the witnessing Dutch book consists of countably many bets; they are *strongly* exploitable if each of the component bets are rationally required; they are *fully* exploitable if the net payoff, in every possible outcome, is negative.

And indeed this is a theorem: if your credences are finitely, but not countably, additive, then they are countably strongly fully exploitable.[2] To see this, suppose $A_1, A_2, \ldots$ is a sequence of disjoint measurable subsets of Ω, that $A = \bigcup_i A_i$, and that $P\left(\bigcup_i A_i\right) \neq \sum_i P(A_i)$. To simplify notation let

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[2] Without assuming finitely additive credences the theorem becomes trivial, as the standard finite Dutch book theorem suffices.

$p = P(\bigcup_i A_i)$ and let $p_i = P(A_i)$. By finite additivity we then have that $p > \sum_i p_i$. Arrange bets as follows: the agent buys a bet on A (a bet that A will occur) for price $\pounds p$ that pays off £1; for each $i \in \mathbb{N}$ the agent sells a bet on A_i for price $\pounds p_i$ that pays off £1. The agent has lost $d = p - \sum_i p_i > 0$ before the bets are settled. Assuming A does not occur, no money changes hands, so the agent has lost $\pounds d$. Assuming A does occur, there is exactly one A_i that occurs; in this case, the £1 won on A is cancelled out by the £1 lost on A_i. So the agent has lost at least $\pounds d$ in every possible $\omega \in \Omega$.

The third and final premiss of Pettigrew's CDBA is as follows:

(DB3CA) If your credences are countably strongly fully exploitable then they are irrational.

This is the premiss that Pettigrew objects to, on the grounds of the counterexample given by McGee. So McGee's example must demonstrate credences that are exploitable by a countable Dutch book but are nonetheless intuitively rational.

This is what McGee's example shows. Suppose you are presented with a coin and a series of wagers on the outcomes of flipping that coin. You believe the coin is fair, so we model your credences as the uniform measure λ on the space 2^ω of infinite binary sequences. That is, given a finite binary string $\sigma \in 2^{<\omega}$ of length n, let $[\sigma] := \{x \in 2^\omega : \sigma \prec x\}$, the set of sequences that extend σ; then you assign credence $\lambda([\sigma]) = 2^{-n}$. By the Carathéodory extension theorem (Ash and Doléans-Dade 2000) your credences uniquely extend to a countably additive probability measure on all Borel sets of 2^ω. In other words, your credences are coherent and countably additive.

Now we consider the following bets.

Bet 1 You lose $\pounds 1$ if the first toss is Heads, and win $\pounds 3$ if it is Tails.

Bet 2 You lose £4 if the first toss is Tails, and you win £12 if the first toss is Heads and the second Toss is Tails; otherwise, the bet is called off.

Bet 3 You lose £12 if the first toss is Heads and the second Tails, and you win £32 if the first two tosses are Heads and the third is Tails; otherwise, the bet is called off.

Bet 4 You lose £32 if the first two tosses are Heads and the third is Tails, and you win £80 if the first three tosses are Heads and the fourth is Tails; otherwise, the bet is called off.

Bet $n + 1$ If the first time Tails appears is on the n^{th} toss, you lose £$(n + 1)2^n$, which is how much you win in the same situation on bet n. If instead Tails appears on the $n + 1^{\text{th}}$ toss, you win £$(n + 2)2^{n+1}$.

Given that you credences are modeled by λ, you take each of these bets to have positive expected value. But in the limit, you are guaranteed to lose £1. So we see that your credences are countably strongly fully exploitable. But, if *any* credences on sequences of coin flips are rational, surely the uniform measure is. So McGee and Pettigrew take this example to be a counterexample to (DB3CA), and thus a counterexample to the CDBA.

3 Freedman's Theorem and Convergence of Random Variables

Mathematically, McGee's Dutch book is airtight. But does it show that a valid CDBA can't be found? How are we meant to square it with the various results (Adams 1962, Williamson 1999) that claim to have established a CDBA? The answer can be found in an unpublished set of lectures notes on Dutch book arguments by the statistician D. A. Freedman (Freedman 2003). Freedman's notes begin with de Finetti's original Dutch book argument for finite additivity, first for finite Ω and working up to the most general case. But the notes end with a theorem that purports to show that there is a valid CDBA for countable additivity.

The theorem is stated as follows. Let $A_1, A_2, \ldots$ be a sequence of (not necessarily disjoint) measurable sets, as before. Let

$$\phi_n = \sum_{i=1}^{n} c_i [1_{A_i} - \mu(A_i)]. \tag{2}$$

These are just the standard assumptions for a Dutch book theorem. We then extend the class of admissible bets as follows. Assume

1. there exists $L < \infty$ such that $|\phi_n(\omega)| < L$ for all n and all $\omega \in \Omega$;
2. $\phi(\omega) := \lim_n \phi_n(\omega)$ for all $\omega \in \Omega$.

