Do we need to know what two is for defining ‘two’? Theory and metatheory: should they align?

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June 11, 2026

Abstract

In a recent debate after a PhD defence, we were involved in the discussion about what the mediaeval philosophers (after Buridan) called the *logica utens*, the logic of common reason. The question was whether, in order to create a non-classical logical system or mathematical theory, we still use our standard logic in the metatheory, which, if pushed sufficiently, will approach *classical logic*. All the contenders agreed that some further discussion was in order. In this note, I sketch my opinion with the intention of fostering discussion; the example of ‘defining two’ serves as a background for the discussion.

Keywords: logica utens, logica docens, constructivism, non-classical logic, metalanguage.

1 Introduction

After a recent defence of a PhD thesis, some of us became slightly involved in a discussion that can be summed up as follows: for defining ‘two’, do we need to know in advance what *two* is? The answer, apparently, is of astonishing triviality: if we don’t know what *two* is, how can we know what we are defining? But things are not so easy. In the beginning of his *Blue Book*, when asking for the meaning of a word, Wittgenstein stressed:

> “Let us attack this question by asking, first, what is an explanation of the meaning of a word; what does the explanation of a word look like?

> The way this question helps us is analogous to the way the question 'how do we measure a length?' helps us to understand the problem 'what is length?'
>
> The questions 'What is length?', 'What is meaning?', 'What is the number one?' etc., produce in us a mental cramp. We feel that we can't point to anything in reply to them and yet ought to point to something. (We are up against one of the great sources of philosophical bewilderment: a substantive makes us look for a thing that corresponds to it.)" (Wittgenstein, 1969, p.1)

Yes, the philosopher touched on a huge question. To start our discussion, we take one of his questions and change 'the number one' to 'the number two' since it seems to be a better example for our aims. Let us explore the point.

2 From intuition to formal systems

Our intuitive or naïve notion of 'two' comes from our experience with two things: a pair of sheep, a pair of pebbles, etc. This achieves many 'natural' numbers and may also give us the idea of negative numbers, so that having an account to pay of two Euros and having nothing in our pocket, we can count as a debit of 'minus two' Euros in our pocket. But this intuitive notion has limitations and is not useful for science and mathematics. So we need an adequate definition of natural, negative, and other numbers.

This intuitive elaboration of the reality surrounding us, dividing it into distinct objects, is what G. Toraldo di Francia called 'objectuation' (*oggettuazione*) and, according to him, "is a primitive act of our mind, that is, logically (and chronologically) previous to all other activities of reasoning." (Toraldo di Francia, 1976, p.315) Classical theories like classical logic, standard mathematics, and classical physics seem to have been elaborated, as Newton da Costa says, "by taking a *static* view of reality." (da Costa, 1980, p.120) my emphasis. It should be remarked that in present day interpretations (or formulations) of quantum physics, mainly from the point of view of quantum field theories, this intuitive image of 'separated' objects must be seen with reservations, since the underlying ontology is that of quantum fields, which spread across all entire universe; they exist everywhere. This, of course, poses huge philosophical questions, but they will not be pursued here.

The fact is that there are several definitions of 'two', and they are not equivalent, although they basically lead to the same properties. This is not a surprising thing, since they are defined precisely to preserve the desired properties. Does any of them represent *the (intuitive) definition* of 'two'? Take, for instance, the so-called Frege-Russell's definition, used also by Cantor (we ignore the differences between these authors);[1] number two is, roughly speaking, the collection of all collections with two objects. Similar definitions are given for all

[1] We leave out other relevant definitions of natural numbers, such as the 'structural' definition of Dedekind, according to which a natural number is an element of a set N with a distinguished

other natural numbers. But how do we know that this collection (the number 'two') has only collections with *two* elements as its members? Apparently, we need to know in advance what 'two' is.[2]

Now suppose von Neumann's definition; having defined zero as the empty set and the 'successor' of a set x as the set $x^+ := x \cup \{x\}$, we have that one (the successor of zero) is the unitary set whose only element is zero, that is, $1_{vN} := \{\emptyset\}$, and two (the successor of one) is the set with zero and one as elements, namely, $2_{vN} := \{\emptyset, \{\emptyset\}\}$. Zermelo took a slightly different route (before von Neumann); Zermelo's zero is still the empty set, and his one is as in von Neumann's definition, but two is this set: $2_Z := \{\{\emptyset\}\}$. Of course, these definitions are not equivalent in the sense that $2_{vN} \neq 2_Z$ and do not have the same properties. For instance, in von Neumann's stance, every natural number is the set of its predecessors, something that does not occur with Zermelo's definition. But are these definitions *really* defining 'two'? The answer is that this question has no answer.

