Disproof of Bell's Theorem by Clifford Algebra Valued Local Variables

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It is shown that Bell's theorem fails for the Clifford algebra valued local realistic variables. This is made evident by exactly reproducing quantum mechanical expectation value for the EPR-Bohm type spin correlations observable by means of a local, deterministic, Clifford algebra valued variable, without necessitating either remote contextuality or backward causation. Since Clifford product of multivector variables is non-commutative in general, the spin correlations derived within our locally causal model violate the CHSH inequality just as strongly as their quantum mechanical counterparts.

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Unlike our basic theories of space and time, quantum mechanics is not a locally causal theory [1]. This fact was famously brought forth by Einstein, Podolsky, and Rosen (EPR) in 1935 [2]. They hoped, however, that perhaps one day quantum mechanics may be "completed" into a realistic, locally causal theory, say by appending the incomplete quantum mechanical description of physical reality with additional "hidden" parameters. Today such hopes of restoring locality in physics while maintaining realism seem to have been severely undermined by Bell's theorem and its variants [3][4], with substantive support from experiments [5][6]. These theorems set out to prove that *no physical theory which is realistic as well as local in a specified sense can reproduce all of the statistical predictions of quantum mechanics* [3][6]. The purpose of this letter is to question the legitimacy of this conclusion by first constructing an exact local realistic model for the EPR-Bell type spin correlations, and then reevaluating Bell's proof of his theorem in the light of this model. In particular, it will be shown that the much studied CHSH inequality in this context [5] is violated within our local model and extended to the extrema of $\pm 2\sqrt{2}$, in *exactly* the same manner as it is within quantum mechanics.

To prove his theorem Bell employed a rather simple argument. His goal was to show that at least some of the predictions of quantum mechanics cannot be mimicked by a locally causal theory. Based on Bohm's spin version of the EPR thought experiment [7], he considered a pair of spin one-half particles, moving freely after production in opposite directions, with particles 1 and 2 subject, respectively, to spin measurements along independently chosen unit directions $\mathbf{a}$ and $\mathbf{b}$, which can be located at a spacelike distance from each other. If initially the pair has vanishing total spin, then its quantum mechanical spin state would be the entangled singlet state

$$|\Psi_{\mathbf{n}}\rangle = \frac{1}{\sqrt{2}}\Big\{|\mathbf{n},\,+\rangle_1\otimes|\mathbf{n},\,-\rangle_2 - |\mathbf{n},\,-\rangle_1\otimes|\mathbf{n},\,+\rangle_2\Big\}\,, \quad (1)$$

with $\mathbf{n}$ indicating an arbitrary unit direction, and

$$\boldsymbol{\sigma}\cdot\mathbf{n}\,|\mathbf{n},\,\pm\rangle \,=\, \pm\,|\mathbf{n},\,\pm\rangle \quad (2)$$

describing the quantum mechanical eigenstates in which the particles have spin "up" or "down" in units of $\hbar = 2$. Here $\boldsymbol{\sigma}$ is the familiar Pauli spin "vector" $(\sigma_x,\,\sigma_y,\,\sigma_z)$. Our interest lies in comparing the quantum predictions of spin correlations between the two remote subsystems with those derived from any locally causal theory.

Now, quantum mechanically the rotational invariance of the state $|\Psi_{\mathbf{n}}\rangle$ ensures that the expectation values of the individual spin observables $\boldsymbol{\sigma}_1\cdot\mathbf{a}$ and $\boldsymbol{\sigma}_2\cdot\mathbf{b}$ are

$$\begin{aligned}\mathcal{E}_{q.m.}(\mathbf{a}) \,&=\, \langle\Psi_{\mathbf{n}}|\,\boldsymbol{\sigma}_1\cdot\mathbf{a}\otimes\mathbb{1}\,|\Psi_{\mathbf{n}}\rangle \,=\, 0 \quad\text{and}\\ \mathcal{E}_{q.m.}(\mathbf{b}) \,&=\, \langle\Psi_{\mathbf{n}}|\,\mathbb{1}\otimes\boldsymbol{\sigma}_2\cdot\mathbf{b}\,|\Psi_{\mathbf{n}}\rangle \,=\, 0\,,\end{aligned} \quad (3)$$

