Random-matrix reduction in projective quantum mechanics

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Abstract

We develop a state-space geometric framework for measurement, classicality, and quantum paradoxes, based on one dynamical conjecture. Classical configuration space and classical phase space for a mechanical system arise as distinguished submanifolds of projective quantum state space. On these submanifolds, the Fubini–Study geometry induces Euclidean classical geometry, and the tangent component of Schrödinger evolution reproduces Newtonian dynamics. Within this framework, interactions with measuring devices and environments are described by random-matrix dynamics on projective state space, generated by matrices drawn from the Gaussian Unitary Ensemble. We show that this random-matrix dynamics yields isotropic diffusion, giving Born-rule transition probabilities in microscopic measurements and stabilizing classical behavior in macroscopic systems. We further argue that the random-matrix conjecture is not an independent ad hoc assumption: under natural translation-invariance assumptions on the distribution of state-space steps originating on the classical submanifold, the unitary lift of homogeneous and isotropic Brownian motion on that submanifold is uniquely given by the Gaussian Unitary Ensemble, up to scale and an irrelevant scalar part. The resulting framework provides a unitary account of measurement and the quantum-to-classical transition and, if accepted, offers a dynamical resolution of standard quantum paradoxes.

Keywords: State reduction, Measurement problem, Random matrices

1 Introduction

Quantum mechanics contains several closely related foundational problems. The Schrödinger equation is linear and unitary, yet measurements produce individual outcomes rather than their superpositions. The probabilities of these outcomes are given by the Born rule, but this rule does not follow directly from Schrödinger dynamics. Macroscopic bodies follow Newtonian trajectories, although their states must also be governed by quantum dynamics. Classical space appears as the arena of classical dynamics and of all observed events, but its relation to quantum state space remains unclear. These problems are usually discussed under separate names: the measurement problem, the origin of the Born rule, the preferred-basis problem, the quantum-to-classical transition, and the emergence of classical space.

The purpose of this paper is to show that these problems have a common origin and admit a common experimentally testable resolution based on a geometric framework and a single dynamical conjecture. The geometric framework follows from the Schrödinger dynamics and from the fact that classical space and classical phase space can be represented by distinguished submanifolds of projective quantum state space. These submanifolds have Euclidean induced geometry and support Newtonian dynamics as the tangent component of Schrödinger evolution. In this sense, classical mechanics is not external to quantum mechanics; it is contained in quantum mechanics as tangent dynamics on a submanifold of projective state space.

The resulting geometry provides the setting for the dynamical conjecture. We assume that, under coarse-graining, the effect of many short and complicated interactions of a particle, or a system of particles, with a measuring device or environment is described by a random-matrix Hamiltonian. More specifically, the effective interaction Hamiltonian is modeled by independent draws from the Gaussian Unitary Ensemble. Under this random-matrix conjecture, denoted below by **(RM)**, the state evolution becomes an isotropic random walk on projective state space.

The conjecture and its consequences are closely analogous to Einstein's derivation of Brownian motion, a process that may be used to describe measurement errors in classical physics. There, too, one does not begin with a complete microscopic description of all individual interactions, but with effective assumptions of small independent increments, homogeneity, and isotropy. In the present framework, Brownian motion on the classical-space submanifold is the restriction of the corresponding state-space random walk. Conversely, under the assumption that the distribution of state-space steps originating on this submanifold is invariant under spatial translations, the unitary lift of a homogeneous and isotropic Gaussian random walk on the classical-space submanifold is generated by random Hermitian matrices from the Gaussian Unitary Ensemble, up to scale and an irrelevant scalar part. Thus **(RM)** is not merely analogous to Brownian motion; under these conditions, it is the unique unitary lift of Brownian measurement dynamics. Additional motivation for **(RM)** comes from the well-known ubiquity of random matrices in the description of fluctuations in quantum systems.

The geometric framework and the conjecture **(RM)** have complementary roles. The geometry explains where classical variables come from and why Schrödinger dynamics has a Newtonian component. The random-matrix conjecture explains why states of macroscopic systems remain close to the classical sector and why microscopic measurements yield individual outcomes with Born probabilities. When the **(RM)**-induced diffusion is restricted to the classical space submanifold, it becomes ordinary Brownian motion and gives the normal distribution of classical measurement errors. In the full projective state space, the same diffusion gives the Born rule. Thus the normal distribution in classical measurement and the Born rule in quantum measurement are two manifestations of a single state-space diffusion process.

Mathematically, the classical space and phase-space submanifolds of state space are formed from equivalence classes of states whose position standard deviation is bounded above by a parameter σ. Physically, σ is related to the resolution of position-measuring devices. A measuring device distinguishes only equivalence classes of states that differ at the given resolution. Thus a classical outcome of a position measurement is not a single wave function, but an equivalence class of sufficiently localized states. This point is essential for explaining why a state undergoing diffusion in projective state space has a nonzero probability of reaching the finite-resolution classical sector, despite the finite dimensionality of the corresponding classical submanifold. It also avoids the usual problem of "tails."

The resulting picture is as follows. Before measurement, a microscopic system evolves according to the usual Schrödinger dynamics. During the measurement interaction, the state may evolve through the full projective state space, where the alternatives associated with different classes can interfere. The (**RM**)-induced isotropic diffusion then leads to a record in one of the detector-defined equivalence classes. The probability of recording a given class is given by the Born rule, in the same sense that a classical diffusion gives the probability of finding a particle in a specified region.

For a macroscopic body, frequent environmental interactions continually return the state to a narrow neighborhood of the classical phase-space submanifold. Between such interactions, the tangent component of the Schrödinger flow gives Newtonian drift. The resulting stochastic process is a random walk in projective state space with intermittent conditioning on returns to detector-defined localized sectors. For macroscopic bodies, this produces a sequence of localized records whose conditional distributions remain narrow around the classical path. The observed motion is therefore a stroboscopic Newtonian trajectory.

It is important that the primary dynamics is not motion of particles in $\mathbb{R}^3$, but motion of states in projective state space. The framework therefore should not be interpreted as a hidden trajectory theory in ordinary space, or as an attempt to visualize quantum motion as particle motion in $\mathbb{R}^3$. In particular, unlike Bohmian mechanics, it does not postulate definite particle positions guided by the wave function. Rather, the state itself evolves in projective state space, which becomes the primary arena for the physical processes described here. Classical motion in $\mathbb{R}^3$ appears only when the state is located near the classical phase-space submanifold of state space.

Assuming that **(RM)** holds as an effective coarse-grained description, and using the geometric representation of classical space and phase space as submanifolds of state space, one obtains strict unitary evolution at the fundamental level, individual outcomes with Born probabilities, and recovery of Newtonian motion within a single framework. These consequences are mathematically precise and physically nontrivial. They also place the standard quantum paradoxes in a new light: many of them arise from assigning classical properties to states that do not lie on the appropriate classical submanifold, or from identifying measurement outcomes with exact rays rather than finite-resolution equivalence classes. The aim of the paper is to make this framework explicit and to show that the parameter regimes required for macroscopic classicality are physically plausible.

The paper is organized as follows. The first part develops the geometric framework and reviews the embedding of classical configuration space and phase space into projective quantum state space. In particular, it shows how the classical action and Newtonian equations arise from the Schrödinger action by restriction to the classical phase-space submanifold. The next part introduces the random-matrix conjecture and explains its relation to Brownian motion, isotropic diffusion, equivalence classes, and the Born rule. This is followed by a formulation of macroscopic classicality as a conditioned stochastic process combining Newtonian tangent drift and **(RM)** diffusion. The subsequent estimates show that physically reasonable parameter choices yield Newtonian behavior for macroscopic bodies and Born-rule reduction for microscopic measurements. The paper then discusses the double-slit experiment, cloud-chamber tracks, Stern–Gerlach measurement, and macroscopic superpositions in this framework. It concludes with a comparison to decoherence, continuous measurement, collapse models, and other approaches to the quantum-to-classical transition.

Some of the material presented here appeared previously in [1, 2] and in earlier publications by the author. The present work, however, gives a more complete and unified treatment. It includes new results, theorems, and physically motivated estimates, as well as a detailed discussion of standard quantum-mechanical paradoxes within this framework and a comparison with existing approaches. A separate companion paper [3] provides extensive numerical simulations supporting the main theoretical results.

2 Classical submanifolds of projective state space

Let the one-particle Hilbert space be $L_2(\mathbb{R}^3)$, and let $\mathbb{CP}^{L_2}$ denote the corresponding projective state space with the Fubini–Study metric. For a small localization parameter $\sigma > 0$, consider states represented by functions of the form

$$\varphi_{\mathbf{a},\mathbf{p}}(\mathbf{x}) = r_{\mathbf{a},\sigma}(\mathbf{x})\, e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}, \tag{1}$$

where

$$r_{\mathbf{a},\sigma}(\mathbf{x}) = \sigma^{-3/2} r\left(\frac{\mathbf{x}-\mathbf{a}}{\sigma}\right). \tag{2}$$

Here $r \in L_2(\mathbb{R}^3)$ is a fixed normalized real-valued sufficiently regular function, centered at the origin, with finite variance normalized to 1. As $\sigma \to 0$, the densities $r^2_{\mathbf{a},\sigma}(\mathbf{x})$ converge in the sense of distributions to $\delta^3(\mathbf{x}-\mathbf{a})$ [4].

Let $M^{\sigma}_{3,3}$ be the image of the map

$$\Omega : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{CP}^{L_2}, \qquad (\mathbf{a},\mathbf{p}) \longmapsto \varphi_{\mathbf{a},\mathbf{p}}. \tag{3}$$

Similarly, let M_3^σ be the image of the corresponding restricted map

$$\omega : \mathbb{R}^3 \longrightarrow \mathbb{CP}^{L_2}, \qquad \mathbf{a} \longmapsto r_{\mathbf{a},\sigma}. \tag{4}$$

Theorem 1. *The images $M_{3,3}^\sigma$ and M_3^σ of the maps Ω and ω are embedded submanifolds of $\mathbb{CP}^{L_2}$. The metrics induced on them by the ambient Fubini–Study metric, equivalently the pull-backs of the Fubini–Study metric under Ω and ω, are Euclidean after an appropriate rescaling of units. Thus $M_{3,3}^\sigma$ and M_3^σ are isometric to $\mathbb{R}^3 \times \mathbb{R}^3$ and $\mathbb{R}^3$, respectively.*

Sketch of proof The maps Ω and ω are differentiable embeddings, and their images are the submanifolds $M_{3,3}^\sigma$ and M_3^σ of $\mathbb{CP}^{L_2}$. The restrictions of these maps provide coordinate parametrizations of the corresponding submanifolds. The tangent space to $M_{3,3}^\sigma$ is spanned by the derivatives

$$\frac{\partial \varphi_{\mathbf{a},\mathbf{p}}}{\partial a^i}, \qquad \frac{\partial \varphi_{\mathbf{a},\mathbf{p}}}{\partial p^j}.$$

The Fubini–Study metric is obtained from the Hilbert inner product after quotienting by the vertical phase direction. Pulling this metric back by Ω, direct computation shows that the $\mathbf{a}$-directions and $\mathbf{p}$-directions are orthogonal and have constant norms. Hence, after rescaling the coordinates, the induced metric on $M_{3,3}^\sigma$ is Euclidean. Accordingly, $M_{3,3}^\sigma$, equipped with the metric induced from the ambient Fubini–Study metric, is isometric to $\mathbb{R}^3 \times \mathbb{R}^3$ in these coordinates. The same pull-back computation for ω gives the corresponding Euclidean metric on M_3^σ, making M_3^σ isometric to $\mathbb{R}^3$. □

We now show that the submanifold $M_{3,3}^\sigma$ also carries the classical dynamics. Consider the action functional

$$S[\varphi] = \int \overline{\varphi}(\mathbf{x}, t) \left[i\hbar \frac{\partial}{\partial t} - \widehat{h} \right] \varphi(\mathbf{x}, t)\, d^3\mathbf{x}\, dt, \tag{5}$$

where

$$\widehat{h} = -\frac{\hbar^2}{2m}\Delta + \widehat{V}(\mathbf{x}, t). \tag{6}$$

Variation of (5) with respect to φ gives the Schrödinger equation.

Theorem 2. *The restriction of the Schrödinger action* (5) *to $M_{3,3}^\sigma$ gives the classical action, up to a total derivative, an additive σ-dependent constant independent of $(\mathbf{a}, \mathbf{p})$, and a potential error that vanishes as $\sigma \to 0$, assuming continuity of V.*

Sketch of proof We restrict φ in (5) to $M_{3,3}^\sigma$ by taking

$$\varphi(\mathbf{x}, t) = r_{\mathbf{a}(t),\sigma}(\mathbf{x}) e^{i\mathbf{p}(t)\cdot\mathbf{x}/\hbar}.$$

For this restricted ansatz, the time-derivative term satisfies

$$i\hbar\langle\varphi, \partial_t\varphi\rangle = \mathbf{p}\cdot\dot{\mathbf{a}} + \frac{d}{dt}(\text{total derivative term}),$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product and the contribution from the real amplitude vanishes by normalization. The kinetic term satisfies

$$\left\langle \varphi, -\frac{\hbar^2}{2m}\Delta\varphi \right\rangle = \frac{\mathbf{p}^2}{2m} + C_\sigma,$$

where

$$C_\sigma = \frac{\hbar^2}{2m\sigma^2}\int_{\mathbb{R}^3} |\nabla r(\mathbf{y})|^2\, d^3\mathbf{y}$$

is independent of $\mathbf{a}$ and $\mathbf{p}$. Thus the non-potential terms already have the classical form, modulo inessential total-derivative and additive constant terms.

The only genuine approximation comes from the potential term

$$\int_{\mathbb{R}^3} V(\mathbf{x},t) r^2_{\mathbf{a},\sigma}(\mathbf{x})\, d^3\mathbf{x}.$$

Since $r^2_{\mathbf{a},\sigma} \rightharpoonup \delta^3(\mathbf{x}-\mathbf{a})$ as $\sigma \to 0$, this term converges to $V(\mathbf{a},t)$ whenever V is continuous at $\mathbf{a}$. Therefore, assuming continuity of the potential and taking σ sufficiently small, the restricted action takes the classical form

$$S = \int \left[\mathbf{p}\cdot\frac{d\mathbf{a}}{dt} - h(\mathbf{p},\mathbf{a},t)\right] dt, \tag{7}$$

where

$$h(\mathbf{p},\mathbf{a},t) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{a},t). \tag{8}$$

□

If V is differentiable, variation of (7) gives Hamilton's equations

$$\frac{d\mathbf{a}}{dt} = \frac{\mathbf{p}}{m}, \qquad \frac{d\mathbf{p}}{dt} = -\nabla V(\mathbf{a},t), \tag{9}$$

which are the Newton equations. In this sense, differentiability of V is needed not for the convergence of the action itself, but for deriving the Newtonian equations of motion from the resulting classical action.

The same result can be formulated directly in terms of the Schrödinger vector field on projective state space. Let

$$X_{\widehat{h}}(\varphi) = -\frac{i}{\hbar}\widehat{h}\varphi$$

denote the Schrödinger velocity. For $\varphi \in M^\sigma_{3,3}$, decompose $X_{\widehat{h}}$ into a component tangent to $M^\sigma_{3,3}$ and an orthogonal component:

$$X_{\widehat{h}} = X_T + X_\perp.$$

Theorem 3. *The tangent component X_T of the Schrödinger vector field on $M^\sigma_{3,3}$ is the Newtonian vector field on classical phase space. In coordinates $(\mathbf{a},\mathbf{p})$, it is given by*

$$X_T = \dot{\mathbf{a}}\cdot\partial_{\mathbf{a}} + \dot{\mathbf{p}}\cdot\partial_{\mathbf{p}},$$

where $\dot{\mathbf{a}} = d\mathbf{a}/dt$ and $\dot{\mathbf{p}} = d\mathbf{p}/dt$ satisfy (9).

Sketch of proof The tangent vectors $\partial_{\mathbf{a}}\varphi$ and $\partial_{\mathbf{p}}\varphi$ span $T_\varphi M^\sigma_{3,3}$. Projecting the Schrödinger velocity onto this tangent space gives precisely the coefficients obtained from the constrained variation of the restricted action. Therefore the tangent component of the Schrödinger flow is the Hamiltonian flow of (8), i.e. Newtonian motion. □

The orthogonal component $X_\perp$ measures the tendency of the state to leave the classical submanifold. For Gaussian representatives with variance σ^2, the norm of the orthogonal component is

$$\|X_\perp\| = \frac{\hbar}{4\sqrt{2}\, m\sigma^2},$$

which is the usual spreading scale of a localized packet. For sufficiently massive bodies, this normal component is small, although its effect can accumulate over time. Thus the geometric construction explains the form of the classical equations, but by itself it does not explain why macroscopic states remain close to the classical submanifold.

The preceding results justify the identification of $\mathbf{a}$ and $\mathbf{p}$ with the classical position and momentum variables. The parameter $\mathbf{a}$ labels the center of localization of the packet, while $\mathbf{p}$ enters through the linear phase factor $e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}$. The localized state is not a momentum eigenstate; rather, $\mathbf{p}$ determines the local wave vector and hence the group velocity of the packet. More generally, for a sufficiently narrow packet with smooth phase $\Theta(\mathbf{x})$, expansion near the center $\mathbf{a}$ gives

$$\Theta(\mathbf{x}) = \Theta(\mathbf{a}) + \nabla\Theta(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + O(|\mathbf{x} - \mathbf{a}|^2).$$

The constant term is projectively irrelevant, while the linear term is represented by $e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}$. Thus a general localized state has a linear phase to leading order, with corrections controlled by the localization scale σ. This justifies the choice of $\varphi_{\mathbf{a},\mathbf{p}}(\mathbf{x})$ in (1) in the regime of sufficiently small σ.

