Locality, Localization, and the Particle Concept: Topics in the Foundations of Quantum Field Theory

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This dissertation reconsiders some traditional issues in the foundations of quantum mechanics in the context of relativistic quantum field theory (RQFT); and it considers some novel foundational issues that arise first in the context of RQFT. The first part of the dissertation considers quantum nonlocality in RQFT. Here I show that the generic state of RQFT displays Bell correlations relative to measurements performed in any pair of spacelike separated regions, no matter how distant. I also show that local systems in RQFT are "open" to influence from their environment, in the sense that it is generally impossible to perform local operations that would remove the entanglement between a local system and any other spacelike separated system.

The second part of the dissertation argues that RQFT does not support a particle ontology — at least if particles are understood to be localizable objects. In particular, while RQFT permits us to describe situations in which a determinate number of particles are present, it does not permit us to speak of the location of any individual particle, nor of the number of particles in any particular region of space. Nonetheless, the absence of localizable particles in RQFT does not threaten the integrity of our commonsense concept of a localized object. Indeed, RQFT itself predicts that descriptions in terms of localized objects can be quite accurate on the macroscopic level.

The third part of the dissertation examines the so-called observer-dependence of the particle concept in RQFT — that is, whether there are any particles present must be relativized to an observer's state of motion. Now, it is not uncommon for modern physical theories to subsume observer-dependent descriptions under a more general observer-independent description of some underlying state of affairs. However, I show that the conflicting accounts concerning the particle content of the field cannot be reconciled in this way. In fact, I argue that these conflicting accounts should be thought of as "complementary" in the same sense that position and momentum descriptions are complementary in elementary quantum mechanics.

Preface

This dissertation derives from a series of articles that I wrote over the past three years. Most of these articles have already appeared in print: An earlier version of chapter 2 appeared as "Generic Bell correlation between arbitrary local algebras in quantum field theory," (jointly authored with R. Clifton) *Journal of Mathematical Physics, 41*, 1711–1717 (2000), an earlier version of chapter 3 appeared as "Entanglement and open systems in algebraic quantum field theory," (with R. Clifton), *Studies in the History and Philosophy of Modern Physics, 32*, 1–31 (2001), and an earlier version of chapter 4 appeared as "Reeh-Schlieder defeats Newton Wigner," *Philosophy of Science, 68*, 111–133 (2001). Furthermore, parts of chapters 6 and 7 are drawn from an article, "Are Rindler quanta real?" (with R. Clifton) that will appear in the September 2001 issue of the *British Journal for the Philosophy of Science*. I would like to thank each of these journals for their permission to reprint material from these articles.

As regards the contents of this dissertation, my greatest debt by far is to my dissertation director, Rob Clifton. As is apparent from the list of articles above, Rob and I have worked together very closely for the past three years — much more closely than is typical for graduate students and their advisors in philosophy. Although the fruitfulness of this collaboration might be attributed in part to a natural chemistry, I think that most of the credit must go to Rob himself. Rob not only displays a highly conscientious commitment to his students' well-being, but he also has more intellectual energy than anyone I have ever known. He always had new projects I could work on, and I never once felt that I would be imposing if I needed to call (even late at night!) to ask for his help in solving a problem. He also provided extensive comments on everything I wrote (very often within a matter of hours), and he never hesitated to publicly acknowledge my contributions to his work, even if they were minor. I count myself one of the most fortunate graduate students to have had Rob as my advisor.

I would also like to thank the other members of my dissertation commit-

tee for reading and commenting on my dissertation, and (in some cases) for travelling great distances to attend my defense. I have a number of other debts to teachers, fellow students, and e-mail correspondents; in fact, far too many debts to mention here. However, I would be remiss if I did not mention Rainer Verch for his constant willingness to answer my mathematical questions; and both Jeremy Butterfield and David Malament for their especially close reading of my work, for their incisive comments, and for their encouragement.

I would like to thank the Pew Younger Scholars Program for three years of financial support during the course of my doctoral studies, and the Rev. Norman and Mrs. Matilda Milbank (better known to me as Gramps and Granny) for supplying funds that enabled me to visit Oxford during the Fall of 1999. I would also like to thank my parents and my parents-in-law for their support — both emotional and financial — during my graduate studies. Finally, to my wife, Keller: I will never be able to thank you adequately for bearing this burden with me.

This dissertation is dedicated to my first philosophy teacher, Dr. Reginald McLelland, whose standards of philosophical rigor and clarity will always be a challenge and inspiration to me.

To the reader: Except for chapter 7, the chapters of this dissertation are self-contained, and can be read independently of each other. Chapter 7 presupposes material from chapter 6 and should be read in conjunction with the latter. In most cases, proofs of theorems are omitted from the main text, and are placed in an appendix at the end of the relevant chapter. One exception is chapter 2, which contains full proofs, with very little discussion of the interpretive significance of the results. Some of the implications of the results obtained in chapter 2 are discussed in chapter 3.

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Chapter 1

Introduction

Many different sorts of inquiries fall under the heading of "philosophy of physics." At one extreme, we have investigations of a metaphysical or epistemological nature that make use—if only tangentially—of facts delivered to us by contemporary physics. At the other extreme, we have almost purely mathematical investigations which might have their original motivation in some philosophical question, but which are ultimately aimed at clarifying the structure of current physical theories. It goes without saying that both sorts of inquiry are essential for gaining an adequate grasp of the foundations of physics, and of the place of contemporary physics in a scientifically informed worldview.

With regard to elementary quantum mechanics, both sorts of foundational inquiry have been carried out extensively over the past 75 years. First, philosophers are indebted to von Neumann, who did more than anyone else to explicate the concepts of quantum mechanics in a mathematically rigorous fashion. Indeed, Bas van Fraassen has gone so far as to claim that, "interpretation of quantum theory became genuinely feasible only after von Neumann's theoretical unification in 1932" (1991, vii). Moreover, as a result of sustained efforts by a generation of philosophers of physics, we now have a quite clear grasp of the interpretive options for elementary quantum mechanics and of their respective advantages and disadvantages.

On the other hand, there has been to date very little work done on the foundations of quantum *field* theory. To make matters worse, the little foundational work that has been done in this area has paid very little attention to "mathematical niceties", and has pressed on rashly in an attempt to get a quick metaphysical payoff. Thus, for the most part, the philosophy of physics community is still in a state of culpable ignorance about quantum field theory. The primary goal of this dissertation is to fill the gap in the philosophical literature left by the absence of proper "foundational" investigations into quantum field theory.

As the above quote from van Fraassen indicates, mathematical rigor can be—and should be—an aid, rather than a hindrance, to the interpreter of physics. Unfortunately, much of what goes by the name of "quantum field theory" still lies in a hazy area between the intuitive and the mathematically rigorous. The approach of this dissertation will be to place a premium on mathematical rigor. Thus, in this dissertation, "quantum field theory" will mean that part of quantum field theory that has been formalized in language acceptable to the current community of mathematicians.

1.1 Nonlocality in quantum field theory

Very early in the development of quantum theory, worries arose about a potential conflict between the correlations predicted by quantum theory and the "physical world view" of classical physics—especially relativistic field theories—in which the state of a local system is independent of the state of distant systems. However, it was John S. Bell (1964) who first elucidated the peculiar nonlocality of quantum theory, by showing rigorously that the correlations predicted by quantum theory cannot be reproduced by any local hidden variable model.

Philosophers of physics have spent a great deal of time analyzing Bell's result, and discussing its implications for the relationship between quantum theory and the theory of relativity. However, these discussions have paid very little regard to what happens in *manifestly* relativistic theories such as relativistic quantum field theory (RQFT). In Part I, I attempt to fill this gap in current discussions of quantum nonlocality and relativity.

One of the central results of (axiomatic) RQFT is the Reeh-Schlieder theorem (Reeh & Schlieder 1961), which shows that any field state with bounded energy (e.g., the vacuum state) is "cyclic" for each local algebra of observables. (i.e., by applying elements of a local algebra to the vector, one can generated the entire state space.) It was very quickly realized that this cyclicity property has *some or other* connection with quantum nonlocality. For example, Segal (1964) expressed his worries about the nonlocality entailed by the Reeh-Schlieder theorem as follows:

> ...this apparently meant that the entire state vector space of the field could be obtained from measurements in an arbitrarily small region of spacetime! (Segal 1964, 140)

However, it was only very recently—with Redhead's (1995a) analysis—that it became clear that cyclicity should have some straightforward connection with the forms of quantum nonlocality that are familiar from elementary quantum mechanics.

In chapter 2, we confirm Redhead's intuition by showing that cyclic states are entangled (Proposition 2.2); i.e., they cannot be thought of as representing our ignorance of the "hidden" state of each local system. Thus, any bounded energy state in RQFT, including the vacuum state, is entangled across *any* two spacelike separated regions.

In chapter 2 we also derive one of the main results of this dissertation, around which much of the discussion of chapter 3 will focus. In particular, we show that for any pair of mutually commuting von Neumann algebras, if both algebras are "of infinite type" then the set of states Bell correlated across these algebras lies dense in the state space (Proposition 2.1). Since algebras of local observables in quantum field theory are always of infinite type, this result shows that for any pair of spacelike separated systems, a dense set of field states violate Bell's inequalities relative to measurements that can be performed on the respective subsystems (Proposition 2.3). (This outcome can be contrasted with elementary quantum mechanics, where "decoherence effects" will most often drive a pair of systems into a classically correlated state.)

In chapter 3, I turn to another form of nonlocality in RQFT. In particular, folklore has it that in *relativistic* quantum field theories, a local system cannot be isolated or shielded from outside effects. First, I make this idea precise by showing that in RQFT, no local operations can remove entanglement between a local system and its environment. I then argue, however, that this robustness of entanglement does not pose the sort of methodological threat that Einstein, among others, thought would result from taking quantum theory to be complete.

On the whole, Part I shows that the familiar sorts of quantum nonlocality return with a vengeance in RQFT. Thus, we have a strong confirmation of the fact that there no fundamental conflict between "quantum nonlocality" and the "locality" required by the special theory of relativity. In particular, the locality principles of special relativity are enforced in RQFT by means of the spectrum condition (which prohibits superluminal energy-momentum transfer) and the microcausality condition (which maintains that observables associated with spacelike separated regions are compatible). However, we will see in Part II that the microcausality assumption itself is incompatible with the existence of *localizable* particles—showing that some aspect of our "classical" concept of locality will have to be abandoned in the move to relativistic quantum theories.

1.2 Localizable particles

When it comes to questions of interpretation, the name "quantum *field* theory" is unfortunate—since it suggests that quantum field theory is a theory about fields. However, we should not assume *apriori* that QFT lends itself more easily to a field ontology than to a particle ontology. For example, the non-relativistic free Bose field can be thought of perfectly well as a system of a variable number of localizable particles.

However, in Part II, we show that there are very good reasons for thinking that *relativistic* QFT is not a theory of localizable particles. First, in chapter 4, we consider a proposal—due to Irving Segal (1964) and Gordon Fleming (2000)—which is supposed to restore the intuitive picture of particle localization to relativistic QFT. However, I show that the Segal-Fleming proposal is able to secure a notion of localizable particles only at the expense of violating the microcausality condition. Furthermore, I defend the "received position" according to which a failure of microcausality *would* entail a genuine conflict with relativistic causality. Thus, the Segal-Fleming approach to localization should be rejected.

In chapter 5, I consider the issue of particle localization in relativistic quantum theories from a more abstract perspective. According to physics folklore, there is some fundamental conflict between relativistic causality and particle localization, so that no *relativistic* quantum theory will permit a notion of localizable particles. This intuition was made rigorous in a theorem by Malament (1996) which shows that there is no nontrivial system of localizing projections that satisfies microcausality. In chapter 5, I generalize this result by giving three no-go theorems (Theorems 5.1, 5.2, and 5.3) which show that there is no nontrivial system of localization operators (including projection operators, effects, and number operators) that satisfies microcausality. Nonetheless, I argue (in section 5.7) that the absence of localizable particles in relativistic QFT does not threaten the integrity of our commonsense—as well as scientifically educated—use of the concept of a localizable object. In particular, RQFT *itself* predicts that descriptions in terms of localized objects, while strictly false, can nonetheless be quite accurate within most familiar contexts.

The upshot of Part II, then, is that no relativistic quantum theory, including RQFT, permits a notion of localizable particles. However, if we dispense with the requirement of locazability, RQFT does permit a notion of "particles" or "objects," as things that can be aggregated or counted. (Formally speaking, although there are no local number operators, there is still a global number operator.) However, we then run into another difficulty—viz., the particle concept is not invariant. In particular, different observers may disagree about whether or not there are particles in a given state of the field. Thus, the objective of Part III is to determine whether we can salvage any aspect of the particle concept given this failure of invariance.

1.3 Inequivalent particle concepts

Part III takes up the the issue of inequivalent particle concepts in RQFT. As a prolegomenon, chapter 6 treats the general issue of inequivalent representations of the canonical commutation relations (CCRs) and, more generally, of inequivalent representations of an abstract C^*-algebra of observables. Philosophers have just recently realized that the existence of inequivalent representations of the CCRs gives rise to a host of interesting foundational questions. However, these inequivalent representations do not fit naturally into the categories of "theoretical relations" discussed by philosophers of science. In particular, inequivalent representations do not correspond to different (inequivalent, or incommensurable) theories; and neither can they be thought of as different—but ultimately equivalent, or intertranslatable—formulations of one and the same theory.

In order to get a handle on inequivalent representations, I provide a four-part classification of the different positions that can be taken on the issue (see p. 123). Each position is distinguished by the ontological significance it attributes to certain elements of the mathematical formalism—viz., whether it attributes physical significance to observables can only be "weakly approximated" by elements in the CCR (Weyl) algebra, and whether it attributes physical significance to states in one, or more than one, folium of the state space of the CCR (Weyl) algebra.

Chapter 7 takes up the special case of the Minkowski and Rindler vacuum representations of the CCRs. Here we are faced with the ontological puzzle that RQFT seems to predict that an observer traveling at a constant (nonzero) rate of acceleration will detect an infinite number of "particles" (commonly called "Rindler quanta") in the Minkowski vacuum state. Various responses have been offered to this "paradox of the observer-dependence of particles." On the one hand, operationalists—such as Davies (1984)—claim that since, "quantum mechanics is an algorithm for computing the results of measurements," we need to bother about whether there *really* are particles in the Minkowski vacuum state. In reaction, some philosophers of a more realist bent have argued against the "reality" of the particles detected by the accelerated observer (cf. Arageorgis 1995; Earman 2001)—thus privileging the description given by the inertial observer over the description given by the accelerating observer.

I argue, however, that the realist response (i.e., to deny the physical significance of Rindler quanta) oversimplifies the problem of inequivalent representations. In particular, the relationship between the Minkowski number observable and Rindler number observable is directly analogous to the relation between the position and momentum observables in elementary quantum mechanics. Thus, the choice of a representation of the Weyl algebra (or any abstract C^*-algebra of observables) should be thought of as directly analogous to a choice of a basis in elementary quantum mechanics; and the claim that there is a privileged representation (say, the Minkowski vacuum representation) is no less controversial than the claim that there is a preferred basis (say, the position basis) in elementary quantum mechanics.

1.4 Operator algebras for quantum theory

Here we give a brief review of some of the basic concepts of operator algebras that will be used throughout the dissertation. Our basic reference is (Kadison & Ringrose 1997), which we will abbreviate hereafter by KR.

An abstract *C^*-algebra* is a Banach $*$-algebra, where the involution and norm are related by $\|A^*A\| = \|A\|^2$. Thus the algebra $\mathbf{B}(\mathcal{H})$ of all bounded operators on a Hilbert space $\mathcal{H}$ is a C^*-algebra, with * taken to be the adjoint operation, and $\|\cdot\|$ the operator norm

$$\|A\| = \sup\left\{ \|Ax\| : x \in \mathcal{H}, \|x\| = 1 \right\}. \tag{1.1}$$

(As is standard practice, we use the same notation for the operator norm on $\mathbf{B}(\mathcal{H})$ and the vector norm on $\mathcal{H}$.) Moreover, any $*$-subalgebra of $\mathbf{B}(\mathcal{H})$ that is closed in the operator norm is a C^*-algebra and, conversely, one can show that every abstract C^*-algebra has a concrete (faithful) representation as a norm-closed $*$-subalgebra of $\mathbf{B}(\mathcal{H})$, for some appropriate Hilbert space $\mathcal{H}$ (KR 1997, Remark 4.5.7).

Let $\mathcal{F}$ be any family of bounded operators acting on some Hilbert space $\mathcal{H}$. Let $\mathcal{F}'$ denote the *commutant* of $\mathcal{F}$, i.e., the set of all operators on $\mathcal{H}$ that commute with *every* operator in $\mathcal{F}$. Observe that $\mathcal{F} \subseteq \mathcal{F}''$, that $\mathcal{F} \subseteq \mathcal{G}$ implies $\mathcal{G}' \subseteq \mathcal{F}'$, and (hence) that $\mathcal{F}' = \mathcal{F}'''$. A C^*-algebra $\mathcal{R}$ acting on $\mathcal{H}$ is called a *von Neumann algebra* just in case $\mathcal{R} = \mathcal{R}''$. This is equivalent, via von Neumann's double commutant theorem (KR 1997, Theorem 5.3.1), to the assertion that $I \in \mathcal{R}$ and $\mathcal{R}$ is closed in the strong operator topology, where $A_i \to A$ strongly just in case $\|(A_i - A)x\| \to 0$ for all $x \in \mathcal{H}$.

If a net $\{A_i\} \subseteq \mathcal{R}$ converges to $A \in \mathcal{R}$ in norm, then since

$$\|(A_i - A)x\| \leq \|A_i - A\| \, \|x\| \, , \tag{1.2}$$

the convergence is also strong. Hence every von Neumann algebra is also a C^*-algebra. However, not every C^*-algebra of operators is a von Neumann algebra. For example, the C^*-algebra $\mathcal{K}$ of all compact operators on an infinite-dimensional Hilbert space $\mathcal{H}$—that is, the norm closure of the $*$-subalgebra of all finite rank operators on $\mathcal{H}$—does *not* contain the identity, nor does $\mathcal{K}$ satisfy $\mathcal{K} = \mathcal{K}''$. Indeed, $\mathcal{K}'' = \mathbf{B}(\mathcal{H})$, because only multiples of the identity commute with all finite-dimensional projections, and of course every operator commutes with multiples of the identity.

If $\mathcal{F}$ is a self-adjoint family of of operators acting on $\mathcal{H}$ (i.e., if $A \in \mathcal{F}$ then $A^* \in \mathcal{F}$), then it is easy to verify that $\mathcal{F}'$ is a von Neumann algebra. Of course, it also follows then that $\mathcal{F}''$ is a von Neumann algebra. If $\mathcal{R}$ is a von Neumann subalgebra of $\mathbf{B}(\mathcal{H})$ such that $\mathcal{F} \subseteq \mathcal{R}$, then $\mathcal{R}' \subseteq \mathcal{F}'$, which in turn entails $\mathcal{F}'' \subseteq \mathcal{R}'' = \mathcal{R}$. Thus $\mathcal{F}''$ is the smallest von Neumann algebra containing $\mathcal{F}$, i.e., the von Neumann generated by $\mathcal{F}$. For example, the von Neumann algebra generated by all finite rank operators is the whole of $\mathbf{B}(\mathcal{H})$.