Then ϕ, so defined, is also an admissible bet. In other words, rather than merely considering bets of the form (2), we include uniformly bounded pointwise limits of bets. Freedman then sketches the proof of the following.

Proposition 1 (Freedman). *Suppose the class of admissible bets contains finite sums of the form (2) and is expanded to include uniformly bounded pointwise limits. Then there exists a Dutch book against μ unless μ is a countably additive probability measure.*

On one hand, we have a theorem asserting that countably additive measures are not exploitable by a countable Dutch book of a certain form; on the other, we have what appears to be a countable Dutch book that exploits a countably additive measure. So the only conclusion is that McGee's Dutch book must not satisfy the assumptions of Proposition 1. Indeed, McGee's Dutch book clearly fails to be uniformly bounded—for any $m < \infty$ there is $n < \infty$ such that

$m < (n+2)2^{n+1}$, the amount you win on bet $n+1$ when Tails appears on the $n+1^{\text{th}}$ toss.

The fact that there is no uniform bound is crucial. For while McGee's Dutch book is the pointwise limit of its sequence of finite approximations, its *expectation* is clearly not the limit of the expectations of its finite approximations. More precisely, let ψ_n be the n^{th} of McGee's bets, let $\phi_n = \sum_{i=1}^{n} \psi_i$, and let $\phi = \sum_{i=1}^{\infty} \psi_i$. McGee notes that each individual bet ψ_i has an expected payoff of 1; that is, the expectation is

$$\mathbb{E}_\lambda[\phi_n] = \mathbb{E}_\lambda\left[\sum_{i=1}^{n} \psi_i\right] = \sum_{i=1}^{n} \mathbb{E}_\lambda[\psi_i] = n.$$

But in the limit, you lose 1 no matter what, so $\mathbb{E}_\lambda[\phi] = -1$. Thus, $\lim_n \mathbb{E}_\lambda[\phi_n] \neq \mathbb{E}_\lambda[\phi]$.

This matters because ϕ, as a countable sum, is really the limit of a sequence of *random variables* ϕ_n, and there are many distinct notions of convergence for random variables. One of them is *pointwise* convergence: given a sequence of random variables $X_1, X_2, \ldots$, we say that the sequence converges everywhere[3] to a random variable X if for all $\omega \in \Omega$ we have $\lim_{n\to\infty} X_n(\omega) = X(\omega)$. Another notion is *convergence in mean*, also known as L^1-*convergence*. We say that the sequence converges in mean to X if $\lim_{n\to\infty} \mathbb{E}[X_n] = \mathbb{E}[X]$.

In McGee's Dutch book we clearly have pointwise convergence—this is guaranteed by the fact that the Dutch book is a countable sum. But we do *not* have convergence in mean; that the ϕ_n do not converge in mean to ϕ is a consequence of the fact that each ϕ_n is a favorable bet but ϕ is guaranteed to lose money. Indeed this is a well-known result in measure theory: convergence everywhere does not imply convergence in mean, and convergence in mean does not im-

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[3] This is the usual notion of pointwise convergence of functions familiar from real analysis. In probability theory we generally consider the weaker notion of *almost-everywhere* convergence: if $(\Omega, \mathcal{F}, \mu)$ is a probability space, we say that the sequence converges almost everywhere if $\lim_{n\to\infty} X_n(\omega) = X(\omega)$ on a set of μ-measure one.

ply convergence everywhere. So McGee's Dutch book connects this result to Dutch book arguments—it shows that even countably additive credences are exploitable by bets that fail to converge in mean.

By contrast Freedman's theorem assumes that admissible bets are either (i) finite sums or (ii) countable sums whose finite initial sums are uniformly bounded. These finite initial sums therefore also converge in mean to the total sum; this is a consequence of the celebrated Dominated Convergence Theorem of measure theory (see e.g. Ash and Doléans-Dade 2000, Theorem 1.6.9). Using our previous notation for bets, we have the following result.

Theorem 1. *Suppose* $(\Omega, \mathcal{F}, \mu)$ *is a finitely additive probability space, i.e,* μ *is a finitely (though perhaps not countably) additive probability measure. Suppose the class of admissible bets contains bets of the form (2).*

1. *If the class of admissible bets is expanded to include all limits of sequences that converge in mean, then there is a Dutch book against* μ *if and only if* μ *is not countably additive.*
2. *If the class of admissible sets is expanded to include all pointwise limits, then there is a Dutch book against* μ.