It follows that $M^\sigma_{3,3}$, equipped with the induced Fubini–Study metric and the tangent Schrödinger dynamics, is identified with the classical phase-space submanifold, while M^σ_3 is its configuration-space part. The normal component of the Schrödinger vector field measures the tendency of the state to leave this classical sector. The free parameter σ controls the localization scale of the submanifolds M^σ_3 and $M^\sigma_{3,3}$. As discussed in the Introduction, we interpret σ physically as the resolution scale of position-measuring devices. This motivates an equivalence-class formulation, in which localized states indistinguishable at resolution σ are identified. The equivalence-class formulation introduced next gives the classical sector its operational meaning in terms of finite-resolution measurements.

2.1 Equivalence-class version of the classical submanifolds

For simplicity, consider first one spatial dimension. Let

$$g_{c,\sigma}(z) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/4} \exp\left[-\frac{(z-c)^2}{4\sigma^2}\right]$$

denote the normalized Gaussian of width σ centered at c. In one dimension, the map

$$\omega : \mathbb{R} \to \mathbb{CP}^{L_2}, \qquad \omega(c) = g_{c,\sigma},$$

parametrizes a one-dimensional submanifold of $\mathbb{CP}^{L_2}$, denoted here by M_1^σ.

For Gaussian representatives in one dimension, one has

$$\cos^2 \rho(g_{c,\sigma}, g_{d,\sigma}) = \exp\left[-\frac{(c-d)^2}{4\sigma^2}\right]. \tag{10}$$

It follows that

$$\rho(g_{c,\sigma}, g_{d,\sigma}) = \frac{|c-d|}{2\sigma} + o\left(\frac{|c-d|}{2\sigma}\right),$$

and hence M_1^σ, equipped with the induced Fubini–Study metric, is isometric to $\mathbb{R}$ with its Euclidean distance rescaled by $1/(2\sigma)$.

Let $\mu_z(\psi)$ and $\delta_z(\psi)$ denote the position expectation value and position standard deviation of a normalized state $\psi \in L_2(\mathbb{R})$. For a fixed resolution σ, define the equivalence class associated with the classical position c by

$$\{g_c\} = \{\psi \in L_2(\mathbb{R}) : \mu_z(\psi) = c, \quad \delta_z(\psi) \leq \sigma\}. \tag{11}$$

The Gaussian state $g_{c,\sigma}$ is a convenient representative of the equivalence class. Each class contains infinitely many mutually orthogonal states. All states within a given class are experimentally indistinguishable by a position-measuring device whose resolution is no finer than σ.

The Fubini–Study distance from a state ψ to an equivalence class is defined by

$$\rho(\psi, \{g_c\}) = \inf_{\chi \in \{g_c\}} \rho(\psi, \chi), \tag{12}$$

and the distance between two equivalence classes is defined by

$$\rho(\{g_c\}, \{g_d\}) = \inf_{\psi \in \{g_c\}} \rho(\psi, \{g_d\}). \tag{13}$$

For nearby classes, the infimum in (13) has the same leading-order asymptotics as the distance between the Gaussian representatives $g_{c,\sigma}$ and $g_{d,\sigma}$. Thus

$$\rho(\{g_c\}, \{g_d\}) = \frac{|c-d|}{2\sigma} + o\left(\frac{|c-d|}{2\sigma}\right), \tag{14}$$

which is the same relation obtained for the representative manifold M_1^σ. Therefore the set of equivalence classes

$$\widetilde{M}_1^\sigma = \{\{g_c\} : c \in \mathbb{R}\}$$

inherits the same Euclidean metric as M_1^σ and is isometric to $\mathbb{R}$.

The corresponding phase-space representative manifold is obtained by adjoining the linear phase factor $e^{ipz/\hbar}$ to the Gaussian representatives. Namely, define

$$M^{\sigma}_{1,1} = \left\{ g_{c,\sigma} e^{ipz/\hbar} : c, p \in \mathbb{R} \right\} \subset \mathbb{CP}^{L_2}.$$

For two representatives in $M^{\sigma}_{1,1}$, direct computation gives

$$\cos^2 \rho \left(g_{a,\sigma} e^{ipz/\hbar}, g_{b,\sigma} e^{iqz/\hbar} \right) = \exp\left[-\frac{(a-b)^2}{4\sigma^2} - \frac{\sigma^2 (p-q)^2}{\hbar^2} \right]. \tag{15}$$

Consequently, for nearby points,

$$\rho^2 \left(g_{a,\sigma} e^{ipz/\hbar}, g_{b,\sigma} e^{iqz/\hbar} \right) = \frac{(a-b)^2}{4\sigma^2} + \frac{\sigma^2 (p-q)^2}{\hbar^2} + o\left(\frac{(a-b)^2}{4\sigma^2} + \frac{\sigma^2 (p-q)^2}{\hbar^2} \right).$$

Thus the induced Fubini–Study metric on $M^{\sigma}_{1,1}$ is

$$ds^2 = \frac{dc^2}{4\sigma^2} + \frac{\sigma^2}{\hbar^2} \, dp^2,$$

where c and p are the position and momentum coordinates, respectively. After rescaling the position and momentum coordinates, $M^{\sigma}_{1,1}$ is isometric to the Euclidean phase plane $\mathbb{R} \times \mathbb{R}$.

The equivalence-class phase-space version is obtained in the same way, by adjoining the phase factor $e^{ipz/\hbar}$ to the equivalence classes of real-valued localized states. Namely, define

$$\widetilde{M}^{\sigma}_{1,1} = \left\{ \{g_c\} e^{ipz/\hbar} : c, p \in \mathbb{R} \right\},$$

where $\{g_c\}$ consists of real-valued states with position expectation c and position standard deviation not exceeding σ. The distance from a state to an equivalence class is defined as above, and the metric on $\widetilde{M}^{\sigma}_{1,1}$ is defined using the Gaussian representatives $g_{c,\sigma} e^{ipz/\hbar}$. Therefore the same formula (15) determines the induced metric on $\widetilde{M}^{\sigma}_{1,1}$. Hence $\widetilde{M}^{\sigma}_{1,1}$ inherits, after the same rescaling of the position and momentum coordinates, the Euclidean metric on the classical phase plane $\mathbb{R} \times \mathbb{R}$.

To avoid confusion, we note that $\widetilde{M}^{\sigma}_{1}$ and $\widetilde{M}^{\sigma}_{1,1}$ differ by the additional phase, or momentum, label. The set of localized states underlying $\widetilde{M}^{\sigma}_{1}$ contains the set underlying $\widetilde{M}^{\sigma}_{1,1}$, since multiplying a localized state by $e^{ipz/\hbar}$ does not change its position expectation or position spread. Conversely, for a sufficiently localized state with smooth phase, the phase is linear to leading order near the center of localization, so it is represented, up to corrections controlled by σ, by such a factor. Thus, for the purposes of localization and position recording, the two underlying localized sectors may be regarded as the same, while the equivalence-class manifold $\widetilde{M}^{\sigma}_{1,1}$ retains the additional momentum information.

The construction extends componentwise to three dimensions. The equivalence-class manifolds

$$\widetilde{M}_3^\sigma, \qquad \widetilde{M}_{3,3}^\sigma$$

are obtained by grouping states with the same position expectation value $\mathbf{a}$ and position spread bounded by σ, with the phase factor $e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}$ added in the phase-space case. They inherit the same induced Euclidean geometry as the representative manifolds M_3^σ and $M_{3,3}^\sigma$.

The reduction of Schrödinger dynamics to Newtonian dynamics also passes to these equivalence-class manifolds. Indeed, the representative calculation above shows that the tangent component of the Schrödinger vector field on $M_{3,3}^\sigma$ is Newtonian. Since all states in a class are indistinguishable at resolution σ, and since the classical variables $(\mathbf{a}, \mathbf{p})$ label the classes, the induced tangent dynamics on $\widetilde{M}_{3,3}^\sigma$ is the same Newtonian dynamics in the variables $(\mathbf{a}, \mathbf{p})$, up to errors below the chosen resolution. Thus the passage from representatives to equivalence classes does not change the classical equations; it gives them their operational meaning.

2.2 Many-particle systems

The same construction applies to systems of many particles. For N distinguishable particles with Hilbert space

$$\mathcal{H}_N = L_2(\mathbb{R}^3)^{\otimes N} \simeq L_2(\mathbb{R}^{3N}),$$

the classical configuration-space sector is represented by tensor products of localized position classes,

$$\widetilde{M}_{3,N}^\sigma = \widetilde{M}_3^{\sigma_1} \otimes \cdots \otimes \widetilde{M}_3^{\sigma_N}. \tag{16}$$

A point of this manifold is labeled by

$$(\mathbf{a}_1, \ldots, \mathbf{a}_N),$$

where $\mathbf{a}_k \in \mathbb{R}^3$ is the position coordinate of the k-th particle. Similarly, the classical phase-space sector is

$$\widetilde{M}_{3,3,N}^\sigma = \widetilde{M}_{3,3}^{\sigma_1} \otimes \cdots \otimes \widetilde{M}_{3,3}^{\sigma_N}, \tag{17}$$

with coordinates

$$(\mathbf{a}_1, \ldots, \mathbf{a}_N; \mathbf{p}_1, \ldots, \mathbf{p}_N).$$

For tensor-product representatives, the Hilbert-space overlap factorizes into the product of the one-particle overlaps. Since the Fubini–Study distance satisfies

$$\cos^2 \rho(\psi, \phi) = |\langle \psi, \phi \rangle|^2,$$

the distance ρ between representatives is determined by this product. In particular, for localized Gaussian representatives one obtains

$$\cos^2 \rho = \exp\left[-\sum_{k=1}^N \frac{(\mathbf{a}_k - \mathbf{b}_k)^2}{4\sigma_k^2} - \sum_{k=1}^N \frac{\sigma_k^2(\mathbf{p}_k - \mathbf{q}_k)^2}{\hbar^2}\right]. \tag{18}$$

Thus, after introducing the corresponding dimensionless coordinates, the induced metric on $\widetilde{M}^{\sigma}_{3,3,N}$ is Euclidean. Consequently,

$$\widetilde{M}^{\sigma}_{3,N} \cong \mathbb{R}^{3N}, \qquad \widetilde{M}^{\sigma}_{3,3,N} \cong \mathbb{R}^{3N} \times \mathbb{R}^{3N},$$

as Riemannian manifolds with the induced metric.

The usual classical picture of N particles in a single copy of physical space $\mathbb{R}^3$ is recovered through the canonical identification

$$(\mathbf{a}_1, \ldots, \mathbf{a}_N) \in \mathbb{R}^{3N} \quad \longleftrightarrow \quad N \text{ points } \mathbf{a}_1, \ldots, \mathbf{a}_N \text{ in one space } \mathbb{R}^3.$$

Thus the tensor product of N localized one-particle classical sectors is the usual N-particle configuration space, equivalently the space of N-point configurations in one classical space.

The dynamical statement also extends. If the N-particle Schrödinger Hamiltonian has the form

$$\widehat{h} = -\sum_{k=1}^{N} \frac{\hbar^2}{2m_k}\Delta_k + \widehat{V}(\mathbf{x}_1, \ldots, \mathbf{x}_N, t), \tag{19}$$

then restriction of the Schrödinger action to $\widetilde{M}^{\sigma}_{3,3,N}$ gives the classical action

$$S = \int \left[\sum_{k=1}^{N} \mathbf{p}_k \cdot \dot{\mathbf{a}}_k - h(\mathbf{a}_1, \ldots, \mathbf{a}_N; \mathbf{p}_1, \ldots, \mathbf{p}_N; t) \right] dt, \tag{20}$$

where

$$h = \sum_{k=1}^{N} \frac{\mathbf{p}_k^2}{2m_k} + V(\mathbf{a}_1, \ldots, \mathbf{a}_N, t). \tag{21}$$

If V is differentiable, constrained variation yields the Newton equations

$$\dot{\mathbf{a}}_k = \frac{\mathbf{p}_k}{m_k}, \qquad \dot{\mathbf{p}}_k = -\nabla_{\mathbf{a}_k} V(\mathbf{a}_1, \ldots, \mathbf{a}_N, t), \qquad k = 1, \ldots, N. \tag{22}$$

Accordingly, the many-particle classical world appears as the tangent Schrödinger dynamics on the corresponding many-particle localized sector of projective state space.

The results of this section are mathematical statements about localized submanifolds of projective quantum state space. They show that classical configuration space and classical phase space, for one particle and for many-particle systems, can be represented inside projective Hilbert space with the correct Euclidean geometry and Newtonian tangent dynamics. These results do not by themselves force an ontological identification. One could regard M^{σ}_3, $M^{\sigma}_{3,3}$, and their equivalence-class versions merely as embedded copies of classical spaces inside $\mathbb{CP}^{L_2}$. However, these submanifolds have precisely the structures expected of classical space and phase space: Euclidean induced geometry, classical variables $(\mathbf{a}, \mathbf{p})$, and Newtonian tangent dynamics. One can, therefore, assume that the classical world is not an additional arena external to quantum mechanics. It is the distinguished localized sector of projective state space.

This assumption is stronger than what is needed merely to derive Newtonian equations from the restricted Schrödinger action or to obtain the later results of the paper. Its role is conceptual and structural. It allows classical records, measurement outcomes, and macroscopic configurations to be described within the same state space in which microscopic quantum superpositions evolve. This identification will be used below when the random-matrix dynamics is added: microscopic measurement, macroscopic classicality, and the standard quantum paradoxes will then be treated as different regimes of one state-space dynamics. However, all of these results can be derived without the identification and we leave it up to the reader to either use the unified state space framework, or to keep the classical picture separate.

3 Randomness, equivalence classes, and the random-matrix conjecture

The preceding section identifies the classical sector of quantum theory: classical configuration space and phase space appear as localized submanifolds of projective state space, and the tangent component of the Schrödinger flow on these submanifolds reproduces Newtonian dynamics. To address why a microscopic measurement yields a single outcome and why the state of a macroscopic body remains close to the classical submanifold rather than wandering into the full projective space, one must add a dynamical mechanism describing the effect of measuring devices and environments.

Brownian motion provides the guiding analogy for introducing such a dynamical mechanism. In classical physics, a measurement of position does not produce an exact point; it produces a recorded value with finite resolution and random error. When the measured particle or the pointer of a measuring device is subject to many small uncontrolled interactions with its environment, these errors may be modeled by Brownian motion. The observed result is then distributed around the classical value, typically according to a normal distribution determined by the diffusion process and the observation period. Thus Brownian motion gives a dynamical model of classical measurement uncertainty.

In Einstein's theory of Brownian motion [5], one does not begin with a detailed account of molecular collisions. Instead, one assumes that the particle undergoes many small, approximately independent random displacements, with a distribution of steps that is homogeneous in space and time and isotropic in the absence of drift. These assumptions are sufficient to derive the diffusion equation and, for a localized initial condition, the normal distribution. The conjecture introduced below is the corresponding state-space analogue: during measurement or environmental interaction, the quantum state undergoes many small random unitary changes, which after coarse-graining produce an isotropic random walk in projective Hilbert space.

3.1 The random-matrix conjecture

Let $\varphi \in \mathbb{CP}^{L_2}$ be the state of the system. During a short interaction interval dt, we write the effective interaction Hamiltonian as a random Hermitian operator $\widehat{h}_{\mathrm{RM}}$. The corresponding infinitesimal evolution is

$$\varphi \longmapsto \exp\left(-\frac{i}{\hbar}\widehat{h}_{\mathrm{RM}}dt\right)\varphi. \tag{23}$$

The conjecture is the following.

(RM) *During measurement or environmental monitoring, the effective interaction Hamiltonians acting on the relevant finite-dimensional coarse-grained subspace are modeled, over successive short time intervals, by independent random matrices drawn from the Gaussian Unitary Ensemble.*

The finite-dimensional restriction in this statement reflects the finite resolution of any actual measurement. A detector distinguishes only a finite number of alternatives at a given resolution and within a finite observation time. The infinite-dimensional projective space is therefore approached through finite-dimensional coarse-grained sectors, with the continuum limit taken only after the induced geometric behavior is identified.

The appearance of the Gaussian Unitary Ensemble in (**RM**) is not arbitrary. It can be motivated from the classical Brownian limit itself. The translated Gaussian states forming M_3^g constitute a complete family in the Hilbert state space $L_2(\mathbb{R}^3)$: if a state is orthogonal to all translated Gaussians, then it is zero in $L_2(\mathbb{R}^3)$. Equivalently, this follows from the standard Fourier-transform argument, since the Fourier transform of a Gaussian is nowhere zero. The corresponding family of tangent vectors, obtained by differentiating the translated Gaussians, is also complete: a state orthogonal to all such tangent vectors would have Fourier transform supported only at $k = 0$, hence would be constant, and therefore zero in $L_2(\mathbb{R}^3)$. Thus, although each tangent space $T_\psi M_3^g$ is finite-dimensional, the family of tangent spaces obtained by translating ψ through M_3^g probes the full state space.

The following result expresses the sense in which (**RM**) is the unitary lift of classical Brownian measurement dynamics. The reason for looking for a unitary lift is the relation between Newtonian and Schrödinger dynamics established above. Newtonian dynamics on the classical phase-space submanifold is obtained by restricting, or projecting, the Schrödinger dynamics to that submanifold. Since Schrödinger evolution is unitary, any state-space process whose restriction gives a stochastic Newtonian process should likewise be represented by a stochastic unitary evolution. Brownian motion, understood as a coarse-grained stochastic process arising from Newtonian dynamics, should therefore appear as the restriction of a unitary stochastic process on projective state space.