Let $\mathcal{A}$ be a C^*-algebra with identity I, and let ω be a linear functional on $\mathcal{A}$. We say that ω is *positive* just in case $\omega(A^*A) \geq 0$ for any $A \in \mathcal{A}$. We say that ω is a *state* just in case ω is positive and $\omega(I) = 1$. For example, if D is a positive, trace-class operator on $\mathcal{H}$, then the mapping

$$\omega(A) = \mathrm{Tr}(DA), \qquad A \in \mathbf{B}(\mathcal{H}), \tag{1.3}$$

is a positive linear functional on $\mathbf{B}(\mathcal{H})$. If $\mathrm{Tr}(D) = 1$, then ω is a state. If $\mathcal{R}$ is a von Neumann algebra and ω is a state on $\mathcal{R}$, we say that ω is *normal* just in case there is some density operator D on $\mathcal{H}$ such that (1.3) holds for all $A \in \mathcal{R}$.

Part I

Nonlocality in Quantum Field Theory

Chapter 2

Generic Bell correlation between arbitrary local algebras in quantum field theory

2.1 Introduction

There are many senses in which the phenomenon of Bell correlation, originally discovered and investigated in the context of elementary nonrelativistic quantum mechanics (Bell 1987; Clauser et al. 1969), is "generic" in quantum field theory models. For example, it has been shown that every pair of commuting nonabelian von Neumann algebras possesses *some* normal state with maximal Bell correlation (Summers 1990; see also Landau 1987b). Moreover, in most standard quantum field models, *all* normal states are maximally Bell correlated across spacelike separated tangent wedges or double cones (Summers 1990). Finally, every bounded energy state in quantum field theory sustains maximal Einstein-Podolsky-Rosen correlations across arbitrary spacelike separated regions (Redhead 1995a), and has a form of nonlocality that may be evinced by means of the state's violation of a conditional Bell inequality (Landau 1987a). (We also note that the study of Bell correlation in quantum field theory has recently borne fruit in the introduction of a new algebraic invariant for an inclusion of von Neumann algebras (Summers & Werner 1995; Summers 1997).)

Despite these numerous results, it remains an open question whether "most" states will have some or other Bell correlation relative to *arbitrary* spacelike separated regions. Our main purpose in this chapter is to verify that this is so: for any two spacelike separated regions, there is an open dense set of states which have Bell correlations across those two regions.

In section 2.2 we prove the general result that for any pair of mutually commuting von Neumann algebras of infinite type, a dense set of vectors will induce states which are Bell correlated across these two algebras. In section 2.3 we introduce, following Werner (1989), a notion of "nonseparability" of states that generalizes, to mixed states, the idea of an entangled pure state vector. We then show that for a pair of nonabelian von Neumann algebras, a vector cyclic for either algebra induces a nonseparable state. Finally, in section 2.4 we apply these results to algebraic quantum field theory.

2.2 Bell correlation between infinite von Neumann algebras

Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{S}$ denote the set of unit vectors in $\mathcal{H}$, and let $\mathbf{B}(\mathcal{H})$ denote the set of bounded linear operators on $\mathcal{H}$. We will use the same notation for a projection in $\mathbf{B}(\mathcal{H})$ and for the subspace in $\mathcal{H}$ onto which it projects. If $x \in \mathcal{S}$, we let ω_x denote the state of $\mathbf{B}(\mathcal{H})$ induced by x. Let $\mathcal{R}_1, \mathcal{R}_2$ be von Neumann algebras acting on $\mathcal{H}$ such that $\mathcal{R}_1 \subseteq \mathcal{R}_2'$, and let $\mathcal{R}_{12}$ denote the von Neumann algebra $\{\mathcal{R}_1 \cup \mathcal{R}_2\}''$ generated by $\mathcal{R}_1$ and $\mathcal{R}_2$. Following (Summers & Werner 1995), we set

$$\begin{aligned} \mathcal{T}_{12} \;\equiv\; & \Big\{(1/2)[A_1(B_1+B_2)+A_2(B_1-B_2)] : \\ & A_i = A_i^* \in \mathcal{R}_1, B_i = B_i^* \in \mathcal{R}_2, -I \le A_i, B_i \le I\Big\}. \end{aligned} \tag{2.1}$$

Elements of $\mathcal{T}_{12}$ are called *Bell operators* for $\mathcal{R}_{12}$. For a given state ω of $\mathcal{R}_{12}$, let

$$\beta(\omega) \;\equiv\; \sup\{\,|\omega(R)| : R \in \mathcal{T}_{12}\}. \tag{2.2}$$

If $\omega = \omega_x|_{\mathcal{R}_{12}}$ for some $x \in \mathcal{S}$, we write $\beta(x)$ to abbreviate $\beta(\omega_x|_{\mathcal{R}_{12}})$. From (2.2), it follows that the map $\omega \to \beta(\omega)$ is norm continuous from the state space of $\mathcal{R}_{12}$ into $[1, \sqrt{2}]$ (Summers & Werner 1995, Lemma 2.1). Since the map $x \to \omega_x|_{\mathcal{R}_{12}}$ is continuous from $\mathcal{S}$, in the vector norm topology, into the (normal) state space of $\mathcal{R}_{12}$, in the norm topology, it also follows that $x \to \beta(x)$ is continuous from $\mathcal{S}$ into $[1, \sqrt{2}]$. If $\beta(\omega) > 1$, we say that ω violates a Bell inequality, or is *Bell correlated*. In this context, Bell's theorem (Bell 1964) is the statement that a local hidden variable model of the correlations that ω dictates between $\mathcal{R}_1$ and $\mathcal{R}_2$ is only possible if $\beta(\omega) = 1$. Note that the set of states ω on $\mathcal{R}_{12}$ that violate a Bell inequality is open (in the norm topology) and, similarly, the set of vectors $x \in \mathcal{S}$ that induce Bell correlated states on $\mathcal{R}_{12}$ is open (in the vector norm topology).

We assume now that the pair $\mathcal{R}_1, \mathcal{R}_2$ satisfies the *Schlieder property*. That is, if $A \in \mathcal{R}_1$ and $B \in \mathcal{R}_2$ such that $AB = 0$, then either $A = 0$ or $B = 0$. Let $V \in \mathcal{R}_1$ and $W \in \mathcal{R}_2$ be nonzero partial isometries.[1] Suppose that the initial space V^*V of V is orthogonal to the final space VV^* of V; or, equivalently, that $V^2 = 0$. Similarly, suppose $W^2 = 0$. Consider the projections

$$E = V^*V + VV^*, \qquad F = W^*W + WW^*. \tag{2.3}$$

We show that there is a Bell operator $\widetilde{R}$ for $\mathcal{R}_{12}$ such that $\widetilde{R}y = \sqrt{2}y$ for some unit vector $y \in EF$, and $\widetilde{R}(I - E)(I - F) = (I - E)(I - F)$.

Let

$$\begin{array}{ll} A_1 = V + V^* & B_1 = W + W^* \\ A_2 = i(V^* - V) & B_2 = i(W^* - W) \\ A_3 = [V, V^*] & B_3 = [W, W^*]. \end{array} \tag{2.4}$$

Note that $A_i^2 = E$, the A_i are self-adjoint contractions in $\mathcal{R}_1$, $A_iE = EA_i = A_i$, and $[A_1, A_2] = 2iA_3$. Similarly, $B_i^2 = F$, the B_i are self-adjoint contractions in $\mathcal{R}_2$, $B_iF = FB_i = B_i$, and $[B_1, B_2] = 2iB_3$. If we let R denote the Bell operator constructed from A_i, B_i, a straightforward calculation shows that (cf. Landau 1987b)

$$R^2 = EF - \frac{1}{4}[A_1, A_2][B_1, B_2] = EF + A_3B_3. \tag{2.5}$$

Note that $P \equiv VV^* \neq 0$ is the spectral projection for A_3 corresponding to eigenvalue 1, and $Q \equiv WW^* \neq 0$ is the spectral projection for B_3 corresponding to eigenvalue 1. Since $\mathcal{R}_1, \mathcal{R}_2$ satisfy the Schlieder property, there is a unit vector $y \in PQ$, and thus $A_3B_3y = y$. Since $PQ < EF$, it follows from (2.5) that $R^2y = 2y$. Thus, we may assume without loss of generality that $Ry = \sqrt{2}y$. (If $Ry \neq \sqrt{2}y$, then interchange B_1, B_2 and replace A_1 with $-A_1$. Note that the resulting Bell operator $R' = -R$ and $R'y_0 = \sqrt{2}y_0$, where $y_0 \equiv (\sqrt{2}y - Ry)/\|\sqrt{2}y - Ry\| \in EF$.)

Now for $i = 1, 2$, let $\widetilde{A}_i = (I - E) + A_i$ and $\widetilde{B}_i = (I - F) + B_i$. It is

[1] A partial isometry V is an operator on a Hilbert space $\mathcal{H}$ that maps some particular closed subspace $C \subseteq \mathcal{H}$ isometrically onto another closed subspace $C' \subseteq \mathcal{H}$, and maps $C^\perp$ to zero.

easy to see that $\widetilde{A}_i^2 = I$ and $\widetilde{B}_i^2 = I$, so the $\widetilde{A}_i$ and $\widetilde{B}_i$ are again self-adjoint contractions in $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively. If we let $\widetilde{R}$ denote the corresponding Bell operator, a straightforward calculation shows that

$$\widetilde{R} = (I - E)(I - F) + (I - E)B_1 + A_1(I - F) + R. \tag{2.6}$$

Since the $\sqrt{2}$ eigenvector y for R lies in EF, we have $\widetilde{R}y = Ry = \sqrt{2}y$. Furthermore, since $A_i(I - E) = 0$ and $B_i(I - F) = 0$, we have $\widetilde{R}(I - E)(I - F) = (I - E)(I - F)$ as required.

A special case of the following result, where $\mathcal{R}_1$ and $\mathcal{R}_2$ are type I_∞ factors, was proved as Proposition 1 of (Clifton, Halvorson, & Kent 2000). Recall that $\mathcal{R}$ is said to be *of infinite type* just in case the identity I is equivalent, in $\mathcal{R}$, to one of its proper subprojections.

Proposition 2.1. *Let $\mathcal{R}_1, \mathcal{R}_2$ be von Neumann algebras acting on $\mathcal{H}$ such that $\mathcal{R}_1 \subseteq \mathcal{R}_2'$, and $\mathcal{R}_1, \mathcal{R}_2$ satisfy the Schlieder property. If $\mathcal{R}_1, \mathcal{R}_2$ are of infinite type, then there is an open dense subset of vectors in $\mathcal{S}$ which induce Bell correlated states for $\mathcal{R}_{12}$.*

Note that the hypotheses of this proposition are invariant under isomorphisms of $\mathcal{R}_{12}$. Thus, by making use of the universal normal representation of $\mathcal{R}_{12}$ (KR 1997, 458), in which all normal states are vector states, it follows that the set of states Bell correlated for $\mathcal{R}_1, \mathcal{R}_2$ is norm dense in the normal state space of $\mathcal{R}_{12}$.

Proof of the proposition: Since $\mathcal{R}_1$ is infinite, there is a properly infinite projection $P \in \mathcal{R}_1$ (KR 1997, Prop. 6.3.7). Since P is properly infinite, we may apply the halving lemma (KR 1997, Lemma 6.3.3) repeatedly to obtain a countably infinite family $\{P_n\}$ of mutually orthogonal projections such that $P_n \sim P_{n+1}$ for all n and $\sum_{n=1}^\infty P_n = P$. (Halve P as $P_1 + F_1$; then halve F_1 as $P_2 + F_2$, and so on. Now replace P_1 by $P - \sum_{n=2}^\infty P_n$; cf. KR 1997, Lemma 6.3.4.) Let $P_0 \equiv I - P$. For each $n \in \mathbb{N}$, let V_n denote the partial isometry with initial space $V_n^* V_n = P_n$ and final space $V_n V_n^* = P_{n+1}$. By the same reasoning, there is a countable family $\{Q_n\}$ of mutually orthogonal projections in $\mathcal{R}_2$ and partial isometries W_n with $W_n^* W_n = Q_n$ and $W_n W_n^* = Q_{n+1}$. For each $n \in \mathbb{N}$, let

$$\begin{array}{ll} A_{1,n} = V_{n+1} + V_{n+1}^* & B_{1,n} = W_{n+1} + W_{n+1}^*, \\ A_{2,n} = i(V_{n+1}^* - V_{n+1}) & B_{2,n} = i(W_{n+1}^* - W_{n+1}), \end{array} \tag{2.7}$$

and let

$$E_n = V_{n+1}^* V_{n+1} + V_{n+1} V_{n+1}^* = P_{n+1} + P_{n+2}, \tag{2.8}$$
$$F_n = W_{n+1}^* W_{n+1} + W_{n+1} W_{n+1}^* = Q_{n+1} + Q_{n+2}. \tag{2.9}$$

Define $\widetilde{A}_{i,n}$ and $\widetilde{B}_{i,n}$ as in the discussion preceding this proposition, let $\widetilde{R}_n$ be the corresponding Bell operator, and let the unit vector $y_n \in E_n F_n$ be the $\sqrt{2}$ eigenvector for $\widetilde{R}_n$.

Now, let x be any unit vector in $\mathcal{H}$. Since $\sum_{i=0}^{n} P_i \leq I - E_n$ and since $\sum_{i=0}^{\infty} P_i = I$, we have $(I - E_n) \to I$ in the strong-operator topology. Similarly, $(I - F_n) \to I$ in the strong-operator topology. Therefore if we let

$$x_n \equiv \frac{(I - E_n)(I - F_n)x}{\|(I - E_n)(I - F_n)x\|}, \tag{2.10}$$

we have

$$x = \lim_n (I - E_n)(I - F_n)x = \lim_n x_n. \tag{2.11}$$

Note that the inner product $\langle x_n, y_n \rangle = 0$, and thus

$$z_n \equiv (1 - n^{-1})^{1/2} x_n + n^{-1/2} y_n \tag{2.12}$$

is a unit vector for all n. Since $\lim_n z_n = x$, it suffices to observe that each z_n is Bell correlated for $\mathcal{R}_{12}$. Recall that $\widetilde{R}_n(I - E_n)(I - F_n) = (I - E_n)(I - F_n)$, and thus $\widetilde{R}_n x_n = x_n$. A simple calculation then reveals that

$$\beta(z_n) \geq \langle \widetilde{R}_n z_n, z_n \rangle = (1 - n^{-1}) + n^{-1}\sqrt{2} > 1. \tag{2.13}$$

□

2.3 Cyclic vectors and entangled states

Proposition 2.1 establishes that Bell correlation is generic for commuting pairs of infinite von Neumann algebras. However, we are given no information about the character of the correlations of particular states. We provide a partial remedy for this in the next proposition, where we show that any vector cyclic for $\mathcal{R}_1$ (or for $\mathcal{R}_2$) induces a state that is not classically correlated; i.e., it is "nonseparable."

Again, let $\mathcal{R}_1, \mathcal{R}_2$ be von Neumann algebras on $\mathcal{H}$ such that $\mathcal{R}_1 \subseteq \mathcal{R}_2'$. Recall that a state ω of $\mathcal{R}_{12}$ is called a *normal product state* just in case ω

is normal, and there are states ω_1 of $\mathcal{R}_1$ and ω_2 of $\mathcal{R}_2$ such that

$$\omega(AB) = \omega_1(A)\omega_2(B), \tag{2.14}$$

for all $A \in \mathcal{R}_1, B \in \mathcal{R}_2$. Werner (1989), in dealing with the case of $\mathbf{B}(\mathbb{C}^n) \otimes \mathbf{B}(\mathbb{C}^n)$, defined a density operator D to be *classically correlated*—the term *separable* is now more commonly used—just in case D can be approximated in trace norm by convex combinations of density operators of form $D_1 \otimes D_2$. Although Werner's definition of nonseparable states directly generalizes the traditional notion of pure entangled states, he showed that a nonseparable mixed state need not violate a Bell inequality; thus, Bell correlation is in general a sufficient, though not necessary condition for a state's being nonseparable. On the other hand, it has since been shown that nonseparable states often possess more subtle forms of nonlocality, which may be indicated by measurements more general than the single ideal measurements which can indicate Bell correlation (Popescu 1995). (See Clifton, Halvorson, & Kent 2000 and Clifton & Halvorson 2000 for further discussion.)

In terms of the linear functional representation of states, Werner's separable states are those in the norm closed convex hull of the product states of $\mathbf{B}(\mathbb{C}^n) \otimes \mathbf{B}(\mathbb{C}^n)$. However, in case of the more general setup—i.e., $\mathcal{R}_1 \subseteq \mathcal{R}_2'$, where $\mathcal{R}_1, \mathcal{R}_2$ are arbitrary von Neumann algebras on $\mathcal{H}$—the choice of topology on the normal state space of $\mathcal{R}_{12}$ will yield in general different definitions of separability. Moreover, it has been argued that norm convergence of a sequence of states can never be verified in the laboratory, and as a result, the appropriate notion of physical approximation is given by the (weaker) weak* topology (Emch 1972; Haag 1992). And the weak* and norm topologies do not generally coincide *even* on the normal state space (Dell'Antonio 1967).

For the next proposition, then, we will define the *separable* states of $\mathcal{R}_{12}$ to be the normal states in the weak* closed convex hull of the normal product states. Note that $\beta(\omega) = 1$ if ω is a product state, and since β is a convex function on the state space, $\beta(\omega) = 1$ if ω is a convex combination of product states (Summers & Werner 1995, Lemma 2.1). Furthermore, since β is lower semicontinuous in the weak* topology (Summers & Werner 1995, Lemma 2.1), $\beta(\omega) = 1$ for any separable state. Contrapositively, any Bell correlated state must be nonseparable.

We now introduce some notation that will aid us in the proof of our result. For a state ω of the von Neumann algebra $\mathcal{R}$ and an operator $A \in \mathcal{R}$, define the state ω^A on $\mathcal{R}$ by

$$\omega^A(X) \equiv \frac{\omega(A^*XA)}{\omega(A^*A)}, \tag{2.15}$$

if $\omega(A^*A) \neq 0$, and let $\omega^A = \omega$ otherwise. Suppose now that $\omega(A^*A) \neq 0$ and ω is a convex combination of states:

$$\omega = \sum_{i=1}^{n} \lambda_i \omega_i. \tag{2.16}$$

Then, letting $\lambda_i^A \equiv \omega(A^*A)^{-1}\omega_i(A^*A)\lambda_i$, ω^A is again a convex combination

$$\omega^A = \sum_{i=1}^{n} \lambda_i^A \omega_i^A. \tag{2.17}$$

Moreover, it is not difficult to see that the map $\omega \to \omega^A$ is weak* continuous at any point ρ such that $\rho(A^*A) \neq 0$. Indeed, let $\mathcal{O}_1 = N(\rho^A : X_1, \ldots, X_n, \epsilon)$ be a weak* neighborhood of ρ^A. Then, taking

$$\mathcal{O}_2 = N(\rho : A^*A, A^*X_1A, \ldots, A^*X_nA, \delta), \tag{2.18}$$

and $\omega \in \mathcal{O}_2$, we have

$$|\rho(A^*X_iA) - \omega(A^*X_iA)| < \delta, \tag{2.19}$$

for $i = 1, \ldots, n$, and

$$|\rho(A^*A) - \omega(A^*A)| < \delta. \tag{2.20}$$

By choosing $\delta < \rho(A^*A) \neq 0$, we also have $\omega(A^*A) \neq 0$, and thus

$$|\rho^A(X_i) - \omega^A(X_i)| < O(\delta) \leq \epsilon, \tag{2.21}$$

for an appropriate choice of δ. That is, $\omega^A \in \mathcal{O}_1$ for all $\omega \in \mathcal{O}_2$ and $\omega \to \omega^A$ is weak* continuous at ρ.