where $\mathbb{1}$ is the identity matrix. The expectation value of the joint observable $\boldsymbol{\sigma}_1\cdot\mathbf{a}\otimes\boldsymbol{\sigma}_2\cdot\mathbf{b}$, on the other hand, is

$$\mathcal{E}_{q.m.}(\mathbf{a},\,\mathbf{b}) \,=\, \langle\Psi_{\mathbf{n}}|\,\boldsymbol{\sigma}_1\cdot\mathbf{a}\,\otimes\,\boldsymbol{\sigma}_2\cdot\mathbf{b}\,|\Psi_{\mathbf{n}}\rangle \,=\, -\,\mathbf{a}\cdot\mathbf{b}\,, \quad (4)$$

regardless of the relative distance between the two remote locations represented by the unit vectors $\mathbf{a}$ and $\mathbf{b}$. The last result can be derived [8] using the well known identity

$$(\boldsymbol{\sigma}\cdot\mathbf{a})\,(\boldsymbol{\sigma}\cdot\mathbf{b}) \,=\, \mathbf{a}\cdot\mathbf{b}\,\mathbb{1} \,+\, i\,\boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{b})\,, \quad (5)$$

which follows from the non-commutativity of products of the Pauli matrices σ_j $(j = x, y, z)$ defined by the algebra

$$\sigma_j\sigma_k \,=\, \delta_{jk}\,\mathbb{1} \,+\, i\,\epsilon_{jkl}\,\sigma_l\,, \quad (6)$$

where δ_{jk} is the Kronecker delta, $i \equiv \sqrt{-1}$ is the unit imaginary, and ϵ_{jkl} is the Levi-Civita alternating symbol.

Suppose now we consider a *complete* specification of the physical state of our two-state system denoted by λ, specifying all of the elements of physical reality of the pair at a suitable instant, in the manner envisaged by EPR [2]. Here the complete state λ can be taken to be discrete or continuous, a single variable or a set of variables, a single function or a set of functions, and can even govern the measurement outcomes of spin stochastically rather than deterministically. For our purposes, however, it would suffice to take λ as a single, continuous variable, fully compatible with deterministic laws of motion. If we now denote by $\rho(\lambda)$ the normalized probability measure on the space Λ of complete states, then the expectation value of the product of the two outcomes of spin measurements, parameterized by **a** and **b** as before, can be written as

$$\mathcal{E}_{h.v.}(\mathbf{a},\,\mathbf{b})\,=\int_{\Lambda} A_{\mathbf{a}}(\lambda)\,B_{\mathbf{b}}(\lambda)\;d\rho(\lambda)\,, \qquad (7)$$

where $A_{\mathbf{a}}(\lambda)$ and $B_{\mathbf{b}}(\lambda)$, with values ±1, represent the possible outcomes of measurements on the subsystems 1 and 2, satisfying the perfect correlations constraint $\mathcal{E}_{h.v.}(\mathbf{n},\,\mathbf{n})=-1$ adapted by EPR. What is crucial to note here is that the function $A_{\mathbf{a}}(\lambda)$ does not depend on the remote context **b**, and likewise the function $B_{\mathbf{b}}(\lambda)$ does not depend on the remote context **a**, thus adhering to the vital condition of locality assumed by EPR. It is important to note also that the probability measure $\rho(\lambda)$ in (7) is required to depend only on λ, and not on either **a** or **b**, which are thereby rendered "freely chosen" detector settings at a later time [1]. Moreover, the factorized form (7) of the joint expectation value of the two outcomes can be derived explicitly as a *consequence* of the relativistic notion of local causality [1]. As a result, Bell's theorem reduces simply to a claim of impossibility to reproduce the quantum mechanical correlations (4), by means of a local realistic expectation value of the form (7).

Before formally proving the theorem, however, Bell provides an illustration of the tension between quantum mechanics and local causality by means of a local model. The idea behind the model is to attempt to reproduce the quantum mechanical correlations (4) for a pair of spin one-half particles with zero total spin, and argue that it cannot be done without admitting remote contextuality. The space Λ of complete states for the model consists of unit vectors $\boldsymbol{\lambda}$ in three-dimensional Euclidean space $\mathbb{E}_3$, with dynamical variables $A_{\mathbf{a}}(\boldsymbol{\lambda})$ and $B_{\mathbf{b}}(\boldsymbol{\lambda})$ defined by