Theorem 4. *Suppose that Brownian motion on M_3^g is represented by a homogeneous and isotropic Gaussian random walk. Suppose also that its lift to projective state space is required to be a Schrödinger evolution, and hence unitary, and that the distribution of state-space steps originating at points of M_3^g is invariant under spatial translations* $\mathbf{a} \to \mathbf{a}+\Delta\mathbf{a}$. *Then the Hamiltonian generating the lifted walk is distributed according to the Gaussian Unitary Ensemble, with scale fixed by the Brownian diffusion coefficient on M_3^g. Scalar multiples of the identity contribute only projectively irrelevant phases.*

Sketch of proof By the assumed unitary lift, an infinitesimal step has the form

$$\psi \mapsto e^{-i\widehat{h}\,dt/\hbar}\psi,$$

where $\widehat{h} = \widehat{h}^*$. For $\psi \in M_3^\sigma$, the tangential component of the infinitesimal step is

$$\mathrm{proj}_{T_\psi M_3^\sigma}\left(-\frac{i}{\hbar}\widehat{h}\psi\,dt\right).$$

By assumption, these tangential increments reproduce the homogeneous and isotropic Gaussian random walk on M_3^σ. Translation invariance of the state-space step distribution means that the same law is sampled at all translated localized states, corresponding to spatial translations

$$\mathbf{a} \mapsto \mathbf{a} + \Delta\mathbf{a}.$$

Since the translated tangent directions of M_3^σ form a complete family in the Hilbert state space, the covariance of the lifted generator is fixed by its action on a complete family of directions. Isotropy of the Brownian motion on M_3^σ forces the covariance of the induced tangent increments to be a scalar multiple of the identity on each $T_\psi M_3^\sigma$. By completeness, the covariance of the lifted generator is likewise fixed as a scalar multiple of the identity on the corresponding state-space directions.

Thus $\widehat{h}$ is a centered Gaussian Hermitian operator with invariant covariance. In a finite-dimensional coarse-grained sector, the corresponding law is the centered Gaussian law on Hermitian matrices invariant under unitary conjugation,

$$\widehat{h} \mapsto U\widehat{h}U^{-1}.$$

This is precisely the Gaussian Unitary Ensemble. Its overall scale is fixed by matching the tangential covariance of the induced walk on M_3^σ with the Brownian diffusion coefficient. The scalar part of $\widehat{h}$, although present in the ensemble, contributes only a global phase and therefore does not affect the induced motion in projective state space. □

The preceding argument can be simplified if one assumes from the outset that the distribution of state-space steps is homogeneous and isotropic with respect to the Fubini–Study geometry. In that case, the covariance of the infinitesimal displacement is already the same at every point and in every tangent direction of projective state space. Therefore, once the scale is fixed by matching the induced Brownian covariance in a single tangent direction at one point of M_3^σ, the entire Gaussian unitary ensemble is determined. The translation-invariance and completeness argument above is needed only to obtain this state-space isotropy from the weaker and more physically transparent assumptions of spatial translation invariance along M_3^σ and isotropy of the Brownian motion on M_3^σ.

A simple finite-dimensional analogy helps clarify the uniqueness of the lift. Consider Brownian motion of a bead constrained to a helix in $\mathbb{R}^3$. The helix is one-dimensional, but its tangent directions form a complete set in $\mathbb{R}^3$: no nonzero vector in $\mathbb{R}^3$ is orthogonal to all of them. If an ambient Gaussian random walk in $\mathbb{R}^3$ has a translation-invariant distribution of steps and induces the given homogeneous Brownian motion along the helix, then the tangential data determine the ambient covariance, and hence the ambient Gaussian walk, up to its diffusion scale. In the present setting, M_3^σ plays the role of the helix, projective Hilbert space plays the role of the ambient space, and unitary Schrödinger evolution replaces arbitrary Euclidean steps by Hermitian generators.

It is useful to distinguish two closely related local descriptions of a random walk on a submanifold. A constrained random walk on a submanifold is defined intrinsically: its small steps lie in the tangent directions of the submanifold, and the resulting path remains on the submanifold. Equivalently, it may be modeled by allowing only those infinitesimal ambient steps that are tangent to the submanifold. A projected random walk begins with steps in the ambient space and then, for each step, retains its tangent component. For homogeneous and isotropic Gaussian small steps, these two descriptions have the same local tangential distribution after the diffusion scale is matched. Thus, at the infinitesimal level relevant for Brownian motion, the constrained walk on the submanifold agrees with the tangent projection of an ambient isotropic Gaussian walk.

In the present setting, Brownian motion on M_3^g is the constrained classical walk, while (**RM**) gives the ambient unitary walk in projective state space. The preceding theorem showed that, under the stated translation-invariance and completeness assumptions, Brownian motion on M_3^g has (**RM**) as its Gaussian unitary lift. Conversely, an (**RM**)-induced state-space walk gives Brownian motion on M_3^g when its infinitesimal displacements are restricted to their tangential components along M_3^g.

Theorem 5. *For initial states $\psi \in M_3^g$, the tangential component of the* (**RM**)*-generated state-space displacement defines a homogeneous and isotropic Gaussian random walk on M_3^g. In the small-step limit, this walk converges to Brownian motion on M_3^g, with diffusion coefficient fixed by the scale of the GUE distribution.*

Sketch of proof Let $\psi \in M_3^g$. Under an infinitesimal (**RM**) step,

$$\psi \mapsto e^{-i\widehat{h}_{\mathrm{RM}}dt/\hbar}\psi,$$

the first-order displacement in projective state space is represented by

$$-\frac{i}{\hbar}\widehat{h}_{\mathrm{RM}}\psi\, dt,$$

after quotienting by the vertical phase direction. Its tangential component along M_3^g is

$$\mathrm{proj}_{T_\psi M_3^g}\left(-\frac{i}{\hbar}\widehat{h}_{\mathrm{RM}}\psi\, dt\right).$$

Since $\widehat{h}_{\mathrm{RM}}$ is drawn from the Gaussian Unitary Ensemble, this tangential displacement is Gaussian. The unitary invariance of the GUE implies that the distribution of the ambient infinitesimal displacement is the same at all points of projective state space and has no preferred projective direction. Consequently, its tangent projection to $T_\psi M_3^g$ has the same covariance at all points $\psi \in M_3^g$. Moreover, the induced covariance on each $T_\psi M_3^g$ is a scalar multiple of the identity with respect to the induced Fubini–Study metric.

Here M_3^g is a real, indeed totally real, submanifold of complex projective state space. The induced increments on $T_\psi M_3^g$ therefore have ordinary real orthogonal isotropy with respect to the induced Fubini–Study metric. At the same time, these increments arise as tangent components of an ambient unitary random walk in projective state space. The ambient walk is generated by Hermitian operators drawn from the GUE, whose unitary invariance gives transitivity on complex projective state space. Thus the real orthogonal isotropy observed on M_3^g is the restriction of the unitary-invariant state-space dynamics to the real classical submanifold.

Thus the tangent projection of the (**RM**)-induced state-space walk has the local covariance of Brownian motion on M_3^g. Passing to the small-step limit gives Brownian motion on M_3^g, with diffusion coefficient fixed by the scale of the GUE distribution. □

The following property of (**RM**)-induced motion is central to the derivation of the Born rule and to the explanation of classical stability.

Theorem 6. *The random unitary steps generated by* (**RM**) *induce a homogeneous and isotropic random walk on projective Hilbert space with respect to the Fubini–Study metric.*

Sketch of proof In the finite-dimensional sector in which (**RM**) is applied, the GUE measure on Hermitian matrices is invariant under

$$\widehat{h}_{\mathrm{RM}} \mapsto U\widehat{h}_{\mathrm{RM}}U^{-1}$$

for every unitary U. Therefore the infinitesimal unitary transformation generated by $\widehat{h}_{\mathrm{RM}}$ has a distribution invariant under the induced action of the unitary group on projective space. Since this action preserves the Fubini–Study metric and is transitive, the displacement distribution is the same at every point of projective space. Moreover, the stabilizer of each point acts transitively on directions of equal Fubini–Study norm in the tangent space, so the displacement law is isotropic in each tangent space. This is precisely homogeneity and isotropy of the induced state-space random walk. □

The Gaussian Unitary Ensemble is therefore singled out in three complementary ways. First, using the completeness of the translated Gaussian states and their tangent vectors, a homogeneous and isotropic Brownian walk on M_3^g has a unique Gaussian unitary lift under spatial translation invariance. This lift is generated by GUE Hamiltonians, with the overall scale fixed by the Brownian diffusion coefficient. Second, once GUE is used as the distribution of random Hamiltonians, the induced tangential walk on M_3^g is Brownian motion on the classical-space submanifold. Third, GUE gives a homogeneous and isotropic random walk in projective Hilbert space. Note that the same conclusion would not hold for the Gaussian Orthogonal Ensemble, because the orthogonal group does not act transitively on complex projective space.

The conjecture **(RM)** is not a microscopic derivation of the interaction Hamiltonian from first principles. However, the preceding results show that its form is strongly constrained, and in this sense nearly forced, by the geometric framework. Once Brownian measurement dynamics on the classical-space submanifold is required to admit a translation-invariant unitary lift to projective state space, the effective Hamiltonian must be drawn from the Gaussian Unitary Ensemble, with scale fixed by the Brownian diffusion coefficient. Thus **(RM)** is not an arbitrary stochastic postulate, but the natural state-space analogue of the assumptions used in Einstein's derivation of Brownian motion. This conclusion is also consistent with the familiar role of random matrices in modeling universal fluctuation statistics in complex quantum systems [6, 7]. Its content is that the coarse-grained effect of many short, complicated, and uncontrollable interaction events is an isotropic random walk in projective state space, whose induced tangential motion on the classical-space submanifold reproduces Brownian measurement dynamics. The physical adequacy of this effective description must ultimately be tested by its consequences and by the plausibility of the resulting parameter values. These estimates will be discussed below.

3.2 Equivalence classes and measurement outcomes

A measuring device cannot distinguish all points of projective Hilbert space. In the present framework, a recorded outcome is therefore not identified with an individual state vector, but with one of the equivalence classes of states indistinguishable by the device at the given resolution. As in Section 2.1, for a position measurement in one spatial direction, let

$$\mu_z(\varphi) = \langle \varphi, \widehat{z}\varphi \rangle$$

denote the expectation value of position, and let

$$\delta_z(\varphi) = \big(\langle \varphi, (\widehat{z} - \mu_z)^2 \varphi \rangle\big)^{1/2}$$

denote the corresponding position standard deviation. A finite-resolution position outcome is represented by a class of states with the same value of μ_z, interpreted as the recorded position at the given resolution, and with position uncertainty bounded by the resolution scale σ, up to an inessential numerical factor. Thus a classical position outcome is represented not by a single wave function, but by an equivalence class localized within the detector resolution.

For the reduction process it is useful to separate the position expectation value and the degree of localization from the remaining directions in state space. Define

$$F(\varphi) = (\mu_z, \delta_z). \tag{24}$$

The level sets

$$\mu_z = \tau, \qquad \delta_z = \lambda$$

form a codimension-two foliation of the relevant regular part of state space. Translating and scaling a suitable initial state produces a two-dimensional surface $M_\varphi \subset \mathbb{CP}^{L_2}$, with coordinates

$$\tau = \mu_z, \qquad s = \ln(\delta_z/\sigma),$$

where the same resolution scale σ is used as reference. Translation changes the mean position τ, while scaling changes the width δ_z, equivalently the coordinate s.

To see the geometry of these coordinates explicitly, one may work in a finite-dimensional approximation of state space generated by a finite superposition of mutually orthogonal, or approximately orthogonal, localized functions. For instance, one may use sufficiently narrow Gaussian states with well-separated centers. In such a representation, the τ- and s-coordinates on M_φ are orthogonal with respect to the induced Fubini–Study metric [1, 2]. The finite-resolution localized sector associated with resolution σ is described by

$$s \leq 0.$$

This is the sector in which the state is sufficiently localized to be assigned to one of the classical position-outcome classes introduced above.

When the walk in (**RM**) is expressed in the (τ, s)-coordinates on M_φ, the τ-component describes motion along the recorded-position direction, while the s-component describes motion toward or away from the localized sector. Since these coordinates are orthogonal and the (**RM**) step distribution is isotropic, the corresponding infinitesimal increments in τ and s are independent Gaussian variables. Motion of the state along the leaves of the foliation does not change μ_z or δ_z, and therefore does not contribute to state reduction.

The state diffuses in projective state space, and a recorded outcome occurs when the path reaches the localized sector $s \leq 0$, corresponding to one of the detector-defined equivalence classes. Since the (**RM**) distribution is isotropic, the induced walk in s is symmetric and has no drift. Hence, after sufficiently many steps, the probability that the state is found in the localized sector at a given observation time is approximately 1/2. As we now show, conditional on reaching the localized sector, the relative probability of reaching a particular outcome sector satisfies the Born rule.

3.3 Normal distribution and Born rule

The same **(RM)**-induced isotropic random walk has two different manifestations, depending on whether it is constrained to a classical submanifold or allowed to evolve in the full projective state space. When constrained to the classical submanifold $M_3^\sigma \simeq \mathbb{R}^3$, the Fubini–Study metric reduces to the Euclidean metric. The **(RM)** process then becomes a random walk in $\mathbb{R}^3$ with independent, isotropic Gaussian increments. In the small-increment limit, this process converges to Brownian motion, and the resulting transition probabilities are normal distributions centered at the classical position. In the full projective state space, the same **(RM)** process is homogeneous and isotropic with respect to the Fubini–Study metric. Hence transition probabilities can depend only on projective distance.

For a single normalized final state ψ, the Born rule may be written as

$$P(\varphi \to \psi) = |\langle \psi, \varphi \rangle|^2 = \cos^2 \rho(\varphi, \psi),$$

where $\rho(\varphi, \psi)$ is the Fubini–Study distance. For an outcome represented by a closed subspace $\mathcal{H}_\alpha$, or by the corresponding projective sector, the probability is

$$P(\alpha) = \|P_\alpha \varphi\|^2,$$

where P_α is the orthogonal projection onto $\mathcal{H}_\alpha$. In particular, for a position measurement in a region $\Delta \subset \mathbb{R}^3$,

$$P(\Delta) = \|P_\Delta \varphi\|^2 = \int_\Delta |\varphi(\mathbf{x})|^2 \, d^3\mathbf{x},$$

where $P_\Delta \varphi(\mathbf{x}) = \chi_\Delta(\mathbf{x}) \varphi(\mathbf{x})$.

This projection rule can be expressed geometrically as a distance to the outcome sector. Let $\mathbb{P}(\mathcal{H}_\alpha)$ denote the projectivization of $\mathcal{H}_\alpha$, and define

$$\rho(\varphi, \mathbb{P}(\mathcal{H}_\alpha)) = \inf_{\psi \in \mathcal{H}_\alpha,\ \|\psi\|=1} \rho(\varphi, \psi).$$

Since $\rho(\varphi, \psi) = \arccos |\langle \psi, \varphi \rangle|$, one obtains

$$\cos^2 \rho(\varphi, \mathbb{P}(\mathcal{H}_\alpha)) = \sup_{\psi \in \mathcal{H}_\alpha,\ \|\psi\|=1} |\langle \psi, \varphi \rangle|^2 = \|P_\alpha \varphi\|^2.$$

Thus the Born probability of an outcome sector is the squared cosine of the Fubini–Study distance from the initial state to that sector. For a position region Δ, this gives

$$P(\Delta) = \cos^2 \rho(\varphi, \mathbb{P}(\mathcal{H}_\Delta)) = \int_\Delta |\varphi(\mathbf{x})|^2 \, d^3\mathbf{x}.$$

The use of sectors is essential. A physical measurement does not require the random path to hit an exact ray or an exact lower-dimensional submanifold. The detector-defined outcome is an equivalence class, or a finite-resolution neighborhood of such a class. The probability of an outcome is therefore the probability of entering the corresponding sector of indistinguishable states, which has nonzero operational thickness.

Theorem 7. *Under the identification of classical space $\mathbb{R}^3$ with the localized-state submanifold M_3^σ, the Born transition law between states in M_3^σ reproduces the normal probability law for the measured position of the corresponding classical particle. Conversely, if transition probabilities in projective state space depend only on Fubini–Study distance, then the normal probability law on M_3^σ determines a unique extension to arbitrary pairs of states. This extension is the Born transition law.*

Sketch of proof The embedding

$$\omega : \mathbb{R}^3 \to \mathbb{CP}^{L_2}, \qquad \omega(\mathbf{a}) = g_{\mathbf{a},\sigma},$$

identifies the classical position $\mathbf{a}$ of a particle with the localized state $g_{\mathbf{a},\sigma} \in M_3^\sigma$. Therefore, the probability of finding a Brownian particle in a small region $W \subset \mathbb{R}^3$ may be equivalently viewed as the probability that the corresponding state lies in the region $\omega(W) \subset M_3^\sigma$. Thus the normal probability distribution for measured positions in $\mathbb{R}^3$ becomes a probability distribution for localized states on M_3^σ.

Let $W \subset \mathbb{R}^3$ be a small region of diameter δ centered at $\mathbf{b}$. The state associated with this finite-resolution region may be represented by a narrow Gaussian $g_{\mathbf{b},\delta}$. Direct computation gives

$$|\langle g_{\mathbf{a},\sigma}, g_{\mathbf{b},\delta} \rangle|^2 = \left(\frac{2\sigma\delta}{\sigma^2 + \delta^2} \right)^3 \exp\left[-\frac{(\mathbf{a} - \mathbf{b})^2}{2(\sigma^2 + \delta^2)} \right].$$

Equivalently,

$$|\langle g_{\mathbf{a},\sigma}, g_{\mathbf{b},\delta} \rangle|^2 = \cos^2 \rho(g_{\mathbf{a},\sigma}, g_{\mathbf{b},\delta}).$$

As δ becomes small, the right-hand side is the normal probability density centered at $\mathbf{a}$, multiplied by the corresponding small volume factor. Hence the normal probability of finding the classical particle near $\mathbf{b}$ agrees with the Born transition probability from $g_{\mathbf{a},\sigma}$ to the localized state representing that region.