Specializing to the case where $\mathcal{R}_1 \subseteq \mathcal{R}_2'$, and $\mathcal{R}_{12} = \{\mathcal{R}_1 \cup \mathcal{R}_2\}''$, it is clear from the above that for any normal product state ω of $\mathcal{R}_{12}$ and for $A \in \mathcal{R}_1$, ω^A is again a normal product state. The same is true if ω is a convex combination of normal product states, or the weak* limit of such combinations. We summarize the results of this discussion in the following lemma:

Lemma 2.1. *For any separable state ω of $\mathcal{R}_{12}$ and any $A \in \mathcal{R}_1$, ω^A is again separable.*

Proposition 2.2. *Let $\mathcal{R}_1, \mathcal{R}_2$ be nonabelian von Neumann algebras such that $\mathcal{R}_1 \subseteq \mathcal{R}_2'$. If x is cyclic for $\mathcal{R}_1$, then ω_x is nonseparable across $\mathcal{R}_{12}$.*

Proof. From Lemma 2.1 of (Summers & Werner 1995), there is a normal state ρ of $\mathcal{R}_{12}$ such that $\beta(\rho) = \sqrt{2}$. But since all normal states are in the (norm) closed convex hull of vector states (KR 1997, Thm 7.1.12), and since β is norm continuous and convex, there is a vector $v \in \mathcal{S}$ such that $\beta(v) > 1$. By the continuity of β (on $\mathcal{S}$), there is an open neighborhood $\mathcal{O}$ of v in $\mathcal{S}$ such that $\beta(y) > 1$ for all $y \in \mathcal{O}$. Since x is cyclic for $\mathcal{R}_1$, there is an $A \in \mathcal{R}_1$ such that $Ax \in \mathcal{O}$. Thus, $\beta(Ax) > 1$ which entails that $\omega_{Ax} = (\omega_x)^A$ is a nonseparable state for $\mathcal{R}_{12}$. This, by the preceding lemma, entails that ω_x is nonseparable. □

Note that if $\mathcal{R}_1$ has at least one cyclic vector $x \in \mathcal{S}$, then $\mathcal{R}_1$ has a dense set of cyclic vectors in $\mathcal{S}$ (Dixmier & Marechal 1971). Since each of the corresponding vector states is nonseparable across $\mathcal{R}_{12}$, Proposition 2.2 shows that if $\mathcal{R}_1$ has a cyclic vector, then the (open) set of vectors inducing nonseparable states across $\mathcal{R}_{12}$ is dense in $\mathcal{S}$. On the other hand, since the existence of a cyclic vector for $\mathcal{R}_1$ is not invariant under isomorphisms of $\mathcal{R}_{12}$, Proposition 2.2 does not entail that if $\mathcal{R}_1$ has a cyclic vector, then there is a norm dense set of nonseparable states in the entire normal state space of $\mathcal{R}_{12}$. (Contrast the analogous discussion preceding the proof of Proposition 2.1.) Indeed, if we let $\mathcal{R}_1 = \mathbf{B}(\mathbb{C}^2) \otimes I$, $\mathcal{R}_2 = I \otimes \mathbf{B}(\mathbb{C}^2)$, then any entangled state vector is cyclic for $\mathcal{R}_1$; but, the set of nonseparable states of $\mathbf{B}(\mathbb{C}^2) \otimes \mathbf{B}(\mathbb{C}^2)$ is *not* norm dense (Clifton & Halvorson 2000; Zyczkowski et al. 1998). However, if in addition to $\mathcal{R}_1$ or $\mathcal{R}_2$ having a cyclic vector, $\mathcal{R}_{12}$ has a separating vector (as is often the case in quantum field theory), then all normal states of $\mathcal{R}_{12}$ are vector states (KR 1997, Thm. 7.2.3), and it follows that the nonseparable states *will* be norm dense in the entire normal state space of $\mathcal{R}_{12}$.

2.4 Applications to algebraic quantum field theory

Let (M, g) be a relativistic spacetime. The basic mathematical object of algebraic quantum field theory (see Haag 1992; Borchers 1996; Dimock 1980) is an association between precompact open subsets O of M and C^*-subalgebras $\mathcal{A}(O)$ of a unital C^*-algebra $\mathcal{A}$. (We assume that each $\mathcal{A}(O)$ contains the identity I of $\mathcal{A}$.) The motivation for this association is the idea that $\mathcal{A}(O)$

represents observables that can be measured in the region O. With this in mind, one assumes

1. *Isotony:* If $O_1 \subseteq O_2$, then $\mathcal{A}(O_1) \subseteq \mathcal{A}(O_2)$.
2. *Microcausality:* $\mathcal{A}(O') \subseteq \mathcal{A}(O)'$.

Here O' denotes the interior of the set of all points of M that are spacelike to every point in O.

In the case where (M, g) is Minkowski spacetime, it is assumed in addition that there is a faithful representation $\mathbf{x} \to \alpha_{\mathbf{x}}$ of the translation group of M in the group of automorphisms of $\mathcal{A}$ such that

3. *Translation Covariance:* $\alpha_{\mathbf{x}}(\mathcal{A}(O)) = \mathcal{A}(O + \mathbf{x})$.
4. *Weak Additivity:* For any $O \subseteq M$, $\mathcal{A}$ is the smallest C^*-algebra containing
$\bigcup_{\mathbf{x} \in M} \mathcal{A}(O + \mathbf{x})$.

The class of physically relevant representations of $\mathcal{A}$ is decided by further desiderata such as—in the case of Minkowski spacetime—a unitary representation of the group of translation automorphisms which satisfies the spectrum condition. Relative to a fixed representation π, we let $\mathcal{R}_\pi(O)$ denote the von Neumann algebra $\pi(\mathcal{A}(O))''$ on the representation space $\mathcal{H}_\pi$. In what follows, we consider only nontrivial representations (i.e., $\dim \mathcal{H}_\pi > 1$), and we let $\mathcal{S}_\pi$ denote the set of unit vectors in $\mathcal{H}_\pi$.

Proposition 2.3. *Let $\{\mathcal{A}(O)\}$ be a net of local algebras over Minkowski spacetime. Let π be any representation in the local quasiequivalence class of some irreducible vacuum representation (e.g. superselection sectors in the sense of Doplicher-Haag-Roberts (1969) or Buchholz-Fredenhagen (1982)). If O_1, O_2 are any two open subsets of M such that $O_1 \subseteq O_2'$, then the set of vectors inducing Bell correlated states for $\mathcal{R}_\pi(O_1), \mathcal{R}_\pi(O_2)$ is open and dense in $\mathcal{S}_\pi$.*

Proof. Let O_3, O_4 be precompact open subsets of M such that $O_3 \subseteq O_1, O_4 \subseteq O_2$, and such that $O_3 + N \subseteq O_4'$ for some neighborhood N of the origin. In an irreducible vacuum representation ϕ, local algebras are of infinite type (Horuzhy 1988, Prop. 1.3.9), and since $O_3 + N \subseteq O_4'$, the Schlieder property holds for $\mathcal{R}_\phi(O_3), \mathcal{R}_\phi(O_4)$ (Schlieder 1969). If π is any representation in the local quasiequivalence class of ϕ, these properties hold for $\mathcal{R}_\pi(O_3), \mathcal{R}_\pi(O_4)$ as well. Thus, we may apply Proposition 2.1 to conclude that the set of vectors inducing Bell correlated states for $\mathcal{R}_\pi(O_3), \mathcal{R}_\pi(O_4)$ is dense in $\mathcal{S}_\pi$.

Finally, note that any state Bell correlated for $\mathcal{R}_\pi(O_3), \mathcal{R}_\pi(O_4)$ is Bell correlated for $\mathcal{R}_\pi(O_1), \mathcal{R}_\pi(O_2)$. □

Proposition 2.4. *Let (M, g) be a globally hyperbolic spacetime, let $\{\mathcal{A}(O)\}$ be the net of local observable algebras associated with the free Klein-Gordon field (Dimock 1980), and let π be the GNS representation of some quasifree Hadamard state (Kay & Wald 1991). If O_1, O_2 are any two open subsets of M such that $O_1 \subseteq O_2'$, then the set of vectors inducing Bell correlated states for $\mathcal{R}_\pi(O_1), \mathcal{R}_\pi(O_2)$ is open and dense in $\mathcal{S}_\pi$.*

Proof. The regular diamonds (in the sense of Verch 1997) form a basis for the topology on M. Thus, we may choose regular diamonds O_3, O_4 such that $\overline{O_3} \subseteq O_1$ and $\overline{O_4} \subseteq O_2$. The nonfiniteness of the local algebras $\mathcal{R}_\pi(O_3), \mathcal{R}_\pi(O_4)$ is established in (Verch 1997, Thm. 3.6.g), and the split property for these algebras is established in (Verch 1997, Thm. 3.6.d). Since the split property entails the Schlieder property, it follows from Proposition 2.1 that the set of vectors inducing Bell correlated states for $\mathcal{R}_\pi(O_3), \mathcal{R}_\pi(O_4)$ [and thereby Bell correlated for $\mathcal{R}_\pi(O_1), \mathcal{R}_\pi(O_2)$] is dense in $\mathcal{S}_\pi$. □

There are many physically interesting states, such as the Minkowski vacuum itself, about which Propositions 2.3 and 2.4 are silent. However, Reeh-Schlieder type theorems entail that many of these physically interesting states are induced by vectors which are cyclic for local algebras, and thus it follows from Proposition 2.2 that these states are nonseparable across any spacelike separated pair of local algebras. In particular, although there is an upper bound on the Bell correlation of the Minkowski vacuum (in models with a mass gap) that decreases exponentially with spacelike separation (Summers & Werner 1995, Prop. 3.2), the vacuum state remains nonseparable (in our sense) at all distances. On the other hand, since nonseparability is only a *necessary* condition for Bell correlation, none of our results decide the question of whether the vacuum state always retains *some* Bell correlation across arbitrary spacelike separated regions.

Chapter 3

Entanglement and open systems in algebraic quantum field theory

. . . despite its conservative way of dealing with physical principles, algebraic QFT leads to a radical change of paradigm. Instead of the Newtonian view of a space-time filled with a material content one enters the reality of Leibniz created by relation (in particular inclusions) between 'monads' ($\sim$ the hyperfinite type III_1 local von Neumann factors $\mathcal{A}(O)$ which as single algebras are nearly void of physical meaning).

— Bert Schroer (1998)

3.1 Introduction

In *PCT, Spin and Statistics, and All That*, Streater and Wightman claim that, as a consequence of the axioms of algebraic quantum field theory (AQFT), "it is difficult to isolate a system described by fields from outside effects" (2000, 139). Haag makes a similar claim in *Local Quantum Physics*: "From the previous chapters of this book it is evidently not obvious how to achieve a division of the world into parts to which one can assign individuality. . . Instead we used a division according to regions in space-time. This leads in general to open systems" (1992, 298). By a field system these authors mean that portion of a quantum field within a specified bounded open region O of spacetime, with its associated algebra of observables $\mathcal{A}(O)$ (constructed in the usual way, out of 'field operators' smeared with test-functions having support in O). The environment of a field system (so construed) is naturally taken to be the field in the region O', the spacelike complement of O. But then the claims above appear, at first sight, puzzling. After all, it is an axiom of AQFT that the observables in $\mathcal{A}(O')$ commute with those in $\mathcal{A}(O)$. And this implies—indeed, is *equivalent* to—the assertion that standard von Neumann measurements performed in O' *cannot* have 'outside effects' on the expectations of observables in O (Lüders 1951). What, then, could the above authors possibly mean by saying that the field in O must be regarded as an open system?

A similar puzzle is raised by a famous passage in which Einstein (1948) contrasts the picture of physical reality embodied in classical field theories with that which emerges when we try to take quantum theory to be complete:

> If one asks what is characteristic of the realm of physical ideas independently of the quantum theory, then above all the following attracts our attention: the concepts of physics refer to a real external world, i.e., ideas are posited of things that claim a "real existence" independent of the perceiving subject (bodies, fields, etc.)...it appears to be essential for this arrangement of the things in physics that, at a specific time, these things claim an existence independent of one another, insofar as these things "lie in different parts of space". Without such an assumption of the mutually independent existence (the "being-thus") of spatially distant things, an assumption which originates in everyday thought, physical thought in the sense familiar to us would not be possible. Nor does one see how physical laws could be formulated and tested without such clean separation. ...For the relative independence of spatially distant things (A and B), this idea is characteristic: an external influence on A has no *immediate* effect on B; this is known as the "principle of local action," which is applied consistently in field theory. The complete suspension of this basic principle would make impossible the idea of the existence of (quasi-)closed systems and, thereby, the establishment of empirically testable laws in the sense familiar to us. (ibid, 321–322; Howard's 1989 translation)

There is a strong temptation to read Einstein's 'assumption of the mutually independent existence of spatially distant things' and his 'principle of local action' as anticipating, respectively, the distinction between separability and locality—or between nonlocal 'outcome-outcome' correlation and 'measurement-outcome' correlation—that some philosophers argue is crucial to unraveling the conceptual implications of Bell's theorem (see, e.g., Howard 1989). However, even in nonrelativistic quantum theory, there is no question of any nonlocal *measurement*-outcome correlation between distinct systems or degrees of freedom, whose observables are always represented as commuting. Making the reasonable assumption that Einstein knew this quite well, what is it about taking quantum theory at face value that he saw as a threat to securing the existence of physically closed systems?

What makes quantum systems open for Einstein, as well as for Streater and Wightman, and Haag, is that quantum systems can occupy entangled states in which they sustain nonclassical EPR correlations with other quantum systems outside their light cones. That is, while it is correct to read Einstein's discussion of the mutually independent existence of distant systems as an implicit critique of the way in which quantum theory typically represents their joint state as entangled, we believe it must be the *outcome-outcome* EPR correlations associated with entangled states that, in Einstein's view, pose a problem for the legitimate testing of the predictions of quantum theory. One could certainly doubt whether EPR correlations really pose any methodological problem, or whether they truly require the existence of physical (or 'causal') influences acting on a quantum system from outside. But the analogy with open systems in thermodynamics that Einstein and the others seem to be invoking is not entirely misplaced.

Consider the simplest toy universe consisting of two nonrelativistic quantum systems, represented by a tensor product of two-dimensional Hilbert spaces $\mathbb{C}^2_A \otimes \mathbb{C}^2_B$, where system A is the 'object' system, and B its 'environment'. Let x be any state vector for the composite system $A+B$, and $D_A(x)$ be the reduced density operator x determines for system A. Then the von Neumann entropy of A, $E_A(x) = -\mathrm{Tr}(D_A(x) \ln D_A(x))$ $(= E_B(x))$, varies with the degree to which A and B are entangled. If x is a product vector with no entanglement, $E_A(x) = 0$, whereas, at the opposite extreme, $E_A(x) = \ln 2$ when x is, say, a singlet or triplet state. The more A and B are entangled, the more 'disordered' A becomes, because it will then have more than one state available to it, and A's probabilities of occupying them will approach equality. In fact, exploiting an analogy to Carnot's heat cycle and the second law of thermodynamics (that it is impossible to construct a *perpetuum mobile*), Popescu and Rohrlich (1997) have shown that the general principle that it is impossible to create entanglement between pairs of systems by local operations on one member of each pair implies that the von Neumann entropy of either member provides the uniquely correct measure of their entanglement when they are in a pure state. Changes in their degree of entanglement, and hence in the entropy of either system A or B, can only come about in the presence of a nontrivial interaction Hamiltonian between them. But the fact remains that there is an intimate connection between a system's entanglement with its environment and the extent to which that system should be thought of as physically closed.

Returning to AQFT, Streater and Wightman, as well as Haag, all intend to make a far stronger claim about quantum field systems—a point that even applies to spacelike-separated regions of a *free* field, and might well have offended Einstein's physical sensibilities even more. The point is that quantum field systems are *unavoidably* and *intrinsically* open to entanglement. Streater and Wightman's comment is made in reference to the Reeh-Schlieder (1961) theorem, a consequence of the general axioms of AQFT. We shall show that this theorem entails severe *practical* obstacles to isolating field systems from entanglement with other field systems. Haag's comment goes deeper, and is related to the fact that the algebras associated with field systems localized in spacetime regions are in all known models of the axioms type III von Neumann algebras. We shall show that this feature of the local algebras imposes a fundamental limitation on isolating field systems from entanglement even *in principle*.

Think again of our toy nonrelativistic universe $A + B$, with Alice in possession of system A, and the state x entangled. Although there are no unitary operations Alice can perform on system A that will reduce its entropy, she can still try to destroy its entanglement with B by performing a standard von Neumann measurement on A. If $P_\pm$ are the eigenprojections of the observable Alice measures, and the initial density operator of $A + B$ is $D = P_x$ (where P_x is the projection onto the ray x generates), then the post-measurement joint state of $A + B$ will be given by the new density operator

$$D \to D' = (P_+ \otimes I)P_x(P_+ \otimes I) + (P_- \otimes I)P_x(P_- \otimes I). \tag{3.1}$$

Since the projections $P_\pm$ are one-dimensional, and x is entangled, there are nonzero vectors $a_x^\pm \in \mathbb{C}_A^2$ and $b_x^\pm \in \mathbb{C}_B^2$ such that $(P_\pm \otimes I)x = a_x^\pm \otimes b_x^\pm$, and a straightforward calculation reveals that D' may be re-expressed as

$$D' = \mathrm{Tr}[(P_+ \otimes I)P_x]P_+ \otimes P_{b_x^+} + \mathrm{Tr}[(P_- \otimes I)P_x]P_- \otimes P_{b_x^-}. \tag{3.2}$$

Thus, regardless of the initial state x, or the degree to which it was entangled, D' will always be a convex combination of product states, and there will no longer be any entanglement between A and B. One might say that Alice's measurement operation on A has the effect of isolating A from any further EPR influences from B. Moreover, this result can be generalized.

Given any finite or infinite dimension for the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, there is always an operation Alice can perform on system A that will destroy its entanglement with B no matter what their initial state D was, pure or mixed. In fact, it suffices for Alice to measure any nondegenerate observable of A with a discrete spectrum. The final state D' will then be a convex combination of product states, each of which is a product density operator obtained by 'collapsing' D using some particular eigenprojection of the measured observable.[1]

The upshot is that if entanglement *does* pose a methodological threat, it can at least be brought under control in nonrelativistic quantum theory. Not so when we consider the analogous setup in quantum field theory, with Alice in the vicinity of one region A, and B any other spacelike-separated field system. We shall see that AQFT puts both practical and theoretical limits on Alice's ability to destroy entanglement between her field system and B. Again, while one can doubt whether this poses any real methodological problem for Alice—an issue to which we shall return in earnest later—we think it is ironic, considering Einstein's point of view, that such limits should be forced upon us once we make the transition to a fully *relativistic* formulation of quantum theory.

