Weyl-type theorems in Galilei and Carroll geometry

Philip K. Schwartz[1,a], James Read[2,b], and Quentin Vigneron[3,c]

[1]*Institute of Theoretical Physics, Leibniz University Hannover,*
Appelstraße 2, 30167 Hannover, Germany
[2]*Faculty of Philosophy, University of Oxford,*
Oxford OX2 6GG, United Kingdom
[3]*Centre de Recherche Astrophysique de Lyon, ENS de Lyon, Université Claude Bernard Lyon 1, CNRS,*
Lyon 69007, France
[a]`philip.schwartz@itp.uni-hannover.de`
[b]`james.read@philosophy.ox.ac.uk`
[c]`quentin.vigneron@ens-lyon.fr`

A classic theorem of Weyl (1921) states that a *Weyl metric*—a natural generalisation of a pseudo-Riemannian metric—is uniquely determined by its conformal and projective structures (i.e. by its conformal structure and its set of unparametrised geodesics). An equivalent formulation of Weyl's result is that a torsion-free linear connection compatible with a pseudo-Riemannian conformal structure is uniquely determined by its projective structure. We discuss analogous results for suitably defined notions of conformal structure for Galilei and Carroll geometry, i.e. for spacetime geometries arising as the 'non-relativistic' and 'ultra-relativistic' limits of Lorentzian geometry.

1. Introduction

In 1921, Hermann Weyl proved that a given 'Weyl metric'—of which a Lorentzian metric is (in a certain sense to be made precise below) a special case—is fixed uniquely by its associated projective and conformal structures (respectively: unparametrised geodesics of the associated connection, and—in the case of indefinite signature—lightcone structure). This result is a uniqueness theorem in the sense that those projective and conformal structures fix the Weyl metric uniquely: there are no *other* Weyl metrics which they can be used to define. Some fifty years later, in 1972, Ehlers et al. (1972, 2012) ('EPS') proved *inter alia* what can understood to be a related existence theorem: given a projective structure and a conformal structure, and given further assumptions regarding their 'compatibility' (roughly: the unparametrised timelike geodesics given by the projective structure all lie inside the timelike lobes of the lightcones given by the conformal structure), they can be used to construct a Weyl metric.

Since that work, many details have been shored up—see Adlam et al. (2025) and references therein for a detailed study. And yet, there also remain technical questions regarding Weyl's theorem which have yet to be answered decisively. In particular, one might ask: is it possible to prove versions of Weyl's theorem in alternative spacetime geometries, in particular the 'non-relativistic' and 'ultra-relativistic' settings of Galilei and Carroll geometries?[1] It is upon such questions that we focus in the present article, giving an affirmative answer for the case of Galilei geometry, but showing that only a weaker result is provable in the case of Carroll geometry.

In the Galilei case, there is some precedent for work on 'non-relativistic' versions of Weyl's theorem. First, Ewen and Schmidt (1989) proved a version of this theorem—but using a notion of Galilean conformal structure which, for reasons discussed by Adlam et al. (2025) and March (2023), is arguably conceptually inadequate. (Essentially, Ewen and Schmidt define conformal structure as spacelike projective structure, which is arguably insufficiently related to a notion of Galilean scale transformations.) Moreover, Ewen and Schmidt market their result as a Galilean version of the construction of Ehlers et al. (1972, 2012), but as pointed out by Adlam et al. (2025, ch. 4) this is to confuse existence and uniqueness results of the kind discussed above. (What Ewen and Schmidt prove is an analogue of the 1921 uniqueness theorem of Weyl, not the 1972 existence result of EPS—see Adlam et al., ch. 4 for further discussion.) After having presented a 'non-relativistic' version of the EPS construction using an improved notion of Galilean conformal structure proposed by March (2023),[2] Adlam et al. (2025, § 4.5.1) assert that for a 'non-relativistic' version of Weyl's theorem 'the proof is exactly analogous to that sketched for the relativistic case', but do not go into further details; these details are filled in by March (2025), but only for Galilei structures on the assumption of metric compatibility.

To expand on this final point, return for a moment to the Lorentzian case. In that setting, a special case of Weyl's result for Lorentzian (rather than Weyl) metrics is presented by Malament (2012, ch. 2)—this special case assumes metric compatibility. Likewise, because March (2025) assumes metric compatibility in the Galilean context, what remains to be accomplished is a more general (and directly analogous) version of Weyl's result for the Galilean setting, in which one shows that a Galilean Weyl structure (suitably defined—see Adlam et al. (2025, ch. 4) and discussion below) is fixed uniquely by its associated

---

[1] We use here the common designations 'non-relativistic' and 'ultra-relativistic' for Galilei and Carroll geometry, even though Galilean/Newtonian and Carrollian physics are relativistic as well, in the sense of satisfying the principle of relativity. The principle is just implemented differently than in special or general relativity (or their modifications), namely by the Galilei or Carroll group, respectively, instead of the Poincaré group.

[2] A precursor to this notion of Galilean conformal structure can be found in Curiel (2015), but it is more specialised than that proposed by March (2023). Curiel also claims to prove a Galilean version of Weyl's theorem, but the proof is erroneous as it omits crucial data needed to fix a connection uniquely with respect to a given Galilei geometry—namely, a choice of timelike vector field. We discuss these extra data in detail in the body of this article.

projective and conformal structures (plus, possibly, other relevant data[3]). Completing this work shores up and completes our understanding of Galilean versions of Weyl's theorem.

With this work on Galilean versions of Weyl's theorem complete, in this article we in addition consider the case of Carroll versions of the result. (Recall that Carrollian spacetime structures are the 'ultra-relativistic' versions of Lorentzian spacetime structures, in which one sends $c \to 0$ and as such 'narrows' the lightcones—see March and Read (2025) for a pedagogical review, and Vigneron et al. (2025) for results regarding general Carrollian connections upon which we draw in this article.) The conjecture here—in parallel with the Lorentzian and Galilean cases—is that one can define a suitable notion of Carrollian conformal structure such that a given Carrollian Weyl structure (again, suitably defined) is fixed uniquely by its associated projective and conformal structures (plus potential further relevant data, as in the Galilean case). In the second half of this article, we show that a version of Weyl's theorem can indeed be proved in the Carroll context, but the result is strictly weaker than the Lorentzian and Galilean Weyl-type theorems: a Carrollian Weyl structure is *not* fixed uniquely by its associated projective and conformal structures (plus potential further relevant data), although this failure of uniqueness can be quantified in a precise way.[4]

As such, the structure of this article is as follows. In section 2, we review the notion of projective structure, since this is in fact the same structure in all of the three cases which we consider. In section 3, we review the definition of Lorentzian conformal structure and recall the essential contours of the original theorem of Weyl (1921). In section 4, we do the same for the Galilean case. In section 5, we define Carrollian conformal structures and prove a (weakened) Carrollian version of Weyl's theorem. We close in section 6 with some reflections on our results and potential directions for future research. (For the impatient, a concise summary of our results may be found in table 1.) In appendices A and B, we discuss some aspects of the classification of connections on Carroll manifolds that are used in the main text.

2. Projective structures

In this section, we review some textbook material on projective structures which will prove essential going forwards. Further discussion of projectively equivalent connections can be found in e.g. Adlam et al. (2025) and Malament (2012).

Definition 2.1. Two linear connections on a differentiable manifold M are *projectively equivalent* if they have the same autoparallels (i.e. unparametrised geodesics). A projective equivalence class of connections is a *projective structure* on M.

---

[3] See footnote 2.

[4] We leave the project of a Carrollian version of the EPS construction—i.e., Carroll constructive axiomatics—for another day. The failure of a Carroll version of Weyl's theorem might, however, be taken to indicate that there will exist serious roadblocks to such a construction.

Remark 2.2. Unlike much of the existing literature on projective structures, here we do *not* demand the connections to be torsion-free. For this reason, we also provide the proof of the following standard result in the general case with possibly non-vanishing torsion:

Proposition 2.3. *Two linear connections on M are projectively equivalent if and only if their difference tensor $D^{\rho}_{\mu\nu}$ satisfies*

$$D^{\rho}_{(\mu\nu)} = \delta^{\rho}_{(\mu}\eta_{\nu)} \tag{2.1}$$

for some one-form $\eta \in \Omega^1(M)$.

Proof. A direct computation shows that if the difference tensor has the stated form, the two connections have the same autoparallels.

For the converse, assume that two connections have the same autoparallels. This implies that for any autoparallel γ we have $D^{\rho}_{\mu\nu}(\gamma(t)))\dot{\gamma}^{\mu}(t)\dot{\gamma}^{\nu}(t) = f(t)\dot{\gamma}^{\rho}(t)$ for some function f. Multiplying with $\dot{\gamma}^{\sigma}$ and antisymmetrising in ρ, σ, we obtain

$$0 = \dot{\gamma}^{[\sigma}D^{\rho]}_{\mu\nu}\dot{\gamma}^{\mu}\dot{\gamma}^{\nu} = D^{[\rho}_{\mu\nu}\delta^{\sigma]}_{\kappa}\dot{\gamma}^{\mu}\dot{\gamma}^{\nu}\dot{\gamma}^{\kappa}. \tag{2.2}$$

Since any vector $v \in TM$ is tangent to some autoparallel, this shows that for all v we have

$$D^{[\rho}_{\mu\nu}\delta^{\sigma]}_{\kappa}v^{\mu}v^{\nu}v^{\kappa} = 0, \tag{2.3}$$

which is equivalent to

$$D^{[\rho}_{(\mu\nu}\delta^{\sigma]}_{\kappa)} = 0. \tag{2.4}$$

Writing out the symmetrisation, this is equivalent to

$$D^{[\rho}_{(\mu\nu)}\delta^{\sigma]}_{\kappa} + D^{[\rho}_{(\mu\kappa)}\delta^{\sigma]}_{\nu} + D^{[\rho}_{(\nu\kappa)}\delta^{\sigma]}_{\mu} = 0. \tag{2.5}$$

Contracting σ and κ, we obtain

$$\begin{aligned} 0 &= D^{\rho}_{(\mu\nu)}\delta^{\sigma}_{\sigma} - D^{\sigma}_{(\mu\nu)}\delta^{\rho}_{\sigma} + D^{\rho}_{(\mu\sigma)}\delta^{\sigma}_{\nu} - D^{\sigma}_{(\mu\sigma)}\delta^{\rho}_{\nu} + D^{\rho}_{(\nu\sigma)}\delta^{\sigma}_{\mu} - D^{\sigma}_{(\nu\sigma)}\delta^{\rho}_{\mu} \\ &= nD^{\rho}_{(\mu\nu)} - D^{\rho}_{(\mu\nu)} + D^{\rho}_{(\mu\nu)} - D^{\sigma}_{(\mu\sigma)}\delta^{\rho}_{\nu} + D^{\rho}_{(\mu\nu)} - D^{\sigma}_{(\nu\sigma)}\delta^{\rho}_{\mu} \\ &= (n+1)D^{\rho}_{(\mu\nu)} - D^{\sigma}_{(\mu\sigma)}\delta^{\rho}_{\nu} - D^{\sigma}_{(\nu\sigma)}\delta^{\rho}_{\mu} \end{aligned} \tag{2.6}$$

where $n = \dim M$. This shows that $D^{\rho}_{(\mu\nu)} = \delta^{\rho}_{(\mu}\eta_{\nu)}$ with $\eta_{\mu} = \frac{2}{n+1}D^{\sigma}_{(\mu\sigma)}$. □

3. The pseudo-Riemannian case

We next recall the definitions of conformal structures and Weyl metrics in the pseudo-Riemannian setting (which includes as a special case the Lorentzian, i.e. relativistic, one). Having presented these details, we then recall the above-mentioned theorem of Weyl (1921). All of this material is standard—see e.g. Adlam et al. (2025) and references therein.

Definition 3.1. Let M be a differentiable manifold.

(i) A *pseudo-Riemannian conformal structure* on M is an equivalence class $[g]$ of pseudo-Riemannian metrics on M under the equivalence relation $g \sim e^{\lambda} g$ for $\lambda \in C^{\infty}(M)$.

(ii) A linear connection ∇ on M is *compatible* with a pseudo-Riemannian conformal structure if for some (and then any) representative g of the conformal structure there is a one-form $\varphi \in \Omega^1(M)$ such that $\nabla g = \varphi \otimes g$.

Remark 3.2. (i) If $\nabla g = \varphi \otimes g$, then for any conformally equivalent metric we have $\nabla(e^{\lambda} g) = e^{\lambda} \nabla g + de^{\lambda} \otimes g = \varphi \otimes e^{\lambda} g + d\lambda \otimes e^{\lambda} g = (\varphi + d\lambda) \otimes e^{\lambda} g$. This shows the 'and then any' part in the definition of compatible connections (and hence that the notion of compatibility is well-defined).

(ii) Further, this shows that a pair $([g], \nabla)$ of a pseudo-Riemannian conformal structure and a compatible connection defines a specific equivalence class $[g, \varphi]$ of pairs of metrics and one-forms:

Definition 3.3. A *Weyl metric* on a differentiable manifold M is an equivalence class $[g, \varphi]$ of pairs of pseudo-Riemannian metrics and one-forms under the equivalence relation $(g, \varphi) \sim (e^{\lambda} g, \varphi + d\lambda)$ for $\lambda \in C^{\infty}(M)$.

Proposition 3.4. *Let M be a differentiable manifold. The natural map from the set of pairs of a pseudo-Riemannian conformal structure and a torsion-free compatible connection on M to the set of Weyl metrics on M, mapping $([g], \nabla)$ to $[g, \varphi]$ where $\nabla g = \varphi \otimes g$, is a bijection.*

Proof. We have seen above that a conformal structure and a compatible connection determine a well-defined Weyl metric. Conversely, any representative (g, φ) of a Weyl metric defines a unique torsion-free connection ∇ satisfying $\nabla g = \varphi \otimes g$, since torsion-free connections are uniquely determined by their non-metricity with respect to a pseudo-Riemannian metric. By construction, this connection is compatible with $[g]$. Furthermore, the above computation shows that ∇ is already determined by the equivalence class (i.e. the Weyl metric). □

Theorem 3.5 (Weyl 1921)**.** *Let M be a differentiable manifold with* $\dim M > 1$. *Let $[g]$ be a pseudo-Riemannian conformal structure on M. If two torsion-free linear connections $\nabla, \tilde{\nabla}$ are compatible with $[g]$ and projectively equivalent, then they agree, $\nabla = \tilde{\nabla}$.*

Put differently: torsion-free connections that are compatible with a given pseudo-Riemannian conformal structure are uniquely determined by their projective structure.

Proof. Fix some representative g of the conformal structure. By compatibility, there are one-forms $\varphi, \tilde{\varphi}$ such that

$$\nabla g = \varphi \otimes g, \tilde{\nabla} g = \tilde{\varphi} \otimes g. \tag{3.1}$$

By projective equivalence, there is a one-form η such that the difference tensor of the connections is given by

$$(\nabla - \tilde{\nabla})^{\rho}_{\mu\nu} = \delta^{\rho}_{(\mu}\eta_{\nu)} \; . \tag{3.2}$$

Note that due to torsion-freeness, this holds without symmetrisation of the difference in its lower indices.