Finally, the Fubini–Study distances between Gaussian states $g_{\mathbf{a},\sigma}$ and $g_{\mathbf{b},\delta}$ range over the full interval of possible projective distances. Therefore, if transition probabilities in projective state space depend only on Fubini–Study distance, the rule determined on M_3^g determines the rule for arbitrary normalized states ψ and ϕ. This gives

$$P(\psi \to \phi) = \cos^2 \rho(\psi, \phi) = |\langle \psi, \phi \rangle|^2,$$

which is the Born rule. □

The same conclusion applies to the equivalence-class manifold $\widetilde{M}_3^g$. The Gaussian representatives determine the induced metric on $\widetilde{M}_3^g$, and distances between classes are computed using these representatives. Hence a classical particle represented by an equivalence class has the same distance relation to other classes as the corresponding Gaussian representative has to the other Gaussian representatives. The normal law for the particle position therefore agrees with the Born transition law between the corresponding representatives. Since this gives the same distance-dependent rule as on M_3^g, the extension to the full projective state space proceeds exactly as before.

Dynamically, the result says that the same state-space diffusion has two complementary descriptions. When the isotropic walk is constrained to the localized classical sector, or equivalently to $\widetilde{M}_3^g$, it becomes the Brownian motion of a measured classical particle, producing the normal distribution of measurement errors. When the same isotropic walk is allowed to evolve through the full projective state space, the transition probabilities are governed by the Fubini–Study distance and therefore by the Born rule. Thus the Gaussian distribution of classical measurement errors and the Born rule of quantum measurement arise from the same geometric source: isotropic diffusion on state space. Classical measurements probe the process through its induced motion on the localized equivalence-class manifold, whereas microscopic measurements involve diffusion through the full projective state space before a detector-defined equivalence class is reached.

Having established the geometric relation between the normal law on the classical submanifold and the Born rule in projective state space, we now turn to the macroscopic regime. The goal is to show that the same (**RM**)-induced stochastic dynamics, combined with the tangent Schrödinger flow on $M_{3,3}^g$, yields stable Newtonian motion when conditioned on finite-resolution environmental records.

4 Macroscopic classicality as a conditioned stochastic process

The previous sections described state reduction as an **(RM)**-induced isotropic random walk on projective state space together with equivalence classes determined by finite detector resolution. We now apply the same mechanism to the motion of macroscopic bodies. The goal is to show, at the mathematical level, how Newtonian motion arises as a stroboscopic process: the state undergoes **(RM)** diffusion in a neighborhood of the classical sector, while repeated returns to the localized sector produce recorded positions distributed narrowly around the Newtonian trajectory. In the next section, we provide physical estimates supporting the validity of these assumptions.

We begin with the one-dimensional case of a macroscopic point particle, since it contains the essential mechanism. As in Section 2.1, let $M^{\sigma}_{1,1}$ denote the phase-space submanifold of localized particle states, with coordinates (a, p), and let $\widetilde{M}^{\sigma}_{1,1}$ denote the corresponding formulation in terms of equivalence classes at position resolution σ. On $M^{\sigma}_{1,1}$, the tangent component of the Schrödinger flow gives

$$\frac{da}{dt} = \frac{p}{M}, \qquad \frac{dp}{dt} = -\frac{dV}{da}. \tag{25}$$

The same equations hold on the equivalence-class manifold $\widetilde{M}^{\sigma}_{1,1}$, since the variables (a, p) label the corresponding classes and the induced tangent dynamics is unchanged. Thus, whenever the state lies on $\widetilde{M}^{\sigma}_{1,1}$, or sufficiently close to the set underlying $\widetilde{M}^{\sigma}_{1,1}$, the free Schrödinger velocity has a well-defined Newtonian tangent component.

Mathematically, the process we aim to derive is a random walk with intermittent conditioning. The **(RM)** term produces isotropic random increments in projective state space, while the Schrödinger flow contributes Newtonian tangent drift whenever the state lies sufficiently close to the set of states underlying $\widetilde{M}^{\sigma}_{1,1}$. When the path reaches the σ-localized sector $\widetilde{M}^{\sigma}_{1,1}$, the particle behaves classically, its coordinate a is recorded as a classical position, and the subsequent walk is conditioned on this recorded value. The observed positions then form a sequence of conditional random variables centered on the Newtonian trajectory. The task is to show that physically reasonable choices of the time step and step variance allow the **(RM)** and Schrödinger contributions to be separated while keeping these conditional distributions narrow on the resolution scale σ. In doing so, we will work with the representative manifold $M^{\sigma}_{1,1}$, keeping in mind that the Newtonian component of the motion can equivalently be obtained on the equivalence-class manifold $\widetilde{M}^{\sigma}_{1,1}$.

4.1 Alternating drift and (RM) diffusion

Let the total effective Hamiltonian during environmental monitoring be written schematically as

$$\widehat{h}_{\text{tot}} = \widehat{h} + \widehat{h}_{\text{RM}}, \tag{26}$$

where $\widehat{h}$ is the usual Hamiltonian of the particle and $\widehat{h}_{\text{RM}}$ is the random-matrix Hamiltonian representing the environmental interaction. The process is considered on a coarse-grained time scale on which $\widehat{h}_{\text{RM}}$ acts during short interaction windows, while $\widehat{h}$ gives the deterministic Schrödinger evolution between them.

The desired separation is

$$e^{-\frac{i}{\hbar}(\widehat{h}+\widehat{h}_{\text{RM}})dt} \approx e^{-\frac{i}{\hbar}\widehat{h}_{\text{RM}}dt} e^{-\frac{i}{\hbar}\widehat{h}dt}, \tag{27}$$

on the dynamically relevant σ-localized sector. This approximation is justified when the deterministic displacement and spreading produced by $\widehat{h}$ during a single interaction window are negligible on the resolution scale. Equivalently, if P_{σ} denotes the local tangent projection onto the σ-localized tube around $M^{\sigma}_{1,1}$, one requires

$$P_{\sigma}[\widehat{h}, \widehat{h}_{\mathrm{RM}}]P_{\sigma} = O(\varepsilon), \qquad \varepsilon \ll 1. \tag{28}$$

Under this condition the evolution may be treated, to the required accuracy, as alternating Newtonian tangent drift and **(RM)** diffusion.

As shown in Section 2, for states on $M^{\sigma}_{1,1}$, the Schrödinger velocity admits the orthogonal decomposition

$$-\frac{i}{\hbar}\widehat{h}\psi = \frac{da}{dt}\,\partial_a\psi + \frac{dp}{dt}\,\partial_p\psi + X_{\perp}, \tag{29}$$

where the first two terms are tangent to $M^{\sigma}_{1,1}$, while $X_{\perp}$ is orthogonal to $M^{\sigma}_{1,1}$ with respect to the Fubini–Study metric. The tangent coefficients are given by (25). For states lying on $M^{\sigma}_{1,1}$, the free-evolution velocity thus decomposes into a tangent component generating classical drift and an orthogonal component responsible for spreading.

The orthogonal component is small for macroscopic bodies on environmental interaction time scales. For Gaussian representatives one obtains the spreading scale

$$\|X_{\perp}\| \sim \frac{\hbar}{M\sigma^2}, \tag{30}$$

or equivalently the spreading time

$$T_{\mathrm{spr}} = \frac{M\sigma^2}{\hbar}. \tag{31}$$

Thus the two basic smallness conditions are

$$\frac{v\,dt}{\sigma} \ll 1, \qquad \frac{dt}{T_{\mathrm{spr}}} \ll 1, \tag{32}$$

where v is the characteristic classical velocity and dt is the duration of an environmental interaction window. The first condition says that the tangent drift during one interaction is below resolution; the second says that wave-packet spreading during one interaction is negligible.

4.2 Conditioning on returns to the localized sector

The **(RM)** steps move the state away from the exact classical submanifold into the surrounding projective state space. The relevant question is therefore not whether the state remains exactly on the set underlying $\widetilde{M}^{\sigma}_{1}$, but whether it returns frequently to this localized sector, where a position record can be assigned.

As in the previous section, let

$$s = \ln(\delta_z/\sigma)$$

be a transverse coordinate measuring the logarithmic width of the state, with σ a fixed reference length. The condition that the state lies in the localized sector is

$$s \leq 0. \tag{33}$$

Under **(RM)** dynamics without drift, the discrete-time process s_n, evaluated at successive **(RM)** steps, is a symmetric random walk. Thus the return problem to $\widetilde{M}_1^\sigma$ becomes a standard hitting problem for a one-dimensional random walk.

Let τ_{ret} be the first return time to the sector $s \leq 0$. By standard recurrence properties of symmetric one-dimensional random walks, the return probability is nonzero and returns occur with high probability on sufficiently long time scales. More quantitatively, the Sparre Andersen theorem gives, for the probability that the walk has not returned to $s \leq 0$ after n steps,

$$\mathbb{P}(\tau_{\text{ret}} > n) = \frac{\binom{2n}{n}}{2^{2n}} \sim \frac{1}{\sqrt{\pi n}}. \tag{34}$$

This estimate shows that frequent returns to the localized sector are typical once the **(RM)** step time is sufficiently small.

Whenever the path reaches $\widetilde{M}_1^\sigma$, the coordinate a is recorded. A particle whose state lies in the set underlying $\widetilde{M}_1^\sigma$ behaves classically, and the recording process can be described by the usual Newtonian dynamics of the particle together with either the measuring device or its environment; see Section 6. The subsequent evolution is then conditioned on the recorded value. In this way, the process generates a sequence

$$a_0, a_1, a_2, \ldots$$

of recorded positions. These are not exact values of an underlying classical trajectory, but conditional random variables associated with repeated returns to the detector-defined localized sector and centered on the Newtonian trajectory.

4.3 Stroboscopic Newtonian motion

Let $a_{\text{N}}(t)$ denote the Newtonian solution generated by (25). The recorded positions a_k define stroboscopic Newtonian motion if their conditional distributions remain sharply concentrated around $a_{\text{N}}(t_k)$ on the resolution scale:

$$\mathbb{E}(a_k \mid a_{k-1}) = a_{\text{N}}(t_k) + o(\sigma), \qquad \text{Var}(a_k \mid a_{k-1}) \ll \sigma^2. \tag{35}$$

The first condition expresses that the deterministic component of the motion is Newtonian. The second expresses that the **(RM)**-induced stochastic deviations do not smear the recorded position beyond detector resolution.

The **(RM)** contribution to the recorded coordinates may be written schematically as

$$\Delta a_{\text{RM}} \sim \xi_a \sqrt{dt}, \qquad \Delta p_{\text{RM}} \sim \xi_p \sqrt{dt}, \tag{36}$$

where ξ_a and ξ_p are zero-mean random variables whose variances are determined by the diffusion coefficients D_a and D_p of the induced diffusion on $M^\sigma_{1,1}$, respectively. Thus over a return interval Δt, the stochastic spread in position is of order

$$\mathrm{Var}(\Delta a_{\mathrm{RM}}) \sim D_a\,\Delta t. \tag{37}$$

The recorded motion is Newtonian on the resolution scale provided

$$D_a\,\Delta t \ll \sigma^2, \tag{38}$$

where Δt is the typical time between returns to $\widetilde{M}^\sigma_1$.

Theorem 8 (Stroboscopic Newtonian motion)**.** *Suppose that the* **(RM)** *step time and step variances are chosen so that:*

1. *the free and* **(RM)** *contributions separate on the σ-localized sector, i.e.*

$$P_\sigma[\widehat{h}, \widehat{h}_{\mathrm{RM}}]P_\sigma = O(\varepsilon), \qquad \varepsilon \ll 1;$$

2. *the deterministic tangent displacement and spreading during one interaction window satisfy*

$$\frac{v\,dt}{\sigma} \ll 1, \qquad \frac{dt}{T_{\mathrm{spr}}} \ll 1;$$

3. *the* **(RM)***-induced stochastic position spread between returns satisfies*

$$D_a\,\Delta t \ll \sigma^2.$$

*Then the sequence of recorded positions obtained by conditioning on returns to $\widetilde{M}^\sigma_1$ is concentrated around the Newtonian trajectory on the resolution scale σ. In this sense, the combined Schrödinger–***(RM)** *process yields stroboscopic Newtonian motion.*

Sketch of proof Between **(RM)** steps, the tangent component of the Schrödinger flow on $\widetilde{M}^\sigma_{1,1}$ is Newtonian. The first two assumptions imply that, during each environmental interaction window, the deterministic tangent displacement is below resolution and the orthogonal spreading is negligible. The commutator estimate allows the total evolution to be treated as alternating free drift and **(RM)** steps.

The **(RM)** contribution has zero mean in the tangent coordinates and variance controlled by D_a and D_p. Conditioning on returns to the localized sector produces recorded positions whose conditional mean follows the deterministic Newtonian drift, while their variance is bounded by the accumulated **(RM)** diffusion between returns. The condition $D_a\Delta t \ll \sigma^2$ ensures that this variance remains below detector resolution. Therefore the recorded positions form a stroboscopic sequence concentrated around the Newtonian trajectory. □

This theorem separates the mathematical and physical parts of the argument. Mathematically, if the time step, step variance, and return-time conditions hold, the conditioned process yields stroboscopic Newtonian motion. The remaining question is whether realistic environmental interactions for macroscopic bodies provide such parameters. This will be addressed in the next section.

5 Physical estimates

The previous section separated the mathematical question from the physical one. Mathematically, the combined Schrödinger–**(RM)** process yields stroboscopic Newtonian motion whenever the drift, spreading, diffusion, and return-time estimates satisfy the inequalities stated in Theorem 8. We now show that these inequalities can be satisfied for physically reasonable parameters. We first consider macroscopic bodies and then microscopic measurements.

5.1 Macroscopic bodies

We now specialize the preceding discussion to a macroscopic body monitored by its environment. The relevant interaction events may be taken to be scattering events with environmental molecules, or with ambient radiation. These events occur over very short time intervals and repeatedly transfer position information to the surroundings. The resulting stochastic contribution is modeled by the (**RM**) term, while the ordinary Schrödinger Hamiltonian provides the deterministic tangent motion.

Consider the center of mass of a macroscopic body of mass M, localized at position resolution σ. For definiteness, take

$$M = 10^{-6}\,\mathrm{kg}, \qquad \sigma = 10^{-6}\,\mathrm{m}. \tag{39}$$

For air molecules at room temperature, a typical thermal speed is $v_{\mathrm{th}} \sim 5 \times 10^{2}\,\mathrm{m/s}$. Taking a molecular interaction length of order

$$\ell \sim 10^{-9}\,\mathrm{m},$$

the duration of a single collision is estimated as

$$dt \sim \frac{\ell}{v_{\mathrm{th}}} \sim \frac{10^{-9}\,\mathrm{m}}{5 \times 10^{2}\,\mathrm{m/s}} \sim 2 \times 10^{-12}\,\mathrm{s}.$$

Thus environmental molecular collisions naturally lead to interaction windows of order

$$\tau \lesssim 10^{-12}\,\mathrm{s}.$$

The corresponding free spreading time is

$$T_{\mathrm{spr}} = \frac{M\sigma^2}{\hbar} \sim \frac{10^{-6} \cdot 10^{-12}}{10^{-34}} \sim 10^{16}\,\mathrm{s}. \tag{40}$$

Therefore, during a single environmental interaction window,

$$\frac{\tau}{T_{\rm spr}} \lesssim 10^{-28} \tag{41}$$

is negligible. The orthogonal component of the free Schrödinger velocity, which is responsible for spreading for states on $M^{\sigma}_{1,1}$, therefore does not move the state appreciably away from the localized sector during a single collision.

The tangent drift is also below the resolution scale during such a short interval. If v is a typical macroscopic velocity, say $v \lesssim 1\,\mathrm{m/s}$, then

$$\frac{v\,\tau}{\sigma} \lesssim \frac{(1\,\mathrm{m/s})(10^{-12}\,\mathrm{s})}{10^{-6}\,\mathrm{m}} = 10^{-6}. \tag{42}$$

Thus the deterministic displacement during a single environmental collision is σ-invisible. Combining (41) and (42), the free evolution during one interaction window differs from a scalar action on the dynamically relevant σ-localized sector only by a small error

$$\varepsilon \sim \frac{v\,dt}{\sigma} + \frac{dt}{T_{\rm spr}} \lesssim 10^{-6}. \tag{43}$$

Let P_σ denote the local tangent projection at the point under consideration, where the relevant tangent space is that of the set underlying $\widetilde{M}^{\sigma}_{1,1}$ in the Hilbert space. For states in this localized sector, the above estimate gives, locally and to leading order,

$$\widehat{h}\psi = E_\psi \psi + O(\varepsilon), \tag{44}$$

where E_ψ is the scalar by which $\widehat{h}$ acts on ψ to leading order. Equivalently, over the short interaction interval considered here, the free evolution acts on the localized sector as an overall phase up to an error $O(\varepsilon)$.

Let $\widehat{h}_{\rm RM}$ denote the Hamiltonian in (**RM**). Since the free evolution is scalar to leading order on the localized sector, its commutator with the (**RM**) contribution is negligible there. In the local projected sense just described, this may be written schematically as

$$P_\sigma[\widehat{h}, \widehat{h}_{\rm RM}]P_\sigma = O(\varepsilon), \qquad \varepsilon \ll 1. \tag{45}$$

Thus, on the dynamically relevant localized sector, the free Schrödinger evolution and the (**RM**) interaction may be treated, to the required accuracy, as effectively separable over individual environmental collision windows. This justifies describing the total evolution as alternating free Schrödinger segments and (**RM**) kicks.

5.2 Momentum diffusion and stochastic corrections

The (**RM**) contribution produces stochastic increments in the classical coordinates when the state is close, in the Fubini–Study metric, to the set underlying $\widetilde{M}^{\sigma}_{1,1}$. We write the increments over a short environmental kick time τ schematically as

$$\Delta a_{\rm RM} \sim \xi_a \sqrt{\tau}, \qquad \Delta p_{\rm RM} \sim \xi_p \sqrt{\tau}, \tag{46}$$

where ξ_a and ξ_p are zero-mean random variables whose variances are determined by the diffusion coefficients D_a and D_p.