We begin in section 3.2 by reviewing the formalism of AQFT, the concept of entanglement between spacelike-separated field systems, and the mathematical representation of an operation performed within a local spacetime region on a field system. In section 3.3, we connect the Reeh-Schlieder theorem with the practical difficulties involved in guaranteeing that a field system is disentangled from other field systems. The language of operations also turns out to be indispensable for clearing up some apparently paradoxical physical implications of the Reeh-Schlieder theorem that have been raised in the literature without being properly resolved. In section 3.4, we discuss differences between type III von Neumann algebras and the standard

[1] The fact that disentanglement of a state can always be achieved in this way does not conflict with the recently established result there can be no 'universal disentangling machine', i.e., no *unitary* evolution that maps an arbitrary $A+B$ state D to an unentangled state with the same reduced density operators as D (Mor 1998; Mor & Terno 1999). Also bear in mind that we have *not* required that a successful disentangling process leave the states of the entangled subsystems unchanged. Finally, though we have written of Alice's measurement 'collapsing' the density matrix D to D', we have *not* presupposed the projection postulate nor begged the question against no-collapse interpretations of quantum theory. What is at issue here is the destruction of entangling correlations between A and B, *not* between the compound system $M + A$, including Alice's measuring device M, and B.

type I von Neumann algebras employed in nonrelativistic quantum theory, emphasizing the radical implications type III algebras have for the ignorance interpretation of mixtures and entanglement. We end section 3.4 by connecting the type III character of the algebra of a local field system with the inability, in principle, to perform local operations on the system that will destroy its entanglement with other spacelike-separated systems. We offer this result as one way to make precise the sense in which AQFT requires a radical change in paradigm—a change that, regrettably, has passed virtually unnoticed by philosophers of quantum theory.

3.2 AQFT, entanglement, and local operations

The basic mathematical object of AQFT on Minkowski spacetime M is an association $O \mapsto \mathcal{A}(O)$ between bounded open subsets O of M and C^*-subalgebras $\mathcal{A}(O)$ of an abstract C^*-algebra $\mathcal{A}$ (Horuzhy 1988; Haag 1992). The motivation for this association is that the self-adjoint elements of $\mathcal{A}(O)$ represent the physical magnitudes, or observables, of the field intrinsic to the region O. We shall see below how the elements of $\mathcal{A}(O)$ can also be used to represent mathematically the physical operations that can be performed within O, and often it is only this latter interpretation of $\mathcal{A}(O)$ that is emphasized (Haag 1992, 104). One naturally assumes

Isotony: If $O_1 \subseteq O_2$, then $\mathcal{A}(O_1) \subseteq \mathcal{A}(O_2)$.

As a consequence, the collection of all local algebras $\mathcal{A}(O)$ defines a net whose limit points can be used to define algebras associated with unbounded regions, and in particular $\mathcal{A}(M)$, which is identified with $\mathcal{A}$ itself.

One of the leading ideas in the algebraic approach to fields is that all of the physics of a particular field theory is encoded in the structure of its net of local algebras.[2] But there are some general assumptions about the net $\{\mathcal{A}(O) : O \subseteq M\}$ that all physically reasonable field theories are held to satisfy. First, one assumes

Microcausality: $\mathcal{A}(O') \subseteq \mathcal{A}(O)'$.

One also assumes that there is a faithful representation $\mathbf{x} \to \alpha_{\mathbf{x}}$ of the spacetime translation group of M in the group of automorphisms of $\mathcal{A}$, satisfying

[2]In particular, though smearing any given field 'algebra' on M defines a unique net, the net underdetermines the field; see (Borchers 1960).

Translation Covariance: $\alpha_{\mathbf{x}}(\mathcal{A}(O)) = \mathcal{A}(O + \mathbf{x})$.

Weak Additivity: For any $O \subseteq M$, $\mathcal{A}$ is the smallest C^*-algebra containing $\bigcup_{\mathbf{x} \in M} \mathcal{A}(O + \mathbf{x})$.

Finally, one assumes that there is some irreducible representation of the net $\{\mathcal{A}(O) : O \subseteq M\}$ in which these local algebras are identified with von Neumann algebras acting on a (nontrivial) Hilbert space $\mathcal{H}$, $\mathcal{A}$ is identified with a strongly dense subset of $\mathbf{B}(\mathcal{H})$, and the following condition holds

Spectrum Condition: The generator of spacetime translations, the energy-momentum of the field, has a spectrum confined to the forward light-cone.

While the spectrum condition itself only makes sense relative to a representation (wherein one can speak, via Stone's theorem, of generators of the spacetime translation group of M—now concretely represented as a strongly continuous group of unitary operators $\{U(\mathbf{x})\}$ acting on $\mathcal{H}$), the requirement that the abstract net *have* a representation satisfying the spectrum condition does not require that one actually *pass* to such a representation to compute expectation values, cross-sections, etc. Indeed, Haag and Kastler (1964) have argued that there is a precise sense in which all concrete representations of a net are physically equivalent, including representations with and without a translationally invariant vacuum state vector Ω. Since we are not concerned with that argument here[3], we shall henceforth take the 'Haag-Araki' approach of assuming that all the local algebras $\{\mathcal{A}(O) : O \subseteq M\}$ are von Neumann algebras acting on some $\mathcal{H}$, with $\mathcal{A}'' = \mathbf{B}(\mathcal{H})$, and there is a translationally invariant vacuum state $\Omega \in \mathcal{H}$.[4]

We turn next to the concept of a state of the field. Generally, a physical state of a quantum system, represented by some von Neumann algebra $\mathcal{R} \subseteq$

[3]See chapter 6 of this dissertation, as well as Arageorgis et al. (2001), for somewhat different criticisms of the Haag-Kastler argument.

[4] Since we do, after all, live in a heat bath at 3 degrees Kelvin, some might think it would be of more immediate *physical* interest if we investigated entanglement in finite temperature "KMS" representations of the net $\{\mathcal{A}(O) : O \subseteq M\}$ that are "disjoint" from the vacuum representation. However, aside from the fact that the vacuum representation is the simplest and most commonly discussed representation, we are interested here only in the conceptual foundations of particle physics, not quantum statistical mechanics. Moreover, much of value can be learned about the conceptual infrastructure of a theory by examining particular classes of its models—whether or not they are plausible candidates for describing our actual world. (In any case, we could hardly pretend to be discussing physics on a cosmological scale by looking at finite temperature representations, given that we would still be presupposing a flat spacetime background!)

$\mathbf{B}(\mathcal{H})$, is given by a normalized linear expectation functional τ on $\mathcal{R}$ that is both positive and countably additive. Positivity is the requirement that τ map any positive operator in $\mathcal{R}$ to a nonnegative expectation (a must, given that positive operators have nonnegative spectra), while countable additivity is the requirement that τ be additive over countable sums of mutually orthogonal projections in $\mathcal{R}$.[5] Every state on $\mathcal{R}$ extends to a state ρ on $\mathbf{B}(\mathcal{H})$ which, in turn, can be represented by a density operator D_ρ on $\mathcal{H}$ via the standard formula $\rho(\cdot) = \mathrm{Tr}(D_\rho \cdot)$ (KR 1997, 462). A pure state on $\mathbf{B}(\mathcal{H})$, i.e., one that is not a nontrivial convex combination or mixture of other states of $\mathbf{B}(\mathcal{H})$, is then represented by a vector $x \in \mathcal{H}$. We shall always use the notation ρ_x for the normalized state functional $\langle x, \cdot\, x\rangle/\|x\|^2$ $(= \mathrm{Tr}(P_x \cdot\,))$. If, furthermore, we consider the restriction $\rho_x|_{\mathcal{R}}$, the induced state on some von Neumann subalgebra $\mathcal{R} \subseteq \mathbf{B}(\mathcal{H})$, we cannot in general expect it to be pure on $\mathcal{R}$ as well. For example, with $\mathcal{H} = \mathbb{C}^2_A \otimes \mathbb{C}^2_B$, $\mathcal{R} = \mathbf{B}(\mathbb{C}^2_A) \otimes I$, and x entangled, we know that the induced state $\rho_x|_{\mathcal{R}}$, represented by $D_A(x) \in \mathbf{B}(\mathbb{C}^2_A)$, is *always* mixed. Similarly, one cannot expect that a pure state ρ_x of the field algebra $\mathcal{A}'' = \mathbf{B}(\mathcal{H})$—which supplies a maximal specification of the state of the field *throughout* spacetime—will have a restriction to a local algebra $\rho_x|_{\mathcal{A}(O)}$ that is itself pure. In fact, we shall see later that the Reeh-Schlieder theorem entails that the vacuum state's restriction to any local algebra is always highly mixed.

There are two topologies on the state space of a von Neumann algebra $\mathcal{R}$ that we shall need to invoke.

One is the metric topology induced by the norm on linear functionals. The norm of a state ρ on $\mathcal{R}$ is defined by

$$\|\rho\| := \sup\{\, |\rho(Z)| : Z = Z^* \in \mathcal{R},\ \|Z\| \leq 1\}\,. \tag{3.3}$$

If two states, ρ_1 and ρ_2, are close to each other in norm, then they dictate close expectation values uniformly for *all* observables. In particular, if both ρ_1 and ρ_2 are vector states, i.e., they are induced by vectors $x_1, x_2 \in \mathcal{H}$ such that $\rho_1 = \rho_{x_1}|_{\mathcal{R}}$ and $\rho_2 = \rho_{x_2}|_{\mathcal{R}}$, then $\|x_1 - x_2\| \to 0$ implies $\|\rho_1 - \rho_2\| \to 0$.[6] More generally, whenever the trace norm distance between two density operators goes to zero, the norm distance between the states they induce on $\mathcal{R}$ goes to zero. Note also that since every state on $\mathbf{B}(\mathcal{H})$ is given by a density

[5]There are also non-countably additive or 'singular' states on $\mathcal{R}$ (KR 1997, 723), but whenever we use the term 'state' we shall mean *countably additive* state.

[6]It is important not to conflate the terms 'vector state' and 'pure state', unless of course $\mathcal{R} = \mathbf{B}(\mathcal{H})$ itself.

operator, which in turn can be decomposed as an infinite convex combination of one dimensional projections (with the infinite sum understood as trace norm convergence), it follows that every state on $\mathcal{R} \subseteq \mathbf{B}(\mathcal{H})$ is the norm limit of convex combinations of vector states of $\mathcal{R}$ (cf. KR 1997, Thm. 7.1.12).

The other topology we shall invoke is the weak* topology: a net of states $\{\rho_i\}$ on $\mathcal{R}$ weak* converges to a state ρ just in case $\rho_i(Z) \to \rho(Z)$ for all $Z \in \mathcal{R}$. This convergence need not be uniform on all elements of $\mathcal{R}$, and is therefore weaker than the notion of approximation embodied by norm convergence. As it happens, any state on the whole of $\mathbf{B}(\mathcal{H})$ that is the weak* limit of a set of states is also their norm limit. However, this is only true for type I von Neumann algebras (Connes & Størmer 1978, Cor. 9).

Next, we turn to defining entanglement in a field. Fix a state ρ on $\mathbf{B}(\mathcal{H})$, and two mutually commuting subalgebras $\mathcal{R}_A, \mathcal{R}_B \subseteq \mathbf{B}(\mathcal{H})$. To define what it means for ρ to be entangled across the algebras, we need only consider the restriction $\rho|_{\mathcal{R}_{AB}}$ to the von Neumann algebra they generate, i.e., $\mathcal{R}_{AB} = (\mathcal{R}_A \cup \mathcal{R}_B)''$, and of course we need a definition that also applies when $\rho|_{\mathcal{R}_{AB}}$ is mixed. A state ω on $\mathcal{R}_{AB}$ is called a *product state* just in case there are states ω_A of $\mathcal{R}_A$ and ω_B of $\mathcal{R}_B$ such that $\omega(XY) = \omega_A(X)\omega_B(Y)$ for all $X \in \mathcal{R}_A$, $Y \in \mathcal{R}_B$. Clearly, product states, or convex combinations of product states, possess only classical correlations. Moreover, if one can *approximate* a state with convex combinations of product states, its correlations do not significantly depart from those characteristic of a classical statistical theory. Therefore, we define ρ to be *entangled* across $(\mathcal{R}_A, \mathcal{R}_B)$ just in case $\rho|_{\mathcal{R}_{AB}}$ is *not* a weak* limit of convex combinations of product states of $\mathcal{R}_{AB}$ (see chapter 2, page 14). Notice that we chose weak* convergence rather than convergence in norm, hence we obtain a strong notion of entanglement. In the case $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, $\mathcal{R}_A = \mathbf{B}(\mathcal{H}_A) \otimes I$, and $\mathcal{R}_B = I \otimes \mathbf{B}(\mathcal{H}_B)$, the definition obviously coincides with the usual notion of entanglement for a pure state (convex combinations and approximations being irrelevant in that case), and also coincides with the definition of entanglement (usually called 'nonseparability') for a mixed density operator that is standard in quantum information theory (Werner 1989; Clifton, Halvorson, & Kent 2000). Further evidence that the definition captures an essentially nonclassical feature of correlations is given by the fact that $\mathcal{R}_{AB}$ will possess an entangled state in the sense defined above if and *only if* both $\mathcal{R}_A$ and $\mathcal{R}_B$ are nonabelian (Bacciagaluppi 1993, Thm. 7; Summers & Werner 1995, Lemma 2.1). Returning to AQFT, it is therefore reasonable to say that a global state of the field ρ on $\mathcal{A}'' = \mathbf{B}(\mathcal{H})$ is entangled across a pair of spacelike-separated regions (O_A, O_B) just in case $\rho|_{\mathcal{A}_{AB}}$, ρ's restriction to $\mathcal{A}_{AB} = [\mathcal{A}(O_A) \cup \mathcal{A}(O_B)]''$, falls outside the weak* closure of the convex hull of $\mathcal{A}_{AB}$'s product states.

3.2.1 Operations, local operations, and entanglement

Our next task is to review the mathematical representation of operations, highlight some subtleties in their physical interpretation, and then discuss what is meant by *local* operations on a system. We then end this section by showing that local operations performed in either of two spacelike-separated regions (O_A,O_B) cannot create entanglement in a state across the regions.

The most general transformation of the state of a quantum system with Hilbert space $\mathcal{H}$ is described by an *operation* on $\mathbf{B}(\mathcal{H})$, defined to be a positive, weak* continuous, linear map $T : \mathbf{B}(\mathcal{H}) \to \mathbf{B}(\mathcal{H})$ satisfying $0 \leq T(I) \leq I$ (Haag & Kastler 1964; Davies 1976; Kraus 1983; Busch et al. 1995; Werner 1987). (The weak* topology on a von Neumann algebra $\mathcal{R}$ is defined in complete analogy to the weak* topology on its state space, viz., $\{Z_n\} \subseteq \mathcal{R}$ weak* converges to $Z \in \mathcal{R}$ just in case $\rho(Z_n) \to \rho(Z)$ for all states ρ of $\mathcal{R}$.) Any such T induces a map $\rho \to \rho^T$ from the state space of $\mathbf{B}(\mathcal{H})$ into itself or 0, where, for all $Z \in \mathbf{B}(\mathcal{H})$,

$$\rho^T(Z) := \begin{cases} \rho(T(Z))/\rho(T(I)) & \text{if } \rho(T(I)) \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

The number $\rho(T(I))$ is the probability that an ensemble in state ρ will respond 'Yes' to the question represented by the positive operator $T(I)$. An operation T is called *selective* if $T(I) < I$, and nonselective if $T(I) = I$. The final state after a selective operation on an ensemble of identically prepared systems is obtained by ignoring those members of the ensemble that fail to respond 'Yes' to $T(I)$. Thus a selective operation involves performing a physical operation on an ensemble followed by a *purely conceptual* operation in which one makes a selection of a subensemble based on the outcome of the physical operation (assigning 'state' 0 to the remainder). Nonselective operations, by contrast, always elicit a 'Yes' response from any state, hence the final state is not obtained by selection but purely as a result of the physical interaction between object system and the device that effects the operation. (We shall shortly discuss some actual physical examples to make this general description of operations concrete.)

An operation T, which quantum information theorists call a superoperator (acting, as it does, on operators to produce operators), "can describe any combination of unitary operations, interactions with an ancillary quantum system or with the environment, quantum measurement, classical communication, and subsequent quantum operations conditioned on measurement results" (Bennett et al. 1999). Interestingly, a superoperator itself can always be represented in terms of operators, as a consequence of the Kraus representation theorem (Kraus 1983, 42): for any operation $T : \mathbf{B}(\mathcal{H}) \to \mathbf{B}(\mathcal{H})$, there exists a (not necessarily unique) countable collection of Kraus operators $\{K_i\} \subseteq \mathbf{B}(\mathcal{H})$ such that

$$T(\cdot) = \sum_i K_i^*(\cdot)K_i, \qquad \text{with } 0 \leq \sum_i K_i^* K_i \leq I\,, \tag{3.5}$$

where both sums, if infinite, are to be understood in terms of weak* convergence. It is not difficult to show that the sum $\sum_i K_i K_i^*$ must also weak* converge, hence we can let T^* denote the operation conjugate to T whose Kraus operators are $\{K_i^*\}$. It then follows (using the linearity and cyclicity of the trace) that if a state ρ is represented by a density operator D on $\mathcal{H}$, ρ^T will be represented by the density operator $T^*(D)$. If the mapping $\rho \to \rho^T$, or equivalently, $D \to T^*(D)$, maps pure states to pure states, then the operation T is called a *pure operation*, and this corresponds to it being representable by a *single* Kraus operator.

More generally, the Kraus representation shows that a general operation is always equivalent to mixing the results of separating an initial ensemble into subensembles to which one applies pure (possibly selective) operations, represented by the individual Kraus operators. To see this, let T be an arbitrary operation performed on a state ρ, where $\rho^T \neq 0$, and suppose T is represented by Kraus operators $\{K_i\}$. Let ρ^{K_i} denote the result of applying to ρ the pure operation given by the mapping $T_i(\cdot) = K_i^*(\cdot)K_i$, and (for convenience) define $\lambda_i = \rho(T_i(I))/\rho(T(I))$. Then, at least when there are finitely many Kraus operators, it is easy to see that T itself maps ρ to the convex combination $\rho^T = \sum_i \lambda_i \rho^{K_i}$. In the infinite case, this sum converges not just weak* but *in norm*, and it is a useful exercise in the topologies we have introduced to see why. Letting ρ_n^T denote the partial sum $\sum_{i=1}^n \lambda_i \rho^{K_i}$, we need to establish that

$$\lim_{n\to\infty} \Big[\sup\big\{\, |\rho^T(Z) - \rho_n^T(Z)| : Z = Z^* \in \mathbf{B}(\mathcal{H}),\ \|Z\| \leq 1\big\}\Big] = 0. \tag{3.6}$$

For *any* $Z \in \mathbf{B}(\mathcal{H})$, we have

$$|\rho^T(Z) - \rho_n^T(Z)| \quad = \quad \rho(T(I))^{-1} \Big| \sum_{i=n+1}^{\infty} \rho(K_i^* Z K_i)\Big|. \tag{3.7}$$

However, $\rho(K_i^*(\,\cdot\,)K_i)$, being a positive linear functional, has a norm that may be computed by its action on the identity (KR 1997, Thm. 4.3.2). Therefore, $|\rho(K_i^* Z K_i)| \leq \|Z\|\rho(K_i^* K_i)$, and we obtain

$$|\rho^T(Z) - \rho_n^T(Z)| \quad \leq \quad \rho(T(I))^{-1}\|Z\| \sum_{i=n+1}^{\infty} \rho(K_i^* K_i). \tag{3.8}$$

However, since $\sum_i K_i^* K_i$ weak* converges, this last summation is the tail set of a convergent series. Therefore, when $\|Z\| \leq 1$, the right-hand side of (3.8) goes to zero independently of Z.