Combining these equations, we obtain

$$\begin{aligned}
\varphi_\rho g_{\mu\nu} &= \nabla_\rho g_{\mu\nu} \\
&= \tilde{\nabla}_\rho g_{\mu\nu} - (\nabla - \tilde{\nabla})^{\kappa}_{\rho\mu} g_{\kappa\nu} - (\nabla - \tilde{\nabla})^{\kappa}_{\rho\nu} g_{\mu\kappa} \\
&= \tilde{\varphi}_\rho g_{\mu\nu} - \delta^{\kappa}_{(\rho}\eta_{\mu)} g_{\kappa\nu} - \delta^{\kappa}_{(\rho}\eta_{\nu)} g_{\mu\kappa} \\
&= \tilde{\varphi}_\rho g_{\mu\nu} - \eta_{(\mu} g_{\rho)\nu} - g_{\mu(\rho}\eta_{\nu)} \\
&= \tilde{\varphi}_\rho g_{\mu\nu} - \eta_\rho g_{\mu\nu} - g_{\rho(\mu}\eta_{\nu)} \; .
\end{aligned} \tag{3.3}$$

Contracting (3.3) with $g^{\mu\nu}$ yields

$$n\varphi_\rho = n\tilde{\varphi}_\rho - n\eta_\rho - g_{\rho\mu}g^{\mu\nu}\eta_\nu = n\tilde{\varphi}_\rho - (n+1)\eta_\rho \; , \tag{3.4}$$

where $n = \dim M$. On the other hand, contracting (3.3) with $g^{\mu\rho}$ yields

$$\varphi_\nu = \tilde{\varphi}_\nu - \eta_\nu - g^{\mu\rho}\frac{1}{2}(g_{\rho\mu}\eta_\nu + g_{\rho\nu}\eta_\mu) = \tilde{\varphi}_\nu - \eta_\nu - \frac{n+1}{2}\eta_\nu = \tilde{\varphi}_\nu - \frac{n+3}{2}\eta_\nu \; . \tag{3.5}$$

Subtracting $n \cdot$ (3.5) from (3.4) (and renaming the free index), we have

$$0 = -(n+1)\eta_\nu + \frac{n(n+3)}{2}\eta_\nu = \frac{n^2+n-2}{2}\eta_\nu \; . \tag{3.6}$$

The expression $n^2 + n - 2$ vanishes if and only if $n \in \{-2, 1\}$. Thus, since we assumed $n > 1$, we obtain $\eta_\nu = 0$, finishing the proof. □

Remark 3.6. (i) Since Weyl metrics are equivalently determined by their conformal structure and their associated torsion-free connection, Weyl's theorem may also be formulated as the statement that *Weyl metrics are uniquely determined by their conformal and projective structures*. This is the original formulation of the theorem as given by Weyl.

(ii) Malament (2012, Proposition 2.1.4) is a special case of this formulation of Weyl's theorem for pseudo-Riemannian metrics: *pseudo-Riemannian metrics are determined by their conformal and projective structure up to multiplication by a locally constant function* (where the projective structure is that of the metric's Levi-Civita connection). This follows from Weyl's theorem since the Levi-Civita connection ∇ of g is the torsion-free connection corresponding to the Weyl metric $[g, 0]$, and an *integrable* Weyl metric—i.e. one such that for all representatives $(\tilde{g}, \tilde{\varphi})$ the one-form $\tilde{\varphi}$ is exact—determines a pseudo-Riemannian metric up to a locally constant factor (namely any metric $\tilde{g}$ such that $(\tilde{g}, 0)$ is a representative).

(iii) Note that for the equivalence between Weyl metrics and pairs of conformal structures and torsion-free compatible connections, it was crucial that a torsion-free connection be uniquely determined by its non-metricity. If this were not the case, several connections could correspond to the same Weyl metric. However, *Weyl's theorem does not need this one-to-one correspondence, as long as it is understood as a statement about uniqueness of connections*. This will become important in the Galilei and Carroll cases: our Weyl-type theorems will be concerned with uniqueness of connections. Only when adding extra data on top of the non-metricity form φ (and part of the torsion, see below), they may be understood as statements on the uniqueness of (suitably defined) analogues of Weyl metrics.

Construction 3.7. We may include torsion into this picture as follows. For two potentially *torsionful* connections ∇ and $\tilde{\nabla}$, if we assume their torsion to agree, $T = \tilde{T}$, this means that their difference tensor is symmetric, $(\nabla - \tilde{\nabla})^{\rho}_{\mu\nu} = (\nabla - \tilde{\nabla})^{\rho}_{(\mu\nu)}$. Since the proof of Weyl's theorem (theorem 3.5) as presented above used the assumption of torsion-freeness only for this equality, it applies in exactly the same manner to the case of two connections with equal torsion.

Thus, we obtain the following torsionful generalisation of Weyl's theorem:

Theorem 3.8 (Weyl's theorem with torsion)**.** *Let M be a differentiable manifold with* $\dim M > 1$. *Let $[g]$ be a pseudo-Riemannian conformal structure on M. Let $\nabla, \tilde{\nabla}$ be two linear connections whose torsions agree, $T = \tilde{T}$. If the connections are compatible with $[g]$ and projectively equivalent, then they agree, $\nabla = \tilde{\nabla}$.*

Put differently: connections that are compatible with a given pseudo-Riemannian conformal structure are uniquely determined by their torsion and their projective structure. □

Remark 3.9. If we define a *torsional Weyl metric* as a pair $([g,\varphi], T)$ of a usual Weyl metric $[g,\varphi]$ and a tensor field $T \in \Gamma(TM \otimes \bigwedge^2 T^*M)$, then these are in natural bijection to pairs $([g], \nabla)$ of pseudo-Riemannian conformal structures and compatible connections (possibly with torsion). The above torsionful generalisation of Weyl's theorem may then be formulated as the statement that *torsional Weyl metrics are uniquely determined by their torsion together with their conformal and projective structures*.

4. The Galilei case

Our next task is to develop a parallel story to the above for the case of Galilei geometry, on which see Cartan (1923, 1924), Dombrowski and Horneffer (1964), Ehlers (1981, 2019), Friedrichs (1928), Hartong et al. (2023), Künzle (1972, 1976), Malament (2012), Schwartz (2026), and Trautman (1963, 1965). This builds upon prior work by Adlam et al. (2025), Curiel (2015), Ewen and Schmidt (1989), and March (2023, 2025).

4.1. Prerequisites

Definition 4.1. Let M be a differentiable manifold with $\dim M > 1$.

(i) A *Galilei structure*[5] on M is a pair (τ, h) of a nowhere-vanishing *clock form* $\tau \in \Omega^1(M)$ and a *space metric* $h \in \Gamma(\bigvee^2 TM)$ which is positive semidefinite of rank $n = \dim M - 1$ such that τ spans the degenerate direction of h, i.e.

$$\tau_\mu h^{\mu\nu} = 0. \tag{4.1}$$

(ii) A *conformal Galilei structure* on M is an equivalence class $[\tau, h]$ of Galilei structures under the equivalence relation $(\tau, h) \sim (e^{\frac{1}{2}\lambda}\tau, e^{-\lambda}h)$ for $\lambda \in C^\infty(M)$.

(iii) A linear connection ∇ on M is *compatible* with a conformal Galilei structure if for some (and then any) representative (τ, h) of the conformal structure there is a one-form $\varphi \in \Omega^1(M)$ such that $\nabla\tau = \frac{1}{2}\varphi \otimes \tau$, $\nabla h = -\varphi \otimes h$.

Remark 4.2. (i) We chose here a definition of conformal equivalence that uses related conformal factors for the clock form and space metric, as it would result from a Newtonian limit of a Lorentzian conformal transformation. This is also the reason for the definition of compatibility using related one-forms.

One could try to relax this relation (i.e. use independent conformal equivalence classes of clock forms and space metrics—see Adlam et al. (2025), Curiel (2015), and March (2023)). However, we try to stay as close as possible to the Lorentzian origin of Galilei geometry here. Furthermore, our Weyl-type theorem needs this stronger notion of conformal structure.

(ii) As in the pseudo-Riemannian case, the one-forms encoding the non-metricity of a compatible connection with respect to different representatives conformally related by λ are related by $\varphi \mapsto \varphi + d\lambda$.

(iii) In Galilei geometry, the non-metricities of a connection need to satisfy two conditions (Schwartz 2025). The one relating only the non-metricities, namely $\tau_\mu \nabla_\rho h^{\mu\nu} = -h^{\mu\nu}\nabla_\rho \tau_\mu$, is satisfied for the non-metricities from the definition of compatibility. The second condition involves the torsion: it reads

$$\tau_\rho T^\rho{}_{\mu\nu} = (d\tau)_{\mu\nu} - 2\nabla_{[\mu}\tau_{\nu]} \,. \tag{4.2a}$$

For the above non-metricity, this becomes

$$\tau(T(\cdot,\cdot)) = d\tau - \frac{1}{2}\varphi \wedge \tau. \tag{4.2b}$$

---

[5] We use here the name 'Galilei structure' for the pair of two 'metric' tensor fields of a Galilei manifold, while this term commonly refers to the reduction of the structure group of the frame bundle induced by them (i.e. the Galilei frame bundle). Since we make no use of frame bundles in this article, this should not cause confusion.

> This shows that connections compatible with a conformal Galilei structure can only be torsion-free if τ satisfies the Frobenius integrability condition, and even then the non-metricity form φ is related to $\mathrm{d}\tau$. Thus in general compatible connections are not torsion-free.

Since—differently to the pseudo-Riemannian case—we cannot in general demand that compatible connections be torsion-free, we need to allow for torsion. However, we can hope for the uniqueness of connections in a Galilean Weyl-type theorem only if we (partially) restrict the torsion, similar to the torsionful version of Weyl's theorem in the pseudo-Riemannian case (theorem 3.8). We are going to fix the 'spatial' part of the torsion with respect to some specified timelike direction, which, according to the classification result of Schwartz (2025), is precisely the freely specifiable part of the torsion:

Definition 4.3. Given a conformal Galilei structure $[\tau, h]$ on a differentiable manifold M, the notion of a *timelike direction* (i.e. a timelike vector field up to rescaling) with respect to $[\tau, h]$ is well-defined: it is an equivalence class $[v]$ of timelike vector fields, $\tau(v) \neq 0$, under the equivalence relation $v \sim \tilde{\lambda} v$ for $\tilde{\lambda} \in C^\infty(M, \mathbb{R} \setminus \{0\})$. Given a timelike direction $[v]$, there is a well-defined unique *spatial projector along* $[v]$, the vector bundle endomorphism P of TM defined by projecting onto $\ker \tau$ (which is invariant under rescaling of τ) and having kernel spanned by v (which is invariant under rescaling of v).

Fixing a representative (τ, h) of the conformal structure, there is a unique representative v of the timelike direction satisfying $\tau(v) = 1$, and in terms of these the spatial projector takes the usual explicit form $P^\mu_\nu = \delta^\mu_\nu - v^\mu \tau_\nu$.

4.2. The Galilei Weyl-type theorem

With these prerequisites, we can now prove a Weyl-type theorem:

Theorem 4.4. *Let M be a differentiable manifold with $\dim M > 2$. Let $[\tau, h]$ be a conformal Galilei structure on M, and $[v]$ a timelike direction with respect to $[\tau, h]$. Let $\nabla, \tilde{\nabla}$ be two linear connections whose spatial torsions with respect to $[v]$ agree, i.e. their torsions satisfy $P(T(\cdot,\cdot)) = P(\tilde{T}(\cdot,\cdot))$ where P is the spatial projector along $[v]$. If the connections are compatible with $[\tau, h]$ and projectively equivalent, then they agree, $\nabla = \tilde{\nabla}$.*

Proof. Fix some representative (τ, h) of the conformal structure, and let v be the corresponding representative of the timelike direction satisfying $\tau(v) = 1$. Denote the difference tensor of the connections by $D := \nabla - \tilde{\nabla}$.

By projective equivalence, there is a one-form η such that the difference tensor satisfies

$$D^\rho_{(\mu\nu)} = \delta^\rho_{(\mu} \eta_{\nu)} \,. \tag{4.3}$$

Furthermore, both connections having the same spatial torsion with respect to $[v]$ is equivalent to

$$P^{\sigma}_{\rho} D^{\rho}_{[\mu\nu]} = 0, \tag{4.4a}$$

implying

$$P^{\sigma}_{\rho} D^{\rho}_{\mu\nu} = P^{\sigma}_{\rho} D^{\rho}_{(\mu\nu)} = P^{\sigma}_{(\mu}\eta_{\nu)} \ . \tag{4.4b}$$

By compatibility of the connections with $[\tau, h]$, there are one-forms $\varphi, \tilde{\varphi}$ such that

$$\nabla\tau = \frac{1}{2}\varphi \otimes \tau, \tilde{\nabla}\tau = \frac{1}{2}\tilde{\varphi} \otimes \tau, \nabla h = -\varphi \otimes h, \tilde{\nabla} h = -\tilde{\varphi} \otimes h. \tag{4.5}$$

Combining the two equations for compatibility with τ, we obtain

$$\begin{aligned} \frac{1}{2}\varphi_\mu \tau_\nu &= \nabla_\mu \tau_\nu \\ &= \tilde{\nabla}_\mu \tau_\nu - D^{\rho}_{\mu\nu}\tau_\rho \\ &= \frac{1}{2}\tilde{\varphi}_\mu \tau_\nu - D^{\rho}_{\mu\nu}\tau_\rho \ . \end{aligned} \tag{4.6}$$

Contracting with $h^{\nu\sigma}$, from this we obtain

$$\tau_\rho D^{\rho}_{\mu\nu} h^{\nu\sigma} = 0. \tag{4.7}$$

Symmetrising the free indices of (4.6) yields

$$\begin{aligned} \frac{1}{2}\varphi_{(\mu}\tau_{\nu)} &= \frac{1}{2}\tilde{\varphi}_{(\mu}\tau_{\nu)} - D^{\rho}_{(\mu\nu)}\tau_\rho \\ &= \frac{1}{2}\tilde{\varphi}_{(\mu}\tau_{\nu)} - \tau_{(\mu}\eta_{\nu)} \ , \end{aligned} \tag{4.8}$$

where we have used (4.3) for the symmetrised difference tensor. Contracting this with $v^\mu v^\nu$, we obtain

$$\frac{1}{2}\varphi(v) = \frac{1}{2}\tilde{\varphi}(v) - \eta(v). \tag{4.9}$$

On the other hand, combining the two equations for compatibility with h, we obtain

$$\begin{aligned} -\varphi_\rho h^{\mu\nu} &= \nabla_\rho h^{\mu\nu} \\ &= \tilde{\nabla}_\rho h^{\mu\nu} + D^{\mu}_{\rho\kappa} h^{\kappa\nu} + D^{\nu}_{\rho\kappa} h^{\mu\kappa} \\ &= -\tilde{\varphi}_\rho h^{\mu\nu} + 2 D^{(\mu}_{\rho\kappa} h^{\nu)\kappa}. \end{aligned} \tag{4.10}$$

In order to simplify this equation, we consider its last term. From (4.7), we know that the expression $D^{\rho}_{\mu\nu} h^{\nu\sigma}$ vanishes when contracted with τ_ρ; i.e. the index ρ is spacelike. Therefore, the expression is equal to its projection onto space along $[v]$ on this index,

$$D^{\rho}_{\mu\nu} h^{\nu\sigma} = P^{\rho}_{\lambda} D^{\lambda}_{\mu\nu} h^{\nu\sigma}. \tag{4.11a}$$

But the spacelike projected difference tensor can be expressed in terms of η according to (4.4b), giving

$$\begin{aligned} D^{\rho}_{\mu\nu} h^{\nu\sigma} &= P^{\rho}_{(\mu}\eta_{\nu)} h^{\nu\sigma} \\ &= \frac{1}{2}(P^{\rho}_{\mu}\eta_{\nu} h^{\nu\sigma} + P^{\rho}_{\nu}\eta_{\mu} h^{\nu\sigma}) \\ &= \frac{1}{2}(P^{\rho}_{\mu}\eta^{\sigma} + \eta_{\mu} h^{\rho\sigma})\,, \end{aligned} \tag{4.11b}$$

where $\eta^{\mu} := h^{\mu\nu}\eta_{\nu}$ is the spacelike vector field corresponding to η. Now using (4.11b), the compatibility equation (4.10) takes the form

$$-\varphi_{\rho} h^{\mu\nu} = -\tilde{\varphi}_{\rho} h^{\mu\nu} + P^{(\mu}_{\rho}\eta^{\nu)} + \eta_{\rho} h^{\mu\nu}. \tag{4.12}$$

Contracting this with v^{ρ} gives

$$-\varphi(v) h^{\mu\nu} = -\tilde{\varphi}(v) h^{\mu\nu} + \eta(v) h^{\mu\nu}, \tag{4.13a}$$

i.e.

$$-\varphi(v) = -\tilde{\varphi}(v) + \eta(v). \tag{4.13b}$$

Adding $2 \cdot$ (4.9) to (4.13b), we obtain

$$0 = -\eta(v). \tag{4.14}$$

Contracting (4.12) with $\gamma_{\mu\nu}$—the components of the covariant space metric[6] with respect to v, defined by the requirements $\gamma_{\mu\nu} = \gamma_{\nu\mu}$, $\gamma_{\mu\nu} v^{\nu} = 0$, and $\gamma_{\mu\nu} h^{\nu\rho} = P^{\rho}_{\mu}$—yields

$$-n\varphi_{\rho} = -n\tilde{\varphi}_{\rho} + P^{\mu}_{\rho}\eta_{\mu} + n\eta_{\rho}\,, \tag{4.15a}$$

where $n = \dim M - 1$. Contracting this with $h^{\rho\nu}$, we obtain

$$-n\varphi^{\nu} = -n\tilde{\varphi}^{\nu} + (n+1)\eta^{\nu}. \tag{4.15b}$$

On the other hand, contracting (4.12) with δ^{ρ}_{μ} yields

$$-\varphi^{\nu} = -\tilde{\varphi}^{\nu} + \delta^{\rho}_{\mu}\frac{1}{2}(P^{\mu}_{\rho}\eta^{\nu} + P^{\nu}_{\rho}\eta^{\mu}) + \eta^{\nu} = -\tilde{\varphi}^{\nu} + \frac{n+1}{2}\eta^{\nu} + \eta^{\nu} = -\tilde{\varphi}^{\nu} + \frac{n+3}{2}\eta^{\nu}. \tag{4.16}$$

Subtracting (4.15b) from $n \cdot$ (4.16), we have

$$0 = \frac{n(n+3)}{2}\eta^{\nu} - (n+1)\eta^{\nu} = \frac{n^2+n-2}{2}\eta^{\nu}. \tag{4.17}$$

The expression $n^2 + n - 2$ vanishes if and only if $n \in \{-2, 1\}$. Thus, since we assumed $n > 1$, we obtain

$$\eta^{\nu} = 0. \tag{4.18}$$

---

[6] In the literature on Galilei geometry / Newton–Cartan gravity, the components of the covariant space metric are commonly denoted by $h_{\mu\nu}$. We choose here the symbol γ instead, in parallel to Carroll geometry; cf. Vigneron et al. (2025).