$$A_{\mathbf{n}}(\boldsymbol{\lambda})\,=\,-\,B_{\mathbf{n}}(\boldsymbol{\lambda})\,=\,sign\,(\boldsymbol{\lambda}\cdot\mathbf{n})\,, \qquad (8)$$

provided $\boldsymbol{\lambda}\cdot\mathbf{n}\neq0$, and otherwise equal to the sign of the first nonzero term from the set $\{n_x,\,n_y,\,n_z\}$. This simply means that $A_{\mathbf{n}}(\boldsymbol{\lambda})=+1$ if the two unit vectors **n** and $\boldsymbol{\lambda}$ happen to point through the same hemisphere centered at the origin of $\boldsymbol{\lambda}$, and $A_{\mathbf{n}}(\boldsymbol{\lambda})=-1$ otherwise. As a visual aid to Bell's model [8] one can think of a bomb at rest exploding into two freely moving fragments with angular momenta $\boldsymbol{\lambda}=\mathbf{J}_1=-\mathbf{J}_2$, with $\mathbf{J}_1+\mathbf{J}_2=0$. The two outcomes $A_{\mathbf{a}}(\mathbf{J}_1)$ and $B_{\mathbf{b}}(\mathbf{J}_2)$ can then be taken as $sign\,(\boldsymbol{\lambda}\cdot\mathbf{a})$ and $sign\,(-\boldsymbol{\lambda}\cdot\mathbf{b})$, respectively. If the initial directions of the two angular momenta are uncontrollable but describable by an isotropic probability distribution $\rho(\boldsymbol{\lambda})$, then the local realistic expectation values of the individual outcomes can be easily worked out to be [8]

$$\mathcal{E}_{h.v.}(\mathbf{n})\,=\,\pm\int_{\mathbb{E}_3} sign\,(\boldsymbol{\lambda}\cdot\mathbf{n})\;d\rho(\boldsymbol{\lambda})\,=\,0\,, \qquad (9)$$

where $\mathbf{n}=\mathbf{a}$ or **b**; and their joint expectation value based on the local form (7) can be similarly worked out to be

$$\mathcal{E}_{h.v.}(\mathbf{a},\,\mathbf{b})\,=\,-1+\frac{2}{\pi}\,\cos^{-1}(\mathbf{a}\cdot\mathbf{b})\,. \qquad (10)$$

Comparing this local realistic correlation function with its quantum mechanical counterpart (4), it is frequently stressed in the established literature [8] that

$$|\,\mathcal{E}_{q.m.}(\mathbf{a},\,\mathbf{b})|\,\geqslant\,|\,\mathcal{E}_{h.v.}(\mathbf{a},\,\mathbf{b})|\,. \qquad (11)$$

Thus quantum correlations are claimed to be stronger than any local realistic possibility, in almost all but the cases where both are either 0 or ±1. Indeed, it is claimed that "quantum phenomena are more disciplined" than their "classical" counterparts [8]. What is more, Bell has surmised [3] that local realistic correlations such as (10) cannot be amended to recover the quantum correlations (4), without necessitating remote contextuality [6].

It is at this stage that our skepticism of Bell's theorem hardens. Although, as yet, we are only at the heuristic stage of the formal proof of his theorem, conceptually it is an important stage, and before reconsidering his proof it is worth investigating whether more realistic models for spin can perhaps replace the inequality (11) with an *exact* equality. After all, spin angular momentum within classical physics is usually represented, not by a polar vector, but by an axial or pseudo vector, composed of a cross product of two polar vectors. Moreover, as we saw above, within quantum mechanics the physics of spin one-half particles is intimately linked to the Pauli algebra (6). Could then incorporating such realistic features into Bell's local model for spin make any difference? As we shall see, even a minimum of such amendments to Bell's model has devastating consequences for his theorem.