Notice that both Zermelo and von Neumann have taken a certain set to *represent* the number two, and the choice will be accurate if the desired properties of 'two' are achieved, which is the case with both choices. So, we don't need to *know* what two is, but just how to operate with it, in a rather formalist approach; in Hilbert's words, the postulates that lead to these representations *implicitly define* the terms; hence, the natural numbers are acknowledged to be described (and hence defined) by Peano's axioms.[3]

Let us move to other situations, summarised with this example: suppose we create a non-classical logical system, for instance, a *paraconsistent logic*. In such a logic, the Explosion Rule is restricted so that two contradictory formulas A and $\neg A$ do not necessarily trivialise the system; that is, their existence as theses of the system may not imply that all formulas of the language are theorems of the system. In particular, in such logics, the Principle of Non-Contradiction (PNC) in the form $A \wedge \neg A$ does not hold fully; see (da Costa et al., 2007) and (Barrio et al., 2018) for a different characterisation of these logics. The question then is: in elaborating such a paraconsistent system, should we use PNC in an essential manner? I mean, when we define formulæ, of course, we don't wish for some expression (finite sequence of symbols of the basic alphabet) to *turn out to be* and *not to be* a formula at the same time. This supposition is grounded in PNC, although in the metalanguage. Reformulating, we can say that we *explain* what happens in our theory by using the tools we are more comfortable with, usually some form of constructive logic (see below).

element (he called it '1', but nowadays it is common to call it '0') and a successor function satisfying suitable conditions; see (Reck, 2025).

[2]For a nice exposition of Russell's Type Theory in its *simple* version (without ramified types), where Russell's conception is formalised, see Copi's book (Copi, 1971); you can also see the nice exposition in (Hatcher, 1982, Chap.4).

[3]For a useful presentation of Hilbert's ideas (which coincide in part with Poincaré's (Poincaré, 1902)), see his discussion with Frege about the nature of the axiomatic method, for instance, in (Shapiro, 2005; Fontanella, 2019). Notice that the resulting natural numbers that arise from Peano's axioms depend on the underlying logic. If it is first-order logic, *non-standard* models appear, with 'natural numbers' that lie outside our intuitive ken; see (Melia, 1995).

In this paper, we shall see that, in this case, PNC is being used in the *metalogic*, which, if pushed sufficiently, tends to be something similar to intuitionistic logic. The subjacent question is: does this metalogic need to be 'classical' (or something similar to it)? Another case we shall consider involves the notion of identity, which, to some philosophers, is essential.

Let us contextualise.

3 *Logica utens* and *logica docens*

Gyula Klima, in his book about John Buridan (1301-1358), tells us that Buridan used to distinguish between *logica utens* and *logica docens*, "that is, logic-in-use and logical doctrine, only the latter of which can be called an art or practical science, whereas the former embodies those operative principles that are spelt out by the latter." The *logica utens* provides a general setting, being operative in all rational activities (Klima, 2009, pp.13-14). This is perhaps the first announcement of a distinction between language (or 'logic') and metalanguage ('metalogic'), something accentuated from the 1930s onwards.

It seems clear from the quotation that, in Buridan's opinion (in Klima's interpretation), the *logica docens* must agree with the *logica utens*, since it 'spells out' the suppositions of the *logica utens*; we could say, following Kunen, whom we shall meet soon, that it 'recreates' the principles of the *logica utens*. But today, this idea has been reconsidered due to the rise of (mainly) non-classical systems of logic and, perhaps, the possibility of having a *logica utens* with different characteristics. We can understand the *logica docens* as the logic (or logics) we learn in school, including classical and non-classical logics which, even departing from 'classical concepts', still use some 'classical' way of reasoning in the metalevel; a typical case is paraconsistent logics. The *logica utens* is *the* logic (or logic-schema) by means of which we make inferences in daily life, including in practicing science; it delineates our intellectual intuition and informal ways of reasoning. In present day contexts, it seems that this logic-in-use is something similar to what we call *metatheory*, generally approaching constructive logic.