For environmental molecular scattering, the momentum diffusion coefficient may be estimated from the collision rate and the typical momentum transfer per collision. If Γ denotes the collision rate and q the characteristic momentum transfer in one collision, then

$$D_p \sim \Gamma q^2. \tag{47}$$

For air at room temperature and for a millimeter-scale body, representative values are

$$\Gamma \sim 10^{22}\,\mathrm{s}^{-1}, \qquad q \sim 2.5 \times 10^{-23}\,\mathrm{kg\,m/s}.$$

Thus

$$D_p \sim \Gamma q^2 \sim 10^{22}\left(2.5 \times 10^{-23}\right)^2 \sim 6 \times 10^{-24}\,\mathrm{kg^2 m^2/s^3}. \tag{48}$$

For a single environmental kick time $\tau \sim 10^{-12}\,\mathrm{s}$, the accumulated stochastic momentum increment is

$$\Delta p_{\mathrm{RM}}(\tau) \sim \sqrt{D_p \tau} \sim \sqrt{(6 \times 10^{-24})(10^{-12})} \sim 2.5 \times 10^{-18}\,\mathrm{kg\,m/s}. \tag{49}$$

For $M = 10^{-6}\,\mathrm{kg}$ and $v \sim 1\,\mathrm{m/s}$, the classical momentum is

$$p_{\mathrm{cl}} = Mv \sim 10^{-6}\,\mathrm{kg\,m/s}.$$

Hence, over a single kick,

$$\frac{\Delta p_{\mathrm{RM}}(\tau)}{p_{\mathrm{cl}}} \sim \frac{2.5 \times 10^{-18}}{10^{-6}} \sim 10^{-12}. \tag{50}$$

This agrees with the estimate that an individual environmental kick is negligible compared with the macroscopic Newtonian momentum.

The corresponding position diffusion coefficient for the center of mass is of order

$$D_a \sim \frac{D_p}{M^2}. \tag{51}$$

For $M = 10^{-6}\,\mathrm{kg}$, this gives

$$D_a \sim \frac{6 \times 10^{-24}}{10^{-12}} \sim 6 \times 10^{-12}\,\mathrm{m^2/s}. \tag{52}$$

Thus, for a macroscopic body, the large value of M strongly suppresses the induced position diffusion.

Over a renewal interval T_{ren}, the accumulated stochastic momentum correction has size

$$\Delta p_{\mathrm{stoch}}(T_{\mathrm{ren}}) \sim \sqrt{D_p T_{\mathrm{ren}}}. \tag{53}$$

Using the renewal time scale estimated in the next subsection,

$$T_{\mathrm{ren}} \sim 3 \times 10^{-4}\,\mathrm{s},$$

one obtains

$$\Delta p_{\rm stoch}(T_{\rm ren}) \sim \sqrt{(6\times10^{-24})(3\times10^{-4})} \sim 4\times10^{-14}\,{\rm kg\,m/s}. \tag{54}$$

Therefore

$$\frac{\Delta p_{\rm stoch}(T_{\rm ren})}{p_{\rm cl}} = \frac{\sqrt{D_p T_{\rm ren}}}{Mv} \sim \frac{4\times10^{-14}}{10^{-6}} \sim 4\times10^{-8} \ll 1. \tag{55}$$

Thus even over a full renewal interval, the accumulated stochastic momentum correction remains negligible compared with the macroscopic Newtonian momentum.

Similarly, the position variance accumulated between successive returns to the localized sector is

$$\mathrm{Var}(\Delta a_{\rm RM}) \sim D_a T_{\rm ren}. \tag{56}$$

With the same representative values,

$$D_a T_{\rm ren} \sim (6\times10^{-12})(3\times10^{-4}) \sim 2\times10^{-15}\,{\rm m}^2. \tag{57}$$

Thus the corresponding root-mean-square displacement is

$$\sqrt{D_a T_{\rm ren}} \sim 4\times10^{-8}\,{\rm m}. \tag{58}$$

For the resolution scale $\sigma = 10^{-6}\,$m, this satisfies

$$D_a T_{\rm ren} \ll \sigma^2, \tag{59}$$

or equivalently

$$\sqrt{D_a T_{\rm ren}} \ll \sigma.$$

Therefore the conditional distribution of the recorded position remains narrow on the detector resolution scale.

5.3 Return to the localized sector

The remaining question is whether the state returns to the localized sector frequently enough. As in Section 4, let

$$s = \ln(\delta_z/\sigma)$$

be the logarithmic transverse coordinate measuring spread, with σ fixed and the localized sector represented by

$$s \le 0.$$

Under (**RM**) dynamics with negligible drift in s, the discrete process s_n, evaluated at successive environmental kicks, is a symmetric random walk.

Let $\tau_{\rm ret}$ denote the first return time to $s \le 0$, measured in the number of kicks. By the Sparre Andersen theorem,

$$\mathbb{P}(\tau_{\rm ret} > n) = \frac{\binom{2n}{n}}{2^{2n}} \sim \frac{1}{\sqrt{\pi n}}. \tag{60}$$

Thus the probability of not returning decreases as $n^{-1/2}$. For a sufficiently small environmental kick time τ, this gives frequent returns to the localized sector on macroscopic observational time scales.

For the representative estimates used here, the environmental kick time was estimated above as

$$\tau \sim 10^{-12}\,\mathrm{s}.$$

A return probability of 0.999968, corresponding to a four-standard-deviation level, requires

$$1 - \frac{1}{\sqrt{\pi n}} \approx 0.999968,$$

and hence

$$n \approx \frac{1}{\pi(1-0.999968)^2} \approx 3.1 \times 10^8. \tag{61}$$

The corresponding renewal time is therefore

$$T_{\mathrm{ren}} \sim n\tau \sim (3.1 \times 10^8)(10^{-12}\,\mathrm{s}) \sim 3 \times 10^{-4}\,\mathrm{s}. \tag{62}$$

Thus, with overwhelming probability, the state returns to the localized sector on a sub-millisecond time scale. During such a time interval, the deterministic Newtonian drift remains well defined, while the stochastic position variance remains below σ^2, as estimated above. Consequently, successive recorded positions form a narrow stochastic band around the Newtonian trajectory.

5.4 Macroscopic and microscopic regimes

The estimates above show that the same Schrödinger–(**RM**) dynamics has two distinct limiting regimes. For macroscopic bodies, the relevant conditions are

$$\frac{v\,\tau}{\sigma} \ll 1, \qquad \frac{\tau}{T_{\mathrm{spr}}} \ll 1, \qquad P_\sigma[\widehat{h}, \widehat{h}_{\mathrm{RM}}]P_\sigma = O(\varepsilon), \quad \varepsilon \ll 1, \tag{63}$$

where τ is the environmental kick time. These conditions are combined with

$$\frac{\Delta p_{\mathrm{RM}}(\tau)}{p_{\mathrm{cl}}} \ll 1, \qquad \frac{\Delta p_{\mathrm{stoch}}(T_{\mathrm{ren}})}{p_{\mathrm{cl}}} \ll 1, \qquad D_a T_{\mathrm{ren}} \ll \sigma^2.$$

Here T_{ren} is the high-probability renewal time for return to the localized sector, estimated above. These inequalities define a combined drift–diffusion–resolution regime: the deterministic tangent displacement during a single environmental kick is below resolution, the orthogonal spreading is negligible, the free and (**RM**) contributions separate to the required accuracy, and the stochastic corrections remain small between successive returns to the localized sector.

Theorem 9. *For macroscopic masses, ordinary localization resolutions, and realistic environmental interaction times, there exist physically reasonable* (**RM**) *step-size and time-step parameters satisfying the conditions of Theorem 8. Consequently, the combined Schrödinger–*(**RM**) *process keeps the state of a macroscopic body close to* *the localized sector associated with $\widetilde{M}^{\sigma}_{1,1}$ and produces stroboscopic Newtonian motion on the resolution scale σ.*

Sketch of proof The drift and spreading estimates (41) and (42) imply that, during individual environmental interaction windows, the additional effect of the free evolution is negligible, apart from the Newtonian tangent drift already accumulated between such windows. The commutator estimate (45) justifies treating the evolution as alternating free segments and (**RM**) kicks. The diffusion and return estimates (59) and (60) show that stochastic corrections remain below detector resolution between returns to the localized sector. The conclusion then follows from Theorem 8. □

The microscopic measurement regime is different. If the measurement interval Δt_{meas} is short enough that the system Hamiltonian is negligible compared with the (**RM**) interaction, then

$$e^{-\frac{i}{\hbar}(\widehat{h}+\widehat{h}_{\text{RM}})\Delta t_{\text{meas}}} \approx e^{-\frac{i}{\hbar}\widehat{h}_{\text{RM}}\Delta t_{\text{meas}}}. \tag{64}$$

There is then no appreciable Newtonian displacement during the measurement interval. The state undergoes (**RM**)-induced isotropic diffusion in projective state space and is recorded in one of the detector-defined equivalence classes.

Theorem 10. *If the measurement interaction time is short enough that the contribution of $\widehat{h}$ is negligible compared with $\widehat{h}_{\text{RM}}$, then the measurement process is governed by isotropic diffusion in projective state space. The probabilities of the detector-defined outcome classes are the Born probabilities.*

Sketch of proof Under the approximation (64), the measurement evolution is generated by $\widehat{h}_{\text{RM}}$. By Theorem 6, this produces homogeneous isotropic diffusion in projective state space. By Theorem 7, the probabilities of reaching detector-defined equivalence classes are the Born probabilities. □

Thus microscopic measurement and macroscopic classical motion are different parameter regimes of the same dynamical model. In the microscopic regime, the (**RM**) term dominates and produces Born-rule state reduction. In the macroscopic regime, frequent environmental (**RM**) interactions produce repeated returns to the localized sector, while the tangent component of the Schrödinger flow supplies Newtonian drift. The distinction is therefore not a difference in fundamental laws, but a difference in regime. We conclude that the same linear Schrödinger dynamics, supplemented by (**RM**), yields Born-rule state reduction for microscopic particles and Newtonian trajectories for macroscopic ones.

6 Standard quantum experiments and paradoxes

We now indicate how the preceding framework applies to several standard quantum experiments and foundational paradoxes. The purpose of this section is not to introduce new assumptions or prove additional theorems. Rather, we use the two ingredients established above—classical submanifolds of projective state space and **(RM)**-induced stochastic unitary evolution with detector-defined equivalence classes—to reinterpret the usual paradoxes in a single language. In each case, the apparent difficulty comes from trying to assign classical properties to states that do not lie on the appropriate classical submanifold, or from identifying measurement outcomes with exact rays rather than with finite-resolution equivalence classes.

6.1 Measurement and state reduction

The measurement problem is the question of how linear Schrödinger evolution can yield definite outcomes governed by the Born rule. In the present framework, measurement is not represented by an additional projection postulate. Instead, during interaction with a measuring device, the state undergoes stochastic but unitary evolution generated by the (**RM**) Hamiltonian. The resulting path in projective state space eventually enters one of the detector-defined equivalence classes corresponding to an outcome.

The outcome is definite because, at a given measurement event, the state enters one detector-defined equivalence class. The stochasticity of the (**RM**) Hamiltonian determines which class is reached, while the homogeneity and isotropy of the induced random walk give the Born probabilities, as explained in Section 3. More precisely, the Born probabilities are the relative probabilities of reaching the possible outcome classes, conditioned on the state reaching the relevant localized sector. Thus what is traditionally called collapse is the approach of the state to an operationally defined outcome sector in projective state space.

This process should be distinguished from the later recording of the outcome. Recording occurs after the state has entered an appropriate equivalence class and proceeds through ordinary macroscopic dynamics of the measuring device and environment. No nonunitary collapse law is added; the underlying evolution remains unitary for each realization of the random Hamiltonian.

The use of equivalence classes is essential. A detector does not determine an exact ray in Hilbert space; it determines a finite-resolution outcome class. This avoids the difficulty that a random path in a high-dimensional state space would generally have zero probability of hitting a prescribed ray or exact lower-dimensional submanifold. The relevant probability is the probability of entering a detector-defined equivalence class, which represents all states indistinguishable by the apparatus at the given resolution.

A superposition of alternatives does not represent several simultaneously realized classical configurations. It represents a single point in projective state space, generally lying away from the relevant classical submanifold. The corresponding classical property becomes well defined only when the state enters the appropriate detector-defined equivalence class. Thus the apparent ambiguity of a superposition arises from assigning classical attributes to a state that does not lie in the classical sector.

6.2 Double-slit experiment

The double-slit experiment asks how a particle can display interference when unobserved by the slits, yet appear localized when measured. In this framework, the question is reformulated in state-space terms. A localized particle state near emission may lie close to the classical submanifold $\widetilde{M}_3^{\sigma}$. After interaction with the slit screen, with both slits open and no which-slit measurement, the state generally leaves $\widetilde{M}_3^{\sigma}$ and evolves through the full projective state space. In fact, the state becomes a superposition of two localized states, representing the particle near each slit. The Fubini–Study distance from such a superposition to $\widetilde{M}_3^{\sigma}$ is not small. In that regime, there is no classical trajectory in $\mathbb{R}^3$, and it is not meaningful to say that the particle passed through one slit or both. Instead, the path of the particle leaves the classical-space submanifold $\widetilde{M}_3^{\sigma}$ and passes "over" the screen in projective state space.

If no which-slit detector is present, the state arriving at the final screen contains coherent contributions from both slits. During propagation, the corresponding state functions spread and overlap, so their superposition contains interference terms. The interaction with the screen then induces (**RM**) evolution, and the state enters one of the localized equivalence classes corresponding to a spot on the screen. Repeating the experiment samples these classes with Born probabilities, producing the interference pattern.

If a which-slit detector is present, the **(RM)** interaction occurs at the slits. The state then enters one of the slit-defined equivalence classes with Born probabilities, and the subsequent evolution is conditioned on the recorded outcome. The later distribution at the screen is no longer the Born distribution of the coherent two-slit state. After which-slit localization, each particle evolves from a single localized state and behaves classically to the relevant accuracy, so repeated runs produce an approximately normal distribution of spots rather than an interference pattern. Interference disappears because the state has already been reduced to a detector-defined class before reaching the screen.

Thus the particle does not need to be described either as passing through both slits as a classical object or as secretly choosing one slit before measurement. Its path is a path in projective state space. When the state lies near $\widetilde{M}_3^{\sigma}$, particle-like localization is meaningful. When the state moves away from this localized sector, wave-like interference phenomena may appear. Thus the distance from $\widetilde{M}_3^{\sigma}$ distinguishes the localized, particle-like regime from the delocalized, wave-like regime.

As explained in Section 2.2, the slit screen can be viewed geometrically as part of the composite system. In a full description, the Hilbert space is the tensor product of the particle Hilbert space and the Hilbert space describing the relevant degrees of freedom of the screen. When both the particle and the screen are in localized classical states, the corresponding joint state lies on a product classical configuration-space sector. In the simplest two-body picture this sector is represented, schematically, by $\widetilde{M}_3^{\sigma_p} \otimes \widetilde{M}_3^{\sigma_s}$, where σ_p and σ_s are the resolution scales associated with the particle and the screen degrees of freedom. At this level, the geometry is the same as the geometry of two classical objects in a single copy of classical space.

The slit screen then selects two spatially separated regions through which the particle state may pass. When the interaction with the screen produces a superposition of alternatives associated with the two slits, the joint state no longer lies on the product classical sector $\widetilde{M}_3^{\sigma_p} \otimes \widetilde{M}_3^{\sigma_s}$. Equivalently, the particle component is no longer represented by a single localized point of $\widetilde{M}_3^{\sigma_p}$. The screen remains macroscopic and localized, but the particle state has moved away from the classical space submanifold. This is the state-space meaning of the statement that, with both slits open and no which-slit measurement, the particle does not pass through one definite slit as a classical object.

6.3 Cloud-chamber and bubble-chamber tracks

A cloud-chamber or bubble-chamber track appears to show a microscopic particle following a classical trajectory. In the present framework, the track is not a continuous microscopic path in $\mathbb{R}^3$. It is a sequence of localized records produced by repeated measurement-like interactions with the medium.

This situation is analogous to the macroscopic case discussed above. A macroscopic body in a natural environment is continuously monitored by surrounding molecules, radiation, and other degrees of freedom. The resulting frequent environmental interactions keep its state close to the localized classical sector and produce a stable Newtonian trajectory. In a cloud chamber or bubble chamber, the measured particle has much smaller mass, but this is compensated by the properties of the medium: the medium supplies frequent, localized, and effectively amplifying interactions, such as ionization or bubble formation. These interactions repeatedly return the state to detector-defined position classes.

Between such interactions, the particle state evolves in projective state space. Each sufficiently strong interaction with the medium defines position-like equivalence classes corresponding to ionization or bubble-formation events. The (**RM**) mechanism drives the state into one of these classes, and the subsequent evolution is conditioned on the recorded event. Repetition produces a sequence

$$a_1, a_2, \ldots, a_N$$

of localized records.

Let ℓ_{rec} denote the typical distance between successive record-forming events in the medium, and let u be the speed of the charged particle. For a visible track, the relevant scale is microscopic; one may take representative values

$$\ell_{\mathrm{rec}} \sim 10^{-6}\text{–}10^{-5}\,\mathrm{m}, \qquad u \sim 10^{7}\text{–}10^{8}\,\mathrm{m/s}.$$

The time between successive localized records is then

$$T_{\mathrm{rec}} \sim \frac{\ell_{\mathrm{rec}}}{u} \sim 10^{-14}\text{–}10^{-12}\,\mathrm{s}. \tag{65}$$

Thus, although the particle is microscopic, the medium provides repeated localization on extremely short time scales.