To appreciate this assertion, let us first recall that Pauli matrices $\{\sigma_j\}$ generating the algebra (6) actually form a matrix representation of the Clifford algebra $Cl_{3,0}$ of the orthogonal directions in the Euclidean space $\mathbb{E}_3$ [9]. It is then hardly surprising that Clifford algebra $Cl_{3,0}$ can be generated also by the set of orthonormal basis vectors $\{\,\mathbf{e}_j\}$ of the physical vector space $\mathbb{E}_3$, defined by

$$\mathbf{e}_j\,\mathbf{e}_k\,:=\,\delta_{jk}\,+\,I\,\epsilon_{jkl}\,\mathbf{e}_l\,\equiv\,\mathbf{e}_j\cdot\mathbf{e}_k+\mathbf{e}_j\wedge\mathbf{e}_k\,, \qquad (12)$$

where "$\cdot$" and "$\wedge$" denote the inner and outer products, and $j=x,\,y,$ or z. The formal similarities between the relations (6) and (12) should not be allowed to obscure the profound differences between them. It is crucial to note that the $\mathbf{e}_j$ appearing in the above definition are not the usual self-adjoint operators on a complex Hilbert space, but are the ordinary 3-vectors in the real physical vector space. Moreover, the $I=\sqrt{-1}$ appearing therein is not the unit imaginary $i=\sqrt{-1}$, but a *real* geometric entity defined by $I:=\mathbf{e}_x\,\mathbf{e}_y\,\mathbf{e}_z$, with $\{\,\mathbf{e}_x,\,\mathbf{e}_y,\,\mathbf{e}_z\}$ being a choice of right-handed frame of orthonormal vectors, and is known variously as a pseudoscalar, a volume form, or a directed volume element [9]. In fact, the algebra of the physical space is spanned by this trivector I, along with a scalar, the vectors $\{\,\mathbf{e}_j\}$, and the bivectors $\{\,\mathbf{e}_j\wedge\mathbf{e}_k\}$. The Euclidean vector space $\mathbb{E}_3$ is then defined simply as a set of all vectors **x** satisfying the equation $I\wedge\mathbf{x}=0$, with the basic product between its elements defined as

$$\mathbf{x}\,\boldsymbol{\xi} := \mathbf{x}\cdot\boldsymbol{\xi} + \mathbf{x}\wedge\boldsymbol{\xi}\,, \tag{13}$$

where $\boldsymbol{\xi}$ is any homogeneous multivector in $Cl_{3,0}$. Thus, by subsuming it as a subsystem, Clifford algebra is said to have "completed" the vector algebra of Gibbs [9].

Returning to Bell's local model, we begin by observing that within Clifford algebra a rotation of a physical object is represented by a *bivector*, which can be visualized as an *oriented* parallelogram, composed of two vectors [9]. Taking aboard this hint, let us then venture to replace the polar vector $\boldsymbol{\lambda}$ of Bell's model with the unit trivector

$$\boldsymbol{\mu} = \mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w} = \pm I \equiv \pm\,\mathbf{e}_x\wedge\mathbf{e}_y\wedge\mathbf{e}_z\,, \tag{14}$$

which can be pictured as a parallelepiped of unit volume, assembled by the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ of finite lengths and arbitrary directions, giving it an unspecified shape and orientation. Here the second of the equalities follows from the fact that every trivector in the algebra $Cl_{3,0}$ differs from I only by its volume and orientation. This allows us to quantify the ambivalence in the orientation of $\boldsymbol{\mu}$ simply by the sign of I. In what follows, we shall take the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$—and hence $\boldsymbol{\mu}$—to be uncontrollable, but describable by an isotropic probability distribution $\boldsymbol{\rho}(\boldsymbol{\mu})$. Thus, in essence, the intrinsic freedom of choice in the *initial* orientation of the unit pseudoscalar $\boldsymbol{\mu}$ would be our "local hidden variable." The local analogue of the spin variable $\boldsymbol{\sigma}\cdot\mathbf{n}$ can then be taken as the projection $\boldsymbol{\mu}\cdot\mathbf{n} = \pm I\,\mathbf{n}$, which turns out to be a *unit* bivector

$$\boldsymbol{\mu}\cdot\mathbf{n} \equiv \pm\{\,n_x\,\mathbf{e}_y\wedge\mathbf{e}_z + n_y\,\mathbf{e}_z\wedge\mathbf{e}_x + n_z\,\mathbf{e}_x\wedge\mathbf{e}_y\}. \tag{15}$$

Since ordinary vectors in $\mathbb{E}_3$, such as the unit vectors $\mathbf{n}$ [being solutions of $\boldsymbol{\mu}\wedge\mathbf{x} = (\pm I)\wedge\mathbf{x} = \pm(I\wedge\mathbf{x}) = 0$], are *insensitive* to the sign ambiguity in $\boldsymbol{\mu}$, the bivector (15) provides us a natural pair of dichotomic observables:

$$A_{\mathbf{n}}(\boldsymbol{\mu}) = B_{\mathbf{n}}(\boldsymbol{\mu}) = \boldsymbol{\mu}\cdot\mathbf{n}\,, \quad \text{with values } \pm 1. \tag{16}$$