In our standard reasoning, we tend to accept (if only intuitively) some of the basic notions of classical logic, such as PNC (Principle of Non-Contradiction), the Law of Identity (every object is *identical* to itself and only to itself), the Law of Excluded Middle, LEM (given a proposition and its negation, one of them is true), the Double Negation Law (the negation of the negation of a proposition is equivalent to the proposition) and several other 'principles'. The LEM, for instance, may be questioned on constructive grounds; given a proposition and its negation, perhaps someone could say that she does not have sufficient information to decide which one is true. In some sense, we can say that classical logic *was erected* precisely for copying standard reasoning, first in mathematics and then for making deductions in the empirical sciences and, in general. Notice that this does not exclude other ways of making inferences, such as *inductions* like *non-monotonic* reasoning. But the general idea remains the same:

the basic principles of classical logic seem to be validated in our discourses. The question is: even with non-classical logics as our *logica docens*, does the *logica utens* remain unaltered (whatever it is)? We shall return to this point at the end.

4 Levels of language

In the 20th century, largely due to Tarski, we distinguished between *language* and *metalanguage*, as well as among other levels (meta-metalanguage, etc.). Below, we shall relate the above discussion to these levels of language. As Tarski has stated,

> "when investigating the language of a formalized deductive science, we must always distinguish clearly between the language *about* which we speak and the language *in* which we speak, as well as between the science which is the object of our investigation and the science in which the investigation is carried out." (Tarski, 1983, p.167)

The 'science in which the investigation is being carried out' is the *theory* (or the *object theory* being investigated). The language *in* which we speak of the object language is the *metalanguage*. Tarski still refers to the *metatheory*, formulated in the metalanguage, where the expressions of the metalanguage and the definitions of the concepts— for instance, those related to the deductive apparatus of the theory, its models, etc.— are defined or referred to (Tarski, 1983, p.167).

Usually, the metatheory is naïve set theory (not formalised); however, if pressed, we can say that it is a set theory such as the ZFC system, which is enough for most of mathematics and also for the disciplines of the empirical sciences when dealt with in an axiomatic way. However, usually this metatheory has a 'classical' aspect, that is, it respects most principles of classical logic, or at least a form of constructive logic, as we shall see next.

5 On the apparent necessity of a constructive reasoning

Let us briefly comment on some philosophical considerations that, in our opinion, may guide us in the discussion: those of Newton da Costa and Kenneth Kunen. For both authors, in the limit, the meta-metalevel of our formal languages is a kind of constructive logic. In this section, we consider da Costa, and in the next, we meet Kunen.

Newton da Costa stressed that present day logic and mathematics (and we could also say the empirical sciences) depend on *intellectual intuition*. We can say (based on his ideas) that there is a necessity for an underlying rational activity of a constructive nature in the elaboration of any rational framework. da Costa says:

> "All relevant intellectual activity in the formal and real sciences presupposes something comparable to intuitionistic mathematics and its corresponding logic." (da Costa, 1980, p.57)

Of course, it is not necessary for it to be *exactly* intuitionistic logic, but any suitable form of constructivism. His Principle of Constructivity stresses that

> "The whole exercise of reason presupposes that reason has a certain intuitive capacity for constructive idealisation, whose regularities are well catalogued by intuitionistic arithmetic (with the addition of its underlying logic)." (id.,ibid.)

In (Krause and Arenhart, 2017, 3.1), the authors discuss da Costa's ideas, and in particular reference to the above principle, they ask: "What does he mean by that?" Their answer is summarised as follows.