The relevant comparison is with the free spreading time

$$T_{\mathrm{spr}} = \frac{m\sigma^2}{\hbar},$$

where m is the particle mass and σ is the localization resolution of the record. For an electron and a proton one finds, for $\sigma = 10^{-6}$ m,

$$T_{\rm spr}^{(e)} \sim \frac{(9 \times 10^{-31})(10^{-12})}{10^{-34}} \sim 10^{-8}\,\text{s},$$

and

$$T_{\rm spr}^{(p)} \sim \frac{(1.7 \times 10^{-27})(10^{-12})}{10^{-34}} \sim 10^{-5}\,\text{s}.$$

Consequently,

$$\frac{T_{\rm rec}}{T_{\rm spr}} \ll 1 \tag{66}$$

for both electrons and heavier charged particles. The spreading between successive record-forming interactions is therefore negligible on the resolution scale of the track.

Between record-forming interactions, the (**RM**) Hamiltonian is absent, and the state evolves under the ordinary Schrödinger Hamiltonian. The only separation estimate needed is therefore during the short ionization or bubble-formation event itself, when the (**RM**) interaction is active. Let $\tau_{\rm int}$ denote the duration of such an event. If $\ell_{\rm int}$ is the microscopic interaction length and u is the particle speed, then

$$\tau_{\rm int} \sim \frac{\ell_{\rm int}}{u}.$$

Taking

$$\ell_{\rm int} \sim 10^{-10}\text{–}10^{-9}\,\text{m}, \qquad u \sim 10^{7}\text{–}10^{8}\,\text{m/s},$$

gives

$$\tau_{\rm int} \sim 10^{-18}\text{–}10^{-16}\,\text{s}.$$

During such a short interaction window,

$$\varepsilon_{\rm ch} \sim \frac{u\tau_{\rm int}}{\sigma} + \frac{\tau_{\rm int}}{T_{\rm spr}} \sim \frac{\ell_{\rm int}}{\sigma} + \frac{\tau_{\rm int}}{T_{\rm spr}} \ll 1.$$

For a localization resolution on the micron scale, $\sigma \sim 10^{-6}$ m, the first term is 10^{-4}–10^{-3}, while the second is negligible relative to the spreading times estimated above. Thus, during the interaction window, the free Schrödinger evolution acts on the localized sector only up to a projectively irrelevant phase and a negligible error. In the local projected sense used above,

$$P_\sigma[\widehat{h}, \widehat{h}_{\rm RM}]P_\sigma = O(\varepsilon_{\rm ch}).$$

Thus the chamber dynamics may be described as alternating ordinary Schrödinger evolution between records and short (**RM**)-dominated kicks during record formation.

The deterministic tangent displacement between records is

$$\Delta a_{\rm cl} \sim uT_{\rm rec} \sim \ell_{\rm rec}.$$

Thus the observed track is naturally stroboscopic: each record is displaced from the preceding one by the classical tangent motion during the short time between interactions. If the recording resolution is comparable to the spacing between successive record-forming events,

$$\sigma \sim \ell_{\rm rec},$$

with

$$\ell_{\rm rec} \sim 10^{-6}\text{–}10^{-5}\,\mathrm{m},$$

then each segment of the track is resolved as a localized classical record, while the sequence of records follows the Newtonian trajectory to within the spatial resolution of the chamber.

The momentum disturbance caused by a single record-forming interaction is also small compared with the momentum of a typical charged particle in a chamber. For a particle with momentum in the range

$$p_{\rm cl} \sim 10^{-20}\text{–}10^{-19}\,\mathrm{kg\,m/s},$$

corresponding to relativistic or semi-relativistic charged-particle tracks, and for an ionization-scale energy transfer

$$\Delta E \sim 10^{1}\text{–}10^{3}\,\mathrm{eV},$$

the associated momentum transfer is of order

$$q \sim \frac{\Delta E}{u} \lesssim 10^{-25}\text{–}10^{-23}\,\mathrm{kg\,m/s}.$$

Hence

$$\frac{q}{p_{\rm cl}} \ll 1. \tag{67}$$

Thus the record-forming interactions localize the particle strongly enough to produce a track, while perturbing the tangent Newtonian motion only weakly from one record to the next.

The estimates in this subsection are only order-of-magnitude estimates. A detailed chamber-specific calculation would require the ionization or bubble-formation rate, the localization scale of each record, the momentum transfer distribution, and the thermodynamic properties of the medium. The purpose here is to show that the required regime is physically plausible: even for microscopic particles, sufficiently frequent localized interactions with an amplifying medium can return the state to the localized sector often enough that the recorded sequence is effectively Newtonian.

6.4 Stern–Gerlach measurement

In a Stern–Gerlach experiment, the magnetic field correlates spin alternatives with spatially separated wave packets. If the incoming spin state is

$$\alpha|+\rangle + \beta|-\rangle,$$

the magnetic field produces a state of the form

$$\alpha|+\rangle|\phi_+\rangle + \beta|-\rangle|\phi_-\rangle,$$

where ϕ_+ and ϕ_- are directed toward distinct detector regions.

The magnetic field separates the alternatives, but the outcome is produced only when the state interacts with the detector. At that stage, the two spatial packets correspond to disjoint detector-defined equivalence classes. The (**RM**) Hamiltonian acts on the full entangled position–spin state, and therefore its random step may include components in both the position and spin degrees of freedom. However, the equivalence classes relevant to the recorded outcome are not spin equivalence classes. The apparatus records the localized position of the particle, or the detector region in which it is found, and is sensitive to spin only through its prior correlation with position.

Thus the same (τ, s)-description used above applies. The coordinate τ labels the recorded position, while s measures localization in the position variable. Motion in spin directions, or in other directions that do not change the recorded position or the localization width, lies along the leaves of the corresponding foliation and does not by itself determine the recorded outcome. The (**RM**)-induced reduction is therefore still a reduction to one of the position-defined detector classes. The Born rule then follows as before, applied to these detector-defined position outcomes.

Consequently, the Stern–Gerlach apparatus converts spin alternatives into position-like detector outcomes. Since the spatial packets ϕ_+ and ϕ_- are correlated with the spin states $|+\rangle$ and $|-\rangle$, the probabilities of the two detector records are

$$P_+ = |\alpha|^2, \qquad P_- = |\beta|^2.$$

The preferred outcome basis is therefore determined by the measurement arrangement, which correlates the spin degree of freedom with macroscopically distinguishable position records.

6.5 Macroscopic superpositions and Schrödinger's cat

Schrödinger's cat paradox raises the question of how quantum theory can allow a macroscopic system to be in a superposition of classically distinct states, such as "alive" and "dead." In the present framework, such a superposition corresponds to a state with components near different macroscopic classical sectors. Under ordinary conditions, a macroscopic system is continually monitored by its environment. Therefore the macroscopic analysis of Section 5 applies to the cat. The (**RM**) mechanism rapidly drives the state into one of the corresponding detector- or environment-defined equivalence classes, while repeated environmental interactions keep returning the state to localized macroscopic sectors.

The cat is not observed in a superposition because the macroscopic degrees of freedom are continually recorded by the environment. These environmental interactions define robust equivalence classes corresponding to stable macroscopic records. Once one such class is reached, subsequent evolution is conditioned on it, and the record persists. Thus the paradox arises from applying the linear superposition principle without accounting for the (**RM**)-environmental dynamics and the finite-resolution equivalence classes that define macroscopic records. The framework does not require a fundamental nonunitary collapse of the universal state, nor does it require many simultaneously realized cats. It gives a single recorded macroscopic outcome through stochastic unitary dynamics in state space.

6.6 Measured system and measuring device as a composite system

In a complete description of a measurement, the measured particle and the macroscopic measuring device may be treated as a single composite system. This composite description must agree with the simpler particle-centered description, in which the measuring device is treated as the source of the (**RM**) interaction and the induced diffusion is described as acting effectively on the particle state.

We now show that the (**RM**) walk on the state space of the composite system provides such a consistent description. The Brownian-lift argument in Theorem 4 applies equally to composite systems. Thus the Gaussian random walk representing Brownian motion of the classical particle–device pair lifts to a unitary random walk in the projective state space of the tensor-product Hilbert space. In a finite-dimensional coarse-grained sector, this lift is again described by (**RM**). Therefore the random Hamiltonian in (**RM**) may be applied to the full particle–device state, not only to the state of the particle considered separately.

Let a denote the particle position coordinate and A the relevant macroscopic pointer, or center-of-mass, coordinate of the device. On the product of the localized classical sectors $M_1^{\sigma_p} \otimes M_1^{\sigma_d}$, Gaussian representatives have the form

$$g_{a,\sigma_p} \otimes G_{A,\sigma_d},$$

where σ_p and σ_d are the resolution scales associated with the particle and device degrees of freedom. For two such representatives,

$$\left|\langle g_{a,\sigma_p} \otimes G_{A,\sigma_d}, g_{b,\sigma_p} \otimes G_{B,\sigma_d}\rangle\right|^2 = \exp\left[-\frac{(a-b)^2}{4\sigma_p^2} - \frac{(A-B)^2}{4\sigma_d^2}\right].$$

Hence, to leading order, the induced Fubini–Study metric on the product classical sector is

$$ds^2 = \frac{da^2}{4\sigma_p^2} + \frac{dA^2}{4\sigma_d^2}. \tag{68}$$

Introduce the Fubini–Study-normalized coordinates

$$u = \frac{a}{2\sigma_p}, \qquad v = \frac{A}{2\sigma_d}.$$

Then (68) becomes

$$ds^2 = du^2 + dv^2.$$

Homogeneous and isotropic Brownian motion on the product classical sector means that, over a small time interval dt, the increments satisfy

$$du \sim N(0, D\,dt), \qquad dv \sim N(0, D\,dt), \qquad \mathbb{E}[du\,dv] = 0.$$

Equivalently, in the original Euclidean coordinates,

$$\mathrm{Var}(da) = 4\sigma_p^2 D\,dt, \qquad \mathrm{Var}(dA) = 4\sigma_d^2 D\,dt, \qquad \mathbb{E}[da\,dA] = 0.$$

Thus isotropy is imposed in the orthonormal Fubini–Study coordinates (u, v), not in the Euclidean coordinates (a, A).

The device scale σ_d is fixed operationally by the Brownian uncertainty of the macroscopic pointer coordinate over the relevant observation time. Let $D_P^{(d)}$ denote the momentum-diffusion coefficient for the device. The corresponding effective displacement scale of the pointer or center-of-mass coordinate is suppressed by the large device mass; schematically,

$$D_A^{(d)} \sim \frac{D_P^{(d)}}{M_d^2},$$

where $D_A^{(d)}$ denotes the induced position-diffusion scale over the measurement interval. Thus, over the measurement time, the device remains localized within its own macroscopic equivalence class.

Under these assumptions, the Brownian-lift argument applies exactly as in Theorem 4. The classical product sector $M_1^{\sigma_p} \otimes M_1^{\sigma_d}$ is a real submanifold of the projective state space of the tensor-product Hilbert space. If the unitary lift of the Gaussian random walk on this sector is required to be homogeneous and isotropic in the Fubini–Study metric of the composite projective state space, then, in each finite-dimensional coarse-grained sector, the corresponding random Hermitian generator has the centered unitarily invariant Gaussian law.

In this formulation the lift is unique up to the overall diffusion scale. The resolution parameters σ_p and σ_d fix the metric identification between Euclidean coordinates and Fubini–Study coordinates, while the scalar diffusion coefficient D fixes the variance of the isotropic Brownian motion in the normalized coordinates (u, v). Once these data are fixed, the homogeneous and isotropic unitary lift is the (**RM**) walk on the finite-dimensional coarse-grained composite state space.

We now use this to relate the composite (**RM**) walk to the effective particle-centered description. For the macroscopic device, the induced Brownian displacement of the pointer coordinate A during the measurement interval is below the resolution σ_d. Thus the Brownian motion of the device coordinate remains, with overwhelming probability, within the same resolution cell corresponding to the recorded macroscopic value of A. In state-space language, this means that the device state remains in the same macroscopic equivalence class, denoted $\{\Psi\}$, up to corrections below the device resolution.

In more detail, let

$$\Phi_0 = \varphi \otimes \Psi$$

be an initial particle–device state, where Ψ is a representative of the macroscopic device equivalence class $\{\Psi\}$ associated with the recorded value of the pointer coordinate A. Let Q denote the coarse-grained projection onto the device sector represented by the equivalence class $\{\Psi\}$, and let $Q^{\perp} = I - Q$. Since the induced Brownian motion of the macroscopic device coordinate remains in $\{\Psi\}$ with overwhelming probability during the measurement interval, one has, after one **(RM)** step,

$$\|(I \otimes Q^{\perp})\Phi_1\|^2 \ll 1.$$

Thus the component of Φ_1 in which the device lies outside its macroscopic equivalence class has negligible weight. Equivalently,

$$\Phi_1 = (I \otimes Q)\Phi_1 + O(\varepsilon), \qquad \varepsilon \ll 1.$$

At the level of detector-defined equivalence classes this gives

$$\{\Phi_1\} \simeq \{\varphi_1\} \otimes \{\Psi\},$$

up to corrections below the device resolution.

Repeating the same argument for the finite sequence of steps involved in the measurement process, and choosing the step size small enough that the accumulated weight outside the device class remains negligible, gives

$$\{\Phi_k\} \simeq \{\varphi_k\} \otimes \{\Psi\}$$

at each step k of the process, until the device records a new macroscopic outcome class. Thus the full composite (**RM**) dynamics preserves the effective product form in the operational sense relevant to measurement: the device remains in its macroscopic record class, while the particle undergoes the effective stochastic evolution leading to one of the detector-defined position equivalence classes.

Consequently, the complete composite-system description reduces, at the operational level, to the particle-centered description. The (**RM**) Hamiltonian acts on the full particle–device state, but the macroscopic device remains in its classical record class, while the microscopic particle undergoes effective (**RM**)-induced reduction to one of the detector-defined position classes. When the particle state lies on, or sufficiently near, its localized phase-space manifold, its interaction with the device is described by the Newtonian tangent dynamics derived above. The device then records the corresponding classical position through ordinary macroscopic dynamics.

6.7 Wigner's friend

The Wigner's friend scenario asks how the friend can record a definite outcome while Wigner may describe the larger laboratory quantum mechanically. In the present framework, the relevant system is composite: it includes the microscopic particle, the measuring apparatus, the friend, and the surrounding laboratory. The analysis of the particle–device system above applies here as well, with the friend and laboratory included among the macroscopic degrees of freedom.

Before the friend's measurement, the relevant state may be written schematically as

$$\Phi_0 = \varphi \otimes \Psi_A \otimes \Psi_F \otimes \Psi_L,$$

where φ is the state of the microscopic particle, while Ψ_A, Ψ_F, Ψ_L represent the macroscopic states of the apparatus, the friend, and the laboratory environment. During the measurement, the (**RM**) dynamics acts on this full composite state. However, as in the particle–device case, the macroscopic factors remain in their classical equivalence-class sectors, while the microscopic alternative being measured is correlated with one of the detector-defined outcome classes.

If the possible outcomes are labeled by c, then after the friend's measurement the realized state belongs, at the level of equivalence classes, to one class of the form

$$\{g_{c,\sigma_p}\} \otimes \{\Psi_A^{(c)}\} \otimes \{\Psi_F^{(c)}\} \otimes \{\Psi_L^{(c)}\}.$$

Here $\{g_{c,\sigma_p}\}$ is the particle outcome class and $\{\Psi_A^{(c)}\}$, $\{\Psi_F^{(c)}\}$, and $\{\Psi_L^{(c)}\}$ are the corresponding macroscopic record classes of the apparatus, friend, and laboratory. The friend's accessible information is precisely this macroscopic record.

This is analogous to an ordinary classical measurement. If a person measures a classical particle in a laboratory, the outcome is recorded by the measuring device and by the person. Adding a friend, or later an outside observer, does not change the already recorded value; it only correlates additional macroscopic systems with the same record. The present framework gives the same structure in state space. The difference is that the microscopic particle is described quantum mechanically until its state enters one of the detector-defined equivalence classes, whereas the apparatus, friend, and laboratory remain in the macroscopic classical regime throughout.

When Wigner subsequently interacts with the laboratory, he becomes part of the same composite measurement process. If Ψ_W denotes Wigner's initial macroscopic state, then the relevant state before his observation is schematically

$$\Phi_W = g_{c,\sigma_p} \otimes \Psi_A^{(c)} \otimes \Psi_F^{(c)} \otimes \Psi_L^{(c)} \otimes \Psi_W,$$

at the level of the realized equivalence class. The interaction with the laboratory correlates Wigner with the already established record, giving

$$\{g_{c,\sigma_p}\} \otimes \{\Psi_A^{(c)}\} \otimes \{\Psi_F^{(c)}\} \otimes \{\Psi_L^{(c)}\} \otimes \{\Psi_W^{(c)}\}.$$

Thus Wigner records the same value c. He does not create a new outcome, nor does his observation erase or alter the macroscopic record already formed inside the laboratory.

In this sense, the final situation is again a particle–device system, but with a larger macroscopic device: the apparatus, friend, Wigner, and laboratory environment together form one large macroscopic record system. The measured particle is the microscopic component, while the rest of the composite system remains in the classical equivalence-class regime. Thus the friend and Wigner do not obtain incompatible facts; they become correlated with the same state-space process at different stages.

The distinction is between the full unitary description of the enlarged system and the equivalence-class description of accessible macroscopic records. Definite records are not observer-relative in the sense of being created by subjective knowledge. They are objective features of the state entering a detector- or environment-defined sector. The consistency of the accounts is due to the fact that macroscopic objects, under **(RM)**, remain in the classical equivalence-class regime and change only through effectively classical correlations. Once the measured particle has entered a localized outcome class, the apparatus, friend, Wigner, and laboratory environment are correlated with that same class as macroscopic record systems. Thus the agreement between the friend and Wigner is, at that stage, an ordinary classical correlation between records.