Moreover, Clifford product of two such bivectors gives

$$\begin{aligned}(\,\boldsymbol{\mu}\cdot\mathbf{a})(\,\boldsymbol{\mu}\cdot\mathbf{b}) &= (\,\boldsymbol{\mu}\,\mathbf{a})(\,\boldsymbol{\mu}\,\mathbf{b}\,) = \boldsymbol{\mu}\,(\,\mathbf{a}\,\boldsymbol{\mu})\,\mathbf{b} = \boldsymbol{\mu}\,(\,\boldsymbol{\mu}\,\mathbf{a})\,\mathbf{b}\\ &= (\,\boldsymbol{\mu}\,\boldsymbol{\mu})(\mathbf{a}\,\mathbf{b}) = -\,\mathbf{a}\,\mathbf{b} = -\,\mathbf{a}\cdot\mathbf{b} - \mathbf{a}\wedge\mathbf{b}\\ &= -\,\mathbf{a}\cdot\mathbf{b} - \boldsymbol{\mu}\cdot(\mathbf{a}\times\mathbf{b}),\end{aligned} \tag{17}$$

where the definition $\boldsymbol{\mu}\wedge\mathbf{x} = 0$, associativity of (13), and duality relation $\mathbf{a}\wedge\mathbf{b} = \boldsymbol{\mu}\,(\mathbf{a}\times\mathbf{b})$ have been used. The last result can be proven also by brute force using (14) and the Clifford analogues of the triple product identities. Unlike the Pauli identity (5), the above result is simply a classical relation among various vector products.

Our next task is to evaluate analogues of the integrals (9) and (10) for the local functions (16). For this purpose the tool we shall be using is that of a *directed measure* $d\boldsymbol{\rho}(\boldsymbol{\mu})$, defined on a smooth orientable *vector* manifold $\mathcal{V}_3$ [9]. In other words, we shall be using a directed measure on a manifold whose "points" are vectors in $\mathbb{E}_3$, obeying the Clifford product (13). It is a matter of indifference relative to which pseudoscalar we choose to evaluate our local integrals, as long as its orientation is single-valued and continuous. An obvious choice is I, which we have defined using a right-handed frame $\{\mathbf{e}_j\}$ of orthonormal vectors. The directed measure on the manifold $\mathcal{V}_3$ can then be written as $d\boldsymbol{\rho}(\boldsymbol{\mu}) = I\,|d\boldsymbol{\rho}(\boldsymbol{\mu})|$, where $|d\boldsymbol{\rho}(\boldsymbol{\mu})|$ is a scalar measure of the Riemann integration. A directed integral is thus an oriented Riemann integral, with the orientation determined by the volume element I (which is constant for us, since $\mathbb{E}_3$ happens to be flat). In general, however, since the Clifford product of any two functions on $\mathcal{V}_3$ can be *non-commutative*, the result of a directed integration may depend on the ordering of its factors.

Using these tools, the isotropically weighted averages of the two outcomes $\boldsymbol{\mu}\cdot\mathbf{a}$ and $\boldsymbol{\mu}\cdot\mathbf{b}$, analogous to the expectation values (9), can be easily worked out, giving

$$\begin{aligned}\mathcal{E}_{c.v.}(\mathbf{n}) &= \int_{\mathcal{V}_3}\boldsymbol{\mu}\cdot\mathbf{n}\;d\boldsymbol{\rho}(\boldsymbol{\mu}) = I^2\!\int_{\mathcal{V}_3}\mathbf{n}\;sign(\boldsymbol{\mu})\;|d\boldsymbol{\rho}(\boldsymbol{\mu})|\\ &= -\,\frac{1}{2}\,\mathbf{n} + \frac{1}{2}\,\mathbf{n} = 0\,,\end{aligned} \tag{18}$$

where $\mathbf{n} = \mathbf{a}$ or $\mathbf{b}$, the subscript *c.v.* stands for Clifford variables, and $\boldsymbol{\rho}(\boldsymbol{\mu})$ is assumed to be normalized on $\mathcal{V}_3$. Similarly, the joint expectation value of the two outcomes in the manifestly local form (7) works out to be