To get a concrete idea of how operations work physically, and to highlight two important interpretational pitfalls, let us again consider our toy universe, with $\mathcal{H} = \mathbb{C}_A^2 \otimes \mathbb{C}_B^2$ and x an entangled state. Recall that Alice disentangled x by measuring an observable of A with eigenprojections $P_\pm$. Her measurement corresponds to applying the nonselective operation T with Kraus operators $K_1 = P_+ \otimes I$ and $K_2 = P_- \otimes I$, resulting in the final state $T^*(P_x) = T(P_x) = D'$, as given in (3.1). If Alice were to further 'apply' the pure selective operation T' represented by the single Kraus operator $P_+ \otimes I$, the final state of her ensemble, as is apparent from (3.2), would be the product state $D'' = P_+ \otimes P_{b_x^+}$. But, as we have emphasized, this corresponds to a conceptual operation in which Alice just throws away all members of the original ensemble that yielded measurement outcome -1.

On the other hand, it is essential not to lose sight of the issue that troubled Einstein. *Whatever* outcome Alice selects for, she will then be in a position to assert that certain B observables—those that have either b_x^+ or b_x^- as an eigenvector, depending on the outcome she favors—have a sharp value in the ensemble she is left with. But prior to Alice performing the first operation T, such an assertion would have contradicted the orthodox interpretation of the entangled superposition x. If, contra Bohr, one were to view this change in B's state as a *real physical* change brought about by one of the operations Alice performs, surely the innocuous conceptual operation T' could not be the culprit—it must have been T which forced B to 'choose' between the alternatives $b_x^\pm$. Unfortunately, this clear distinction between the physical operation T and conceptual operation T' is not reflected well in the formalism of operations. For we could equally well have represented Alice's final product state $D'' = P_+ \otimes P_{b_x^+}$, not as the result of successively applying the operations T and T', but as the outcome of applying the single composite operation $T' \circ T$, which is just the mapping T'. And *this* T' now needs to be understood, not purely as a conceptual operation, but as also involving a physical operation, with possibly real nonlocal effects on B, depending on one's view of the EPR paradox.[7]

There is a second pitfall that concerns interpreting the result of *mixing* subensembles, as opposed to singling out a particular subensemble. Consider an alternative method available to Alice for disentangling a state x. For concreteness, let us suppose that x is the singlet state $1/\sqrt{2}(a^+ \otimes b^- - a^- \otimes b^+)$. Alice applies the nonselective operation with Kraus representation

$$T(\cdot) = \frac{1}{2}(\sigma_a \otimes I)(\cdot)(\sigma_a \otimes I) + \frac{1}{2}(I \otimes I)(\cdot)(I \otimes I), \tag{3.9}$$

where σ_a is the spin observable with eigenstates $a^{\pm}$. Since $\sigma_a \otimes I$ maps x to the triplet state $1/\sqrt{2}(a^+ \otimes b^- + a^- \otimes b^+)$, T^* $(= T)$ will map P_x to an equal mixture of the singlet and triplet, which admits the following convex decomposition into product states

$$D' = \frac{1}{2}P_{a^+ \otimes b^-} + \frac{1}{2}P_{a^- \otimes b^+}. \tag{3.10}$$

Has Alice truly disentangled A from B? Technically, Yes. Yet all Alice has done, physically, is to separate the initial A ensemble into two subensembles in equal proportion, left the second subensemble alone while performing a (pure, nonselective) unitary operation $\sigma_a \otimes I$ on the first that maps all its $A + B$ pairs to the triplet state, and then remixed the ensembles. Thus, notwithstanding the above decomposition of the final density matrix D', Alice *knows quite well* that she is in possession of an ensemble of A systems each of which is entangled either via the singlet or triplet state with the corresponding B systems. This will of course be recognized as one aspect of the problem with the ignorance interpretation of mixtures. We have two different ways to decompose D'—as an equal mixture of the singlet and triplet or of two product states—but which is the correct way to understand how the ensemble is *actually* constituted? The definition of entanglement is just not sensitive to the answer.[8] Nevertheless, we are inclined to think the destruction of the singlet's entanglement that Alice achieves by applying the operation in (3.9) is an artifact of her mixing process, in which she is repre-

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[7] In particular, keep in mind that you are taking the first step on the road to conceding the incompleteness of quantum theory if you attribute the change in the state of B brought about by T' in this case to a mere change in Alice's *knowledge* about B's state.

[8] It is exactly this insensitivity that is at the heart of the recent dispute over whether NMR quantum computing is correctly understood as implementing genuine *quantum* computing that cannot be simulated classically (Braunstein et al. 1999; Laflamme 1998).

sented as simply forgetting about the history of the A systems. And this is the view we shall take when we consider similar possibilities for destroying entanglement between field systems in AQFT.

In the two examples considered above, Alice applies operations whose Kraus operators lie in the subalgebra $\mathbf{B}(\mathcal{H}_A) \otimes I$ associated with system A. In the case of a nonselective operation, this is clearly sufficient for her operation not to have any effect on the expectations of the observables of system B. However, it is also necessary. The point is quite general.

Let us define a nonselective operation T to be (*pace* Einstein!) *local* to the subsystem represented by a von Neumann subalgebra $\mathcal{R} \subseteq \mathbf{B}(\mathcal{H})$ just in case $\rho^T|_{\mathcal{R}'} = \rho|_{\mathcal{R}'}$ for all states ρ. Thus, we require that T leave the expectations of observables outside of $\mathcal{R}$, as well as those in its center $\mathcal{R} \cap \mathcal{R}'$, unchanged. Since distinct states of $\mathcal{R}'$ cannot agree on all expectation values, this means T must act like the identity operation on $\mathcal{R}'$. Now fix an arbitrary element $Y \in \mathcal{R}'$, and suppose T is represented by Kraus operators $\{K_i\}$. A straightforward calculation reveals that

$$\sum_i [Y, K_i]^*[Y, K_i] = T(Y^2) - T(Y)Y - YT(Y) + YT(I)Y. \tag{3.11}$$

Since $T(I) = I$, and T leaves the elements of $\mathcal{R}'$ fixed, the right-hand side of (3.11) reduces to zero. Thus each of the terms in the sum on the left-hand side, which are positive operators, must individually be zero. Since Y was an arbitrary element of $\mathcal{R}'$, it follows that $\{K_i\} \subseteq (\mathcal{R}')' = \mathcal{R}$. So we see that nonselective operations local to $\mathcal{R}$ *must* be represented by Kraus operators taken from the subalgebra $\mathcal{R}$.

As for selective operations, we have already seen that they *can* ‘change’ the global statistics of a state ρ outside the subalgebra $\mathcal{R}$, particularly when ρ is entangled. However, a natural extension of the definition of local operation on $\mathcal{R}$ to a cover the case when T is selective is to require that $T(Y) = T(I)Y$ for all $Y \in \mathcal{R}'$. This implies $\rho^T(Y) = \rho(T(I)Y)/\rho(T(I))$, and so guarantees that T will leave the statistics of any observable in $\mathcal{R}'$ the same *modulo* whatever correlations that observable might have had in the initial state with the Yes/No question represented by the positive operator $T(I)$. Further motivation is provided by the fact this definition is equivalent to requiring that T factor across the algebras $(\mathcal{R}, \mathcal{R}')$, in the sense that $T(XY) = T(X)Y$ for all $X \in \mathcal{R}$, $Y \in \mathcal{R}'$ (Werner 1987, Lemma). If there exist product states across $(\mathcal{R}, \mathcal{R}')$ (an assumption we shall later see does *not* usually hold when $\mathcal{R}$ is a local algebra in AQFT), this guarantees that any local selective operation on $\mathcal{R}$, when the global state is an entirely uncorrelated product state, will leave the statistics of that state on $\mathcal{R}'$ unchanged. Finally, observe that $T(Y) = T(I)Y$ for all $Y \in \mathcal{R}'$ implies that the right-hand side of (3.11) again reduces to zero. Thus it follows (as before) that selective local operations on $\mathcal{R}$ must also be represented by Kraus operators taken from the subalgebra $\mathcal{R}$.

Applying these considerations to field theory, any local operation on the field system within a region O, whether or not the operation is selective, is represented by a family of Kraus operators taken from $\mathcal{A}(O)$. In particular, each individual element of $\mathcal{A}(O)$ represents a pure operation that can be performed within O (cf. Haag & Kastler 1964, 850). We now need to argue that local operations performed by two experimenters in spacelike-separated regions cannot create entanglement in a state across the regions where it had none before. This point, well-known by quantum information theorists working in nonrelativistic quantum theory, in fact applies quite generally to any two commuting von Neumann algebras $\mathcal{R}_A$ and $\mathcal{R}_B$.

Suppose that a state ρ is not entangled across $(\mathcal{R}_A, \mathcal{R}_B)$, local operations T_A and T_B are applied to ρ, and the result is nonzero (i.e., some members of the initial ensemble are not discarded). Since the Kraus operators of these operations commute, it is easy to check that $(\rho^{T_A})^{T_B} = (\rho^{T_B})^{T_A}$, so it does not matter in which order we take the operations. It is sufficient to show that ρ^{T_A} will again be unentangled, for then we can just repeat the same argument to obtain that neither can $(\rho^{T_A})^{T_B}$ be entangled. Next, recall that a general operation T_A will just produce a mixture over the results of applying a countable collection of pure operations to ρ; more precisely, the result will be the norm, and hence weak*, limit of finite convex combinations of the results of applying pure operations to ρ. If the states that result from ρ under those pure operations are themselves not entangled, ρ^{T_A} itself could not be either, because the set of unentangled states is by definition convex and weak* closed. Without loss of generality, then, we may assume that the local operation T_A is pure and, hence, given by $T_A(\cdot) = K^*(\cdot)K$, for some *single* Kraus operator $K \in \mathcal{R}_A$. As before, we shall denote the resulting state ρ^{T_A} by ρ^K $(\equiv \rho(K^* \cdot K)/\rho(K^*K))$.

Next, suppose that ω is any product state on $\mathcal{R}_{AB}$ with restrictions to $\mathcal{R}_A$ and $\mathcal{R}_B$ given by ω_A and ω_B, and such that $\omega^K \neq 0$. Then, for any

$X \in \mathcal{R}_A$, $Y \in \mathcal{R}_B$,

$$\omega^K(XY) = \frac{\omega(K^*(XY)K)}{\omega(K^*K)} \tag{3.12}$$

$$= \frac{\omega(K^*XKY)}{\omega(K^*K)} \tag{3.13}$$

$$= \frac{\omega_A(K^*XK)}{\omega_A(K^*K)}\omega_B(Y) = \omega_A^K(X)\omega_B(Y). \tag{3.14}$$

It follows that K maps product states of $\mathcal{R}_{AB}$ to product states (or to zero). Suppose, instead, that ω is a convex combination of states on $\mathcal{R}_{AB}$, i.e., $\omega = \sum_{i=1}^n \lambda_i\omega_i$. Then, setting $\lambda_i^K = \omega_i(K^*K)/\omega(K^*K)$, it is easy to see that $\omega^K = \sum_{i=1}^n \lambda_i^K\omega_i^K$, hence K preserves convex combinations of states on $\mathcal{R}_{AB}$ as well. It is also not difficult to see that the mapping $\omega \mapsto \omega^K$ is weak* continuous at any point where $\omega^K \neq 0$ (cf. section 3 of chapter 2).

Returning to our original state ρ, our hypothesis is that it is not entangled. Thus, there is a net of states $\{\omega_n\}$ on $\mathcal{R}_{AB}$, each of which is a convex combination of product states, such that $\omega_n \to \rho|_{\mathcal{R}_{AB}}$ in the weak* topology. It follows from the above considerations that $\omega_n^K \to \rho^K|_{\mathcal{R}_{AB}}$, where each of the states $\{\omega_n^K\}$ is again a convex combination of product states. Hence, by definition, $\rho^K|_{\mathcal{R}_{AB}}$ is not entangled either.

In summary, we have shown:

> *If $\mathcal{R}_A$ and $\mathcal{R}_B$ are any two commuting von Neumann algebras, and ρ is any unentangled state across $(\mathcal{R}_A, \mathcal{R}_B)$, then operations on ρ, local to either or both of A and B, cannot produce an entangled state.*

3.3 The operational implications of the Reeh-Schlieder theorem

Again, let $\mathcal{R} \subseteq \mathbf{B}(\mathcal{H})$ be any von Neumann algebra. A vector $x \in \mathcal{H}$ is called cyclic for $\mathcal{R}$ if the norm closure of the set $\{Ax : A \in \mathcal{R}\}$ is the *whole* of $\mathcal{H}$. In AQFT, the Reeh-Schlieder (RS) theorem connects this formal property of cyclicity to the physical property of a field state having bounded energy.[9] A pure global state x of the field has bounded energy just in case $E([0, r])x = x$ for some $r < \infty$, where E is the spectral measure for the global Hamiltonian

[9] More generally, the connection is between cyclicity and field states that are 'analytic' in the energy.

of the field. In other words, the probability in state x that the field's energy is confined to the bounded interval $[0, r]$ is unity. In particular, the vacuum Ω is an eigenstate of the Hamiltonian with eigenvalue 0, and hence trivially has bounded energy. The RS theorem implies that

> *If x has bounded energy, then x is cyclic for any local algebra $\mathcal{A}(O)$.*

Our first order of business is to explain Streater and Wightman's comment that the RS theorem entails "it is difficult to isolate a system described by fields from outside effects" (2000, 139).

A vector x is called *separating* for a von Neumann algebra $\mathcal{R}$ if $Ax = 0$ implies $A = 0$ whenever $A \in \mathcal{R}$. It is an elementary result of von Neumann algebra theory that x is cyclic for $\mathcal{R}$ if and only if x is separating for $\mathcal{R}'$ (KR 1997, Cor. 5.5.12). To illustrate with a simple example, take $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. If $\dim \mathcal{H}_A \geq \dim \mathcal{H}_B$, then there is a vector $x \in \mathcal{H}$ that has Schmidt decomposition $\sum_i c_i a_i \otimes b_i$ where $|c_i|^2 \neq 0$ for $i = 1, \ldots, \dim \mathcal{H}_B$. If we act on such an x by an operator in the subalgebra $I \otimes \mathbf{B}(\mathcal{H}_B)$, of form $I \otimes B$, then $(I \otimes B)x = 0$ only if B itself maps all the basis vectors $\{b_i\}$ to zero, i.e., $I \otimes B = 0$. Thus such vectors are separating for $I \otimes \mathbf{B}(\mathcal{H}_B)$, and therefore cyclic for $\mathbf{B}(\mathcal{H}_A) \otimes I$. Conversely, it is easy to convince oneself that $\mathbf{B}(\mathcal{H}_A) \otimes I$ possesses a cyclic vector—equivalently, $I \otimes \mathbf{B}(\mathcal{H}_B)$ has a separating vector—*only if* $\dim \mathcal{H}_A \geq \dim \mathcal{H}_B$. So, to take another example, each of the A and B subalgebras have a cyclic and a separating vector just in case $\mathcal{H}_A$ and $\mathcal{H}_B$ have the same dimension (cf. the proof of Clifton et al. 1998, Thm. 4).

Consider, now, a local algebra $\mathcal{A}(O)$ with $O' \neq \emptyset$, and a field state x with bounded energy. The RS theorem tells us that x is cyclic for $\mathcal{A}(O')$, and therefore, separating for $\mathcal{A}(O')'$. But by microcausality, $\mathcal{A}(O) \subseteq \mathcal{A}(O')'$, hence x must be separating for the subalgebra $\mathcal{A}(O)$ as well. Thus it is an immediate corollary to the RS theorem that

> *If x has bounded energy, then x is separating for any local algebra $\mathcal{A}(O)$ with $O' \neq \emptyset$.*

It is this corollary that prompted Streater and Wightman's remark. But what has it got to do with thinking of the field system $\mathcal{A}(O)$ as isolated? For a start, we can now show that the local restriction $\rho_x|_{\mathcal{A}(O)}$ of a state with bounded energy is always a highly 'noisy' mixed state. Recall that a state ω on a von Neumann algebra $\mathcal{R}$ is said to be a *component* of another state ρ if there is a third state τ such that $\rho = \lambda\omega + (1 - \lambda)\tau$ with $\lambda \in (0, 1)$

(van Fraassen 1991, 161). We are going to show that $\rho_x|_{\mathcal{A}(O)}$ has a *norm* dense set of components in the state space of $\mathcal{A}(O)$.

Once again, the point is quite general. Let $\mathcal{R}$ be any von Neumann algebra, x be separating for $\mathcal{R}$, and let ω be an arbitrary state of $\mathcal{R}$. We must find a sequence $\{\omega_n\}$ of states of $\mathcal{R}$ such that each ω_n is a component of $\rho_x|_{\mathcal{R}}$ and $\|\omega_n - \omega\| \to 0$. Since $\mathcal{R}$ has a separating vector, it follows that every state of $\mathcal{R}$ is a vector state (KR 1997, Thm. 7.2.3).[10] In particular, there is a nonzero vector $y \in \mathcal{H}$ such that $\omega = \omega_y$. Since x is separating for $\mathcal{R}$, x is cyclic for $\mathcal{R}'$, therefore we may choose a sequence of operators $\{A_n\} \subseteq \mathcal{R}'$ so that $A_n x \to y$. Since $\|A_n x - y\| \to 0$, $\|\omega_{A_n x} - \omega_y\| \to 0$ (see page 26). We claim now that each $\omega_{A_n x}$ is a component of $\rho_x|_{\mathcal{R}}$. Indeed, for any positive element $B^*B \in \mathcal{R}$, we have:

$$\begin{aligned} \langle A_n x, B^*BA_n x\rangle &= \langle x, A_n^*A_n B^*Bx\rangle = \langle Bx, A_n^*A_n Bx\rangle && (3.15)\\ &\leq \|A_n^*A_n\|\langle Bx, Bx\rangle = \|A_n\|^2\langle x, B^*Bx\rangle. && (3.16)\end{aligned}$$

Thus,

$$\omega_{A_n x}(B^*B) = \frac{\langle A_n x, B^*BA_n x\rangle}{\|A_n x\|^2} \leq \frac{\|A_n\|^2}{\|A_n x\|^2}\,\rho_x(B^*B). \qquad (3.17)$$

If we now take $\lambda = \|A_n x\|^2/\|A_n\|^2 \in (0,1)$, and consider the linear functional τ on $\mathcal{R}$ given by $\tau = (1-\lambda)^{-1}(\rho_x|_{\mathcal{R}} - \lambda\omega_{A_n x})$, then (3.17) implies that τ is a state (in particular, positive), and we see that $\rho_x|_{\mathcal{R}} = \lambda\omega_{A_n x} + (1-\lambda)\tau$ as required.[11]

So bounded energy states are, locally, highly mixed. And such states are far from special—they lie norm dense in the pure state space of $\mathbf{B}(\mathcal{H})$. To see this, recall that it is part of the spectral theorem for the global Hamiltonian that $E([0,n])$ converges strongly to the identity as $n \to \infty$. Thus we may approximate any vector $y \in \mathcal{H}$ by the sequence of bounded energy states $\{E([0,n])y/\|E([0,n])y\|\}_{n=0}^{\infty}$. Since there are so many bounded energy states of the field, that are locally so 'noisy', Streater and Wightman's comment is entirely warranted. But somewhat more can be said. As we saw

[10] That this should be so is not as surprising as it sounds. Again, if $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, and $\dim \mathcal{H}_A \geq \dim \mathcal{H}_B$, then as we have seen, the B subalgebra possesses a separating vector. But it is also easy to see, in this case, that every state on $I \otimes \mathbf{B}(\mathcal{H}_B)$ is the reduced density operator obtained from a pure state on $\mathbf{B}(\mathcal{H})$ determined by a vector in $\mathcal{H}$.