4 An Airtight CDBA

Theorem 1 clarifies the relationship between the countable Dutch book theorems in the literature and McGee's Dutch book. In particular, it shows that Pettigrew's conclusion is too quick. The fact that infinitely many bets are in play is not the problem. The problem is that the expected winnings of the total set of bets is not the limit of the expected winnings of each finite approximation. But this is no reason to conclude that agents simply cannot make rational decisions over countably many bets.

A comparison with real analysis is instructive. Some countable sequences of real numbers have no finite sum despite the fact that each finite initial subsequence does. For example, the sequence $1, 2, 3, \ldots$ clearly satisfies $\sum_{n \leq m} n < \infty$ for every $m < \infty$, but $\sum_{n \in \mathbb{N}} n$ diverges. One should *not* conclude that we simply cannot take infinite sums. Instead one should conclude that we must be more careful about infinite sums than about finite sums. One can show that some sequences have finite sums when the sum is taken in a specific order, but not in other orders (these are called *conditionally convergent* sums); for example, the harmonic sequence $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges to $\ln(2)$, but if we permute the order of terms the sum can diverge. One can also show that some sequences have finite sums regardless of the order in which the sum is taken. These are the *absolutely convergent* sequences, and a sequence a_n is absolutely convergent if and only if there is $M < \infty$ such that, for all $n < \infty$, $\sum_{i=1}^{n} |a_i| \leq M$.

Our current situation is analogous. If a countable bet is such that its finite partial sums converge in mean to the total sum, call it a *mean-convergent bet.* Mean-convergent bets are the most natural variety of countably infinite bet. A bet that is mean-convergent involves payoffs that are all less than some fixed bound. By contrast a bet that is not mean-convergent requires us to permit in principle *arbitrarily large* sums of money to exchange hands. Put another way, for any price *£n*, there is some outcome where the payoff is greater than *£n*. This is obviously unrealistic, since any real betting scenario involves a fixed finite amount of cash for gambling. Even Vegas has its limits.

Moreover we see that no rational betting plan is possible when the bet is not mean-convergent, because the fair price of those finite bets does not converge to the fair price of the total. But rational betting plans *are* possible for mean-convergent bets. We can therefore reformulate Pettigrew's premisses for the CDBA as follows. Say that your credences are *strongly mean-exploitable* if there is a mean-convergent Dutch book against you. This condition therefore replaces *countable* exploitability.

(DB2M) If your credences are not countably additive then they are strongly fully mean-exploitable.

(DB3M) If your credences are strongly fully mean-exploitable then they are irrational.

Premiss (DB2M) is part 1 of Theorem 1 in the format of Pettigrew's schema. What can be said in favor of premiss (DB3M)?

I argue that it is defensible on precisely the same pragmatic grounds that motivate the classical Dutch book argument for finite additivity and that also motivate Pettigrew's and McGee's rejection of the validity of a CDBA. Generally we gloss the force of a Dutch book argument as: it is irrational to accept as fair (or favorable) a set of bets that is guaranteed to lose. In other words, one *should not* accept such an arrangement. But clearly this holds only if such an arrangement is avoidable. McGee's Dutch book purports to show that such an arrangement is unavoidable when we allow countable collections of bets, even if your credences are satisfy all the axioms of probability. As he says: "in situations in which there can be infinitely many bets over an unbounded utility scale, no rational plan of action is available" (McGee 1999, 257). And since it is unavoidable, it cannot be irrational: as Pettigrew concludes, "[McGee's example] seems to show that being countably exploitable is no marker of irrationality" (Pettigrew 2020, 30).

But a kind of exploitability *is* avoidable for countably additive credences, namely, mean-exploitability. The mean-convergent bets are a natural class of functions that precisely separate the finitely additive measures from the countably additive measures. If credences are irrational when they lead to *avoidable* sure losses, then finitely additive credences are irrational. Countably additive credences, by contrast, do not lead to avoidable sure losses, but only to unavoidable ones. We can agree with Pettigrew that this is no marker of irrationality.

In fact I agree with Pettigrew and McGee that (DB3CA) is false. But as I have shown, (DB3CA) is not the only way to conclude a CDBA. Indeed the issue lies with (DB2CA): we should not consider *all* countable collections of bets, but only those that converge in the right way. In conclusion we have the following CDBA.

(DB1) If you have credence p in proposition A, then

  (i) you are rationally permitted to pay x for a S bet on A for any $x \leq pS$,

  (ii) you are rationally required to pay x for a S bet on A for any $x < pS$.

(DB2M) If your credences are not countably additive then they are strongly fully mean-exploitable.

(DB3M) If your credences are strongly fully mean-exploitable then they are irrational.

Thus we conclude that if your credences are not countably additive, then they are irrational.

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