$$\begin{aligned}\mathcal{E}_{c.v.}(\mathbf{a},\,\mathbf{b}) &= \int_{\mathcal{V}_3}(\,\boldsymbol{\mu}\cdot\mathbf{a})(\,\boldsymbol{\mu}\cdot\mathbf{b}\,)\;d\boldsymbol{\rho}(\boldsymbol{\mu})\\ &= -\,\mathbf{a}\cdot\mathbf{b} - \int_{\mathcal{V}_3}\boldsymbol{\mu}\cdot(\mathbf{a}\times\mathbf{b})\;d\boldsymbol{\rho}(\boldsymbol{\mu})\\ &= -\,\mathbf{a}\cdot\mathbf{b} + 0\,,\end{aligned} \tag{19}$$

where we have used the results (17) and (18).

It is crucial to note here that in our derivation above the measure $d\boldsymbol{\rho}(\boldsymbol{\mu})$ remains independent of the detector settings $\mathbf{a}$ and $\mathbf{b}$, chosen "freely" at a later time. Nor have we at any time let the local variables $A_{\mathbf{a}}(\boldsymbol{\mu})$ and $B_{\mathbf{b}}(\boldsymbol{\mu})$ depend on the remote settings $\mathbf{b}$ and $\mathbf{a}$, respectively [1].

The above result is of course exactly what is predicted by quantum mechanics. Thus, contrary to Bell's claim, a local realistic model can indeed be constructed to exactly reproduce quantum mechanical correlations (4), *without necessitating remote contextuality or backward causation.* This fact immediately raises a question: what has gone wrong with Bell's proof of his theorem? The answer to this question is not difficult to discern. In Bell's proof there is a tacit assumption that alternative functions such as $A_{\mathbf{a}}(\lambda)$ and $A_{\mathbf{a}'}(\lambda)$ always commute with each other. This, however, is not true for the Clifford algebra valued functions of the multivector variables $\boldsymbol{\xi}$, satisfying the non-commutative product relations defined in (13).

To appreciate this explicitly, let us reconsider the much studied CHSH string of expectation values [5][6][10]:

$$\mathcal{E}(\mathbf{a},\,\mathbf{b}) + \mathcal{E}(\mathbf{a},\,\mathbf{b}') + \mathcal{E}(\mathbf{a}',\,\mathbf{b}) - \mathcal{E}(\mathbf{a}',\,\mathbf{b}'). \tag{20}$$

For a generic multivector $\boldsymbol{\xi}$, this can be rewritten as

$$\int_{\mathcal{V}_3} \mathcal{F}_{c.v.}(\boldsymbol{\xi})\; d\boldsymbol{\rho}(\boldsymbol{\xi})\,, \tag{21}$$

with the local realistic function $\mathcal{F}_{c.v.}(\boldsymbol{\xi})$ defined as

$$\begin{aligned}\mathcal{F}_{c.v.}(\boldsymbol{\xi}) := A_{\mathbf{a}}(\boldsymbol{\xi})\,\{\,B_{\mathbf{b}}(\boldsymbol{\xi}) + B_{\mathbf{b'}}(\boldsymbol{\xi})\,\}\\ + A_{\mathbf{a'}}(\boldsymbol{\xi})\,\{\,B_{\mathbf{b}}(\boldsymbol{\xi}) - B_{\mathbf{b'}}(\boldsymbol{\xi})\,\}\,.\end{aligned} \tag{22}$$

If we now use the fact that, by definition, the functions $A^2_{\mathbf{a}}(\boldsymbol{\xi})$, $A^2_{\mathbf{a'}}(\boldsymbol{\xi})$, $B^2_{\mathbf{b}}(\boldsymbol{\xi})$, and $B^2_{\mathbf{b'}}(\boldsymbol{\xi})$ are all equal to $\pm$ unity, then the square of the function $\mathcal{F}_{c.v.}(\boldsymbol{\xi})$ simplifies to

$$\mathcal{F}^2_{c.v.}(\boldsymbol{\xi}) \,=\, 4 \,+\, [\,A_{\mathbf{a}}(\boldsymbol{\xi}),\, A_{\mathbf{a'}}(\boldsymbol{\xi})\,]\,[\,B_{\mathbf{b'}}(\boldsymbol{\xi}),\, B_{\mathbf{b}}(\boldsymbol{\xi})\,], \tag{23}$$

provided we assume that both of the A's commute with both of the B's, and vice versa:

$$[\,A_{\mathbf{n}}(\boldsymbol{\xi}),\, B_{\mathbf{n'}}(\boldsymbol{\xi})\,] \,=\, 0\,, \quad \forall\ \mathbf{n} \text{ and } \mathbf{n'}. \tag{24}$$