[11] This result also holds more generally for states ρ of $\mathcal{R}$ that are faithful, i.e., $\rho(Z) = 0$ entails $Z = 0$ for any positive $Z \in \mathcal{R}$; see the proof of Theorem 2.1 of (Summers & Werner 1988).

with our toy example in section 3.1, when a local subsystem of a global system in a pure state is itself in a mixed state, this is a sign of that subsystem's entanglement with its environment. And there is entanglement lurking in bounded energy states too. But, first, we want to take a closer look at the operational implications of local cyclicity.

If a vector x is cyclic for $\mathcal{R}$, then for any $y \in \mathcal{H}$, there is a sequence $A_n \in \mathcal{R}$ such that $A_n x \to y$. Thus for any $\epsilon > 0$ there is an $A \in \mathcal{R}$ such that $\|\rho_{Ax} - \rho_y\| < \epsilon$. However, ρ_{Ax} is just the state one gets by applying the pure operation given by the Kraus operator $K = A/\|A\| \in \mathcal{R}$ to ρ_x. It follows that if x is cyclic for $\mathcal{R}$, one can get arbitrarily close in norm to any other pure state of $\mathbf{B}(\mathcal{H})$ by applying an appropriate pure local operation in $\mathcal{R}$ to ρ_x. In particular, pure operations on the vacuum Ω within a local region O, no matter how small, can prepare essentially any global state of the field. As Haag emphasizes, to do this the operation must "judiciously exploit the small but nonvanishing long distance correlations which exist in the vacuum" (1992, 102). This, as Redhead (1995a) has argued by analogy to the singlet state, is made possible by the fact that the vacuum is highly entangled (cf. Clifton et al. 1998).

3.3.1 Physical versus conceptual operations

The first puzzle we need to sort out is that it looks as though entirely *physical* operations in O can change the global state, in particular the vacuum Ω, to any desired state![12]

Redhead's analysis of the cyclicity of the singlet state $x = 1/\sqrt{2}(a^+ \otimes b^- - a^- \otimes b^+)$ for the subalgebra $\mathbf{B}(\mathbb{C}^2_A) \otimes I$ is designed to remove this puzzle (ibid, 128).[13] Redhead writes:

> ...we want to distinguish clearly two senses of the term "operation". Firstly there are physical operations such as making measurements, selecting subensembles according to the outcome of measurements, and mixing ensembles with probabilistic weights, and secondly there are the mathematical operations of producing superpositions of states by taking linear combinations of pure

[12]For example, Segal and Goodman (1965) have called this "bizarre" and "physically quite surprising", sentiments echoed recently by Fleming who calls it "amazing!" (2000) and Fleming and Butterfield who think it is "hard to square with naïve, or even educated, intuitions about localization!" (1999, 161).

[13]Note that in this simple 2×2-dimensional case, Redhead could equally well have chosen *any* entangled state, since they are all separating for $I \otimes \mathbf{B}(\mathbb{C}^2_B)$.

> states produced by appropriate selective measurement procedures. These superpositions are of course quite different from the mixed states whose preparation we have listed as a physical operation. (1995a, 128–129)

Note that, in stark contrast to our discussion in the previous section, Redhead counts selecting subensembles and mixing as physical operations; it is only the operation of superposition that warrants the adjective 'mathematical'. When he explains the cyclicity of the singlet state, Redhead first notes (ibid, 129) that the four basis states

$$a^+ \otimes b^-,\ a^- \otimes b^-,\ a^- \otimes b^+,\ a^+ \otimes b^-, \tag{3.18}$$

are easily obtained by the physical operations of applying projections and unitary transformations to the singlet state, and exploiting the fact that the singlet strictly correlates σ_a with σ_b. He goes on:

> But *any* state for the joint system is some linear combination of these four states, so by the *mathematical* operation of linear combination, we can see how to generate an arbitrary state in $\mathcal{H}_1 \otimes \mathcal{H}_2$ from physical operations performed on particle one. But all the operations we have described can be represented in the algebra of operators on $\mathcal{H}_1$ (extended to $\mathcal{H}_1 \otimes \mathcal{H}_2$). (ibid, 129)

Now, while Redhead's explanation of why it is mathematically possible for x to be cyclic is perfectly correct, he actually misses the mark when it comes to the physical interpretation of cyclicity. The point is that superposition *of states* is a red-herring. Certainly a superposition of the states in (3.18) could not be prepared by physical operations confined to the A system. But, as Redhead himself notes in the final sentence above, one can get the same *effect* as superposing those states by acting on x with an operator of form $A \otimes I$ in the subalgebra $\mathbf{B}(\mathbb{C}^2_A) \otimes I$—an operator that is itself a 'superposition' of other operators in that algebra. What Redhead neglects to point out is that the action of this operator on x *does have a local physical interpretation*: as we have seen, it is a Kraus operator that represents the outcome of a generalized positive operator-valued measurement on the A system. The key to the puzzle is, rather, that this positive operator-valued measurement will generally have to be *selective*. For one certainly could never, with nonselective operations on A alone, get as close as one likes to any state vector in $\mathbb{C}^2_A \otimes \mathbb{C}^2_B$ (otherwise all state vectors would induce the same state on $I \otimes \mathbf{B}(\mathbb{C}^2_B)$!).

We conclude that the correct way to view the physical content of cyclicity is that changes in the global state are partly due to an experimenter's ability to perform a generalized measurement on A, and partly due (*pace* Redhead) to the purely conceptual operation of selecting a subensemble based on the outcome of the experimenter's measurement together with the consequent 'change' in the state of B via the EPR correlations between A and B.[14]

One encounters the same interpretational pitfall concerning the cyclicity of the vacuum in relation to localized states in AQFT. A global state of the field is said to be *localized* in O if its expectations on the algebra $\mathcal{A}(O')$ agree with vacuum expectation values (Haag 1992, 102). Thus localized states are 'excitations' of the vacuum confined to O. In particular, $U\Omega$ is a localized state whenever U is a unitary operator taken from $\mathcal{A}(O)$ (since unitary operations are nonselective). But every element of a C^*-algebra is a finite linear combination of unitary operators (KR 1997, Thm. 4.1.7). Since Ω is cyclic for $\mathcal{A}(O)$, this means we must be able to approximate any global state by linear superpositions of vectors describing states localized in O—even approximate states that are localized in regions spacelike separated from O! Haag, rightly cautious, calls this a "(superficial) paradox" (1992, 254; parenthesis his), but he neglects to put his finger on its resolution: while unitary operations are nonselective, a local operation in $\mathcal{A}(O)$ given by a Kraus operator that is a linear combination of local unitary operators will generally be *selective*.[15]

The (common) point of the previous two paragraphs is perhaps best summarized as follows. Both Redhead and Haag would agree that unitary Kraus operators in $\mathcal{A}(O)$ give rise to purely physical operations in the local region O. But there are many Kraus operators in $\mathcal{A}(O)$ that do not represent purely physical operations in O insofar as they are selective. Since every Kraus operator is a linear superposition of unitary operators, it follows that "superposition of local operations" does not preserve (pure) physicality (so

[14] In fairness to Redhead, we would like to add that in his first book (1989, 58) he includes an exceptionally clear discussion of the difference between nonselective and selective measurements. In particular, while we have dubbed the latter 'conceptual' operations, he uses the term 'mental', without implying anything mystical is involved. As he puts it, while a nonselective operation *can* have a physical component—like the physical action of throwing some subensemble of particles into a box for further examination—it is the *decision* to focus on a particular subensemble to the exclusion of the rest that is not dictated by the physics.

[15] Haag *does* make the interesting point that only a proper subset of the state space of a field can be approximated if we restrict ourselves to local operations that involve a physically reasonable expenditure of energy. But we do not share the view of Schroer (1999) that this point by itself reconciles the RS theorem with 'common sense'.

to speak). Redhead is right that the key to diffusing the paradox is in noting that superpositions are involved—but it is essential to understand these superpositions as occurring locally in $\mathcal{A}(O)$, not in the Hilbert space.

3.3.2 Cyclicity and entanglement

Our next order of business is to supply the rigorous argument behind Redhead's intuition about the connection between cyclicity and entanglement. The point, again, is quite general (cf. chapter 2):

> *For any two commuting nonabelian von Neumann algebras $\mathcal{R}_A$ and $\mathcal{R}_B$, and any state vector x cyclic for $\mathcal{R}_A$ (or $\mathcal{R}_B$), ρ_x is entangled across the algebras.*

For suppose, in order to extract a contradiction, that ρ_x is *not* entangled. Then as we have seen, operations on ρ_x that are local to $\mathcal{R}_A$ cannot turn that state into an entangled state across $(\mathcal{R}_A, \mathcal{R}_B)$. Yet, by the cyclicity of x, we know that we can apply pure operations to ρ_x, that are local to $\mathcal{R}_A$ (or $\mathcal{R}_B$), and approximate in norm (and hence weak* approximate) any other vector state of $\mathcal{R}_{AB}$. It follows that no vector state of $\mathcal{R}_{AB}$ is entangled across $(\mathcal{R}_A, \mathcal{R}_B)$, and the same goes for all its mixed states (which lie in the the norm closed convex hull of the vector states). But this means that $\mathcal{R}_{AB}$ would possess *no* entangled states at all—in flat contradiction with the fact that neither $\mathcal{R}_A$ nor $\mathcal{R}_B$ is abelian (see page 27).

Returning to the context of AQFT, if we now consider *any* two spacelike separated field systems, $\mathcal{A}(O_A)$ and $\mathcal{A}(O_B)$, then the argument we just gave establishes that the dense set of field states bounded in the energy will *all* be entangled across the regions (O_A, O_B).[16] However, by itself this result does not imply that Alice cannot destroy a bounded energy state x's entanglement across (O_A, O_B) by performing local operations in O_A. In fact, Borchers (1965, Cor. 7) has shown that any state of the field induced by a vector of form Ax, for any nontrivial $A \in \mathcal{A}(O_A)$, *never* has bounded energy.[17] So it might seem that all Alice needs to do is perform any pure operation

[16]Note that the fact that $\mathcal{A}(O_A)$ and $\mathcal{A}(O_B)$ are nonabelian is *itself* a consequence of the RS theorem. For if, say, $\mathcal{A}(O_A)$ were abelian, then since by the RS theorem that algebra possesses a cyclic vector, it must be a maximal abelian subalgebra of $\mathbf{B}(\mathcal{H})$ (KR 1997, Cor. 7.2.16). The same conclusion would have to follow for any subregion $\tilde{O}_A \subset O_A$ whose closure is a proper subset of O_A. And this, by isotony, would lead to the absurd conclusion that $\mathcal{A}(\tilde{O}_A) = \mathcal{A}(O_A)$, which is readily shown to be inconsistent with the axioms of AQFT (Horuzhy 1988, Lemma 1.3.10).

[17]Nor will the state be 'analytic' in the energy (see note 9).

within O_A and the resulting state, because it is no longer subject to the RS theorem, need no longer be entangled across (O_A, O_B).

However, the RS theorem gives only a sufficient, *not* a necessary, condition for a state x of the field to be cyclic for $\mathcal{A}(O_A)$. And notwithstanding that no pure operation Alice performs can preserve boundedness in the energy, *almost all* the pure operations she could perform *will* preserve the state's cyclicity! The reason is, once again, quite general.

Again let $\mathcal{R}_A$ and $\mathcal{R}_B$ be two commuting nonabelian von Neumann algebras, suppose x is cyclic for $\mathcal{R}_A$, and consider the state induced by the vector Ax where $A \in \mathcal{R}_A$. Now every element in a von Neumann algebra is the strong limit of invertible elements in the algebra (Dixmier & Marechal 1971, Prop. 1). Therefore, there is a sequence of invertible operators $\{\tilde{A}_n\} \subseteq \mathcal{R}_A$ such that $\tilde{A}_n x \to Ax$, i.e., $\|\rho_{\tilde{A}_n x} - \rho_{Ax}\| \to 0$. Notice, however, that since each $\tilde{A}_n$ is invertible, each vector $\tilde{A}_n x$ is again cyclic for $\mathcal{R}_A$, because we can 'cycle back' to x by applying to $\tilde{A}_n x$ the inverse operator $\tilde{A}_n^{-1} \in \mathcal{R}_A$, and from there we know, by hypothesis, that we can cycle with elements of $\mathcal{R}_A$ arbitrarily close to any other vector in $\mathcal{H}$. It follows that, even though Alice may *think* she has applied the pure operation given by some Kraus operator $A/\|A\|$ to x, she could well have *actually* applied an invertible Kraus operation given by one of the operators $\tilde{A}_n/\|\tilde{A}_n\|$ in a strong neighborhood of $A/\|A\|$. And if she actually did this, then she certainly would *not* disentangle x, because she would not have succeeded in destroying the *cyclicity* of the field state for her local algebra.

We could, of course, give Alice the freedom to employ more general mixing operations in O_A. But as we saw in the last section, it is far from clear whether a mixing operation should count as a successful disentanglement when all the states that are mixed by her operation are themselves entangled—or at least not *known* by Alice to be disentangled (given her practical inability to specify exactly which Kraus operations go into the pure operations of her mixing process).

Besides this, there is a more fundamental practical limitation facing Alice, even if we allow her any local operation she chooses. If, as we have seen, we can approximate the result of acting on x with any given operator A in von Neumann algebra $\mathcal{R}$ by acting on x with an invertible operator that preserves x's cyclicity, then the set of all such 'invertible actions' on x must itself produce a dense set of vector states, given that $\{Ax : A \in \mathcal{R}\}$ is dense. It follows that if a von Neumann algebra possesses even just one cyclic vector, it must possess a dense set of them (Dixmier & Marechal 1971, Lemma 4; cf. Clifton et al. 1998).

Now consider, again, the general situation of two commuting nonabelian algebras $\mathcal{R}_A$ and $\mathcal{R}_B$, where either algebra possesses a cyclic vector, and hence a dense set of such. If, in addition, the algebra $\mathcal{R}_{AB} = (\mathcal{R}_A \cup \mathcal{R}_B)''$ possesses a separating vector, then *all* states of that algebra are vector states, a *norm* dense set of which are therefore entangled across $(\mathcal{R}_A, \mathcal{R}_B)$. And since the entangled states of $\mathcal{R}_{AB}$ are open in the weak* topology, they must be open in the (stronger) norm topology too—so we are dealing with a truly generic set of states. Thus, it follows—quite independently of the RS theorem—that

> **Generic Result:** *If $\mathcal{R}_A$ and $\mathcal{R}_B$ are commuting nonabelian von Neumann algebras either of which possesses a cyclic vector, and $\mathcal{R}_{AB}$ possesses a separating vector, then the generic state of $\mathcal{R}_{AB}$ is entangled across $(\mathcal{R}_A, \mathcal{R}_B)$.*

The role that the RS theorem plays is to guarantee that the antecedent conditions of this Generic Result are satisfied whenever we consider spacelike-separated regions (and corresponding algebras) satisfying $(O_A \cup O_B)' \neq \emptyset$. This is a very weak requirement, which is satisfied, for example, when we assume both regions are bounded in spacetime. In that case, in order to be *certain* that her local operation in O_A (pure or mixed) produced a disentangled state, Alice would need the extraordinary ability to distinguish the state of $\mathcal{A}_{AB}$ which results from her operation from the generic set of states of $\mathcal{A}_{AB}$ that are entangled!

Finally, while we noted in our introduction the irony that limitations on disentanglement arise precisely when one considers *relativistic* quantum theory, the practical limitations we have just identified—as opposed to the *intrinsic* limits on disentanglement which are the subject of the next section—are not characteristic of AQFT alone. In particular, the existence of locally cyclic states does not depend on field theory. As we have seen, both the A and B subalgebras of $\mathbf{B}(\mathcal{H}_A) \otimes \mathbf{B}(\mathcal{H}_B)$ possess a cyclic vector just in case $\dim \mathcal{H}_A = \dim \mathcal{H}_B$. Indeed, operator algebraists so often find themselves dealing with von Neumann algebras that, together with their commutants, possess a cyclic vector, that such algebras are said by them to be in 'standard form'. So we should not think that local cyclicity is somehow peculiar to the states of local quantum fields.

Neither is it the case that our Generic Result above finds its only application in quantum *field* theory. For example, consider the infinite-by-infinite state space $\mathcal{H}_A \otimes \mathcal{H}_B$ of any two nonrelativistic particles, ignoring their spin degrees of freedom. Take the tensor product with a third auxiliary infinite-dimensional Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. Then obviously

$\infty = \dim \mathcal{H}_C \geq \dim(\mathcal{H}_A \otimes \mathcal{H}_B) = \infty$, whence the C subalgebra possesses a cyclic vector, which is therefore separating for the $A + B$ algebra. On the same dimensional grounds, both the A and B subalgebras possess cyclic vectors of their own. So our Generic Result applies immediately yielding the conclusion that a typical state of $A + B$ is entangled.

Nor should we think of local cyclicity or the applicability of our Generic Result as peculiar to standard *local* quantum field theory. After noting that the local cyclicity of the vacuum in AQFT was a "great, counterintuitive, surprise" when it was first proved, Fleming (2000, 4) proposes, instead, to build up local algebras associated with bounded open spatial sets within hyperplanes from raising and lowering operators associated with nonlocal Newton-Wigner position eigenstates—a proposal that goes back at least as far as Segal (1964). Fleming then observes, as did Segal (1964, 143), that the resulting vacuum state is *not* entangled, nor cyclic for any such local algebra. Nevertheless, as Segal points out, each Segal-Fleming local algebra will be isomorphic to the algebra $\mathbf{B}(\mathcal{H})$ of all bounded operators on an *infinite*-dimensional Hilbert space $\mathcal{H}$, and algebras associated with spacelike-separated regions in the same hyperplane commute. It follows that if we take any two spacelike-separated bounded open regions O_A and O_B lying in the same hyperplane, then $[\mathcal{A}(O_A) \cup \mathcal{A}(O_B)]''$ is naturally isomorphic to $\mathbf{B}(\mathcal{H}_A) \otimes \mathbf{B}(\mathcal{H}_B)$ (Horuzhy 1988, Lemma 1.3.28), and the result of the previous paragraph applies. So Fleming's 'victory' over the RS theorem of standard local quantum field theory rings hollow. Even though the Newton-Wigner vacuum is not itself entangled or locally cyclic across the regions (O_A, O_B), it is indistinguishable from globally pure states of the Newton-Wigner field that are![18]

On the other hand, generic entanglement is certainly not to be expected in every quantum-theoretic context. For example, if we ignore external degrees of freedom, and just consider the spins of two particles with joint state space $\mathcal{H}_A \otimes \mathcal{H}_B$, where both spaces are nontrivial and *finite*-dimensional, then the Generic Result no longer applies. Taking the product with a third auxiliary Hilbert space H_C does not work, because in order for the $A + B$ subalgebra to have a separating vector we would need $\dim H_C \geq \dim H_A \dim H_B$, but for either the A or B subalgebras to possess a cyclic vector we would *also* need that either $\dim H_A \geq \dim H_B \dim H_C$ or $\dim H_B \geq \dim H_A \dim H_C$—both of which contradict the fact $\mathcal{H}_A$ and $\mathcal{H}_B$

[18]For further critical discussion of the Segal-Fleming approach to quantum fields, see chapter 4.

are nontrivial and finite-dimensional.[19]

The point is that while the conditions for generic entanglement may or may not obtain in *any* quantum-theoretical context—depending on the observables and dimensions of the state spaces involved—the beauty of the RS theorem is that it allows us to deduce that generic entanglement between bounded open spacetime regions *must* obtain just by making some very general and natural assumptions about what should count as a physically reasonable relativistic quantum field theory.