Since we can express η as $\eta_\mu = (v^\rho \tau_\mu + P^\rho_\mu)\eta_\rho = \eta(v)\tau_\mu + \gamma_{\mu\nu} h^{\nu\rho} \eta_\rho = \eta(v)\tau_\mu + \gamma_{\mu\nu}\eta^\nu$, together with (4.14) this shows $\eta = 0$. Hence the symmetrised part of the difference tensor vanishes,

$$D^\rho_{(\mu\nu)} = 0, \tag{4.19}$$

and due to (4.12) we have

$$\varphi = \tilde{\varphi}. \tag{4.20}$$

The antisymmetric part of the difference D is given by (one half of) the difference of the torsions of the two connections,

$$D^\rho_{[\mu\nu]} = \frac{1}{2}(T^\rho{}_{\mu\nu} - \tilde{T}^\rho{}_{\mu\nu}). \tag{4.21a}$$

Due to the assumption of agreeing spatial torsion with respect to $[v]$, this is given by

$$D^\rho_{[\mu\nu]} = \frac{1}{2} v^\rho \tau_\sigma (T^\sigma{}_{\mu\nu} - \tilde{T}^\sigma{}_{\mu\nu}), \tag{4.21b}$$

i.e. by the difference of the temporal torsions of the two connections. However, according to (4.2b) the temporal torsions may be expressed in terms of $\mathrm{d}\tau$ and $\varphi, \tilde{\varphi}$. Thus, due to $\varphi = \tilde{\varphi}$, the temporal torsions of ∇ and $\tilde{\nabla}$ agree. This shows $D = 0$, finishing the proof. □

Remark 4.5. Note that the proof of the vanishing of the spacelike part of η was directly analogous to the pseudo-Riemannian case. In fact, if we had assumed $\tau \wedge \mathrm{d}\tau = 0$ (which is invariant under conformal rescaling of τ), we could have appealed directly to the Riemannian version of Weyl's theorem for this conclusion: $\tau \wedge \mathrm{d}\tau = 0$ is equivalent to the spacelike distribution $\ker \tau$ being integrable by n-dimensional 'spatial leaves' Σ, on each of which $[h]$ then induces a Riemannian conformal structure. The connections $\nabla, \tilde{\nabla}$ then induce 'spatial connections' on these leaves, both of which are compatible with the Riemannian conformal structure. Further, the spatial connections' geodesics are precisely the spacelike geodesics of the spacetime connections, which agree up to parametrisation; i.e. on each spatial leaf the two spatial connections are projectively equivalent. Thus, Weyl's theorem implies that the spatial connections agree.

4.3. A Galilei analogue of Weyl metrics

We may define an analogue of Weyl metrics for Galilei geometry as follows.

Construction 4.6. Fixing a Galilei structure (τ, h) on M, with respect to a choice of unit timelike vector field $v \in \Gamma(TM)$, $\tau(v) = 1$, any linear connection ∇ on M is uniquely determined by—and uniquely determines, vice versa—its τ-non-metricity $\nabla\tau$, the spatial part $P^\rho_\mu P^\sigma_\nu \nabla_\kappa h^{\mu\nu}$ of its h-non-metricity, its spatial torsion $(PT)^\sigma{}_{\mu\nu} := P^\sigma_\rho T^\rho{}_{\mu\nu}$, and its Newton–Coriolis form $\Omega_{\mu\nu} = 2(\nabla_{[\mu} v^\rho)\gamma_{\nu]\rho}$; see Schwartz (2025). Assuming the non-metricities to be of the form $\nabla\tau = \frac{1}{2}\varphi \otimes \tau$, $\nabla h = -\varphi \otimes h$, such that ∇ is compatible with the conformal Galilei structure $[\tau, h]$, we already know that fixing the connection ∇, under a conformal rescaling $(\tau, h) \mapsto (\mathrm{e}^{\frac{1}{2}\lambda}\tau, \mathrm{e}^{-\lambda}h)$ with $\lambda \in C^\infty(M)$ the one-form φ transforms as $\varphi \mapsto \varphi + \mathrm{d}\lambda$. Further, fixing the timelike direction $[v]$, the spatial projector P along it stays the same, such that the spatial torsion PT stays invariant. Rescaling $v \mapsto \mathrm{e}^{-\frac{1}{2}\lambda}$ such that it remains unit timelike with respect to the new representative of the conformal structure, a direct computation shows that the Newton–Coriolis form of ∇ with respect to v transforms as $\Omega \mapsto \mathrm{e}^{-\frac{1}{2}\lambda}\Omega$.

Therefore, a conformal Galilei structure $[\tau, h]$ on M together with a compatible linear connection is equivalently characterised by an equivalence class $[\tau, h, \varphi, v, PT, \Omega]$ of Galilei structures (τ, h), one-forms φ, unit timelike vector fields v with respect to τ, tensor fields $PT \in \Gamma(\ker\tau \otimes \bigwedge^2 T^*M)$, and two-forms Ω on M under the equivalence relation

$$(\tau, h, \varphi, v, PT, \Omega) \sim (\mathrm{e}^{\frac{1}{2}\lambda}\tau, \mathrm{e}^{-\lambda}h, \varphi + \mathrm{d}\lambda, \mathrm{e}^{-\frac{1}{2}\lambda}v, PT, \mathrm{e}^{-\frac{1}{2}\lambda}\Omega) \tag{4.22}$$

for $\lambda \in C^\infty(M)$.

Alternatively, since v is uniquely determined by the timelike direction $[v]$ once τ is fixed, and PT is invariant, these are the same data as a triple $([\tau, h, \varphi, \Omega], [v], PT)$ consisting of (i) an equivalence class of Galilei structures, one-forms, and two-forms on M under

$$(\tau, h, \varphi, \Omega) \sim (\mathrm{e}^{\frac{1}{2}\lambda}\tau, \mathrm{e}^{-\lambda}h, \varphi + \mathrm{d}\lambda, \mathrm{e}^{-\frac{1}{2}\lambda}\Omega); \tag{4.23}$$

(ii) a timelike direction $[v]$ with respect to the conformal Galilei structure $[\tau, h]$; and (iii) a tensor field $PT \in \Gamma(\ker\tau \otimes \bigwedge^2 T^*M)$.

Thus, we may reformulate our Weyl-type theorem for Galilei geometry (theorem 4.4) as the statement that *such a 'Galilei Weyl metric'* $([\tau, h, \varphi, \Omega], [v], PT)$ *is uniquely determined by its conformal and projective structures together with its timelike direction* $[v]$ *and its spatial torsion* PT *with respect to* $[v]$.

Note that differently to the pseudo-Riemannian case, for a 'Galilei Weyl metric' we always include the freely specifiable torsion PT as part of the data, since its vanishing is not a meaningful invariant statement: the same connection may have $PT = 0$ with respect to some choice $[v]$ of timelike direction, but $\tilde{P}T \neq 0$ with respect to some other direction $[\tilde{v}]$.

Remark 4.7. The above formulation of the notion of a 'Galilei Weyl metric' corrects and generalises the notion of an 'NR–Weyl manifold' presented by Adlam et al. (2025, p. 170), in which the data v, PT, Ω are not included.

4.4. Necessity of the dimensionality assumption

We are now going to show that the 'additional' assumption in theorem 4.4 as compared to the torsionful version of Weyl's theorem in the pseudo-Riemannian case, namely that $\dim M > 2$, is necessary for the conclusion of agreeing connections: we are going to show that theorem 4.4 fails to apply if $\dim M = 2$, and in fact precisely classify *how* it fails.

Proposition 4.8. *Let M be a differentiable manifold with* $\dim M = 2$. *Let $[\tau, h]$ be a conformal Galilei structure on M, and $[v]$ a timelike direction with respect to $[\tau, h]$.*

(i) Let $\nabla, \tilde{\nabla}$ be two linear connections whose spatial torsions with respect to $[v]$ agree, i.e. their torsions satisfy $P(T(\cdot,\cdot)) = P(\tilde{T}(\cdot,\cdot))$ where P is the spatial projector along $[v]$.

If the connections are compatible with $[\tau, h]$ and projectively equivalent, then their difference tensor $D := \nabla - \tilde{\nabla}$ is given by

$$D^{\rho}_{\mu\nu} = \eta_{\mu}\delta^{\rho}_{\nu} - \eta_{[\mu}P^{\rho}{}_{\nu]} \tag{4.24a}$$

for a one-form $\eta \in \Omega^1(M)$ that is spacelike with respect to $[v]$, i.e. that satisfies

$$\eta(v) = 0. \tag{4.24b}$$

(ii) Conversely, let ∇ be a linear connection that is compatible with $[\tau, h]$. Then for any *choice of η with $\eta(v) = 0$, the connection $\tilde{\nabla} := \nabla - D$ with difference D given by (4.24a) is compatible with $[\tau, h]$ and has the same spatial torsion with respect to $[v]$ as ∇, and the connections are projectively equivalent.*

Proof. Fix some representative (τ, h) of the conformal structure, and let v be the corresponding representative of the timelike direction satisfying $\tau(v) = 1$.

(i) As in the proof of theorem 4.4, we denote the one-form determining the symmetric part of the difference tensor $D = \nabla - \tilde{\nabla}$ by η, and the one-forms determining the non-metricities of ∇ and $\tilde{\nabla}$ by φ and $\tilde{\varphi}$, respectively.

In the proof of theorem 4.4, we did not use any assumption on $\dim M$ in order to show that

$$\eta(v) = 0 \text{ and } \varphi(v) = \tilde{\varphi}(v). \tag{4.25a}$$

Therefore, these still hold in the present case $\dim M = 2$. However, the spatial part $\eta^{\mu} = h^{\mu\nu}\eta_{\nu}$ of η need not vanish in the case $n = \dim M - 1 = 1$: both (4.15b) and (4.16) become

$$\varphi^{\mu} = \tilde{\varphi}^{\mu} - 2\eta^{\mu} \tag{4.25b}$$

in this case. Together, these equations show that for $\dim M = 2$ we have

$$\varphi = \tilde{\varphi} - 2\eta, \tag{4.26}$$

with η spacelike with respect to $[v]$.

Further, using (4.2b) this implies that the difference of the temporal torsions of ∇ and $\tilde{\nabla}$ is given by

$$\tau_{\rho}(T^{\rho}{}_{\mu\nu} - \tilde{T}^{\rho}{}_{\mu\nu}) = -(\varphi_{[\mu} - \tilde{\varphi}_{[\mu})\tau_{\nu]} = 2\eta_{[\mu}\tau_{\nu]}\ . \tag{4.27}$$

Therefore, using (4.21b), the antisymmetric part of the difference tensor is $D^\rho_{[\mu\nu]} = v^\rho \eta_{[\mu}\tau_{\nu]}$. Together with the equation determining the symmetric part of D in terms of η, we thus obtain

$$D^\rho_{\mu\nu} = \delta^\rho_{(\mu}\eta_{\nu)} + \eta_{[\mu}v^\rho\tau_{\nu]} = \delta^\rho_{(\mu}\eta_{\nu)} + \eta_{[\mu}(\delta^\rho_{\nu]} - P^\rho_{\nu]}) = \eta_\mu\delta^\rho_\nu - \eta_{[\mu}P^\rho_{\nu]} \tag{4.28}$$

as stated.

(ii) Again, we denote the one-form determining the non-metricities of ∇ by φ, such that we have

$$\nabla\tau = \frac{1}{2}\varphi \otimes \tau, \nabla h = -\varphi \otimes h. \tag{4.29}$$

Let $\eta \in \Omega^1(M)$ be an arbitrary one-form satisfying $\eta(v) = 0$, and define $\tilde{\nabla} := \nabla - D$ with D given by (4.24a).

Directly from (4.24a), we obtain

$$D^\rho_{(\mu\nu)} = \delta^\rho_{(\mu}\eta_{\nu)} , \tag{4.30}$$

showing projective equivalence of ∇ and $\tilde{\nabla}$. Further, the difference of their spatial torsions with respect to $[v]$ is

$$P^\sigma_\rho D^\rho_{[\mu\nu]} = P^\sigma_\rho\eta_{[\mu}(\delta^\rho_{\nu]} - P^\rho_{\nu]}) = P^\sigma_\rho\eta_{[\mu}v^\rho\tau_{\nu]} = 0, \tag{4.31}$$

i.e. the spatial torsions agree.