In quantum field theory the operator analogue of the last relation—which would state that the operators acting on different subsystems should commute if the subsystems happen to be spacelike separated—follows from the usual assumption of "local commutativity" [1]. However, in our equation above the commutation relation is between two ordinary functions, with a sharper geometrical meaning. It would hold whenever the Clifford product between the two functions happens to be symmetric [9], at least after average [1], as in the result (19) of our local model.

Returning to the square of the function $\mathcal{F}_{c.v.}(\boldsymbol{\xi})$ in (23), let us first note that if either of the two commutators in it were to vanish identically, then we would be led to the standard CHSH inequality with bounds $\pm\,2$ [5]. However, neither of the two commutators can vanish in general, because of the dependence of the local functions $A_{\mathbf{a}}(\boldsymbol{\xi})$ and $B_{\mathbf{b}}(\boldsymbol{\xi})$ on the Clifford algebra valued variables. One can easily see an explicit instance of this by once again using our local model, but now setting $A_{\mathbf{a}}(\boldsymbol{\mu}) = I\,\mathbf{e}_x$ and $A_{\mathbf{a'}}(\boldsymbol{\mu}) = -\,I\,\mathbf{e}_y$. The relations (12) then at once leads to the non-vanishing of the commutator $[\,A_{\mathbf{a}}(\boldsymbol{\mu}),\, A_{\mathbf{a'}}(\boldsymbol{\mu})\,]$. What is more, even when both commutators individually vanish upon average, their Clifford product within (21) before and after the average may not vanish in general, which can again be checked by simple examples. Once these facts are appreciated, it is easy to establish that

$$\mathcal{F}^2_{c.v.}(\boldsymbol{\xi}) \,\leqslant\, 4 \,+\, 2\times 2 \,=\, 8\,, \tag{25}$$

since each of the two commutators in (23) can reach a maximum value of $+\,2$ (which follows from the fact that each product such as $A_{\mathbf{a}}(\boldsymbol{\xi})A_{\mathbf{a'}}(\boldsymbol{\xi})$ can equal to either $+\,1$ or $-\,1$). Consequently, we arrive at the inequality

$$|\mathcal{F}_{c.v.}(\boldsymbol{\xi})| \,\leqslant\, 2\sqrt{2}\,. \tag{26}$$

Since this inequality holds for all values of $\boldsymbol{\xi}$, using (21) we finally arrive at the violation of CHSH inequality:

$$|\,\mathcal{E}(\mathbf{a},\,\mathbf{b}) \,+\, \mathcal{E}(\mathbf{a},\,\mathbf{b'}) \,+\, \mathcal{E}(\mathbf{a'},\,\mathbf{b}) \,-\, \mathcal{E}(\mathbf{a'},\,\mathbf{b'})\,| \,\leqslant\, 2\sqrt{2}\,.$$

This is of course exactly the same result as that obtained within standard quantum mechanics [10]. The difference, however, is that the above inequality is obtained within an entirely classical, local realistic framework of Bell.

It is worth noting here that, although for definiteness we have used the language of spin for our analysis above, it can be easily extended to any two-state system. Hence we can conclude that Bell inequalities must be violated, with precisely the same characteristics as they are indeed violated in experiments, not only by quantum mechanics, but by *any* theory that correctly implements the algebra of orthogonal directions in the physical space, namely the Clifford algebra $Cl_{3,0}$ of 3D space within nonrelativistic domain, or the Clifford algebra $Cl_{1,3}$ of spacetime within relativistic domain. Indeed, it is clear from our analysis that the often cited Tsirel'son bound [10] simply reflects the algebraic properties of the physical space, and hence should not be taken as characterizing a purely quantum mechanical feature of the EPR-Bohm correlations.

Finally, what can we say about some of the variants of Bell's theorem, such as the Greenberger-Horne-Zeilinger, or Hardy's variant [4]? Can our analysis be extended to these theorems which do not involve any inequalities? No attempt has been made here to address this question. We believe, however, that such an extension is possible.

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