3.4 Type III von Neumann algebras and intrinsic entanglement

Though it is not known to follow from the general axioms of AQFT (cf. Kadison 1963), all known concrete models of the axioms are such that the local algebras associated with bounded open regions in M are type III factors (Horuzhy 1988, 29, 35; Haag 1992, Sec. V.6). We start by reviewing what precisely is meant by the designation 'type III factor'.

A von Neumann algebra $\mathcal{R}$ is a factor just in case its center $\mathcal{R} \cap \mathcal{R}'$ consists only of multiples of the identity. It is easy to verify that this is equivalent to $(\mathcal{R} \cup \mathcal{R}')'' = \mathbf{B}(\mathcal{H})$. Thus, $\mathcal{R}$ induces a 'factorization' of the total Hilbert space algebra $\mathbf{B}(\mathcal{H})$ into two subalgebras which together generate that algebra.

To understand what 'type III' means, a few further definitions need to be absorbed. A *partial isometry* V is an operator on a Hilbert space $\mathcal{H}$ that maps some closed subspace $C \subseteq \mathcal{H}$ isometrically onto another closed subspace $C' \subseteq \mathcal{H}$, and maps $C^{\perp}$ to zero. (Think of V as a 'hybrid' unitary/projection operator.) Given the set of projections in a von Neumann algebra $\mathcal{R}$, we can define the following equivalence relation on this set: $P \sim Q$ just in case there is a partial isometry $V \in \mathcal{R}$ that maps the range of P onto the range of Q.[20] For example, any two infinite-dimensional projections in $\mathbf{B}(\mathcal{H})$ are equivalent (when $\mathcal{H}$ is separable), including projections one of whose range is properly contained in the other (cf. KR 1997, Cor. 6.3.5). A nonzero projection $P \in \mathcal{R}$ is called *abelian* if the von Neumann algebra $P\mathcal{R}P$ acting on the subspace $P\mathcal{H}$ (with identity P) is abelian. One can

[19] In fact, it can be shown that the spins of any pair of particles are *not* generically entangled, unless of course we ignore their mixed spin states; see (Clifton & Halvorson 2000) for further discussion.

[20] It is important to notice that this definition of equivalence is relative to the particular von Neumann algebra $\mathcal{R}$ that the projections are considered to be members of.

show that the abelian projections in a factor $\mathcal{R}$ are exactly the atoms in its projection lattice (KR 1997, Prop. 6.4.2). For example, the atoms of the projection lattice of $\mathbf{B}(\mathcal{H})$ are all its one-dimensional projections, and they are all (trivially) abelian, whereas it is clear that higher-dimensional projections are not. Finally, a projection $P \in \mathcal{R}$ is called *infinite* (relative to $\mathcal{R}$) when it is equivalent to another projection $Q \in \mathcal{R}$ such that $Q < P$, i.e., Q projects onto a proper subspace of the range of P. One can also show that any abelian projection in a von Neumann algebra is *finite*, i.e., not infinite (KR 1997, Prop. 6.4.2).

A von Neumann factor $\mathcal{R}$ is said to be *type* I just in case it possesses an abelian projection. For example, $\mathbf{B}(\mathcal{H})$ for any Hilbert space $\mathcal{H}$ is type I; and, indeed, every type I factor is isomorphic to $\mathbf{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ (KR 1997, Thm. 6.6.1). On the other hand, a factor is type III if all its nonzero projections are infinite and equivalent. In particular, this entails that the algebra itself is not abelian, nor could it even possess an abelian projection—which would have to be finite. And since a type III factor contains no abelian projections, its projection lattice has no atoms. Another fact about type III algebras (acting on a separable Hilbert space) is that they *always* possess a vector that is both cyclic and separating (Sakai 1971, Cor. 2.9.28). Therefore we know that type III algebras will always possess a dense set of cyclic vectors, and that all their states will be vector states. *Notwithstanding this*, type III algebras possess *no* pure states, as a consequence of the fact that they lack atoms.

To get some feeling for why this is the case—and for the general connection between the failure of the projection lattice of an algebra to possess atoms and its failure to possess pure states—let $\mathcal{R}$ be any non-atomic von Neumann algebra possessing a separating vector (so all of its states are vector states), and let ρ_x be any state of $\mathcal{R}$. We shall need two further definitions. The *support projection* S_x of ρ_x in $\mathcal{R}$ is defined to be the meet of all projections $P \in \mathcal{R}$ such that $\rho_x(P) = 1$. (So S_x is the smallest projection in $\mathcal{R}$ that ρ_x 'makes true'.) The *left-ideal* $\mathcal{I}_x$ of ρ_x in $\mathcal{R}$ is defined to be the set of all $A \in \mathcal{R}$ such that $\rho_x(A^*A) = 0$. Now since S_x is not an atom, there is some nonzero $P \in \mathcal{R}$ such that $P < S_x$. Choose any vector y in the range of P (noting it follows that $S_y \leq P$). We shall first show that $\mathcal{I}_x$ is a proper subset of $\mathcal{I}_y$. So let $A \in \mathcal{I}_x$. Clearly this is equivalent to saying that $Ax = 0$, or that x lies in the range of $N(A)$, the projection onto the null-space of A. $N(A)$ itself lies $\mathcal{R}$ (KR 1997, Lemma 5.1.5 and Prop. 2.5.13), thus, $\rho_x(N(A)) = 1$, and accordingly $S_x \leq N(A)$. But since $S_y \leq P < S_x$, we also have $\rho_y(N(A)) = 1$. Thus, y too lies in the range of $N(A)$, i.e., $Ay = 0$, and therefore $A \in \mathcal{I}_y$. To see that the inclusion

$\mathcal{I}_x \subseteq \mathcal{I}_y$ is proper, note that since $\langle y, S_y y\rangle = 1$, $\langle y, [I - S_y]^2 y\rangle = 0$, and thus $I - S_y \in \mathcal{I}_y$. However, certainly $I - S_y \notin \mathcal{I}_x$, for the contrary would entail that $\langle x, S_y x\rangle = 1$, in other words, $S_x \leq S_y \leq P < S_x$, which is a contradiction. We can now see, finally, that ρ_x is not pure. For, a pure state of a von Neumann algebra $\mathcal{R}$ determines a maximal left-ideal in $\mathcal{R}$ (KR 1997, Thm. 10.2.10), whereas we have just shown (under the assumption that $\mathcal{R}$ is non-atomic) that $\mathcal{I}_x \subset \mathcal{I}_y$.

The fact that every state of a type III algebra $\mathcal{R}$ is mixed throws an entirely new wrench into the works of the ignorance interpretation of mixtures.[21] Not only is there no preferred way to pick out components of a mixture, but the components of states of $\mathcal{R}$ are always mixed states. Thus, it is impossible to understand the physical preparation of such a mixture in terms of mixing pure states—the states of $\mathcal{R}$ are always irreducibly or what we shall call *intrinsically* mixed. Note, however, that while the states of type III factors fit this description, so do the states of certain *abelian* von Neumann algebras. For example, the 'multiplication' algebra $\mathcal{M} \subseteq \mathbf{B}(L_2(\mathbb{R}))$ of all bounded functions of the position operator for a single particle lacks atomic projections because position has no eigenvectors. Moreover, all the states of $\mathcal{M}$ are vector states, because any state vector that corresponds to a wavefunction whose support is the whole of $\mathbb{R}$ is separating for $\mathcal{M}$. Thus the previous paragraph's argument applies equally well to $\mathcal{M}$.

Of course no properly *quantum* system has an abelian algebra of observables, and, as we have already noted, systems with abelian algebras are never entangled with other systems (Bacciagaluppi 1993, Thm. 3). This makes the failure of a type III factor $\mathcal{R}$ to have pure states importantly different from that failure in the case of an abelian algebra. Because $\mathcal{R}$ is *non*abelian, and taking the commutant preserves type (KR 1997, Thm. 9.1.3) so that $\mathcal{R}'$ will also be nonabelian, one suspects that any pure state of $(\mathcal{R} \cup \mathcal{R}')'' = \mathbf{B}(\mathcal{H})$—which must restrict to an intrinsically mixed state on both subalgebras $\mathcal{R}$ and $\mathcal{R}'$—has to be *intrinsically entangled* across $(\mathcal{R}, \mathcal{R}')$. And that intuition is exactly right. Indeed, one can show that there are not even any *product* states across $(\mathcal{R}, \mathcal{R}')$ (Summers 1990, 213). And, of course, if there are no unentangled states across $(\mathcal{R}, \mathcal{R}')$, then the infamous distinction, some have argued is important to preserve, between so-called 'improper' mixtures that arise by restricting an entangled state to a subsystem, and 'proper' mixtures that do not, becomes *irrelevant*.

Even more interesting is the fact that in all known models of AQFT, the

[21] To our knowledge, van Aken (1985) is the only philosopher of quantum theory to have noticed this.

local algebras are 'type III_1' (cf. Haag 1992, 267). It would take us too far afield to explain the standard sub-classification of factors presupposed by the subscript '1'. We wish only to draw attention to an equivalent characterization of type III_1 algebras established by Connes and Størmer (1978, Cor. 6): A factor $\mathcal{R}$ acting standardly on a (separable) Hilbert space is type III_1 just in case for *any two* states ρ, ω of $\mathbf{B}(\mathcal{H})$, and any $\epsilon > 0$, there are unitary operators $U \in \mathcal{R}$, $U' \in \mathcal{R}'$ such that $\|\rho - \omega^{UU'}\| < \epsilon$. Notice that this result immediately implies that there are no unentangled states across $(\mathcal{R}, \mathcal{R}')$; for, if some ω were not entangled, it would be impossible to act on this state with local unitary operations in $\mathcal{R}$ and $\mathcal{R}'$ and get arbitrarily close to the states that *are* entangled across $(\mathcal{R}, \mathcal{R}')$. Furthermore—and this is the interesting fact—the Connes-Størmer characterization immediately implies the impossibility of distinguishing in any reasonable way between the different degrees of entanglement that states might have across $(\mathcal{R}, \mathcal{R}')$. For it is a standard assumption in quantum information theory that all reasonable measures of entanglement must be *invariant* under unitary operations on the separate entangled systems (cf. Vedral, Plenio, Rippin, & Knight 1997), and presumably such a measure should assign close degrees of entanglement to states that are close to each other in norm. In light of the Connes-Størmer characterization, however, imposition of both these requirements forces triviality on any proposed measure of entanglement across $(\mathcal{R}, \mathcal{R}')$.[22]

The above considerations have particularly strong physical implications when we consider local algebras associated with *diamond regions* in M, i.e., regions given by the intersection of the timelike future of a given spacetime point p with the timelike past of another point in p's future. When $\diamondsuit \subseteq M$ is a diamond, it can be shown in many models of AQFT, including for *non*interacting fields, that $\mathcal{A}(\diamondsuit') = \mathcal{A}(\diamondsuit)'$ (Haag 1992, Sec. III.4.2). Thus every global state of the field is *intrinsically* entangled across $(\mathcal{A}(\diamondsuit), \mathcal{A}(\diamondsuit'))$, and it is never possible to think of the field system in a diamond region $\diamondsuit$ as disentangled from its spacelike complement. Though he does not use the language of entanglement, this is precisely the reason for Haag's remark that field systems are always open. In particular, Alice would have *no hope whatsoever* of using local operations in $\diamondsuit$ to disentangle that region's state

[22] Of course, the standard von Neumann entropy measure we discussed in Section 1 is norm continuous, and, because of the unitary invariance of the trace, this measure is invariant under unitary operations on the component systems. But in the case of a type III factor $\mathcal{R}$, that measure, as we should expect, is *not* available. Indeed, the state of a system described by $\mathcal{R}$ cannot be represented by any density operator *in* $\mathcal{R}$ because $\mathcal{R}$ cannot contain compact operators, like density operators, whose spectral projections are all finite!

from that of the rest of the world.

Suppose, however, that Alice has only the more limited goal of disentangling a state of the field across some isolated pair of *strictly* spacelike-separated regions (O_A, O_B), i.e., regions which remain spacelike separated when either is displaced by an arbitrarily small amount. It is also known that in many models of AQFT the local algebras possess the *split property*: for any bounded open $O \subseteq M$, and any larger region $\tilde{O}$ whose interior contains the closure of O, there is a type I factor $\mathcal{N}$ such that $\mathcal{A}(O) \subset \mathcal{N} \subset \mathcal{A}(\tilde{O})$ (Buchholz 1974; Werner 1987). This implies that the von Neumann algebra generated by a pair of algebras for strictly spacelike-separated regions is isomorphic to their tensor product and, as a consequence, that there *are* product states across $(\mathcal{A}(O_A), \mathcal{A}(O_B))$ (cf. Summers 1990, 239–240). Since, therefore, not every state of $\mathcal{A}_{AB}$ is entangled, we might hope that whatever the global field state is, Alice could *at least in principle* perform an operation in O_A on that state that disentangles it across (O_A, O_B). However, we are now going use the fact that $\mathcal{A}(O_A)$ lacks abelian projections to show that a norm dense set of entangled states of $\mathcal{A}_{AB}$ cannot be disentangled by any pure local operation performed in $\mathcal{A}(O_A)$.

Let ρ_x be any one of the norm dense set of entangled states of $\mathcal{A}_{AB}$ induced by a vector $x \in \mathcal{H}$ cyclic for $\mathcal{A}(O_B)$, and let $K \in \mathcal{A}(O_A)$ be an arbitrary Kraus operator. (Observe that $\rho_x^K \neq 0$ because x is separating for $\mathcal{A}(O_B)'$—which includes $\mathcal{A}(O_A)$—and $K^*K \in \mathcal{A}(O_A)$ is positive.) Suppose for reductio ad absurdum that ω_x^K is not entangled. Let Ky, with $y \in \mathcal{H}$, be any nonzero vector in the range of K. Then, since x is cyclic for $\mathcal{A}(O_B)$, we have, for some sequence $\{B_i\} \subseteq \mathcal{A}(O_B)$, $Ky = K(\lim B_i x) = \lim(B_i K x)$, which entails $\|(\omega_x^K)^{B_i/\|B_i\|} - \omega_{Ky}\| \to 0$. Since ω_x^K is not entangled across $(\mathcal{A}(O_A), \mathcal{A}(O_B))$, and the local pure operations on $\mathcal{A}(O_B)$ given by the Kraus operators $B_i/\|B_i\|$ cannot create entanglement, we see that ω_{Ky} is the norm (hence weak*) limit of a sequence of unentangled states and, as such, is not itself entangled either. Since y was arbitrary, it follows that every nonzero vector in the range of K induces an unentangled state across $(\mathcal{A}(O_A), \mathcal{A}(O_B))$. Obviously, the same conclusion follows for any nonzero vector in the range of $R(K)$—the projection onto the range of K—since the range of the latter lies dense in that of the former.

Next, consider the von Neumann algebra

$$\mathcal{C}_{AB} \equiv [R(K)\mathcal{A}(O_A)R(K) \cup R(K)\mathcal{A}(O_B)R(K)]'' \tag{3.19}$$

acting on the Hilbert space $R(K)\mathcal{H}$. Since $K \in \mathcal{A}(O_A)$, $R(K) \in \mathcal{A}(O_A)$ (KR 1997, 309), and thus the subalgebra $R(K)\mathcal{A}(O_A)R(K)$ cannot be abelian—

on pain of contradicting the fact that $\mathcal{A}(O_A)$ has no abelian projections. And neither is $R(K)\mathcal{A}(O_B)R(K)$ abelian. For since $\mathcal{A}(O_B)$ itself is nonabelian, there are $Y_1, Y_2 \in \mathcal{A}(O_B)$ such that $[Y_1, Y_2] \neq 0$. And because our regions (O_A, O_B) are strictly spacelike-separated, they have the Schlieder property: $0 \neq A \in \mathcal{A}(O_A), 0 \neq B \in \mathcal{A}(O_B)$ implies $AB \neq 0$ (Summers 1990, Thm. 6.7). Therefore,

$$[R(K)Y_1R(K), R(K)Y_2R(K)] = [Y_1, Y_2]R(K) \neq 0. \tag{3.20}$$

So we see that neither algebra occurring in $\mathcal{C}_{AB}$ is abelian; yet they commute, and so there must be at least one entangled state across those algebras (see page 27). But this conflicts with the conclusion of the preceding paragraph! For the vector states of $\mathcal{C}_{AB}$ are precisely those induced by the vectors in the range of $R(K)$, and we concluded above that these all induce unentangled states across $(\mathcal{A}(O_A), \mathcal{A}(O_B))$. Therefore, by restriction, they all induce unentangled states across the algebra $\mathcal{C}_{AB}$. But if none of $\mathcal{C}_{AB}$'s vector states are entangled, it can possess *no* entangled states at all.

The above argument still goes through under the weaker assumption that Alice applies any mixed *projective* operation, i.e., any operation T corresponding to a standard von Neumann measurement associated with a mutually orthogonal set $\{P_i\} \in \mathcal{A}(O_A)$ of projection operators. For suppose, again for reductio ad absurdum, that $\rho_x^T = \sum_i \lambda_i \rho_x^{P_i}$ is not entangled across the regions. Then, since entanglement cannot be created by a further application to ρ_x^T of the local projective operation given by (say) $T_1(\cdot) = P_1(\cdot)P_1$, it follows that $(\rho_x^T)^{T_1} = (\rho_x^{T_1 \circ T}) = \rho_x^{P_1}$ must again be unentangled, and the above reasoning to a contradiction goes through *mutatis mutandis* with $K = P_1$. This is to be contrasted to the nonrelativistic case we considered in section 3.1, where Alice *was* able to disentangle an arbitrary state of $\mathbf{B}(\mathcal{H}_A) \otimes \mathbf{B}(\mathcal{H}_B)$ by a nonselective projective operation on A. And a moment's reflection will reveal that that was possible precisely because of the availability of abelian projections in the algebra of her subsystem A.

We have not, of course, shown that the above argument covers *arbitrary* mixing operations Alice might perform in O_A; in particular, positive operator-valued mixings, where the Kraus operators $\{K_i\}$ of a local operation T in O_A do not have mutually orthogonal ranges. However, although it would be interesting to know how far the result could be pushed, we have already expressed our reservations about whether arbitrary mixing operations should count as disentangling when none of the pure operations of which they are composed could possibly produce disentanglement on their own.

In summary:

> *There are many regions of spacetime within which no local operations can be performed that will disentangle that region's state from that of its spacelike complement, and within which no pure or projective operation on any one of a norm dense set of states can yield disentanglement from the state of any other strictly spacelike-separated region.*

Clearly the advantage of the formalism of AQFT is that it allows us to see clearly just how much more deeply entrenched entanglement is in *relativistic* quantum theory. At the very least, this should serve as a strong note of caution to those who would quickly assert that quantum nonlocality cannot peacefully exist with relativity!

3.4.1 Neutralizing the methodological worry

What then becomes of Einsteinian worries about the possibility of doing science in such a highly entangled world? As we shall now explain, *for all practical purposes* the split property of local algebras neutralizes Einstein's main methodological worry.[23]

Let us suppose Alice wants to prepare some state ρ on $\mathcal{A}(O_A)$ for subsequent testing. By the split property, there is a type I factor $\mathcal{N}$ satisfying $\mathcal{A}(O_A) \subset \mathcal{N} \subset \mathcal{A}(\tilde{O}_A)$ for any super-region $\tilde{O}_A$ that contains the closure of O_A. Since ρ is a vector state (when we assume $(O_A)' \neq \emptyset$), its vector representative defines a state on $\mathcal{N}$ that extends ρ and is, therefore, represented by some density operator D_ρ in the type I algebra $\mathcal{N}$. Now D_ρ is a convex combination $\sum_i \lambda_i P_i$ of mutually orthogonal atomic projections in $\mathcal{N}$ satisfying $\sum_i P_i = I$ with $\sum_i \lambda_i = 1$. But each such projection is equivalent, in the type III algebra $\mathcal{A}(\tilde{O}_A)$, to the identity operator. Thus, for each i, there is a partial isometry $V_i \in \mathcal{A}(\tilde{O}_A)$ satisfying $V_i V_i^* = P_i$ and $V_i^* V_i = I$.