Using (4.24a), we may compute

$$\begin{aligned}\tilde{\nabla}_\mu\tau_\nu &= \nabla_\mu\tau_\nu + D^\rho_{\mu\nu}\tau_\rho \\ &= \frac{1}{2}\varphi_\mu\tau_\nu + (\eta_\mu\delta^\rho_\nu - \eta_{[\mu}P^\rho_{\nu]})\tau_\rho \\ &= \frac{1}{2}(\varphi_\mu + 2\eta_\mu)\tau_\nu\end{aligned} \tag{4.32a}$$

and

$$\begin{aligned}\tilde{\nabla}_\mu h^{\rho\sigma} &= \nabla_\mu h^{\rho\sigma} - D^\rho_{\mu\nu}h^{\nu\sigma} - D^\sigma_{\mu\nu}h^{\rho\nu} \\ &= -\varphi_\mu h^{\rho\sigma} - (\eta_\mu\delta^\rho_\nu - \eta_{[\mu}P^\rho_{\nu]})h^{\nu\sigma} - (\eta_\mu\delta^\sigma_\nu - \eta_{[\mu}P^\sigma_{\nu]})h^{\rho\nu} \\ &= -\varphi_\mu h^{\rho\sigma} - \eta_{(\mu}P^\rho_{\nu)}h^{\nu\sigma} - \eta_{(\mu}P^\sigma_{\nu)}h^{\rho\nu} \\ &= -\varphi_\mu h^{\rho\sigma} - \frac{1}{2}(\eta_\mu h^{\rho\sigma} + \eta^\sigma P^\rho_\mu) - \frac{1}{2}(\eta_\mu h^{\rho\sigma} + \eta^\rho P^\sigma_\mu) \\ &= -(\varphi_\mu + \eta_\mu)h^{\rho\sigma} - P^{(\rho}_\mu\eta^{\sigma)} .\end{aligned} \tag{4.32b}$$

Due to $\dim M = 2$, i.e. there being only one spatial direction, and using that η is spacelike, we have $P^{(\rho}_\mu\eta^{\sigma)} = \eta_\mu h^{\rho\sigma}$. Hence, the previous equation becomes

$$\tilde{\nabla}_\mu h^{\rho\sigma} = -(\varphi_\mu + 2\eta_\mu)h^{\rho\sigma} . \tag{4.32c}$$

This shows compatibility of $\tilde{\nabla}$ with $[\tau, h]$, as claimed. □

Remark 4.9. In the situation of proposition 4.8, a direct computation shows that the two connections ∇ and $\tilde{\nabla}$ with difference tensor (4.24a) have agreeing Newton–Coriolis forms $\Omega = \tilde{\Omega}$ with respect to a choice v of representative of the timelike direction. Thus, we have shown that *on a 2-dimensional manifold with a conformal Galilei structure* $[\tau, h]$*, when fixing the spatial torsion with respect to* $[v]$ *and the projective structure of a connection compatible with* $[\tau, h]$*, the remaining freedom is precisely the spatial part* $P^{\mu}_{\nu}\varphi_{\mu}$ *of the one-form* φ *encoding the connection's non-metricities.*

5. The Carroll case

Here we develop a parallel story for the case of Carroll geometry, on which see Bergshoeff et al. (2014, 2017), Duval et al. (2014b), and March and Read (2025).

5.1. Prerequisites

Definition 5.1. Let M be a differentiable manifold with $\dim M > 1$.

(i) A *Carroll structure* on M is a pair (v, γ) of a nowhere-vanishing vector field $v \in \Gamma(TM)$ and a positive-semidefinite $\gamma \in \Gamma(\bigvee^2 T^*M)$ of rank $n = \dim M - 1$ such that v spans the degenerate direction of γ, i.e.

$$v^{\mu}\gamma_{\mu\nu} = 0. \tag{5.1}$$

(ii) A *conformal Carroll structure* on M is an equivalence class $[v, \gamma]$ of Carroll structures under the equivalence relation $(v, \gamma) \sim (\mathrm{e}^{-\frac{1}{2}\lambda}v, \mathrm{e}^{\lambda}\gamma)$ for $\lambda \in C^{\infty}(M)$.

(iii) A linear connection ∇ on M is *compatible* with a conformal Carroll structure if for some (and then any) representative (v, γ) of the conformal structure there is a one-form $\varphi \in \Omega^1(M)$ such that $\nabla v = -\frac{1}{2}\varphi \otimes v$, $\nabla\gamma = \varphi \otimes \gamma$.

Remark 5.2. (i) As in the Galilei case, we chose a definition of conformal equivalence that uses related factors for v and γ, as this would result from the Carroll limit of a Lorentzian conformal transformation. This is also the reason for the definition of compatibility using related one-forms. This definition of conformal Carroll structures is also used, e.g., by Herfray (2022).

As in the Galilei case, one could seek to relax this assumption.

(ii) As in the pseudo-Riemannian and Galilei cases, the one-forms encoding the non-metricity of a compatible connection with respect to different representatives conformally related by λ are related by $\varphi \mapsto \varphi + \mathrm{d}\lambda$.

(iii) As in the Galilei case, the non-metricities of a connection with respect to a Carroll structure need to satisfy two conditions, see Vigneron et al. (2025). The one relating only the non-metricities, namely $v^\mu \nabla_\rho \gamma_{\mu\nu} = -\gamma_{\mu\nu} \nabla_\rho v^\mu$, is satisfied for the non-metricities from the definition of compatibility. The second condition involves the torsion: it reads

$$T_{(\mu\nu)\rho} v^\rho = \frac{1}{2} v^\rho \nabla_\rho \gamma_{\mu\nu} + \gamma_{\rho(\mu} \nabla_{\nu)} v^\rho - \frac{1}{2} (\mathcal{L}_v \gamma)_{\mu\nu} \,, \tag{5.2a}$$

where the first index on the torsion has been lowered with γ. For the above non-metricities, this becomes

$$T_{(\mu\nu)\rho} v^\rho = \frac{1}{2} \varphi(v) \gamma_{\mu\nu} - \frac{1}{2} (\mathcal{L}_v \gamma)_{\mu\nu} \,. \tag{5.2b}$$

Therefore, as in the Galilei case, we have to allow for torsionful connections in order to formulate a Carrollian analogue of Weyl's theorem.

In order to fix the free parts of the torsion, we again need a notion of timelike direction, or more specifically a co-direction in this case:

Definition 5.3. Given a conformal Carroll structure $[v, \gamma]$ on a differentiable manifold M, the notion of a *timelike co-direction* (i.e. a timelike one-form up to rescaling) with respect to $[v, \gamma]$ is well-defined: it is an equivalence class $[\tau]$ of timelike one-forms, $\tau(v) \neq 0$, under the equivalence relation $\tau \sim \tilde{\lambda} \tau$ for $\tilde{\lambda} \in C^\infty(M, \mathbb{R} \setminus \{0\})$. Given a timelike co-direction $[\tau]$, there is a well-defined unique *spatial projector along* $[\tau]$, the vector bundle endomorphism P of T^*M defined by projecting onto $\ker v$ (which is invariant under rescaling of v) and having kernel spanned by τ (which is invariant under rescaling of τ).

Fixing a representative (v, γ) of the conformal structure, there is a unique representative τ of the timelike co-direction satisfying $\tau(v) = 1$, and in terms of these the spatial projector takes the usual explicit form $P^\mu_\nu = \delta^\mu_\nu - v^\mu \tau_\nu$.

Construction 5.4. Let $[v, \gamma]$ be a conformal Carroll structure on M, and $[\tau]$ a timelike co-direction with respect to it. Further, let P be the spatial projector along $[\tau]$. Fix some representative (v, γ) of the conformal structure.

By (5.2), the components of a connection's torsion that are determined by the non-metricities are those of the form $T_{(\mu\nu)\rho} v^\rho$: we lower the first index with γ, symmetrise the first two indices, and take the temporal component on the third index (with respect to (v, γ)). All other components of the torsion not determined by these may be freely specified in the definition of a connection (see appendix A for a detailed discussion of the classification of connections on Carroll manifolds, based on Vigneron et al. (2025)).

The first set of freely specifiable torsion components is given by

$$T_{[\mu\nu]\rho} v^\rho \,, \tag{5.3a}$$

where we still take the temporal component on the third index, but lower the first index with γ and then *anti*symmetrise the first two indices. Together with the components $T_{(\mu\nu)\rho} v^\rho$ determined by the non-metricities, these are all torsion components of the form

$T_{\mu\nu\rho}v^\rho$, i.e. spatial on the first index and mixed spatial–temporal on the second and third index. The missing freely specifiable torsion components are thus the second set

$$P^\rho_\mu P^\sigma_\nu T_{\kappa\rho\sigma} \tag{5.3b}$$

which is spatial on all three indices, and the third set

$$(\delta^\rho_\sigma - P^\rho_\sigma)T^\sigma{}_{\mu\nu} \tag{5.3c}$$

which is temporally projected (with respect to $[\tau]$) on the first index.

Combined, we have thus seen that the freely specifiable torsion components with respect to a choice of timelike co-direction $[\tau]$ are

$$\left(T_{[\mu\nu]\rho}v^\rho \ , \ P^\rho_\mu P^\sigma_\nu T_{\kappa\rho\sigma} \ , \ (\delta^\rho_\sigma - P^\rho_\sigma)T^\sigma{}_{\mu\nu}\right) . \tag{5.4}$$

This field may be understood as a section of the bundle

$$\left(\bigwedge\nolimits^2 \ker v\right) \oplus \left(\ker v \otimes \bigwedge\nolimits^2 \ker v\right) \oplus \left(\operatorname{span}\{v\} \otimes \bigwedge\nolimits^2 T^*M\right) . \tag{5.5}$$

Note that the form of the free torsion components as given in (5.4) depends on the choice of representative (v, γ) of the conformal Carroll structure. For a discussion of the free torsion in a conformally invariant way, see appendix B.

5.2. The Carroll Weyl-type theorem

With these prerequisites, we can now prove a Weyl-type theorem, which is however significantly weaker than in the case of Galilei geometry:

Theorem 5.5. *Let M be a differentiable manifold with $\dim M > 1$. Let $[v, \gamma]$ be a conformal Carroll structure on M, and $[\tau]$ a timelike co-direction with respect to $[v, \gamma]$.*

(i) Let $\nabla, \tilde\nabla$ be two linear connections whose free torsion components with respect to $[\tau]$ agree, i.e. their torsions satisfy $T_{[\mu\nu]\rho}v^\rho = \tilde T_{[\mu\nu]\rho}v^\rho$, $P^\rho_\mu P^\sigma_\nu T_{\kappa\rho\sigma} = P^\rho_\mu P^\sigma_\nu \tilde T_{\kappa\rho\sigma}$, and $(\delta^\rho_\sigma - P^\rho_\sigma)T^\sigma{}_{\mu\nu} = (\delta^\rho_\sigma - P^\rho_\sigma)\tilde T^\sigma{}_{\mu\nu}$, where P is the spatial projector along $[\tau]$.[7] If the connections are compatible with $[v, \gamma]$ and projectively equivalent, then their difference tensor $D := \nabla - \tilde\nabla$ is given by

$$D^\rho_{\mu\nu} = f\tau_\mu \delta^\rho_\nu \tag{5.6}$$

for a function $f \in C^\infty(M)$ and a representative τ of the timelike co-direction.

(ii) Conversely, let ∇ be a linear connection that is compatible with $[v, \gamma]$. Then for any *choice of f and τ, the connection $\tilde\nabla := \nabla - D$ with difference D given by (5.6) is compatible with $[v, \gamma]$ and has the same free torsion components with respect to $[\tau]$ as ∇, and the connections are projectively equivalent.*

---

[7] Note that the condition of agreeing free torsion is invariant under change of representative (v, γ) of the conformal structure.

Remark 5.6. Note that even though the assumptions of part (i) of this theorem are the direct analogues of those in the Galilei case (theorem 4.4), we cannot infer complete equality of the connections: we cannot prove covariant derivatives in temporal direction to agree. Part (ii) shows that this conclusion cannot be improved: any difference tensor of the form inferred in part (i) can indeed be realised. Note also that this bears some similarity to the failure of the Galilean theorem for $\dim M = 2$ (proposition 4.8).

Proof of theorem 5.5. Fix some representative (v, γ) of the conformal structure, and let τ be the corresponding representative of the timelike co-direction satisfying $\tau(v) = 1$.

(i) By projective equivalence, there is a one-form η such that the difference tensor satisfies

$$D^{\rho}_{(\mu\nu)} = \delta^{\rho}_{(\mu}\eta_{\nu)} \,. \tag{5.7}$$

Furthermore, both connections having the same free torsion components with respect to $[\tau]$ implies in particular

$$P^{\rho}_{\mu}P^{\sigma}_{\nu}\gamma_{\kappa\lambda}D^{\lambda}_{[\rho\sigma]} = 0, \tag{5.8a}$$

implying

$$P^{\rho}_{\mu}P^{\sigma}_{\nu}\gamma_{\kappa\lambda}D^{\lambda}_{\rho\sigma} = P^{\rho}_{\mu}P^{\sigma}_{\nu}\gamma_{\kappa\lambda}D^{\lambda}_{(\rho\sigma)} = P^{\rho}_{\mu}P^{\sigma}_{\nu}\gamma_{\kappa(\rho}\eta_{\sigma)} \,. \tag{5.8b}$$

It also implies

$$\tau_{\rho}D^{\rho}_{[\mu\nu]} = 0, \tag{5.9a}$$

implying

$$\tau_{\rho}D^{\rho}_{\mu\nu} = \tau_{\rho}D^{\rho}_{(\mu\nu)} = \tau_{(\mu}\eta_{\nu)} \,. \tag{5.9b}$$

By compatibility of the connections with $[v, \gamma]$, there are one-forms $\varphi, \tilde{\varphi}$ such that

$$\nabla v = -\frac{1}{2}\varphi \otimes v, \tilde{\nabla} v = -\frac{1}{2}\tilde{\varphi} \otimes v, \nabla\gamma = \varphi \otimes \gamma, \tilde{\nabla}\gamma = -\tilde{\varphi} \otimes \gamma. \tag{5.10}$$

Combining the two equations for compatibility with v, we obtain

$$\begin{aligned} -\frac{1}{2}\varphi_{\mu}v^{\rho} &= \nabla_{\mu}v^{\rho} \\ &= \tilde{\nabla}_{\mu}v^{\rho} + D^{\rho}_{\mu\nu}v^{\nu} \\ &= -\frac{1}{2}\tilde{\varphi}_{\mu}v^{\rho} + D^{\rho}_{\mu\nu}v^{\nu}. \end{aligned} \tag{5.11}$$

From this, we will now derive three equations. First, contracting with $\gamma_{\rho\sigma}$, we obtain

$$\gamma_{\rho\sigma}D^{\rho}_{\mu\nu}v^{\nu} = 0. \tag{5.12}$$

Second, contracting (5.11) with v^μ, we obtain

$$\begin{aligned} -\frac{1}{2}\varphi(v)v^\rho &= -\frac{1}{2}\tilde{\varphi}(v)v^\rho + D^\rho_{(\mu\nu)}v^\mu v^\nu \\ &= -\frac{1}{2}\tilde{\varphi}(v)v^\rho + \eta(v)v^\rho, \end{aligned} \tag{5.13}$$

where we have used (5.7). This directly implies

$$-\frac{1}{2}\varphi(v) = -\frac{1}{2}\tilde{\varphi}(v) + \eta(v). \tag{5.14}$$

Third, contracting (5.11) with $2\tau_\rho$ yields

$$-\varphi_\mu = -\tilde{\varphi}_\mu + 2\tau_{(\mu}\eta_{\nu)}v^\nu = -\tilde{\varphi}_\mu + \eta(v)\tau_\mu + \eta_\mu \ , \tag{5.15}$$

where we used (5.9b). Contracting this with $h^{\mu\nu}$—the components of the contravariant space metric with respect to τ, defined by the requirements $h^{\mu\nu} = h^{\nu\mu}$, $h^{\mu\nu}\tau_\nu = 0$, and $h^{\mu\nu}\gamma_{\nu\rho} = P^\mu_\rho$—yields

$$-\varphi^\nu = -\tilde{\varphi}^\nu + \eta^\nu, \tag{5.16}$$

where $\varphi^\nu := h^{\mu\nu}\varphi_\mu$ is the τ-spacelike vector field corresponding to φ, and analogously for $\tilde{\varphi}$ and η.