In developing a formal system such as ZFC set theory, we use expressions such as 'infinite set of individual variables', which presuppose a previous notion of the concept of 'infinity'; hence, the construction of set theory as a formal theory apparently presupposes at least part of the very mathematics it intends to ground, since later we shall *define* infinite sets within ZFC. da Costa stresses:

> "Formal disciplines are essentially discursive. But the discourse develops itself on different levels, and each one of them must be understood or intuited, as already noted by Descartes. Even if one reasons symbolically and formally, the different elementary steps of the evolution of the discourse need to be clear and evident; otherwise, there would be no reasoning, and one would not know what she is doing." (op.cit., p.50)

In trying to systematise a parcel of empirical reality, we make use of intuition and other resources, such as previously learnt theories, personal experiences, memory, imagination, expertise, and insights. In a certain sense, the present-day stage of the evolution of science is a result of not only our biological and cultural characteristics but also of previous theories and scientific backgrounds. But intuition is not enough. We need to systematise our intuitions, and in mathematics, the axiomatic method became the (apparently) best methodological tool, extending to empirical science since the beginning of the 20^{th} century. Well-developed disciplines, even when not completely axiomatised or formalised, can be treated, from a mathematical perspective, in the same way as the formal sciences, being also essentially discursive. In other words, scientific knowledge, being essentially conceptual knowledge, needs discourse, and as da Costa says, our 'discourse' needs language and symbolism (op.cit.,p.35); we could add that our inferences depend on logic, although not necessarily deductive logic.

In a certain sense, we may say that our knowledge of a specific scientific domain is given by means of the elaboration of structures; we 'structure' the domain by gathering concepts, which may be relations and operations over a specific domain (or domains) of objects in which we are interested. Even if we do it only informally, for instance, in psychology, when concepts like *id* and *ego* are used, we are elaborating structures that relate these concepts, usually called *theories*.[4] Thus, we may say that our scientific knowledge is conceptual *and* structural. Sometimes, recall Krause and Arenhart, these 'postulates' are not even formulated explicitly; in mathematics (and much more so in the empirical sciences), a certain field may not be presented axiomatically in the standard sense, such as Analytic Geometry or Differential and Integral Calculus, as presented in a first undergraduate course. These disciplines are not grounded on axioms but are developed from definitions to theorems. The justification is that every axiomatic theory can be transformed into a 'definitional' theory, where definitions play the role of axioms, comprising just definitions and theorems, as seen before.

Going back to da Costa, we recall, as is well known, that in intuitionistic mathematics, we have an intuitive 'visualisation' of the entities that interest us (op.cit., p.50). This is essentially an intellectual intuition—an expression that intends to capture the idea that

> "there cannot be immediate and evident knowledge without contemplation, without a look at the objects that interest us or, at least, at the conceptual relations that define them; in an analogous way, there is no intellectual contemplation that does not enable us to formulate direct judgments, linked to different levels of evidence." (da Costa, 1980, p.51)

The reference to intuitionistic mathematics is grounded in the fact that, according to da Costa, it provides an intuitive 'visualisation' of the entities that interest us (op.cit., p.52), contrary to standard mathematics, where such an intuition does not need to be available (op.cit., p.54). He illustrates the issues with examples of the following kind: we have no clear 'vision' either of transfinite cardinals or of the totality of the real numbers, but only an intuition of the system of relations that implicitly define these concepts by means of axiomatic systems (*idem, ibidem*). This 'intuitive visualisation' provides us with an intuitive pragmatic nucleus, and we may say that, based on this nucleus, we articulate a kind of algebra (da Costa doesn't use this word in this context) that enables us to compose them, operate with them, and so on, moving towards more sophisticated and sometimes not intuitively evident conceptualisations. This way, we go beyond the intuitive nucleus, and in flying so high, the axiomatic method is our 'autopilot', which enables us to navigate domains where our intuitions do not help us much. This process of arriving at and justifying axioms by means of our intuitive notions looks similar to Georg Kreisel's

[4] Certainly, different scientists can choose different concepts and formulate different theories about the same domain.

notion of *informal rigour* (Kreisel, 1967), but this is a point to be further analysed.

In this sense, we begin by describing a formal system using this intuitive nucleus of finitist and constructive nature; as da Costa says, "it is today universally accepted that there cannot be formalised arithmetic without intuitive arithmetic" (op.cit., p.57). It is this informal and intuitive handling of symbols and concepts that, at first glance, enables us to refer to the tools we need to characterise our axiomatic/formalised theories. Thus, it is in this sense that we formulate the language of ZF(C) and *within it* we can state what a denumerable infinite collection of individual variables is.

6 "Formal logic must be developed twice."