6.8 EPR correlations and nonlocality

EPR and Bell-type experiments demonstrate that spatially separated systems can display correlations stronger than any local hidden-variable model allows. In the present framework, an entangled pair is represented by a single point in the joint projective state space, not by two independent points in ordinary space. As in the particle–device case considered above, the (**RM**) random walk acts on the joint state, producing a state-space trajectory. The difference is that here both components of the pair are microscopic, so neither component is fixed in a macroscopic record class before measurement.

For a two-particle state in one spatial dimension, the Hilbert space is $L_2(\mathbb{R})\otimes L_2(\mathbb{R})$, and the relevant projective state space is the projectivization of this tensor-product Hilbert space. The classical configuration-space sector $\widetilde{M}_1^{\sigma_1} \otimes \widetilde{M}_1^{\sigma_2}$ is represented by products of localized equivalence classes of the form

$$\{g_{c,\sigma_1}\} \otimes \{g_{d,\sigma_2}\},$$

corresponding to the classical situation of two particles with positions c and d. Here σ_1 and σ_2 are the corresponding detector resolutions.

An entangled initial state is a superposition of alternatives associated with these classes,

$$\Phi = \sum_k \alpha_k \, g_{c_k,\sigma_1} \otimes g_{d_k,\sigma_2}.$$

Such a state does not lie on the classical configuration-space submanifold $\widetilde{M}_1^{\sigma_1} \otimes \widetilde{M}_1^{\sigma_2}$, except in the special case in which the two particles already have definite localized positions.

Measurement corresponds to the joint state entering one equivalence class associated with a pair of detector outcomes, $\{g_{c,\sigma_1}\} \otimes \{g_{d,\sigma_2}\}$. The Born rule for the joint state is derived from the **(RM)** walk in the same way as for a single measured system, but now applied to the joint projective state space. It gives the usual correlations between the recorded outcomes. Thus, if the measurement process brings the first particle into the class $\{g_{c,\sigma_1}\}$, then the joint state enters the corresponding product class, and the second particle is recorded in the correlated class $\{g_{d,\sigma_2}\}$. The correlated outcome is therefore not produced by a signal or physical influence propagating through classical space. It follows from the motion of the joint state toward a single detector-defined equivalence class of the pair.

Bell's theorem constrains local hidden-variable models formulated in spacetime. The present framework is not such a model. The correlations arise from the position of the initial state in state space, from the geometry of the classical configuration-space submanifold $\widetilde{M}_1^{\sigma_1} \otimes \widetilde{M}_1^{\sigma_2}$, and from the Born-rule transition probabilities generated by (**RM**). The evolution toward $\widetilde{M}_1^{\sigma_1} \otimes \widetilde{M}_1^{\sigma_2}$ is unitary and local in time. It is also local in state space: infinitesimal time increments produce infinitesimal displacements in the Fubini–Study metric. Relativistic no-signaling is preserved because the marginal probabilities remain those of standard quantum mechanics, so the correlations cannot be used to transmit information superluminally.

In the single-particle case, wave-like properties were associated with the particle's state being away from a classical space manifold, such as $\widetilde{M}_1^{\sigma}$. Similarly, for an entangled pair, what appears as nonlocality in classical space reflects the fact that the joint state is a single point in the projective state space of $L_2(\mathbb{R}) \otimes L_2(\mathbb{R})$, lying outside the classical configuration-space submanifold $\widetilde{M}_1^{\sigma_1} \otimes \widetilde{M}_1^{\sigma_2}$. Accordingly, EPR correlations and Bell-inequality violations are reinterpreted as geometric features of state space, its classical submanifolds, and the stochastic unitary dynamics generated by (**RM**), rather than as evidence for nonlocal dynamics in classical space.

6.9 Preferred basis and classical observables

The preferred-basis problem is the question of why measurements yield outcomes in particular bases, especially position, rather than in arbitrary superpositions. In the present framework, the (**RM**) dynamics itself is homogeneous and isotropic in projective state space and therefore does not select a preferred basis by hand. The relevant basis is selected by the measurement arrangement, by the observable-related classical submanifold, and by the equivalence relation induced by finite detector resolution.

Position has a distinguished role because macroscopic measuring devices ultimately record positions or configurations of matter. Other observables are measured by correlating them with position through the design of the apparatus. For example, a Stern–Gerlach device correlates spin alternatives with spatially separated detector regions, and a spectrometer correlates momentum values with positions on a detection screen. Thus the measured observable is determined by the apparatus through a mapping from the alternatives of that observable to detector-defined position sectors.

Equivalently, each measurement defines a family of outcome sectors in state space. For position measurement, these sectors are the localized equivalence classes associated with a classical space submanifold, such as $\widetilde{M}_1^{\sigma}$. For other observables, the corresponding submanifold is obtained from the physical and mathematical transformation relating that observable to position records. For instance, the manifold of approximate momentum eigenstates may be obtained from the position manifold by applying the Fourier transform.

In this sense, a preferred basis is not a separate postulate. It is determined by the geometry of the relevant outcome sectors, the design of the measuring device, and the equivalence classes of states indistinguishable at the detector resolution. An observable becomes classical precisely when the state reaches one of the corresponding detector-defined equivalence classes, so that a macroscopic record can be formed.

6.10 Quantum Zeno effect

The quantum Zeno effect is usually described as the inhibition of evolution by frequent measurements, often formulated in terms of repeated projection. In the present framework, no separate projection postulate is required to account for the effect. Repeated measurements correspond to repeated (**RM**)-induced returns of the state to the same detector-defined equivalence class, or to a small neighborhood of that class. Each return conditions the subsequent evolution on the recorded class.

As the interval between measurement interactions decreases, the probability that the state escapes the corresponding operational neighborhood becomes small. The effect traditionally attributed to repeated projection is therefore described as the dynamical stabilization of a state-space path by repeated stochastic unitary interactions and conditioning on records. In this sense, the process that returns the state to the equivalence class plays the role usually assigned to projection.

As explained in the next section, recording does not introduce a new dynamical law. It only fixes the conditional information used to describe subsequent evolution. The suppression of transitions follows from the (**RM**) dynamics, the geometry of the relevant equivalence class, and the repeated conditioning on recorded returns to that class.

6.11 Recording

It is important to distinguish state reduction from the recording of an outcome. In the present framework, collapse is the dynamical approach of the state to a detector-defined equivalence class through the (**RM**) process. For a position measurement, this means that the state reaches a class such as $\{g_{c,\sigma}\}$, or the corresponding phase-space class when momentum information is also relevant. At that point, the particle has a well-defined position at the resolution of the measuring device.

The subsequent recording of this fact is not an additional collapse process. Once the state belongs to a localized position equivalence class, the situation has entered the classical regime relevant to the detector. Namely, as explained in Section 2.2, the Schrödinger dynamics of the system becomes equivalent to its Newtonian dynamics. The particle, measuring device, and nearby environment can then interact through ordinary macroscopic dynamics. A pointer may move, an atom may be ionized, a bubble may form, a scintillation event may occur, or a stable mark may be produced. These are classical correlations between an already localized particle state and macroscopic degrees of freedom of the apparatus.

Thus the detector does not create localization merely by recording the result. Rather, the (**RM**)-induced state-space dynamics brings the state to a detector-defined equivalence class, and the detector records that this has happened. The recording process is then governed by the Newtonian dynamics of the particle–device–environment system, because the relevant state is already in, or sufficiently close to, the classical sector.

This also explains why recording is stable. The apparatus is macroscopic and is continually monitored by its environment. Its state therefore remains in a macroscopic equivalence-class sector, in close analogy with the dynamical stabilization described in connection with the quantum Zeno effect, and the recorded outcome becomes encoded in many correlated degrees of freedom. Once such a record is formed, subsequent evolution is conditioned on it. The record supplies the initial data for the next stage of the effective stochastic evolution, but it does not introduce a new dynamical law.

In this sense, recording is a classical amplification and stabilization of an outcome that has already been selected dynamically in state space. Collapse is the approach to the relevant equivalence class; recording is the ordinary macroscopic process by which membership in that class becomes a persistent classical fact.

6.12 Irreversibility and the arrow of time

Although each realization of the evolution in the present framework is unitary, the effective description of measurement is irreversible. This irreversibility does not arise from a fundamental nonunitary collapse law. It arises from the combination of stochastic state-space dynamics, the use of equivalence classes, and the conditioning of subsequent evolution on recorded outcomes.

First, the **(RM)**-driven motion is a stochastic motion in projective state space of large, and in the idealized case infinite, dimension. A typical realized path explores new regions of state space, and the probability of reconstructing its past from its later position is negligible. In finite dimensions, recurrence may occur in principle, but in very large dimensions the recurrence times are overwhelmingly long. In infinite-dimensional state space, the probability of recurrence to a given small neighborhood is effectively absent. The evolution is local in time and unitary along each realization, but the stochastic sequence of independently drawn Hamiltonians gives a direction to typical state-space trajectories. In this sense, an arrow of time is already present at the level of realized **(RM)** paths.

Second, the ensemble used in **(RM)** is the Gaussian Unitary Ensemble. The complex random Hamiltonians in this ensemble generate unitary evolution while scrambling phases and directions in state space. Although each short step is unitary, the sequence of independently drawn Hamiltonians is not a reversible record of its own past. Equivalently, the Hamiltonians in the ensemble are not, in general, invariant under time reversal. Reversing a realized trajectory would require the corresponding reversed sequence of Hamiltonians, which is not supplied by the forward stochastic dynamics. Thus the random-matrix dynamics produces irreversibility at the level of typical paths, and already at the level of individual steps of the walk, even though each step remains unitary.

Third, equivalence classes introduce a finite-resolution coarse-graining. A detector does not distinguish all vectors in Hilbert space. It distinguishes only classes of states that agree within the resolution of the apparatus. In passing from exact rays to detector-defined equivalence classes, microscopic information about phases and about distinctions between states inside a class is no longer part of the operational description. The exact state may continue to evolve unitarily, but the corresponding description in terms of classical records has discarded information that is not recoverable from the record.

The role of records was described in the preceding section. Recording does not cause collapse and does not introduce a new dynamical law. Once the state has entered a position equivalence class, the subsequent formation of a record is an ordinary macroscopic process governed by Newtonian dynamics. However, after such a record is formed, the effective description of later evolution is conditioned on that record. The recorded class becomes the starting point for the next stage of the stochastic evolution.

This conditioning gives the operational arrow of time. The past is represented by a sequence of recorded equivalence classes, while the future remains probabilistic. Later probabilities are computed relative to the records already formed; they are not computed from a superposition of all counterfactual unrecorded alternatives. Repeated environmental monitoring amplifies this asymmetry, since macroscopic records become correlated with many degrees of freedom and are stabilized by continual return to classical equivalence-class sectors.

Thus the arrow of time in the framework has a layered origin. The underlying dynamics remains unitary for each realization, but typical stochastic state-space paths are not operationally reversible; equivalence classes discard microscopic distinctions inaccessible to the detector; and records condition all subsequent effective evolution. The result is an irreversible classical history emerging from stochastic yet unitary dynamics in projective state space.

6.13 Summary

The standard paradoxes of quantum mechanics are reformulated in this framework as questions about the relation between state-space dynamics, classical submanifolds, detector-defined equivalence classes, and records. Superposition and entanglement occur when the state evolves away from the relevant classical submanifold rather than being confined to it. Measurement occurs when the (**RM**) dynamics drives the state into a detector-defined equivalence class. The preferred basis is fixed not by an additional postulate, but by the measurement arrangement, the relevant observable-related submanifold, and the finite-resolution equivalence relation defined by the apparatus. EPR correlations arise from the geometry of the joint state space rather than from signals or physical influences propagating in ordinary space.

Classical motion occurs when frequent environmental interactions keep the state near the localized phase-space submanifold, so that the tangent component of Schrödinger evolution reproduces Newtonian dynamics. Recording is not a separate collapse process; once the state has entered a localized equivalence class, the relevant dynamics is effectively classical, and ordinary macroscopic dynamics correlates this localized state with a stable detector or environmental record. Repeated monitoring then stabilizes the record, maintains Newtonian behavior, and gives the effective irreversibility of the observed classical history.

Thus the same mechanism—unitary Schrödinger evolution supplemented by **(RM)**-induced stochastic dynamics and finite-resolution equivalence classes—accounts for state reduction, Born probabilities, classical trajectories, interference, preferred outcome sectors, macroscopic definiteness, record formation, Zeno stabilization, irreversibility, and nonlocal correlations without adding nonunitary collapse or observer-dependent rules.

7 Relation to existing approaches

The framework developed here overlaps with several existing approaches to the quantum-to-classical transition, but it differs from them in the role assigned to state-space geometry and to the effective unitary measurement dynamics formulated on state space. The dynamics is not formulated primarily on classical configuration space, nor on a reduced density matrix alone, but on projective Hilbert space $\mathbb{CP}^{L_2}$ with its Fubini–Study metric. Classical configuration space and phase space appear as submanifolds, and measurement outcomes are represented by finite-resolution equivalence classes. The conjecture (**RM**) then supplies an effective isotropic stochastic dynamics on this same state space. This section compares this structure with several standard approaches.

7.1 Decoherence and open-system dynamics

In standard decoherence theory, a system S coupled to an environment E is described by a joint density operator $\rho_{SE}(t)$ on the tensor-product Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_E$. The joint system evolves unitarily,

$$\rho_{SE}(t) = U(t)\rho_{SE}(0)U(t)^{\dagger},$$

where $U(t)$ is the unitary evolution operator on $\mathcal{H}_S \otimes \mathcal{H}_E$. The reduced state of the system alone is obtained by tracing over the environmental degrees of freedom:

$$\rho_S(t) = \mathrm{Tr}_E\, \rho_{SE}(t).$$

Here Tr_E denotes the partial trace over $\mathcal{H}_E$. For suitable system–environment interactions, the reduced density matrix becomes approximately diagonal in a preferred pointer basis,

$$\rho_S(t) \approx \sum_{\alpha} p_\alpha |\alpha\rangle\langle\alpha|,$$

with off-diagonal terms suppressed. This explains the practical disappearance of interference between macroscopically distinct alternatives and the stability of localized or pointer-like states [8].

The present framework is compatible with this mechanism but addresses a different mathematical problem. Decoherence produces an improper mixture in the reduced density matrix; it does not by itself specify which individual outcome occurs in a single run. In the present framework, the state follows a stochastic unitary path in projective state space,

$$\varphi(t+dt) = \exp\left(-\frac{i}{\hbar}\widehat{h}_{\mathrm{RM}}dt\right)\varphi(t),$$

and the outcome is the detector-defined equivalence class reached by this path. Thus the object carrying outcome information in the present framework is not the reduced density matrix alone, but the state-space path together with the equivalence-class structure determined by detector resolution. In particular, decoherence helps explain the stability and robustness of macroscopic records, while the present framework distinguishes this recording process from the state-space reduction that selects one equivalence class.

At the level of scales, the two descriptions should agree. For example, in collisional decoherence the reduced density matrix in the position representation often has the form

$$\rho(x, x', t) = \rho(x, x', 0) \exp[-\Lambda t(x - x')^2],$$

in a suitable short-distance approximation. The coefficient Λ is determined by environmental scattering rates and momentum transfers. In the present framework, the same environmental data enter the (**RM**) diffusion coefficients. For example, Section 4 estimated the coefficients

$$D_p \sim \Gamma q^2, \qquad D_a \sim \frac{D_p}{M^2},$$

where Γ is a collision rate and q is a typical momentum transfer. Thus decoherence theory and **(RM)** should agree at the level of environmental diffusion scales, even though they answer different questions: decoherence explains suppression of interference in ρ_S, while **(RM)** supplies a stochastic unitary mechanism for reaching one detector-defined equivalence class.

7.2 Caldeira–Leggett-type microscopic bath models

A representative microscopic model is the Caldeira–Leggett model, in which a distinguished coordinate x is linearly coupled to a bath of harmonic oscillators [9]. The Hamiltonian has the schematic form

$$H = \frac{p^2}{2M} + V(x) + \sum_j \left(\frac{p_j^2}{2m_j} + \frac{1}{2} m_j \omega_j^2 q_j^2 \right) - x \sum_j c_j q_j + x^2 \sum_j \frac{c_j^2}{2m_j \omega_j^2}.$$

After tracing out the bath under suitable approximations, one obtains a master equation for the reduced density matrix of the system. In the high-temperature Markovian limit, this has the typical structure

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_S, \rho] - \frac{i\gamma}{\hbar}[x, \{p, \rho\}] - \frac{D}{\hbar^2}[x, [x, \rho]] + \cdots,$$

where γ is a damping coefficient and D is a diffusion coefficient. The double commutator suppresses spatial coherences and gives decoherence in the position basis.

The role of **(RM)** is different. It is not proposed as a rival microscopic bath Hamiltonian. Rather, it is an effective conjecture about the coarse-grained action of many short and complicated environmental interactions on the state itself. In Caldeira–Leggett-type models, the stochastic or dissipative structure appears after reduction to the system density matrix. In the present framework, the effective randomness is placed at the level of the Hamiltonian generating unitary motion in projective state space. The comparison is therefore between a microscopic bath model that yields reduced open-system dynamics and an effective state-space diffusion model. The two should be compatible when the **(RM)** diffusion coefficients are chosen to match environmental diffusion scales derived from microscopic models.

This is analogous to the relation between Einstein's theory of Brownian motion, the related Langevin equation, and microscopic molecular mechanics. Einstein's assumptions determine the diffusion equation without tracking every molecular collision, while the Langevin equation gives an effective stochastic description of the same underlying microscopic process. Similarly, **(RM)** identifies the structural assumptions – small independent random unitary steps, homogeneity, and isotropy in projective state space, needed to derive the Born rule and the macroscopic stochastic process, without requiring a complete microscopic derivation of every interaction event.