On the other hand, combining the two equations for compatibility with γ, we obtain

$$\begin{aligned} \varphi_\rho\gamma_{\mu\nu} &= \nabla_\rho\gamma_{\mu\nu} \\ &= \tilde{\nabla}_\rho\gamma_{\mu\nu} - D^\kappa_{\rho\mu}\gamma_{\kappa\nu} - D^\kappa_{\rho\nu}\gamma_{\mu\kappa} \\ &= \tilde{\varphi}_\rho\gamma_{\mu\nu} - 2D^\kappa_{\rho(\mu}\gamma_{\nu)\kappa} \ . \end{aligned} \tag{5.17}$$

Contracting this with $h^{\rho\sigma}$ yields

$$\varphi^\sigma\gamma_{\mu\nu} = \tilde{\varphi}^\sigma\gamma_{\mu\nu} - 2h^{\rho\sigma}D^\kappa_{\rho(\mu}\gamma_{\nu)\kappa} \ . \tag{5.18}$$

In order to simplify (5.18), we consider its last term. From (5.12), we know that the expression $\gamma_{\rho\sigma}D^\rho_{\mu\nu}$ vanishes when contracted with v^ν; i.e. the index ν is spacelike. Therefore, the expression is equal to its projection onto space along $[\tau]$ on this index,

$$\gamma_{\rho\sigma}D^\rho_{\mu\nu} = \gamma_{\rho\sigma}P^\lambda_\nu D^\rho_{\mu\lambda} \ . \tag{5.19a}$$

Contracting with $h^{\mu\kappa}$, we have

$$\begin{aligned} \gamma_{\rho\sigma}h^{\mu\kappa}D^\rho_{\mu\nu} &= \gamma_{\rho\sigma}h^{\mu\kappa}P^\lambda_\nu D^\rho_{\mu\lambda} \\ &= h^{\kappa\alpha}\gamma_{\rho\sigma}P^\mu_\alpha P^\lambda_\nu D^\rho_{\mu\lambda} \ . \end{aligned} \tag{5.19b}$$

But the totally spacelike projected difference tensor can be expressed in terms of η according to (5.8b), giving

$$\begin{aligned}\gamma_{\rho\sigma} h^{\mu\kappa} D^{\rho}_{\mu\nu} &= h^{\kappa\alpha} P^{\mu}_{\alpha} P^{\lambda}_{\nu} \gamma_{\sigma(\mu}\eta_{\lambda)} \\ &= h^{\kappa\mu} P^{\lambda}_{\nu} \gamma_{\sigma(\mu}\eta_{\lambda)} \\ &= \frac{1}{2} h^{\kappa\mu} P^{\lambda}_{\nu} (\gamma_{\sigma\mu}\eta_{\lambda} + \gamma_{\sigma\lambda}\eta_{\mu}) \\ &= \frac{1}{2} (P^{\kappa}_{\sigma} P^{\lambda}_{\nu} \eta_{\lambda} + \gamma_{\sigma\nu}\eta^{\kappa}). \end{aligned} \tag{5.19c}$$

Now using (5.19c), the compatibility equation (5.18) takes the form

$$\varphi^{\sigma}\gamma_{\mu\nu} = \tilde{\varphi}^{\sigma}\gamma_{\mu\nu} - P^{\sigma}_{(\nu} P^{\lambda}_{\mu)} \eta_{\lambda} - \gamma_{\mu\nu}\eta^{\sigma}. \tag{5.20}$$

Contracting this with $h^{\mu\nu}$ yields

$$n\varphi^{\sigma} = n\tilde{\varphi}^{\sigma} - \eta^{\sigma} - n\eta^{\sigma} = n\tilde{\varphi}^{\sigma} - (n+1)\eta^{\sigma}, \tag{5.21}$$

where $n = \dim M - 1$. Adding $n \cdot$ (5.16) to (5.21) (and renaming the free index), we obtain

$$0 = -\eta^{\rho}, \tag{5.22a}$$

i.e.

$$\eta = \eta(v)\tau. \tag{5.22b}$$

Inserting this result, using (5.7) we may write the difference tensor as

$$D^{\rho}_{\mu\nu} = D^{\rho}_{(\mu\nu)} + D^{\rho}_{[\mu\nu]} = \delta^{\rho}_{(\mu}\eta_{\nu)} + \frac{1}{2}(\Delta T)^{\rho}{}_{\mu\nu} = \eta(v)\delta^{\rho}_{(\mu}\tau_{\nu)} + \frac{1}{2}(\Delta T)^{\rho}{}_{\mu\nu}\,, \tag{5.23}$$

where $\Delta T := T - \tilde{T}$ denotes the difference between the two connections' torsions. By performing a spacelike–timelike decomposition (i.e. by using $\delta^{\mu}_{\nu} = P^{\mu}_{\nu} + v^{\mu}\tau_{\nu}$), a straightforward calculation shows that the torsions may be expressed in terms of their free components with respect to $[\tau]$ and their constrained components as

$$T^{\rho}{}_{\mu\nu} = (\delta^{\rho}_{\sigma} - P^{\rho}_{\sigma})T^{\sigma}{}_{\mu\nu} + \big(h^{\rho\sigma}T_{(\sigma\mu)\kappa}v^{\kappa}\tau_{\nu} + h^{\rho\sigma}T_{[\sigma\mu]\kappa}v^{\kappa}\tau_{\nu} - (\mu \leftrightarrow \nu)\big) + h^{\rho\sigma}P^{\kappa}_{\nu}P^{\lambda}_{\mu}T_{\sigma\lambda\kappa}\,. \tag{5.24}$$

Since we assumed the free torsion components with respect to $[\tau]$ to agree, the torsion difference is thus given by

$$(\Delta T)^{\rho}{}_{\mu\nu} = h^{\rho\sigma}(\Delta T)_{(\sigma\mu)\kappa}v^{\kappa}\tau_{\nu} - (\mu \leftrightarrow \nu). \tag{5.25}$$

Now by (5.2b) the constrained torsion components are given by

$$\overset{(\sim)}{T}_{(\sigma\mu)\kappa}v^{\kappa} = \frac{1}{2}\overset{(\sim)}{\varphi}(v)\gamma_{\sigma\mu} - \frac{1}{2}(\mathcal{L}_{v}\gamma)_{\sigma\mu}\,. \tag{5.26}$$

Hence their difference is

$$\begin{aligned}(\Delta T)_{(\sigma\mu)\kappa} v^\kappa &= \frac{1}{2}(\varphi(v) - \tilde{\varphi}(v))\gamma_{\sigma\mu} \\ &= -\eta(v)\gamma_{\sigma\mu}\,,\end{aligned} \tag{5.27}$$

where we used (5.14). Inserting this into (5.25), the torsion difference evaluates to

$$\begin{aligned}(\Delta T)^\rho{}_{\mu\nu} &= -h^{\rho\sigma}\eta(v)\gamma_{\sigma\mu}\tau_\nu - (\mu \leftrightarrow \nu) \\ &= -\eta(v)P^\rho_\mu\tau_\nu - (\mu \leftrightarrow \nu) \\ &= 2\eta(v)P^\rho_{[\nu}\tau_{\mu]} \\ &= 2\eta(v)(\delta^\rho_{[\nu} - v^\rho\tau_{[\nu})\tau_{\mu]} \\ &= 2\eta(v)\delta^\rho_{[\nu}\tau_{\mu]}\,.\end{aligned} \tag{5.28}$$

Inserting this into (5.23), the difference tensor is given by

$$D^\rho_{\mu\nu} = \eta(v)\delta^\rho_{(\mu}\tau_{\nu)} + \eta(v)\delta^\rho_{[\nu}\tau_{\mu]} = \eta(v)\delta^\rho_\nu\tau_\mu\,. \tag{5.29}$$

Thus, we have proved that it takes the form from the theorem statement with $f = \eta(v)$.

(ii) Since ∇ is compatible with $[v, \gamma]$, there is a one-form φ such that

$$\nabla v = -\frac{1}{2}\varphi \otimes v, \nabla\gamma = \varphi \otimes \gamma. \tag{5.30}$$

Let $f \in C^\infty(M)$ be an arbitrary smooth function, and define $\tilde{\nabla} := \nabla - D$ for

$$D^\rho_{\mu\nu} = f\tau_\mu\delta^\rho_\nu\,. \tag{5.31}$$

We have to show that $\tilde{\nabla}$ is compatible with $[v, \gamma]$, that its free torsion components with respect to $[\tau]$ agree with those of ∇, and that the two connections are projectively equivalent. All of this follows by direct computations, which we will spell out in the following.

First, considering projective equivalence, by definition we directly obtain

$$D^\rho_{(\mu\nu)} = f\tau_{(\mu}\delta^\rho_{\nu)} = \delta^\rho_{(\mu}\eta_{\nu)} \tag{5.32}$$

with $\eta := f\tau$, showing that ∇ and $\tilde{\nabla}$ are projectively equivalent.

Next, we turn to compatibility with the conformal structure. We compute

$$\begin{aligned}\tilde{\nabla}_\mu v^\rho &= \nabla_\mu v^\rho - D^\rho_{\mu\nu}v^\nu \\ &= -\frac{1}{2}\varphi_\mu v^\rho - f\tau_\mu\delta^\rho_\nu v^\nu \\ &= -\frac{1}{2}(\varphi_\mu + 2f\tau_\mu)v^\rho\end{aligned} \tag{5.33}$$

and

$$\begin{aligned}\tilde{\nabla}_\rho\gamma_{\mu\nu} &= \nabla_\rho\gamma_{\mu\nu} + D^\kappa_{\rho\mu}\gamma_{\kappa\nu} + D^\kappa_{\rho\nu}\gamma_{\mu\kappa} \\ &= \varphi_\rho\gamma_{\mu\nu} + f\tau_\rho\delta^\kappa_\mu\gamma_{\kappa\nu} + f\tau_\rho\delta^\kappa_\nu\gamma_{\mu\kappa} \\ &= (\varphi_\rho + 2f\tau_\rho)\gamma_{\mu\nu}\ . \end{aligned} \tag{5.34}$$

This means

$$\tilde{\nabla}v = -\frac{1}{2}\tilde{\varphi}\otimes v, \tilde{\nabla}\gamma = \tilde{\varphi}\otimes\gamma \tag{5.35}$$

with $\tilde{\varphi} = \varphi + 2f\tau$, showing that $\tilde{\nabla}$ is compatible with $[v, \gamma]$.

Finally, the difference $\Delta T := T - \tilde{T}$ between the connections' torsions is given by

$$(\Delta T)^\rho{}_{\mu\nu} = 2D^\rho_{[\mu\nu]} = 2f\tau_{[\mu}\delta^\rho_{\nu]}\ . \tag{5.36}$$

This directly implies

$$(\delta^\rho_\sigma - P^\rho_\sigma)(\Delta T)^\sigma{}_{\mu\nu} = v^\rho\tau_\sigma 2f\tau_{[\mu}\delta^\sigma_{\nu]} = 2fv^\rho\tau_{[\mu}\tau_{\nu]} = 0. \tag{5.37}$$

Further, we have

$$\begin{aligned}(\Delta T)_{\sigma\mu\nu} &= \gamma_{\sigma\rho}(\Delta T)^\rho{}_{\mu\nu} \\ &= \gamma_{\sigma\rho}2f\tau_{[\mu}\delta^\rho_{\nu]} \\ &= 2f\tau_{[\mu}\gamma_{\nu]\sigma} \\ &= f\tau_\mu\gamma_{\nu\sigma} - f\tau_\nu\gamma_{\mu\sigma}\ , \end{aligned} \tag{5.38}$$

from which we obtain

$$(\Delta T)_{[\mu\nu]\rho}v^\rho = (f\tau_{[\nu}\gamma_{|\rho|\mu]} - f\tau_\rho\gamma_{[\nu\mu]})v^\rho = 0 \tag{5.39}$$

as well as

$$P^\mu_\lambda P^\nu_\kappa(\Delta T)_{\sigma\mu\nu} = P^\mu_\lambda P^\nu_\kappa(f\tau_\mu\gamma_{\nu\sigma} - f\tau_\nu\gamma_{\mu\sigma}) = 0. \tag{5.40}$$

But (5.37), (5.39), and (5.40) mean that $(\delta^\rho_\sigma - P^\rho_\sigma)T^\sigma{}_{\mu\nu} = (\delta^\rho_\sigma - P^\rho_\sigma)\tilde{T}^\sigma{}_{\mu\nu}$, $T_{[\mu\nu]\rho}v^\rho = \tilde{T}_{[\mu\nu]\rho}v^\rho$, and $P^\mu_\lambda P^\nu_\kappa T_{\sigma\mu\nu} = P^\mu_\lambda P^\nu_\kappa\tilde{T}_{\sigma\mu\nu}$—i.e. the connections' free torsion components with respect to $[\tau]$ agree. □

5.3. A Carroll analogue of Weyl metrics

We may define an analogue of Weyl metrics for Carroll geometry as follows, in parallel to the Galilei case.

Construction 5.7. Fixing a Carroll structure (v, γ) on M, with respect to a choice of unit timelike one-form $\tau \in \Omega^1(M)$, $\tau(v) = 1$, any linear connection ∇ on M is uniquely determined by—and uniquely determines, vice versa—its v-non-metricity ∇v, the spatial part $P^\rho_\mu P^\sigma_\nu \nabla_\kappa \gamma_{\rho\sigma}$ of its γ-non-metricity, its free torsion components T_{free} with respect to $[\tau]$ as discussed in construction 5.4, and the field $\hat{\chi}_{\mu\nu} = P^\rho_\mu P^\sigma_\nu \nabla_{(\rho}\tau_{\sigma)}$; see Vigneron et al. (2025) and our discussion in appendix A. Assuming the non-metricities to be of the form $\nabla v = -\frac{1}{2}\varphi \otimes v$, $\nabla\gamma = \varphi \otimes \gamma$, such that ∇ is compatible with the conformal Carroll structure $[v, \gamma]$, we already know that fixing the connection ∇, under a conformal rescaling $(v, \gamma) \mapsto (\mathrm{e}^{-\frac{1}{2}\lambda} v, \mathrm{e}^{\lambda}\gamma)$ with $\lambda \in C^\infty(M)$ the one-form φ transforms as $\varphi \mapsto \varphi + \mathrm{d}\lambda$. Further, the free torsion may be written in a conformally invariant way, see appendix B. Rescaling $\tau \mapsto \mathrm{e}^{\frac{1}{2}\lambda}$ such that it remains unit timelike with respect to the new representative of the conformal structure, a direct computation shows that the $\hat{\chi}$ field of ∇ with respect to τ transforms as $\hat{\chi} \mapsto \mathrm{e}^{\frac{1}{2}\lambda}\hat{\chi}$.

Therefore, a conformal Carroll structure $[v, \gamma]$ on M together with a compatible linear connection is equivalently characterised by a triple $([v, \gamma, \varphi, \hat{\chi}], [\tau], T_{\text{free}})$ consisting of (i) an equivalence class of Carroll structures (v, γ), one-forms φ, and symmetric tensor fields $\hat{\chi} \in \bigvee^2 \ker v$ on M under the equivalence relation

$$(v, \gamma, \varphi, \hat{\chi}) \sim (\mathrm{e}^{-\frac{1}{2}\lambda} v, \mathrm{e}^{\lambda}\gamma, \varphi + \mathrm{d}\lambda, \mathrm{e}^{\frac{1}{2}\lambda}\hat{\chi}) \tag{5.41}$$

for $\lambda \in C^\infty(M)$; (ii) a timelike co-direction $[\tau]$ with respect to the conformal Carroll structure $[v, \gamma]$; and (iii) a section T_{free} of the bundle in which the conformally invariant free torsion takes its values, as discussed in construction B.3, in particular (B.12).

In order to reformulate our Weyl-type theorem for Carroll geometry (theorem 5.5) in terms of such a 'Carroll Weyl metric' $([v, \gamma, \varphi, \hat{\chi}], [\tau], T_{\text{free}})$, we have to examine the influence of the non-uniqueness in the theorem statement on the fields defining the connection. A direct computation shows that, fixing $[v, \gamma]$ and $[\tau]$, a change of the connection according to $\nabla \mapsto \nabla - D$ as allowed by the theorem—i.e. with $D^\rho_{\mu\nu} = f\tau_\mu \delta^\rho_\nu$—leaves $\hat{\chi}$ invariant. In the proof of theorem 5.5 we have further seen that and how φ changes—namely by addition of $2f\tau$.

Thus, we may reformulate theorem 5.5 as the statement that *a 'Carroll Weyl metric' $([v, \gamma, \varphi, \hat{\chi}], [\tau], T_{\text{free}})$ is uniquely determined by its conformal and projective structures together with its timelike co-direction $[\tau]$ and its free torsion T_{free} with respect to $[\tau]$, up to addition of (functional) multiples of τ to φ.*

6. Conclusions

In this article, we have proved versions of one famous theorem of Weyl (1921)—that a Weyl metric is fixed uniquely by its associated projective and conformal structures—for the cases of Galilei and Carroll geometries; for both of the latter two cases, we used the appropriate notion of conformal structure as would arise in the relevant 'non-relativistic' or 'ultra-relativistic' limit. Our Galilean result generalises and extends prior work by Curiel (2015), Ewen and Schmidt (1989), and March (2025); our Carrollian result is entirely new to the literature—although also interestingly weaker than those for the pseudo-Riemannian and Galilei cases, as the equivalent uniqueness result does *not* hold in the Carroll context. A concise summary of our results may be found in table 1.