The sentence in the title of this section came from Kenneth Kunen's book, *The Foundations of Mathematics* (Kunen, 2009, p.191), where he presents similar ideas. Speaking about how set theory is developed, Kunen says that

> "[y]ou don't need any knowledge about infinite sets; you could learn about these as the axioms are being developed; but you do need to have some basic understanding of finite combinatorics even to understand what statements are and are not axioms. [...] This basic finitistic reasoning, which we do not analyze formally, is called the metatheory. In this metatheory, we explain various notions such as what a formula is and which formulas are axioms of our formal theory, which here is ZFC." (Kunen, 2009, pp.28-9); see also (Krause and Arenhart, 2017)

In the same way as before, we can interpret these words as indicating that we start with intuitive notions necessary, say, to distinguish between two different symbols (such as a and b). So we are presupposing things like the intuitive meaning of the number two as the idea of 'different' things. Then we start to combine these symbols in a way that enables us to elaborate on sophisticated mathematical languages, such as ZFC. Then, in a second stage, we work inside ZF(C) and, with its tools, we can reconstruct the steps we have gone through previously only at an informal level and, in particular, explain rigorously what two may be taken to be. In this sense, as Kunen says, "formal logic must be developed twice" (ibid., p.191). The metatheory, of course, can also be treated formally, but in order to do that, we would need to have available once again, as a prerequisite, an informal metametatheory and so on. But let us turn now to ZF and ZFC set theories.

7 Bohr and the *physics utens*

Niels Bohr, one of the founding fathers of quantum physics, acknowledged that this discipline brings notions that depart from those of classical physics.

Notions such as entangled systems, contextuality, non-locality, and the necessity of considering *completely* indistinguishable things are well-known quantum novelties.[5] But Bohr also thought that our minds work classically. This can be seen from his insistence that to understand quantum notions and to communicate the results to other people, they must be explained in classical terms; as Max Jammer stressed, to Bohr, "classical physics and quantum theory, although asymptotically connected by the correspondence principle, seemed irreconcilable" (Jammer, 1974, p.122). The measurement apparatuses, he said, are classical and work according to classical physics; even if we describe these apparatuses and their interactions with quantum systems 'quantically', they would lose their characteristics as *measurement apparatuses* (Pinto Neto, 2010, Chap.3).

Bohr's insistence on using classical physics to communicate the results of quantum mechanics suggests that he is using such physics as the quantum metatheory or as a *physical utens* (I owe this term to Raony W. Arroyo). In our opinion, this is not *necessarily* so; it being a contingent fact due to the knowledge we have of classical physics, where we feel 'comfortable'. Maybe in other situations, like after a sufficient advance in our familiarity with quantum physics, we will start to reason in quantum physics proper, even at a metamathematical level.

8 Another *logica utens*?

As we have argued above, the *logica utens* was interpreted as the metalogic, or metatheory of a certain theory. According to its standards, it encompasses most classical principles, although it being of a constructive nature. Until the beginning of the 20th century, few scholars considered non-classical systems of logic. Metalogic seems to be as described above; however, let us analyse a possible case where this metalogic could be different.

Part of what follows comes from (Krause, 2025). The Azande (or Zande) are a people who live in Central Africa between the Nile and Congo rivers (more specifically, in present-day South Sudan). They were studied by the anthropologist E. E. Evans-Pritchard, who published a book, *Witchcraft, Oracles, and Magic Among the Azande* relating his experience (Evans-Pritchard, 1976).[6] To summarise one of the most relevant points, the fact is that the Azande believe in witchcraft; according to them, some people are witches and can injure others by a psychic act. They distinguish between witches and sorcerers, and someone can consult an oracle to know if another person is injuring him. If the

[5]Although John Dalton, the father of modern chemistry, stated in 1808 that atoms of the same substance must be completely indistinguishable; that time, elementary quantum entities were not known. Dalton (1808). Today we need to qualify such a statement by restricting it to isomers of the same kind.