7.3 Continuous measurement and quantum trajectories

Continuous-measurement theory describes monitored quantum systems by stochastic master equations or stochastic Schrödinger equations [10, 11]. For example, continuous measurement of an observable A may be described by a conditioned stochastic equation of the schematic form

$$d\rho = -\frac{i}{\hbar}[H,\rho]\,dt - k[A,[A,\rho]]\,dt + \sqrt{2k}\,(A\rho + \rho A - 2\langle A\rangle\rho)\,dW_t,$$

where dW_t is a Wiener increment and k is the measurement strength. Equivalently, one may write a stochastic Schrödinger equation for the conditioned pure state. Such conditioned equations are generally nonlinear in the normalized state and nonunitary at the level of the conditioned system evolution. They successfully describe measurement records, feedback, and quantum trajectories.

The present framework has a similar conditioning structure, but it has a different geometric and dynamical origin and preserves unitary evolution. In continuous-measurement theory, the measured observable A is specified in advance, and the stochastic equation is constructed to describe conditioning on the corresponding measurement record. In the present framework, the measured classical variables arise from the geometry of submanifolds such as M_3^σ and $M_{3,3}^\sigma$, while the stochasticity is generated by random Hamiltonians:

$$d\varphi = -\frac{i}{\hbar}\widehat{h}_{\mathrm{RM}}\varphi\,dt, \qquad \widehat{h}_{\mathrm{RM}} \in \mathrm{GUE}.$$

For each realization of $\widehat{h}_{\mathrm{RM}}$, the evolution remains unitary. The conditioning occurs only when the recording interaction assigns the state to a detector-defined equivalence class.

Thus the formal role of dW_t in continuous measurement is replaced here by isotropic random unitary motion in projective state space. The outcome sectors are not merely exact eigenspaces of a preassigned operator; they are finite-resolution equivalence classes determined by the apparatus and often associated with localized classical submanifolds. This is why the same stochastic process gives both classical Brownian errors on M_3^σ and Born-rule probabilities in $\mathbb{CP}^{L_2}$, without introducing fundamental nonlinear or nonunitary dynamics.

7.4 Ehrenfest theorem and semiclassical limits

The Ehrenfest theorem [12] gives

$$\frac{d}{dt}\langle\widehat{x}\rangle = \frac{1}{m}\langle\widehat{p}\rangle, \qquad \frac{d}{dt}\langle\widehat{p}\rangle = -\langle\nabla V(\widehat{x})\rangle,$$

where $\widehat{x}$ and $\widehat{p}$ are the position and momentum operators, m is the particle mass, V is the potential, and $\langle\cdot\rangle$ denotes expectation value in the quantum state. If the state remains sufficiently localized and V is sufficiently regular, then

$$\langle\nabla V(\widehat{x})\rangle \approx \nabla V(\langle\widehat{x}\rangle),$$

and the expectation values approximately satisfy Newton's equations. This is an important consistency condition, but it does not explain why localization persists, why a single outcome occurs, or why the Born rule governs outcome statistics.

The present construction is different. Newtonian dynamics of a particle is obtained by restricting the action

$$S[\varphi] = \int \overline{\varphi}\left(i\hbar\partial_t - \widehat{h}\right)\varphi\, d^3x\, dt$$

to $M_{3,3}^\sigma$. The non-potential terms reduce to the classical form, and the potential term converges to $V(a,t)$ under continuity of V. If V is differentiable, the restricted action yields

$$\dot{a} = \frac{p}{m}, \qquad \dot{p} = -\nabla V(a).$$

Equivalently, the tangent component of the Schrödinger vector field on $M_{3,3}^\sigma$ is the Newtonian vector field. In this restricted dynamics, the commutator form of the Heisenberg equations reduces to the corresponding Poisson-bracket form of classical Hamiltonian dynamics. The result is therefore a geometric reduction of the dynamics, not merely an approximation for expectation values.

The persistence of localization is then supplied by the **(RM)**/environmental mechanism.This is the role of the drift–diffusion–resolution conditions

$$\frac{v\,dt}{\sigma} \ll 1, \qquad \frac{dt}{T_{\text{spr}}} \ll 1, \qquad D_a\Delta t \ll \sigma^2,$$

discussed in Section 4. Thus the Ehrenfest theorem is consistent with the framework, but it is not the source of the explanation.

7.5 Objective-collapse models

Objective-collapse models modify the Schrödinger equation by adding stochastic nonlinear terms. In continuous spontaneous localization (CSL) models, for example, the state vector satisfies an equation of the general form

$$d|\psi_t\rangle = \left[-\frac{i}{\hbar}H\,dt + \sqrt{\lambda}\int d^3x\,\big(N(x) - \langle N(x)\rangle_t\big)dW_t(x) - \frac{\lambda}{2}\int d^3x\,\big(N(x) - \langle N(x)\rangle_t\big)^2 dt\right]|\psi_t\rangle,$$

where λ is a collapse rate, $N(x)$ is a smeared mass-density operator, and $dW_t(x)$ is a classical noise field [13, 14]. The nonlinear terms drive localization and produce single outcomes, but the dynamics is no longer strictly linear unitary Schrödinger evolution.

The present framework also produces single outcomes, but the mechanism is different. The state evolves by

$$|\psi_{t+dt}\rangle = \exp\left[-\frac{i}{\hbar}\widehat{h}_{\mathrm{RM}}dt\right]|\psi_t\rangle$$

during an **(RM)** step. For each realization of $\widehat{h}_{\mathrm{RM}}$, the evolution is unitary. Reduction means that the path in projective state space enters a detector-defined equivalence class; it is not a fundamental nonunitary transformation of the state vector.

This distinction changes the empirical interpretation of the parameters. Collapse models predict effects associated with fundamental noise, such as spontaneous heating or radiation, and are strongly constrained by interferometry, heating bounds, and radiation bounds. In the present framework, there is no universal collapse noise acting independently of measurement or environmental interaction. The empirical content instead lies in the effective random-matrix character of complex interactions and in the resulting diffusion scales. Thus the parameters of **(RM)** are environmental and contextual, rather than universal collapse constants.

7.6 Coarse-graining and finite-resolution measurements

Several approaches emphasize that classical behavior may emerge when measurements are coarse-grained [15–18]. In such approaches, limited resolution prevents observation of fine quantum interference structure, and classical-looking statistics may result. Mathematically, this often means replacing sharp projectors by coarse-grained projectors or POVM elements,

$$P_\Delta = \int_\Delta |x\rangle\langle x|\,dx, \qquad P(\Delta) = \|P_\Delta\psi\|^2.$$

The present framework incorporates finite resolution more structurally. A detector outcome is an equivalence class or finite-resolution sector in projective state space. For a position region Δ, the corresponding probability is

$$P(\Delta) = \int_\Delta |\psi(x)|^2\,dx = \cos^2\rho\big(\psi, \mathbb{P}(\mathcal{H}_\Delta)\big),$$

where

$$\rho\big(\psi, \mathbb{P}(\mathcal{H}_\Delta)\big) = \inf_{\chi \in \mathcal{H}_\Delta,\ \|\chi\|=1} \rho(\psi, \chi),$$

and $\mathcal{H}_\Delta$ is the subspace of states supported, to the detector resolution, in the region Δ. Thus finite resolution is not only an observational limitation; it determines the outcome sectors to which the **(RM)** random walk can return with nonzero probability.

This also resolves a dimensionality issue. In a high- or infinite-dimensional state space, a random path would generally have zero probability of hitting a prescribed exact ray. But an actual outcome is not an exact ray. It is an equivalence class or finite-resolution neighborhood. The relevant event is entrance into that sector, and its probability is computed by the projection rule above.

7.7 Center-of-mass classicalization

Another route to classical behavior derives the classical motion of the center of mass of a large quantum system through many-particle concentration or central-limit-type arguments [19, 20]. For a system of N particles with center-of-mass coordinate

$$R = \frac{1}{M} \sum_{i=1}^{N} m_i x_i,$$

one typically finds that the center-of-mass distribution becomes sharply peaked as N grows. Under broad assumptions, the width of the center-of-mass distribution decreases like

$$\Delta R \sim \frac{1}{\sqrt{N}},$$

or by an analogous mass-weighted concentration estimate. This makes the center of mass increasingly classical.

This is complementary to, but distinct from, the present approach. In our construction, classicality is not derived primarily from the large-N concentration of a configuration-space distribution. Instead, the main object is the localized phase-space submanifold and Newtonian dynamics is the tangent component of Schrödinger evolution on this submanifold. Gaussian representatives are convenient and physically natural, especially for macroscopic center-of-mass states, but the geometric derivation of the classical action does not require exact Gaussianity. It extends to equivalence classes of sufficiently localized states.

The central-limit picture nevertheless supports the physical plausibility of Gaussian representatives in macroscopic regimes. If a macroscopic center of mass is influenced by many weakly correlated microscopic degrees of freedom, then smooth approximately Gaussian localization is expected. Thus center-of-mass concentration helps explain why Gaussian representatives are typical, while the present framework explains why Gaussianity is not structurally essential for the transition to classicality or for measurement dynamics.

7.8 Bohmian mechanics

Bohmian mechanics [21, 22] supplements the wave function

$$\Psi = \Psi(q_1, \dots, q_N, t)$$

with actual particle positions $Q_1(t), \dots, Q_N(t)$. These positions are guided by

$$\dot{Q}_k(t) = \frac{\hbar}{m_k} \operatorname{Im} \frac{\nabla_k \Psi(q_1, \dots, q_N, t)}{\Psi(q_1, \dots, q_N, t)}\bigg|_{q_j = Q_j(t)}.$$

This gives definite trajectories in configuration space, but requires additional hidden variables and a guiding equation that is nonlocal on configuration space for entangled states.

The present framework introduces no additional particle positions. The fundamental object is the state as a point in projective Hilbert space. Classical positions appear when the state lies in, or is recorded as belonging to, localized equivalence classes associated with a classical space submanifold. Thus the classical trajectory is not a hidden path $Q(t)$ in $\mathbb{R}^3$, but a sequence of records produced by returns of the state-space path to the classical sector.

7.9 Everettian approaches

Everettian approaches [23, 24] preserve unitary evolution by interpreting measurement as branching into multiple realized alternatives,

$$\sum_\alpha c_\alpha |\alpha\rangle |A_0\rangle \longrightarrow \sum_\alpha c_\alpha |\alpha\rangle |A_\alpha\rangle.$$

The present framework also preserves unitary evolution for each realization of the dynamics, but it does not interpret all terms in the superposition as simultaneously realized classical worlds. The (**RM**) Hamiltonian selects a stochastic state-space path, and an observed outcome is the detector-defined equivalence class reached by that path. Thus outcome definiteness is obtained through stochastic unitary dynamics and conditioning, rather than through branching.

7.10 Summary

The comparison can be summarized as follows. Decoherence explains the suppression of off-diagonal terms in reduced density matrices. Caldeira–Leggett-type models derive open-system coefficients from microscopic bath Hamiltonians. Continuous-measurement theory gives stochastic conditioned equations once the measured observable is specified. Collapse models add fundamental nonunitary noise. Coarse-graining approaches emphasize finite resolution, and center-of-mass classicalization explains the concentration of large systems. Bohmian mechanics introduces additional particle positions guided by the wave function, while Everettian approaches interpret measurement in terms of branching into multiple realized alternatives.

The present framework combines different geometric and dynamical ingredients. Classical space and phase space are submanifolds of projective Hilbert space. The Schrödinger action restricted to the phase-space submanifold becomes the classical action. Measurement outcomes are finite-resolution equivalence classes. Finally, (**RM**) supplies isotropic stochastic unitary dynamics on projective state space. This combination yields, within one framework, normal distributions for classical measurement errors, Born probabilities for microscopic measurements, and stroboscopic Newtonian motion for macroscopic bodies.

8 Conclusion

We have proposed a unified geometric and stochastic framework for understanding measurement, state reduction, and the quantum-to-classical transition. The geometric part of the framework identifies classical configuration space and classical phase space with localized submanifolds of projective quantum state space. The additional dynamical ingredient is the conjecture **(RM)**: complex interactions with measuring devices and environments are effectively described, after coarse-graining, by random-matrix Hamiltonians generating isotropic stochastic unitary motion on projective state space.

The geometric construction explains how classical mechanics is contained in quantum mechanics. The manifolds M_3^σ and $M_{3,3}^\sigma$ carry the Euclidean geometry of the classical configuration space and phase space of a particle through the metric induced by the Fubini–Study metric. When the Schrödinger action is restricted to $M_{3,3}^\sigma$, it becomes the classical action, and the tangent component of the Schrödinger vector field reproduces Newtonian dynamics. Thus Newtonian motion appears not as an independent postulate, but as the tangent dynamics of quantum states constrained to the classical phase-space submanifold.

The conjecture **(RM)** explains how measurement and state reduction can arise within unitary dynamics. It produces homogeneous and isotropic stochastic motion in projective state space. When restricted to the classical submanifold $M_3^\sigma \simeq \mathbb{R}^3$, this motion becomes ordinary Brownian-type diffusion of a particle and yields the normal distribution of classical measurement errors. In the full projective state space, the same isotropy makes transition probabilities depend only on Fubini–Study distance. Since the distance-dependent transition law is fixed on M_3^σ, where it agrees with the Born probability for localized states, it extends uniquely to the Born rule on the full projective state space. The construction extends to arbitrary systems of particles.

Equivalence classes of detector-indistinguishable states play an essential role. A measurement outcome is not an exact ray in Hilbert space, but a finite-resolution outcome sector determined by the measuring device. The probability of an outcome is the probability that, at the recording time, the **(RM)**-driven state is assigned to the corresponding sector. This avoids the difficulty of requiring a random path in an infinite-dimensional space to hit an exact state or exact lower-dimensional submanifold. It also clarifies why wave-function tails do not prevent single outcomes: what is recorded is membership in an equivalence class, not pointwise disappearance of the wave function elsewhere.

The same mechanism yields different physical regimes. For microscopic measurements, the **(RM)** interaction can dominate over the free Hamiltonian during the short measurement interval. The state then undergoes essentially isotropic diffusion in projective state space during the measurement interval and is recorded in one of the detector-defined outcome classes, with Born probabilities. For macroscopic bodies, the free Schrödinger flow supplies Newtonian tangent drift, while frequent environmental **(RM)** interactions continually return the state to a narrow neighborhood of $M^{g}_{3,3}$. Under the drift–diffusion–resolution conditions derived above, the resulting recorded positions form a stroboscopic Newtonian trajectory.

The framework therefore gives a single account of several phenomena that are often treated separately: classical space geometry, Newtonian dynamics, normal distributions in classical measurement, Born probabilities in quantum measurement, single outcomes, and the stability of macroscopic classical behavior. It also gives a common interpretation of standard experiments and paradoxes. Interference and entanglement occur when the state evolves outside the relevant classical submanifold. Measurement occurs when the recording interaction assigns the state to a detector-defined outcome class. Macroscopic definiteness results from repeated environmental stabilization of classical sectors. EPR correlations arise from the geometry of joint state space rather than from signals propagating in classical space.

The status of **(RM)** is stronger than that of an arbitrary random-matrix ansatz. Classical measurement errors can be modeled by Brownian motion arising from a Newtonian bath. At the same time, Newtonian dynamics is the restriction of Schrödinger dynamics to the classical phase-space submanifold. It follows that the Brownian bath dynamics admits a unitary lift to projective Hilbert space. According to Theorem 4, in the Gaussian random-walk approximation, translation invariance of the total step distribution on M^{g}_{3} fixes the corresponding unitary lift, which is generated by the GUE random Hamiltonians of **(RM)**. In this sense, **(RM)** is the natural unitary lift of classical Brownian measurement dynamics, and is unique under the assumptions of Theorem 4.

This result is also supported by the widespread role of random Hamiltonians and random matrices in the description of fluctuations in complex quantum systems. Moreover, the choice of the GUE ensemble is imposed upon us in this framework. The Gaussian Orthogonal Ensemble would not be sufficient for the present purpose: GOE is invariant only under real orthogonal transformations and therefore selects a preferred real structure in Hilbert space. It does not give the full unitary-invariant distribution of random Hamiltonian directions needed for isotropic diffusion in complex projective Hilbert space with the Fubini–Study geometry, as required in the framework.

It is not claimed here that the random-matrix form of the interaction Hamiltonian has already been derived from a complete microscopic model of every measuring apparatus or environment. Rather, the point is that the remaining physical assumption is narrower: complex environmental and measurement interactions must realize, after coarse-graining, the unitary lift of classical Brownian measurement dynamics described by Theorem 4. Under this assumption, the GUE form of (**RM**) is fixed rather than chosen independently. The resulting consequences are mathematically precise and physically nontrivial. They include unitary evolution for each realization, single Born-distributed outcomes, and recovery of Newtonian motion for macroscopic systems.

Several directions remain open. First, one should derive the **(RM)** form from more microscopic models of environmental interaction, or at least identify broad dynamical conditions under which GUE statistics emerge. Second, the parameter estimates should be refined and tested for concrete experimental systems, including mesoscopic interferometers, optomechanical devices, and continuous-measurement setups. Third, the extension to relativistic fields and quantum field theory should be developed systematically, with classical field configurations and source dynamics represented as appropriate submanifolds of field-state space. Finally, possible deviations from ideal **(RM)** behavior may provide additional experimental tests of the proposed mechanism.

The main conclusion is that the classical world need not be added to quantum mechanics from outside. Classical space, classical phase space, and Newtonian dynamics arise inside projective quantum state space, while measurement outcomes and Born probabilities arise from stochastic unitary motion between detector-defined equivalence classes. If the random-matrix conjecture **(RM)** is physically correct, then quantum mechanics contains within its own geometry and unitary dynamics the mechanisms needed for state reduction, classicality, and the quantum-to-classical transition.

Declaration of interest statement

The author declares no competing interests.

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