There are various payoffs of this work. From a conceptual point of view, one merit of undertaking this work is that one comes to understand better the 'sub-metrical' constituents of both Galilei and Carroll geometries. This is particularly pertinent since in recent years there has been heightened interest in conformal structures in both the Galilei and Carroll contexts. One motivation for this interest has been due to the application of these non-relativistic conformal geometries in the context of holography; the work has mostly so far had to do with Galilean conformal geometry—see e.g. Bagchi and Gopakumar (2009) and Bagchi and Mandal (2009). Another motivation has had to do with the study of non-relativistic field theories more generally—see e.g. Alishahiha et al. (2009), Bagchi et al. (2014, 2018), Chen and Liu (2023), Duval and Horvathy (2011), and Hagen (1972) for conformal structures in Galilean field theory, and e.g. Bergshoeff et al. (2026), Chen et al. (2024), Dutta (2024), Duval et al. (2014a), and Gupta and Suryanarayana (2021) for conformal structures in Carrollian field theory. A further motivation has to do with the study of asymptotic symmetries and the BMS group; to these issues it is Carroll conformal structures which are most pertinent—see e.g. Ciambelli et al. (2019). Finally, conformal Galilei structures have found application in work on non-relativistic twistor theory—see Dunajski and Gundry (2016), Dunajski and Penrose (2023), and March (2023)—which raises the question of whether conformal Carroll structures of the kind discussed in this article could find application to ultra-relativistic versions of twistor theory.

One might, indeed, anticipate that the results presented in this article could interact quite directly with the above contexts. For example, if one is in the context of working with a conformal Galilei or Carroll structure (perhaps e.g. working with holography or asymptotics), then the introduction of a preferred class of observers would *ipso facto* introduce a projective structure, which would in turn suffice to derive a Galilei/Carroll Weyl spacetime. As such, this work might well serve as a simple illustration of the 'emergence of spacetime' which has aroused physicists' interests in (for example) the—admittedly much more advanced!—context of 'dressing to the observer' in algebraic quantum field theory.[8]

There are also more foundational/conceptual payoffs of this work. As mentioned in the introduction, taking the lead from Weyl (1921), Ehlers et al. (1972) (EPS) sought to provide a *constructive axiomatisation* (in the sense of Reichenbach (1969); for more on constructive axiomatic approaches to spacetime theories see Adlam et al. (2025)) of (the kinematical structures of) the general theory of relativity, which is to say that they sought to (i) build up projective and (relativistic) conformal structures from elementary,

---

[8] See e.g. Chandrasekaran et al. (2023) and Witten (2022).

Table 1: Summary of definitions and properties for conformal and projective structures in the pseudo-Riemannian, Galilean, and Carrollian cases. We let $\lambda, f \in C^\infty(M)$, $\tilde{\lambda} \in C^\infty(M, \mathbb{R} \setminus \{0\})$, and $\eta, \varphi \in \Omega^1(M)$.

Quantities	**pseudo-Riemannian geometry**	**Galilei geometry**	**Carroll geometry**
		DEFINITIONS	
Projective structure $\mathcal{P} = [\nabla]$	$\nabla \sim \nabla + D$ with $D^\rho_{(\mu\nu)} = \delta^\rho_{(\mu}\eta_{\nu)}$	same	same
Metric structure	g	(τ, h) with $\tau_\mu h^{\mu\nu} = 0$	(v, γ) with $v^\mu \gamma_{\mu\nu} = 0$
Conformal structure	$[g]$ with $g \sim e^\lambda g$	$[\tau, h]$ with $(\tau, h) \sim (e^{\frac{1}{2}\lambda}\tau, e^{-\lambda}h)$	$[v, \gamma]$ with $(v, \gamma) \sim (e^{-\frac{1}{2}\lambda}v, e^\lambda \gamma)$
Conformally compatible connection	$\nabla g = \varphi \otimes g$	$\nabla \tau = \frac{1}{2}\varphi \otimes \tau, \quad \nabla h = -\varphi \otimes h$	$\nabla v = -\frac{1}{2}\varphi \otimes v, \quad \nabla \gamma = \varphi \otimes \gamma$
Timelike (co-)direction	–	$[v]$ with $\tau(v) \neq 0$, $v \sim \tilde{\lambda} v$	$[\tau]$ with $\tau(v) \neq 0$, $\tau \sim \tilde{\lambda}\tau$
Spatial projector	–	$P^\mu_\nu = \delta^\mu_\nu - \hat{v}^\mu \tau_\nu$ where $\hat{v} \in [v]$ such that $\tau(\hat{v}) = 1$	$P^\mu_\nu = \delta^\mu_\nu - v^\mu \hat{\tau}_\nu$ where $\hat{\tau} \in [\tau]$ such that $\hat{\tau}(v) = 1$
Free parts of the torsion (T_{free}) given fixed non-metricities	$T^\rho{}_{\mu\nu}$	$P^\rho_\sigma T^\rho{}_{\mu\nu}$ with respect to a choice of $[v]$	$\gamma_{\kappa[\mu} T^\kappa{}_{\nu]\rho}\, v^\rho$, $P^\rho_\mu P^\sigma_\nu \gamma_{\kappa\alpha} T^\alpha{}_{\rho\sigma}$, $(\delta^\rho_\sigma - P^\rho_\sigma) T^\sigma{}_{\mu\nu}$ with respect to a choice of $[\tau]$
		PROPERTIES	
Weyl-type theorem	Theorem 3.8: For $\dim M > 1$, given $[g]$ and $\mathcal{P}$ with fixed T, $\nabla \in \mathcal{P}$ compatible with $[g]$ is unique.	Theorem 4.4: For $\dim M > 2$, given $[\tau, h]$ and $\mathcal{P}$ with fixed T_{free}, $\nabla \in \mathcal{P}$ compatible with $[\tau, h]$ is unique.	Theorem 5.5: Given $[v, \gamma]$ and $\mathcal{P}$ with fixed T_{free}, $\nabla \in \mathcal{P}$ compatible with $[v, \gamma]$ is unique up to $D^\rho_{\mu\nu} = f \tau_\mu \delta^\rho_\nu$.
Connection and (torsional) Weyl metric	$([g], \nabla) \longleftrightarrow ([g, \varphi], T_{\text{free}})$ with $(g, \varphi) \sim (e^\lambda g, \varphi + d\lambda)$	$([\tau, h], \nabla) \longleftrightarrow ([\tau, h, \varphi, \Omega], [v], T_{\text{free}})$ with $(\tau, h, \varphi, \Omega) \sim (e^{\frac{1}{2}\lambda}\tau, e^{-\lambda}h, \varphi + d\lambda, e^{-\frac{1}{2}\lambda}\Omega)$	$([v, \gamma], \nabla) \longleftrightarrow ([v, \gamma, \varphi, \hat{\chi}], [\tau], T_{\text{free}})$ with $(v, \gamma, \varphi, \hat{\chi}) \sim (e^{-\frac{1}{2}\lambda}v, e^\lambda \gamma, \varphi + d\lambda, e^{\frac{1}{2}\lambda}\hat{\chi})$

empirically-informed axioms, and then (ii) show how those structures, together with further 'compatibility' assumptions, yield the familiar Lorentzian geometries of general relativity. In this article, we have provided the Weyl-style 'uniqueness' result as a complement to the EPS-style 'existence' result for Galilean physics provided by Adlam et al. (2025, ch. 4); moreover, the failure of the Carrollian 'uniqueness' result might raise some *prima facie* concerns about the possibility of Carroll constructive axiomatics.[9]

Another foundational question in the vicinity of our work has to do with the logic of experimental tests of spacetime theories. Recently, it has been argued by Hansen et al. (2019) that many of the classic experimental tests of general relativity would also be met by modified Newtonian theories of gravity (with, generically, torsion and/or non-metricity); Wolf et al. (2024) analyse these claims and consider whether these experimental tests are in fact best understood as testing projective/conformal structures. Given that we now have better control over e.g. conformal Carroll structures, it would be worthwhile to consider the extent to which these claims regarding experimental tests carry over to the Carroll context.

And there is yet more work to be done besides the above. Having now thoroughly assessed the status of Weyl's *theorem* in non-relativistic contents, one might now turn to consideration of the behaviour of the Weyl *tensor* in such settings. How does the object behave in the relevant limits? Does it remain an invariant of the relevant conformal structure in each case? Can it be used to classify non-relativistic spacetimes, *à la* the Petrov classification of relativistic spacetimes? Although some work has already been undertaken in this vicinity (see in particular Dewar and Read (2020), Dewar and Weatherall (2018), and Ehlers and Buchert (2009)), the results of this article should pave the way to a more systematic treatment of the Galilei and Carroll cases.

In the end, all this triangulates that there remains much work to be done, along many axes, which draws upon the results of this article. It is our hope that what we have achieved here lays the groundwork for these future investigations.

Acknowledgements

P.K.S. and Q.V. thank Pembroke College, Oxford, for hospitality during a stay where the present research was conducted. Q.V. was supported in part by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement ERC advanced grant 740021-ARTHUS, principal investigator Thomas Buchert). P.K.S. thanks Domenico Giulini for helpful discussions. J.R. thanks Niels Linnemann, Eleanor March, and the audience of a talk at UC Irvine, for helpful discussions.

---

[9] Cf. footnote 4. There are other *prima facie* worries about the construction in the Carroll case—for example, Ehlers et al. (1972) make substantial use of light-and-mirror constructions, but how to make sense of these when all signals are tachyonic, as is the case in Carroll spacetimes?

A. The classification of connections on Carroll manifolds

Here, we will discuss the classification of connections on Carroll manifolds in terms of freely specifiable tensor fields, based on Vigneron et al. (2025).

Construction A.1. Let (v, γ) be a Carroll structure on M. In Vigneron et al. (2025), it was shown that any linear connection ∇ on M satisfies the two conditions

$$v^\mu \nabla_\rho \gamma_{\mu\nu} = -\gamma_{\mu\nu} \nabla_\rho v^\mu \tag{A.1a}$$

and

$$T_{(\mu\nu)\rho} v^\rho = \frac{1}{2} v^\rho \nabla_\rho \gamma_{\mu\nu} + \gamma_{\rho(\mu} \nabla_{\nu)} v^\rho - \frac{1}{2} (\mathcal{L}_v \gamma)_{\mu\nu} \tag{A.1b}$$

where T is the torsion, whose first index has been lowered with γ. Further it was shown that, fixing a unit timelike one-form τ (i.e. $\tau \in \Omega^1(M)$, $\tau(v) = 1$), the difference of any connection ∇ to a reference connection $\overset{v,\tau}{\nabla}$ (fully determined by v, γ, τ) can be expressed algebraically in terms of

1. v, γ, τ, and the contravariant space metric h with respect to τ,
2. the connection's non-metricities $\nabla\gamma$, ∇v and torsion T (subject to the above identities), as well as
3. the field $\nabla_{(\mu}\tau_{\nu)}$,

where the dependence on $\nabla\gamma$, ∇v, T, and $\nabla_{(\mu}\tau_{\nu)}$ is affine. (An explicit formula may be found in Vigneron et al. (2025, Proposition 2.1).)

However, Vigneron et al. (2025) did not study to a full extent which components of these fields may be specified independently of each other, i.e. how to fully *classify* a connection in terms of these fields. We are now going to deduce this classification. For this discussion, denote by P the spatial projector along τ.

First considering the identity (A.1a), we see that when arbitrarily fixing the v-non-metricity

$$\nabla_\rho v^\mu, \tag{A.2a}$$

this determines the temporal part $v^\mu \nabla_\kappa \gamma_{\mu\nu}$ of the γ-non-metricity, leaving free precisely its purely spatial part (w.r.t. τ) on its last two indices,

$$P^\rho_\mu P^\sigma_\nu \nabla_\kappa \gamma_{\rho\sigma} \; . \tag{A.2b}$$

With the non-metricities fixed, the identity (A.1b) fixes the part $T_{(\mu\nu)\rho} v^\rho$ of the torsion. As discussed in the main text in construction 5.4, the remaining free components of the torsion then are those of the form

$$\left(T_{[\mu\nu]\rho} v^\rho \; , \; P^\rho_\mu P^\sigma_\nu T_{\kappa\rho\sigma} \; , \; (\delta^\rho_\sigma - P^\rho_\sigma) T^\sigma{}_{\mu\nu} \right) . \tag{A.2c}$$

Finally, regarding the field $\nabla_{(\mu}\tau_{\nu)}$, one can show that given the non-metricities and torsion, it is fully determined by knowing only its purely spatial part[10]

$$\hat{\chi}_{\mu\nu} := P^\rho_\mu P^\sigma_\nu \nabla_{(\rho}\tau_{\sigma)}\ . \tag{A.2d}$$

These fields may be understood as sections of the following vector bundles: the v-non-metricity is a section of

$$T^*M \otimes TM; \tag{A.3a}$$

the purely spatial part of the γ-non-metricity on its last two indices is a section of

$$T^*M \otimes \bigvee^2 \ker v; \tag{A.3b}$$

the free torsion is a section of

$$E_{\text{free torsion}} := \left(\bigwedge^2 \ker v\right) \oplus \left(\ker v \otimes \bigwedge^2 \ker v\right) \oplus \left(\operatorname{span}\{v\} \otimes \bigwedge^2 T^*M\right); \tag{A.3c}$$

and the purely spatial part $\hat{\chi}$ of $\nabla_{(\mu}\tau_{\nu)}$ is a section of

$$\bigvee^2 \ker v. \tag{A.3d}$$

Thus, the collection of all these freely specifiable fields

$$\left(\nabla_\rho v^\mu\ ,\ P^\rho_\mu P^\sigma_\nu \nabla_\kappa \gamma_{\rho\sigma}\ ,\ T_{[\mu\nu]\rho} v^\rho\ ,\ P^\rho_\mu P^\sigma_\nu T_{\kappa\rho\sigma}\ ,\ (\delta^\rho_\sigma - P^\rho_\sigma) T^\sigma{}_{\mu\nu}\ ,\ \hat{\chi}_{\mu\nu}\right) \tag{A.4a}$$

is a section of the vector bundle

$$E_{\text{free fields}} := (T^*M \otimes TM) \oplus \left(T^*M \otimes \bigvee^2 \ker v\right) \oplus E_{\text{free torsion}} \oplus \bigvee^2 \ker v. \tag{A.4b}$$

Now considering in addition the affine bundle $\operatorname{Conn}(TM)$ of linear connections on M, we have an affine bundle homomorphism

$$\Psi_\tau \colon \operatorname{Conn}(TM) \longrightarrow E_{\text{free fields}}\ , \tag{A.5}$$

mapping a connection ∇ to the field (A.4a). Conversely, as shown by Vigneron et al. (2025), any connection ∇ can be expressed affinely in terms of $\nabla\gamma$, ∇v, T, $\nabla_{(\mu}\tau_{\nu)}$; by our above construction, these may further be expressed solely via (A.4a). This provides an affine bundle homomorphism

$$\Phi_\tau \colon E_{\text{free fields}} \longrightarrow \operatorname{Conn}(TM) \tag{A.6}$$

---

[10] This may be seen as follows. First, performing a spacelike–timelike decomposition, $\nabla_{(\mu}\tau_{\nu)}$ is determined by its purely spatial part $P^\rho_\mu P^\sigma_\nu \nabla_{(\rho}\tau_{\sigma)}$, its purely temporal part $v^\mu v^\nu \nabla_\mu \tau_\nu$, and its mixed part $P^\rho_\mu v^\nu \nabla_{(\rho}\tau_{\nu)} = P^\rho_\mu v^\nu \nabla_\rho \tau_\nu - P^\rho_\mu v^\nu \nabla_{[\rho}\tau_{\nu]}$. Next, covariantly differentiating $\tau_\mu v^\mu = 1$, we see that the non-metricity ∇v determines the expression $v^\mu \nabla_\rho \tau_\mu$, and hence the purely temporal part and the first term of the mixed part. Finally, the antisymmetric covariant derivative $\nabla_{[\rho}\tau_{\nu]}$ is determined by the exterior derivative $\mathrm{d}\tau$ and the torsion expression $\tau_\kappa T^\kappa{}_{\rho\nu}$, thus fixing the second term of the mixed part.

of which we know $\Phi_\tau \circ \Psi_\tau = \mathrm{id}_{\mathrm{Conn}(TM)}$. We are now going to show that the ranks of $E_{\text{free fields}}$ and $\mathrm{Conn}(TM)$ are equal, such that for dimensional reasons Φ_τ and Ψ_τ are mutually inverse isomorphisms.