[6]The name of this group of people is 'Azande', but 'Zande' (or 'zande') is also used to designate either the people or a particular person or belief, which is why we also find references to 'Zande logic', 'Zande beliefs', etc. See also da Costa et al. (1998); Jennings (1989), (Evans-Pritchard, 1976, Introduction).

oracle says yes, then the suspect is a witch, but if it says no, nothing is inferred in this regard. It seems that they do not see that someone either is a witch or is not; if he is not one, then the other possibility should hold, at least according to the disjunctive syllogism $p \vee q, \neg p \vdash q$, known already by the Stoics. But it is important to remark that this is *our* way of reasoning; it is *our logica utens*, probably (as we prefer to interpret) not theirs.

In our opinion, a better interpretation would be that they *refuse* to proceed to the conclusion as we do. For the Azande, witchcraft is caused by a substance in the body of the witch that can contain various small objects, and this substance is sometimes extracted by autopsy (Evans-Prichard reported that he never saw an exemplar). According to their beliefs, the substance is inherent and transmitted by unilinear descent from male to male; the son of a witch is a witch, but the daughters are not. A clan is formed by direct descent and not by adoption.

The interesting fact is that, in our standard reasoning, it follows that if someone is found to be a witch, then the whole clan is composed of witches. The argument can be put in the following terms (in our present-day way of reasoning) (Jennings, 1989):

(1) All and only witches have witchcraft substance.

(2) Witchcraft-substance is always inherited by the same-sex children of a witch.

(3) The Zande clan is a group of individuals related biologically to one another through the male line.

(4) Man A of clan C is a witch.

(5) The conclusion, not considered by the Azande: Every man in clan C is a witch.

However, as Evans-Pritchard says,

> "[the] Azande see the sense of this argument, but they do not accept its conclusion, and it would involve the whole notion of witchcraft in contradiction were they to do so."

He continues:

> "Azande do not perceive the contradiction as we perceive it because they have no theoretical interest in the subject, and the situations in which they express their beliefs in witchcraft do not force the problem upon them." (Evans-Pritchard, 1976, p.3).

The Azande stop the supposed deduction, not going to its end; that is, they did not go to step (5) above. One of the reasons is that, in these cases, they regard the witch as not 'really' being part of the clan, but as a bastard, the son of another father, introduced into the clan artificially by a sin of the mother, who will now suffer for that even if this is a false accusation (according to our standards). This is just a way to 'save' the clan by introducing another hypothesis into the deduction in typical non-monotonic reasoning.

But even without introducing new hypotheses (as could be required by Popper),[7] similar forms of reasoning appear among us in our daily lives and even in the practice of science; we continue with our hypotheses until we get something we wish for, trying to solve the problems we face even in the presence of inconsistencies, 'ignoring' that the continuation turns them into anomalies which may lead to a contradiction and sometimes finding excuses for them. By the way, perhaps the Azande did not know the notion of 'contradiction' as we do, and surely they don't care about the laws of 'our' logic. The Zande paradigm is preserved even in the presence of such inconsistencies. Their *logica utens* can be a different kind of logic; some guess that it may be a paraconsistent logic (da Costa et al., 1998). I don't agree with these philosophers; I think that the Azande simply reason differently (from us).

The presence of inconsistencies does not necessarily give us a contradiction if *we do not go there*, using Wittgenstein's claim about arithmetic:

> "Can one find a contradiction in a certain system? One might say, 'It depends on you.' —One might say, 'Finding a contradiction in a system, like finding a germ in an otherwise healthy body, shows that the whole system or body is diseased.' —Not at all. The contradiction does not even falsify anything. Let it lie. *Do not go there*. (Diamond, 1976, p.138) (my emphasis)

This way of reasoning entails that one can understand a theory (or a 'paradigm') not as a whole but as a species of constructive thing, despite the underlying suppositions (or 'logic') that may conform to classical settings; that is, it can be a strong system such as ZFC set theory, where non-constructive notions can be formulated. That is, a theory may contain inconsistencies not yet perceived in the metatheory, perhaps because they do not matter, as seems to be the Zande case. This way of treating inconsistencies, by 'isolating' them and working in the 'consistent' parts of the theory, is what we have termed 'Wittgenstein's way' (Krause, 2025).

9 Conclusions

I wrote in the beginning that this paper is a provocation for further discussion. The subject is important and philosophically significant. I hope the paper achieves its aim.

[7] According to Popper, once an anomaly has been detected, we have grounds for refusing a theory Popper (2002).

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