Writing $\dim M = n + 1$, the ranks of the direct summands of $E_{\text{free fields}}$ are

$$\mathrm{rk}(T^*M \otimes TM) = (n+1)^2, \tag{A.7a}$$

$$\mathrm{rk}\left(T^*M \otimes \textstyle\bigvee^2 \ker v\right) = (n+1) \cdot \frac{n(n+1)}{2} = \frac{n(n+1)^2}{2}, \tag{A.7b}$$

$$\begin{aligned}\mathrm{rk}(E_{\text{free torsion}}) &= \frac{n(n-1)}{2} + n \cdot \frac{n(n-1)}{2} + 1 \cdot \frac{(n+1)n}{2} \\ &= (1+n) \cdot \frac{n(n-1)}{2} + \frac{n(n+1)}{2} \\ &= \frac{n(n^2-1)}{2} + \frac{n(n+1)}{2} \\ &= \frac{n^2(n+1)}{2},\end{aligned} \tag{A.7c}$$

$$\mathrm{rk}(\textstyle\bigvee^2 \ker v) = \frac{n(n+1)}{2}, \tag{A.7d}$$

respectively. Thus, we have

$$\begin{aligned}\mathrm{rk}(E_{\text{free fields}}) &= (n+1)^2 + \frac{n(n+1)^2}{2} + \frac{n^2(n+1)}{2} + \frac{n(n+1)}{2} \\ &= (n+1)^2 + \frac{n(n+1)^2}{2} + \frac{n(n+1)^2}{2} \\ &= (n+1)^3.\end{aligned} \tag{A.8}$$

This is indeed equal to the rank of $\mathrm{Conn}(TM)$. Therefore we may indeed conclude that Ψ_τ is an isomorphism of affine bundles.

That is, choosing a unit timelike one-form τ, *any linear connection* ∇ *on* M *is uniquely determined by, and uniquely determines, the corresponding field* (A.4a)*, which is a section of* $E_{\text{free fields}}$ (A.4b).

Remark A.2. Note that regarding the independent degrees of freedom in the definition of a linear connection, it is crucial that with fixed non-metricities and torsion *only the purely spatial part* $\hat{\chi}_{\mu\nu} = P^\rho_\mu P^\sigma_\nu \nabla_{(\rho}\tau_{\sigma)}$ of $\nabla_{(\mu}\tau_{\nu)}$ may be freely specified. This was not explicitly realised in the discussion in Vigneron et al. (2025), where instead the whole quantity $\chi_{\mu\nu} = \nabla_{(\mu}\tau_{\nu)}$ was considered.

The explicit counting of dimensions that we performed above shows that the field (A.4a) captures all the freedom in the definition of a linear connection, expressed with respect to the Carroll structure, without any 'double-counting'.

B. The free torsion of connections on conformal Carroll manifolds in conformally invariant form

Here, we will discuss how the freely specifiable torsion components of connections on manifolds with a conformal Carroll structure may be written in a conformally invariant way.

Construction B.1. Let $[v, \gamma]$ be a conformal Carroll structure on M, and $[\tau]$ a timelike co-direction with respect to it. Further, let P be the spatial projector along $[\tau]$.

In construction 5.4, we have seen that the components of the torsion that may be freely specified when defining a connection in terms of its non-metricities may be written in the form

$$\left(T_{[\mu\nu]\rho}v^\rho \;,\; P^\rho_\mu P^\sigma_\nu T_{\kappa\rho\sigma} \;,\; (\delta^\rho_\sigma - P^\rho_\sigma)T^\sigma{}_{\mu\nu}\right) \tag{B.1}$$

with respect to $[\tau]$ (cf. (5.4)), where a choice of representative (v, γ) of the conformal structure was made. In the first two terms, the first index on T was lowered with γ. We are now going to write the free torsion components in a form that is *independent* of the choice of representative of the conformal structure, i.e. conformally invariant.

For that, let h be the contravariant space metric of (v, γ) with respect to $[\tau]$, defined by the requirements[11]

$$h^{\mu\nu} = h^{\nu\mu}, \quad h^{\mu\nu}\tau_\nu = 0, \quad h^{\mu\nu}\gamma_{\nu\rho} = P^\mu_\rho\ . \tag{B.2}$$

The conformally invariant form of the free torsion components is then given by

$$\left(h^{\mu\kappa}\gamma_{\lambda[\kappa}T^\lambda{}_{\nu]\sigma}\,(\delta^\sigma_\rho - P^\sigma_\rho)\;,\; P^\lambda_\kappa P^\rho_\mu P^\sigma_\nu T^\kappa{}_{\rho\sigma} \;,\; (\delta^\rho_\sigma - P^\rho_\sigma)T^\sigma{}_{\mu\nu}\right). \tag{B.3}$$

The second and third terms in (B.3) are manifestly conformally invariant, since the spatial projector P along $[\tau]$ is. Further, under a conformal rescaling $(v, \gamma) \mapsto (\mathrm{e}^{-\frac{1}{2}\lambda}v, \mathrm{e}^\lambda\gamma)$ the contravariant space metric transforms as $h \mapsto \mathrm{e}^{-\lambda}h$. Hence, the first term in (B.3) is conformally invariant as well. Of course, however, the free torsion components (B.3) still depend on the choice of timelike co-direction $[\tau]$.

Next, we are going to discuss the space where the conformally invariant form of the free torsion lives. For this, we need a few prerequisites:

Construction B.2. (i) Let V be a finite-dimensional real vector space and $g \in \bigvee^2 V^*$ a non-degenerate inner product on it (of any signature).

For a linear map $X\colon V \to V$, the condition of being self-adjoint or anti-self-adjoint with respect to g,

$$\forall v, w \in V : g(v, X(w)) = \pm g(X(v), w), \tag{B.4}$$

---

[11] Note that the second of these equations depends only on $[\tau]$.

is invariant under rescaling of g: a linear map X is (anti-)self-adjoint with respect to g if and only if it is (anti-)self-adjoint with respect to $\mathrm{e}^{\lambda} g$, for any $\lambda \in \mathbb{R}$. Therefore, *the notion of (anti-)self-adjointness depends only on the conformal class* $[g]$ of the inner product. In particular, given only a conformal class $[g]$ of non-degenerate inner products on V, the space

$$\mathfrak{so}(V,[g]) = \{X\colon V \longrightarrow V \text{ linear} \mid X \text{ anti-self-adjoint w.r.t. } [g]\} \tag{B.5}$$

is well-defined.[12]

Further, this implies that *the decomposition of a linear endomorphism* $X\colon V \longrightarrow V$ *into its self-adjoint and anti-self-adjoint parts with respect to a conformal class* $[g]$ *is well-defined*. Concretely, taking any non-degenerate inner product $g \in \bigvee^2 V^*$, with components g_{ab}, and denoting its inverse by $g^{-1} \in \bigvee^2 V$, with components g^{ab}, the decomposition of X is given by

$$\begin{aligned} X^a{}_b &= g^{ac} g_{dc} X^d{}_b \\ &= g^{ac} \left(g_{d(c} X^d{}_{b)} + g_{d[c} X^d{}_{b]} \right) \\ &= g^{ac} g_{d(c} X^d{}_{b)} + g^{ac} g_{d[c} X^d{}_{b]} \end{aligned} \tag{B.6a}$$

with the first summand being the self-adjoint and the second being the anti-self-adjoint part. Equivalently, the decomposition can be written as

$$X^a{}_b = g_{bd} g^{c(d} X^{a)}{}_c + g_{bd} g^{c[d} X^{a]}{}_c \,. \tag{B.6b}$$

Scaling g with e^{λ}, the inverse g^{-1} is scaled with $\mathrm{e}^{-\lambda}$, such that the decomposition stays invariant.

(ii) Let M be a differentiable manifold of dimension $n+1$, $n \geq 1$, and (v, γ) a Carroll structure on it. γ induces a natural positive-definite bundle metric ${}^{(n)}\gamma$ on the vector bundle $\ker v \subset T^*M$ of spacelike covectors. Concretely, choosing any timelike co-direction $[\tau]$ and considering the contravariant space metric h of (v, γ) with respect to $[\tau]$ (defined by (B.2)), ${}^{(n)}\gamma$ can be expressed as

$${}^{(n)}\gamma(\alpha, \beta) = h^{\mu\nu} \alpha_\mu \beta_\nu \tag{B.7}$$

for $\alpha, \beta \in \Gamma(\ker v)$ (i.e. ${}^{(n)}\gamma$ is the restriction of h to $\ker v$).

---

[12] Similarly, the condition of a linear map $R\colon V \longrightarrow V$ being orthogonal with respect to g,

$$\forall v, w \in V : g(R(v), R(w)) = g(v, w),$$

is also invariant under rescaling of g, and hence only depends on the conformal class $[g]$. Therefore, the orthogonal group $\mathrm{O}(V,[g])$ with respect to a conformal class of inner products is well-defined, and $\mathfrak{so}(V,[g])$ is its Lie algebra.

Conformally rescaling $(v, \gamma) \mapsto (\mathrm{e}^{-\frac{1}{2}\lambda} v, \mathrm{e}^{\lambda}\gamma)$, this bundle metric gets rescaled according to ${}^{(n)}\gamma \mapsto \mathrm{e}^{-\lambda} \cdot {}^{(n)}\gamma$. In particular, a conformal Carroll structure $[v, \gamma]$ on M induces a positive definite conformal bundle metric $[{}^{(n)}\gamma]$ on $\ker v$. Thus, according to our previous discussion, the bundle

$$\mathfrak{so}(\ker v, [{}^{(n)}\gamma]) \subset \mathrm{End}(\ker v) \tag{B.8}$$

is well-defined, given a conformal Carroll structure $[v, \gamma]$.[13] Any vector bundle endomorphism of $\ker v$ may then be decomposed into its self-adjoint and anti-self-adjoint parts with respect to $[{}^{(n)}\gamma]$, and the anti-self-adjoint part is a section of $\mathfrak{so}(\ker v, [{}^{(n)}\gamma])$.

Given a timelike co-direction $[\tau]$, we may identify vector bundle endomorphisms of $\ker v$ with vector bundle endomorphisms of T^*M which are *purely spatial with respect to* $[\tau]$, i.e. which take values in $\ker v$ and vanish on $\mathrm{span}\{\tau\}$.[14] Under this identification, the decomposition of such a purely spatial map $X\colon T^*M \to T^*M$ into its self-adjoint and anti-self-adjoint parts with respect to $[{}^{(n)}\gamma]$ reads

$$X_\mu{}^\nu = h^{\nu\sigma}\gamma_{\rho(\sigma}X_{\mu)}{}^\rho + h^{\nu\sigma}\gamma_{\rho[\sigma}X_{\mu]}{}^\rho \,, \tag{B.9}$$

where $h^{\mu\nu}$ is the contravariant space metric with respect to $[\tau]$.

Further, under transposition (i.e., abstractly speaking, taking dual maps), purely spatial (w.r.t. $[\tau]$) endomorphisms of T^*M correspond to endomorphisms of TM which are purely spatial w.r.t. $[\tau]$ in the sense of vanishing on $\mathrm{span}\{v\}$ and taking values in $\ker \tau$. Therefore, the decomposition of a purely spatial map $Y\colon TM \to TM$ according to

$$Y^\mu{}_\nu = h^{\mu\sigma}\gamma_{\rho(\sigma}Y^\rho{}_{\nu)} + h^{\mu\sigma}\gamma_{\rho[\sigma}Y^\rho{}_{\nu]} \tag{B.10}$$

—lowering the first index with γ, decomposing into symmetric and antisymmetric parts, and raising the first index again with h—corresponds to the decomposition of endomorphisms of $\ker v$ into self-adjoint and anti-self-adjoint parts with respect to $[{}^{(n)}\gamma]$, under this identification.[15]

Construction B.3. Let $[v, \gamma]$ be a conformal Carroll structure on M, and $[\tau]$ a timelike co-direction with respect to it. Further, let P be the spatial projector along $[\tau]$.

---

[13] Note that $\ker v$ is invariant under rescaling of v, and hence well-defined given only a conformal Carroll structure.

[14] Concretely, the identification map reads

$$\mathrm{End}(\ker v) \ni \hat{X} \overset{\cong}{\longmapsto} \iota \circ \hat{X} \circ \tilde{P} \in \left\{ X \in \mathrm{End}(T^*M) \;\middle|\; X|_{\mathrm{span}\{\tau\}} = 0,\ \mathrm{im}(X) \subseteq \ker v \right\},$$

where $\iota\colon \ker v \to T^*M$ is the inclusion and $\tilde{P}\colon T^*M \to \ker v$ is the co-restriction of P, the spatial projector along $[\tau]$, to $\ker v$. The inverse to this map is restriction and co-restriction of X to $\ker v$.

[15] That is, under the identification of purely spatial endomorphisms of TM first with purely spatial endomorphisms of T^*M via transposition, and then with endomorphisms of $\ker v$ by restriction and co-restriction to $\ker v$.

In construction B.1, we have seen that when defining a connection on M in terms of its non-metricities w.r.t. $[v, \gamma]$, the freely specifiable torsion components w.r.t. $[\tau]$ may be written in the conformally invariant form (B.3)

$$\left(h^{\mu\kappa} \gamma_{\lambda[\kappa} T^{\lambda}{}_{\nu]\sigma} \, (\delta^{\sigma}_{\rho} - P^{\sigma}_{\rho}) \; , \; P^{\lambda}_{\kappa} P^{\rho}_{\mu} P^{\sigma}_{\nu} T^{\kappa}{}_{\rho\sigma} \; , \; (\delta^{\rho}_{\sigma} - P^{\rho}_{\sigma}) T^{\sigma}{}_{\mu\nu} \right) . \tag{B.11}$$

We will now discuss the space in which this form of the free torsion lives.

The second term in (B.11)—the 'purely spatial' projection of the torsion w.r.t. $[\tau]$—may be seen as a section of the vector bundle $\ker \tau \otimes \bigwedge^2 \ker v$. Similarly, the third term—the temporal projection w.r.t. $[\tau]$ on the first index—is a section of $\operatorname{span}\{v\} \otimes \bigwedge^2 T^*M$.

In the first term in (B.11), the third index of the torsion is temporally projected. Hence, due to the anti-symmetry of the torsion, the second index becomes purely spatial (i.e. contractions of the second index with elements of $\operatorname{span}\{v\}$ vanish). The first index of the torsion is spatially projected by contraction with γ. Thus, comparing to (B.10), we see that we may interpret the first term in (B.11) as follows: we temporally project the third index of the torsion, and on the first two indices, we spatially project and then take the anti-self-adjoint part w.r.t. the conformal bundle metric $[{}^{(n)}\gamma]$ on $\ker v$, in the sense of construction B.2 (ii). Therefore, implicitly identifying purely spatial endomorphisms of TM with endomorphisms of $\ker v$ as in construction B.2 (ii), the first term in (B.11) is a section of $\mathfrak{so}(\ker v, [{}^{(n)}\gamma]) \otimes \operatorname{span}\{\tau\}$.[16]

Combined, we have thus seen that and how the conformally invariant form of the free torsion (B.11) of a connection on a conformal Carroll manifold with respect to a timelike co-direction $[\tau]$ is a section of the bundle

$$\left(\mathfrak{so}(\ker v, [{}^{(n)}\gamma]) \otimes \operatorname{span}\{\tau\} \right) \oplus \left(\ker \tau \otimes \textstyle\bigwedge^2 \ker v \right) \oplus \left(\operatorname{span}\{v\} \otimes \textstyle\bigwedge^2 T^*M \right) . \tag{B.12}$$

Note that this value space (B.12) of the conformally invariant form of the free torsion depends on the choice of timelike co-direction $[\tau]$. (This is different to the case of Galilei geometry (Schwartz 2025), where the *value* $P^{\rho}_{\sigma} T^{\sigma}{}_{\mu\nu}$ of the free torsion depends on the choice of timelike direction $[v]$, but the *space* $\ker \tau \otimes \bigwedge^2 T^*M$ in which it takes values does not.) We may get rid of this $[\tau]$ dependence of the value space, however at the cost of giving up conformal invariance—see the discussion in construction 5.4.

References

Adlam, Emily, Linnemann, Niels, and Read, James (2025). *Constructive axiomatics for spacetime physics*. Oxford University Press. ISBN: 978-0-19-892237-7. DOI: `10.1093/9780198922391.001.0001`.

---

[16] Note that the *constrained* part of the torsion, which according to (5.2) is $T_{(\mu\nu)\rho} v^{\rho}$ or, in conformally invariant form, $h^{\mu\kappa} \gamma_{\lambda(\kappa} T^{\lambda}{}_{\nu)\sigma} \, (\delta^{\sigma}_{\rho} - P^{\sigma}_{\rho})$, is the corresponding *self-adjoint* part w.r.t. $[{}^{(n)}\gamma]$ of the spatial projection of the torsion on its first two indices, with the third index temporally projected.

Alishahiha, Mohsen, Davody, Ali, and Vahedi, Ali (2009). 'On AdS/CFT of Galilean Conformal Field Theories'. *J. High Energy Phys.* 2009.08, p. 022. DOI: 10.1088/1126-6708/2009/08/022. arXiv: 0903.3953 [hep-th].

Bagchi, Arjun, Basu, Rudranil, and Mehra, Aditya (2014). 'Galilean Conformal Electrodynamics'. *J. High Energy Phys.* 2014 (11), p. 061. DOI: 10.1007/JHEP11(2014)061. arXiv: 1408.0810 [hep-th].

Bagchi, Arjun, Chakrabortty, Joydeep, and Mehra, Aditya (2018). 'Galilean field theories and conformal structure'. *J. High Energy Phys.* 2018 (04), p. 144. DOI: 10.1007/JHEP04(2018)144. arXiv: 1712.05631 [hep-th].

Bagchi, Arjun and Gopakumar, Rajesh (2009). 'Galilean conformal algebras and AdS/CFT'. *J. High Energy Phys.* 2009 (07), p. 037. DOI: 10.1088/1126-6708/2009/07/037. arXiv: 0902.1385 [hep-th].

Bagchi, Arjun and Mandal, Ipsita (2009). 'On representations and correlation functions of Galilean conformal algebras'. *Phys. Lett. B* 675.3, pp. 393–397. DOI: 10.1016/j.physletb.2009.04.030. arXiv: 0903.4524 [hep-th].

Bergshoeff, Eric, Gomis, Joaquim, and Longhi, Giorgio (2014). 'Dynamics of Carroll particles'. *Class. Quantum Gravity* 31.20, p. 205009. DOI: 10.1088/0264-9381/31/20/205009. arXiv: 1405.2264 [hep-th].

Bergshoeff, Eric et al. (2017). 'Carroll versus Galilei gravity'. *J. High Energy Phys.* 2017 (03), p. 165. DOI: 10.1007/JHEP03(2017)165. arXiv: 1701.06156 [hep-th].

Bergshoeff, Eric A. et al. (2026). 'Applied Conformal Carroll Geometry'. In: *Geometric Science of Information. GSI 2025*. Ed. by Frank Nielsen and Frédéric Barbaresco. Vol. 16034. Lect. Notes Comput. Sci. Cham: Springer, pp. 323–331. DOI: 10.1007/978-3-032-03921-7_33.

Cartan, E (1923). 'Sur les variétés à connexion affine et la théorie de la relativité généralisée (première partie)'. *Ann. Sci. Éc. Norm. Supér.* 40, pp. 325–412. DOI: 10.24033/asens.751.

– (1924). 'Sur les variétés à connexion affine et la théorie de la relativité généralisée (première partie) (Suite)'. *Ann. Sci. Éc. Norm. Supér.* 41, pp. 1–25. DOI: 10.24033/asens.753. English translation available as (Cartan 1986).

Cartan, Elie (1986). *On Manifolds with an Affine Connection and the Theory of General Relativity*. Trans. by Anne Magnon and Abhay Ashtekar. With a forew. by Andrzej Trautman. Monographs and Textbooks in Physical Science. Napoli: Bibliopolis. ISBN: 9788870880861. URL: https://bibliopolis.it/shop/on-manifolds-with-an-affine-connection-and-the-theory-of-general-relativity/. English translation of original article series (Cartan 1923, 1924).

Chandrasekaran, Venkatesa et al. (2023). 'An algebra of observables for de Sitter space'. *J. High Energy Phys.* 2023 (02), p. 82. DOI: 10.1007/JHEP02(2023)082. arXiv: 2206.10780 [hep-th].

Chen, Bin and Liu, Reiko (2023). 'The shadow formalism of Galilean CFT$_2$'. *J. High Energy Phys.* 2023 (05), p. 224. DOI: 10.1007/JHEP05(2023)224. arXiv: 2203.10490 [hep-th].

Chen, Bin, Sun, Haowei, and Zheng, Yu-fan (2024). 'Quantization of Carrollian conformal scalar theories'. *Phys. Rev. D* 110 (12), p. 125010. DOI: `10.1103/PhysRevD.110.125010`. arXiv: `2406.17451 [hep-th]`.

Ciambelli, Luca et al. (2019). 'Carroll structures, null geometry, and conformal isometries'. *Phys. Rev. D* 100 (4), p. 046010. DOI: `10.1103/PhysRevD.100.046010`. arXiv: `1905.02221 [hep-th]`.

Curiel, Erik (2015). *A Weyl-Type Theorem for Geometrized Newtonian Gravity*. Withdrawn due to an error in the proof of the main result. arXiv: `1510.02089 [gr-qc]`.

Dewar, Neil and Read, James (2020). 'Conformal Invariance of the Newtonian Weyl Tensor'. *Found. Phys.* 50 (11), pp. 1418–1425. DOI: `10.1007/s10701-020-00386-w`. arXiv: `2009.09733 [gr-qc]`.

Dewar, Neil and Weatherall, James Owen (2018). 'On Gravitational Energy in Newtonian Theories'. *Found. Phys.* 48 (5), pp. 558–578. DOI: `10.1007/s10701-018-0151-6`. arXiv: `1707.00563 [physics.hist-ph]`.

Dombrowski, Heinz Dieter and Horneffer, Klaus (1964). 'Die Differentialgeometrie des Galileischen Relativitätsprinzips'. *Math. Z.* 86 (4), pp. 291–311. DOI: `10.1007/BF01110404`.

Dunajski, Maciej and Gundry, James (2016). 'Non-Relativistic Twistor Theory and Newton–Cartan Geometry'. *Commun. Math. Phys.* 342.3, pp. 1043–1074. DOI: `10.1007/s00220-015-2557-8`. arXiv: `1502.03034 [hep-th]`.

Dunajski, Maciej and Penrose, Roger (2023). 'Quantum state reduction, and Newtonian twistor theory'. *Ann. Phys. (N. Y.)* 451, p. 169243. DOI: `10.1016/j.aop.2023.169243`. arXiv: `2203.08567 [gr-qc]`.

Dutta, Sudipta (2024). 'Stress tensors of 3d Carroll CFTs'. *Phys. L. B* 853, p. 138672. DOI: `10.1016/j.physletb.2024.138672`. arXiv: `2212.11002 [hep-th]`.

Duval, C., Gibbons, G. W., and Horvathy, P. A. (2014a). 'Conformal Carroll groups'. *J. Phys. A* 47.33, p. 335204. DOI: `10.1088/1751-8113/47/33/335204`. arXiv: `1403.4213 [hep-th]`.

Duval, C. et al. (2014b). 'Carroll versus Newton and Galilei: two dual non-Einsteinian concepts of time'. *Class. Quantum Gravity* 31.8, p. 085016. DOI: `10.1088/0264-9381/31/8/085016`. arXiv: `1402.0657 [gr-qc]`.

Duval, Christian and Horvathy, Peter (2011). 'Conformal Galilei groups, Veronese curves, and Newton–Hooke spacetimes'. *J. Phys. A* 44, p. 335203. DOI: `10.1088/1751-8113/44/33/335203`. arXiv: `1104.1502 [hep-th]`.

Ehlers, Jürgen (1981). 'Über den Newtonschen Grenzwert der Einsteinschen Gravitationstheorie'. In: *Grundlagenprobleme der modernen Physik: Festschrift für Peter Mittelstaedt zum 50. Geburtstag*. Ed. by J. Nitsch, J. Pfarr, and E.-W. Stachow. Mannheim, Wien, Zürich: Bibliographisches Institut, pp. 65–84. Republished as (Ehlers 2019).

– (2019). 'On the Newtonian limit of Einstein's theory of gravitation'. *Gen. Relativ. Grav.* 51 (12), p. 163. DOI: `10.1007/s10714-019-2624-0`. URL: `https://doi.org/10.1007/s10714-019-2624-0`. Republication of original article (Ehlers 1981) as 'Golden Oldie'.

Ehlers, Jürgen and Buchert, Thomas (2009). 'On the Newtonian Limit of the Weyl Tensor'. *Gen. Relativ. Grav.* 41 (9), pp. 2153–2158. DOI: 10.1007/s10714-009-0855-1. arXiv: 0907.2645 [gr-qc].

Ehlers, Jürgen, Pirani, Felix A. E., and Schild, Alfred (1972). 'The geometry of free fall and light propagation'. In: *General Relativity. Papers in honour of J. L. Synge*. Ed. by L. O'Reifeartaigh. Oxford: Clarendon Press, pp. 63–84. Republished as (Ehlers, Pirani, and Schild 2012).

– (2012). 'The geometry of free fall and light propagation'. *Gen. Relativ. Grav.* 44 (6), pp. 1587–1609. DOI: 10.1007/s10714-012-1353-4. Republication of original article (Ehlers, Pirani, and Schild 1972) as 'Golden Oldie'.

Ewen, Holger and Schmidt, Heinz Jürgen (1989). 'Geometry of free fall and simultaneity'. *J. Math. Phys.* 30.7, pp. 1480–1486. DOI: 10.1063/1.528279.

Friedrichs, Kurt (1928). 'Eine invariante Formulierung des Newtonschen Gravitationsgesetzes und des Grenzüberganges vom Einsteinschen zum Newtonschen Gesetz'. *Math. Ann.* 98, pp. 566–575. DOI: 10.1007/BF01451608.

Gupta, Nishant and Suryanarayana, Nemani V. (2021). 'Constructing Carrollian CFTs'. *J. High Energy Phys.* 2021 (03), p. 194. DOI: 10.1007/JHEP03(2021)194. arXiv: 2001.03056 [hep-th].

Hagen, C. R. (1972). 'Scale and Conformal Transformations in Galilean-Covariant Field Theory'. *Phys. Rev. D* 5 (2), pp. 377–388. DOI: 10.1103/PhysRevD.5.377.

Hansen, Dennis, Hartong, Jelle, and Obers, Niels A. (2019). 'Gravity between Newton and Einstein'. *Int. J. Mod. Phys. D* 28.14, p. 1944010. DOI: 10.1142/S0218271819440103. arXiv: 1904.05706 [gr-qc].

Hartong, Jelle, Obers, Niels A., and Oling, Gerben (2023). 'Review on non-relativistic gravity'. *Front. Phys* 11, p. 1116888. DOI: 10.3389/fphy.2023.1116888. arXiv: 2212.11309 [gr-qc].

Herfray, Yannick (2022). 'Carrollian manifolds and null infinity: a view from Cartan geometry'. *Class. Quantum Gravity* 39.21, p. 215005. DOI: 10.1088/1361-6382/ac635f. arXiv: 2112.09048 [gr-qc].

Künzle, H. P. (1972). 'Galilei and Lorentz structures on space-time : Comparison of the corresponding geometry and physics'. *Ann. Inst. Henri Poincaré* 17.4, pp. 337–362. URL: http://www.numdam.org/item/AIHPA_1972__17_4_337_0/.

– (1976). 'Covariant Newtonian Limit of Lorentz Space-Times'. *Gen. Relativ. Gravit.* 7.5, pp. 445–457. DOI: 10.1007/BF00766139.

Malament, David B. (2012). *Topics in the Foundations of General Relativity and Newtonian Gravitation Theory*. Chicago Lectures in Physics. Chicago: The University of Chicago Press. ISBN: 9780226502458. URL: https://press.uchicago.edu/ucp/books/book/chicago/T/bo12893557.html. available at the author's website under https://www.socsci.uci.edu/~dmalamen/bio/GR.pdf.

March, Eleanor (2023). 'Non-relativistic twistor theory: Newtonian limits and gravitational collapse'. PhilSci-Archive:22924. URL: https://philsci-archive.pitt.edu/22924/.

– (2025). 'A note on Newton-Cartan Weyl-type theorems'. Unpublished note.

March, Eleanor and Read, James (2025). 'A primer on Carroll gravity'. *Class. Quantum Gravity* 42.5, p. 055004. DOI: 10.1088/1361-6382/adaf03. arXiv: 2409.12200 [physics.hist-ph].

Reichenbach, Hans (1969). *Axiomatization of the Theory of Relativity*. Trans. by Maria Reichenbach. With a forew. by Wesley C. Salmon. Berkeley, Los Angeles: University of California Press.

Schwartz, Philip K. (2025). 'The classification of general affine connections in Newton–Cartan geometry: Towards metric-affine Newton–Cartan gravity'. *Class. Quantum Gravity* 42.1, p. 015010. DOI: 10.1088/1361-6382/ad922f. arXiv: 2403.15460 [math-ph].

– (2026). *Newton–Cartan Gravity: A Modern Introduction to Geometrised Newtonian Gravity*. Vol. 1044. Lect. Notes Phys. Cham: Springer. ISBN: 978-3-032-03967-5. DOI: 10.1007/978-3-032-03967-5.

Trautman, Andrzej (1963). 'Sur la théorie newtonienne de la gravitation'. *C. R. Acad. Sci.* 257, pp. 617–620. URL: https://gallica.bnf.fr/ark:/12148/bpt6k4007z/f639.item.

– (1965). 'Foundations and current problems of general relativity'. In: *Lectures on General Relativity*. Ed. by Stanley Deser and Kenneth W. Ford. Englewood Cliffs, N. J.: Prentice-Hall, pp. 1–248. URL: http://trautman.fuw.edu.pl/publications/Books/21.pdf.

Vigneron, Quentin, Barzegar, Hamed, and Read, James (2025). 'Affine connections for Galilean and Carrollian structures: a unified perspective'. *Class. Quantum Gravity* 42.17, p. 175018. DOI: 10.1088/1361-6382/adfc1e. arXiv: 2506.03936 [gr-qc].

Weyl, H. (1921). 'Zur Infinitesimalgeometrie: Einordnung der projektiven und der konformen Auffassung'. *Nachr. Gött. Ges. Wiss., Math.-Phys. Kl.* 1921, pp. 99–112. URL: https://eudml.org/doc/59096.

Witten, Edward (2022). 'Gravity and the crossed product'. *J. High Energy Phys.* 2022 (10), p. 8. DOI: 10.1007/JHEP10(2022)008. arXiv: 2112.12828 [hep-th].

Wolf, William J., Sanchioni, Marco, and Read, James (2024). 'Underdetermination in Classic and Modern Tests of General Relativity'. *Eur. J. Philos. Sci.* 14.4, p. 57. DOI: 10.1007/s13194-024-00617-1. arXiv: 2307.10074 [physics.hist-ph].