Next, consider the nonselective operation T on $\mathcal{A}(\tilde{O}_A)$ given by Kraus operators $K_i = \sqrt{\lambda_i} V_i$. We claim that $T(X) = \rho(X)I$ for all $X \in \mathcal{A}(O_A)$. Indeed, because each P_i is abelian in $\mathcal{N} \supseteq \mathcal{A}(O_A)$, the operator $P_i X P_i$ acting on $P_i\mathcal{H}$ can only be some multiple, c_i, of the identity operator P_i on $P_i\mathcal{H}$, and taking the trace of both sides of the equation

$$P_i X P_i = c_i P_i \tag{3.21}$$

immediately reveals that $c_i = \mathrm{Tr}(P_i X)$. Moreover, acting on the left of (3.21)

[23]The following arguments are essentially just an amplification of the reasoning in Werner (1987) and Summers (1990, Thm. 3.13).

with V_i^* and on the right with V_i, we obtain $V_i^* X V_i = \mathrm{Tr}(P_i X)I$, which yields the desired conclusion when multiplied by λ_i and summed over i.

Finally, since $T(X) = \rho(X)I$ for all $X \in \mathcal{A}(O_A)$, obviously $\omega^T = \rho$ for all initial states ω of $\mathcal{A}(O_A)$. Thus, once we allow Alice to perform an operation like T that is *approximately* local to $\mathcal{A}(O_A)$ (choosing $\tilde{O}_A$ to approximate O_A as close as we like), she has the freedom to prepare any state of $\mathcal{A}(O_A)$ that she pleases!

Notice that, ironically, testing the theory is actually *easier* here than in nonrelativistic quantum theory. For we were able to exploit above the type III character of $\mathcal{A}(\tilde{O}_A)$ to show that Alice can always prepare her desired state on $\mathcal{A}(O_A)$ *nonselectively*, i.e., without ever having to sacrifice any members of her ensemble! Also observe that the result of her preparing operation T, because it is local to $\mathcal{A}(\tilde{O}_A)$, will always produce a product state across (O_A, O_B) when $O_B \subseteq (\tilde{O}_A)'$. That is, for any initial state ω across the regions, and all $X \in \mathcal{A}(O_A)$ and $Y \in \mathcal{A}(O_B)$, we have

$$\omega^T(XY) = \omega(T(X)Y) = \omega(\rho(X)Y) = \rho(X)\omega(Y). \tag{3.22}$$

x So as soon as we allow Alice to perform *approximately* local operations on her field system, she *can* isolate it from entanglement with other strictly spacelike-separated field systems, while simultaneously preparing its state as she likes and with relative ease. *God is subtle, but not malicious.*

Part II

Localizable Particles

Chapter 4

Reeh-Schlieder defeats Newton-Wigner: On alternative localization schemes in relativistic quantum field theory

4.1 Introduction

Relativistic quantum theory presents us with a set of peculiar interpretive difficulties over and above the traditional ones of elementary quantum mechanics. For example, while the notion of a "localized object" has a transparent mathematical counterpart in elementary quantum mechanics, it appears that not every aspect of our common-sense notion of localization can be maintained in the context of relativistic quantum theory (cf. Malament 1996). Many of the thorny issues involving localization in relativistic quantum *field* theory have a common formal root in the so-called "Reeh-Schlieder theorem." Thus, it is of particular philosophical interest that I. E. Segal (1964) and, more recently, G. Fleming (2000) have shown that it is possible—at least on a purely formal level—to avoid the Reeh-Schlieder theorem, and thereby its counterintuitive consequences, by means of a judicious reworking of the standard association between observables and regions of space.[1]

I am not convinced, however, that Segal and Fleming's "Newton-Wigner" localization scheme offers any satisfying resolution for the "problem" of localization in relativistic quantum field theory. In particular, the Newton-Wigner localization scheme is itself subject to variants of the Reeh-Schlieder theorem which are no less counterintuitive than the original version of the theorem. Furthermore, under the only defensible interpretation of the Newton-Wigner localization scheme, its empirical predictions come into direct conflict with special relativity.

The context of the Reeh-Schlieder theorem is the axiomatic (or algebraic) approach to quantum field theory. This approach singles out a family of postulates that apply quite generally to "physically reasonable" quantum field models, and these postulates are used as a starting point for further structural investigations. One might expect, then, that Segal and Fleming would attempt to undercut the Reeh-Schlieder theorem by questioning one of the assumptions it makes concerning which models are "physically reasonable." However, Segal and Fleming do not discuss the Reeh-Schlieder theorem at this level of generality; rather, their discussion of the Reeh-Schlieder theorem is restricted to a concrete field model, viz., the free Bose field.

I begin then in section 4.2 with a brief review of the global structure of the free Bose field model. In section 4.3, I present the standard recipe for assigning observables to regions in space, and I explicate the counterintuitive consequences—stemming from the Reeh-Schlieder theorem—of this standard localization scheme. In section 4.4, I present the Newton-Wigner localization scheme and show how it "avoids" the counterintuitive consequences of the Reeh-Schlieder theorem. Finally, in sections 4.5 and 4.6, I argue that Reeh-Schlieder has the final word against the Newton-Wigner localization scheme.

4.2 The free Bose field

In this section, I briefly review the mathematical formalism for the quantum theory of the free Bose field. Although my presentation differs from Fleming's (2000) in being more abstract and in its emphasis on mathematical rigor, I take it that all parties agree concerning the *global* structures of the

[1] Saunders (1992) provides an extensive discussion of Segal's approach, although with different points of emphasis than the current presentation.

free field model (at least in the absence of measurement interactions). That is, we agree on our answers to the following four questions:

1. What is the state space?
2. What are the observables (i.e., physical quantities)?
3. When no measurements are being made, how does the system evolve in time? In other words, what is the (free) Hamiltonian?
4. What is the ground (i.e., vacuum) state?

Disputes arise only at the level of the *local* structure of the free field model; e.g., which states are "localized" in this region of space? In this section, I spell out the answers to questions 1–4. In section 4.3, I take up questions concerning localization.

Recall that in its heuristic formulation, the free scalar quantum field is described by an "operator-valued field" Φ on Minkowski spacetime that solves the Klein-Gordon equation

$$\frac{\partial^2\Phi}{\partial t^2} + m^2\Phi = \nabla^2\Phi, \tag{4.1}$$

and that satisfies the appropriate (equal-time) canonical commutation relations. As is well-known, however, there are mathematical difficulties with understanding Φ as an operator-valued function. A more rigorous approach takes Φ as an "operator-valued distribution." That is, for each smooth, real-valued test-function f on Minkowski spacetime, $\Phi(f)$ can be defined as an operator on some Hilbert space.

For my purposes here, it will be more convenient to turn to another (mathematically equivalent) representation of the field Φ. Let $C_0^\infty(\mathbb{R}^3)$ denote the vector space of smooth, compactly supported functions from $\mathbb{R}^3$ into $\mathbb{R}$, and let

$$S = C_0^\infty(\mathbb{R}^3) \oplus C_0^\infty(\mathbb{R}^3). \tag{4.2}$$

Recall now that a scalar-valued solution ϕ of the Klein-Gordon equation is uniquely determined by its Cauchy data (i.e., its values, and the values of its first derivative) at any fixed time. Thus, there is a one-to-one correspondence between elements of S and (a certain subset of) the space of solutions of the Klein-Gordon equation. Moreover, the conserved four-vector current $\phi \overleftrightarrow{\partial}_\mu \psi$ gives rise to a symplectic form σ on S:

$$\sigma(u_0 \oplus u_1, v_0 \oplus v_1) = \int_{\mathbb{R}^3} (u_0 v_1 - u_1 v_0)\, d^3\mathbf{x}. \tag{4.3}$$

We let D_t denote the natural (inertial) symplectic flow on S; i.e., D_t maps the time-zero Cauchy data of ϕ to the time-t Cauchy data of ϕ. The triple (S, σ, D_t) contains the essential information specifying the classical theory of the scalar field of mass m.

A representation of the Weyl form of the canonical commutation relations (CCRs) is a mapping $f \mapsto W(f)$ of S into unitary operators acting on some Hilbert space $\mathcal{K}$ such that $W(0) = I$ and

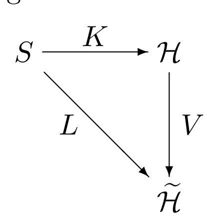
$$W(f)W(g) = e^{-i\sigma(f,g)}W(f+g). \tag{4.4}$$

I will now sketch the construction of the unique (up to unitary equivalence) "Minkowski vacuum representation" of the CCRs. This construction proceeds in two steps. In *first quantization*, we "Hilbertize" the classical phase space S, and we "unitarize" the classical dynamical group D_t. More precisely, suppose that $\mathcal{H}$ is a Hilbert space, and that U_t is a weakly continuous one-parameter group of unitary operators acting on $\mathcal{H}$. Suppose also that the infinitesimal generator A of U_t is a positive operator; i.e., $(f, Af) \geq 0$ for all f in the domain of A. If there is a one-to-one real-linear mapping K of S into $\mathcal{H}$ such that

1. $K(S) + iK(S)$ is dense in $\mathcal{H}$,
2. $2\text{Im}(Kf, Kg) = \sigma(f, g)$,
3. $U_t K = K D_t$,

then we say that the triple $(K, \mathcal{H}, U_t)$ is a *one-particle structure* over (S, σ, D_t). Constructing a one-particle structure over (S, σ, D_t) is a mathematically rigorous version of "choosing the subspace of positive frequency solutions" of the space of complex solutions to the Klein-Gordon equation.

If there is a one-particle structure over (S, σ, D_t), then it is unique up to unitary equivalence (Kay, 1979). That is, suppose that $(K, \mathcal{H}, U_t)$ and $(L, \widetilde{\mathcal{H}}, \widetilde{U}_t)$ are one-particle structures over (S, σ, D_t). Then, $L \circ K^{-1}$ extends uniquely to a unitary mapping V from $\mathcal{H}$ onto $\widetilde{\mathcal{H}}$.



It is also not difficult to see that V intertwines the unitary groups on the respective Hilbert spaces, i.e., $VU_t = \widetilde{U}_t V$. This uniqueness result can be interpreted as showing that the choice of time evolution in the classical phase space suffices to determine uniquely the (first) quantization of the classical system.

I will construct two (unitarily equivalent) versions of the one-particle structure over (S,σ,D_t). First, we may complete S relative to the unique Hilbert space norm in which time-evolution (given by D_t) is an isometry. Specifically, let H denote the linear operator $(-\nabla^2+m^2)^{1/2}$ on $C_0^\infty(\mathbb{R}^3)$,[2] and define a real inner-product μ on S by

$$\mu(u_0\oplus u_1,v_0\oplus v_1) = (1/2)\big((u_0,Hv_0)+(u_1,H^{-1}v_1)\big) \tag{4.5}$$

$$= (1/2)\left(\int_{\mathbb{R}^3} u_0(Hv_0)\,d^3\mathbf{x}+\int_{\mathbb{R}^3} u_1(H^{-1}v_1)\,d^3\mathbf{x}\right) \tag{4.6}$$

Now let $\mathcal{H}_\mu$ denote the completion of S relative to the inner-product μ.[3] Define an operator J on $\mathcal{H}_\mu$ by setting

$$J(u_0\oplus u_1) = -H^{-1}u_1\oplus Hu_0\,, \tag{4.7}$$

on the dense subset S of $\mathcal{H}_\mu$. Clearly $J^2=-I$, i.e., J is a "complex structure" on $\mathcal{H}_\mu$. Thus, $\mathcal{H}_\mu$ becomes a complex vector space when we define scalar multiplication by $(a+ib)f=af+J(bf)$, and is a complex Hilbert space relative to the inner-product

$$(f,g)_\mu = \mu(f,g)+i\mu(Jf,g) \tag{4.8}$$

$$= \mu(f,g)+(i/2)\sigma(f,g). \tag{4.9}$$

Finally, it can be shown that $[J,D_t]=0$, so that D_t extends uniquely to a weakly continuous one-parameter group of *unitary* operators (denoted again by D_t) on the complex Hilbert space $\mathcal{H}_\mu$. Therefore, $(\iota,\mathcal{H}_\mu,D_t)$, with ι the identity mapping, is a one-particle structure over (S,σ,D_t).

It may not be immediately obvious—especially to those accustomed to non-relativistic quantum mechanics—how to tie the physics of localization to the mathematical structure of the Hilbert space $\mathcal{H}_\mu$. (For example, which

[2] The mathematically rigorous definition of H is as follows: Define the operator $A=-\nabla^2+m^2$ on $C_0^\infty(\mathbb{R}^3)$. Then, A is essentially self-adjoint, and the self-adjoint closure $\overline{A}$ of A is a positive operator with spectrum in $[m^2,\infty)$. Using the functional calculus for unbounded operators, we may define $H=\overline{A}^{1/2}$, and it follows that the spectrum of H is contained in $[m,\infty)$.

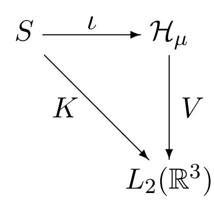
[3] If $\mathcal{L}^\pm(\mathbb{R}^3)$ denotes the completion of $C_0^\infty(\mathbb{R}^3)$ relative to the inner product $(\,\cdot\,,H^{\pm1}\,\cdot\,)$, then $\mathcal{H}_\mu=\mathcal{L}^+(\mathbb{R}^3)\oplus\mathcal{L}^-(\mathbb{R}^3)$.

vectors in $\mathcal{H}_\mu$ are localized in a given spatial region?) The Newton-Wigner one-particle structure brings us back to familiar territory by using the space $L_2(\mathbb{R}^3)$ as the concrete representation of the one-particle space. In particular, define the mapping $K : S \mapsto L_2(\mathbb{R}^3)$ by

$$K(u_0 \oplus u_1) \;=\; 2^{-1/2}(H^{1/2}u_0 + iH^{-1/2}u_1). \tag{4.10}$$

It is then straightforward to check that the complex-linear span of $K(S)$ is dense in $L_2(\mathbb{R}^3)$, and that K preserves (modulo a factor of 2) the symplectic form σ. Moreover, it can be shown that K intertwines D_t with the one parameter unitary group $U_t = e^{-itH}$ on $L_2(\mathbb{R}^3)$. Therefore, $(K, L_2(\mathbb{R}^3), U_t)$ is a one-particle structure over (S, σ, D_t).

Since $(\iota, \mathcal{H}_\mu, D_t)$ and $(K, L_2(\mathbb{R}^3), U_t)$ are one-particle structures over (S, σ, D_t), it follows that $(K \circ \iota^{-1}) = K$ extends uniquely to a unitary operator V from $\mathcal{H}_\mu$ onto $L_2(\mathbb{R}^3)$:



Thus, the one-particle spaces $(\mathcal{H}_\mu, D_t)$ and $(L_2(\mathbb{R}^3), U_t)$ are mathematically, and hence physically, equivalent. On the other hand, the two spaces certainly *suggest* different notions of localization.

4.2.1 Second quantization

Once we have a one-particle space $(\mathcal{H}, U_t)$ in hand, the movement to a quantum *field* theory (i.e., "second quantization") is mathematically straightforward and uniquely determined.[4] In particular, let $\mathcal{F}(\mathcal{H})$ denote the "Fock space" over $\mathcal{H}$. That is,

$$\mathcal{F}(\mathcal{H}) = \mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H}^2 \oplus \mathcal{H}^3 \oplus \cdots \,, \tag{4.11}$$

where $\mathcal{H}^n$ is the n-fold symmetric tensor product of $\mathcal{H}$. As usual we let

$$\Omega = 1 \oplus 0 \oplus 0 \oplus \cdots \tag{4.12}$$

[4]For a more detailed exposition, see Bratteli & Robinson 1996, Section 5.2.

denote the vacuum vector in $\mathcal{F}(\mathcal{H})$. For each $f \in \mathcal{H}$, we define the creation $a^+(f)$ and annihilation $a(f)$ operators on $\mathcal{F}(\mathcal{H})$ as usual, and we let $\Phi(f)$ denote the self-adjoint closure of the unbounded operator

$$2^{-1/2}(a(f) + a^+(f)). \tag{4.13}$$

If we let $W(f) = \exp\{ i\Phi(f)\}$, then the $W(f)$ satisfy the Weyl form of the canonical commutation relations:

$$W(f)W(g) = e^{-i\,\mathrm{Im}(f,g)/2}W(f + g), \tag{4.14}$$

and vacuum expectation values are given explicitly by

$$\langle \Omega, W(f)\Omega\rangle = \exp\left(-\|f\|^2/4\right). \tag{4.15}$$

The dynamical group on $\mathcal{F}(\mathcal{H})$ is given by the "second quantization" $\Gamma(U_t) = e^{itd\Gamma(H)}$ of the dynamical group $U_t = e^{itH}$ on $\mathcal{H}$, and the vacuum vector Ω is the unique eigenvector of the Hamiltonian $d\Gamma(H)$ with eigenvalue 0.

4.3 Local algebras and the Reeh-Schlieder theorem

To this point we have only discussed the global structure of the free Bose field model. The physical observables for the free Bose field are given by the self-adjoint operators on Fock space $\mathcal{F}(\mathcal{H})$. We equip this model with a *local structure* when we define a correspondence between regions in space and "subalgebras" of observables. This labelling may be done for various purposes, but the traditional motivation was to indicate those observables that can (in theory) be measured in that region of space.

Now, each real-linear subspace E of the one-particle space $\mathcal{H}$ gives rise naturally to a subalgebra of operators, viz., the algebra generated by the Weyl operators $\{W(f) : f \in E\}$. Thus, a localization scheme needs only to determine which real-linear subspace of $\mathcal{H}$ should be taken as corresponding to a region G in physical space. *It is on this point that the Newton-Wigner localization scheme disagrees with the standard localization scheme.* In the remainder of this section, I discuss the standard localization scheme and its consequences.

The standard localization scheme assigns to the spatial region G the subset $S(G) \subseteq \mathcal{H}_\mu$ of Cauchy data localized in G. That is, if $C^\infty(G)$ denotes the subspace of $C_0^\infty(\mathbb{R}^3)$ of functions with support in G, then

$$S(G) = C^\infty(G) \oplus C^\infty(G), \tag{4.16}$$

is a real-linear subspace of $\mathcal{H}_\mu$. (Note that $S(G)$ is not closed nor, as we shall soon see, complex-linear.) Thus, in the Newton-Wigner representation, the classical localization scheme assigns G to the real-linear subspace $V(S(G))$ of $L_2(\mathbb{R}^3)$. When no confusion can result, I will suppress reference to the unitary operator V and simply use $S(G)$ to denote the pertinent subspace in either concrete version of the one-particle space.

Note that the correspondence $G \mapsto S(G)$ is monotone; i.e., if $G_1 \subseteq G_2$ then $S(G_1) \subseteq S(G_2)$. Moreover, if $G_1 \cap G_2 = \emptyset$, then $S(G_1)$ and $S(G_2)$ are "symplectically orthogonal." That is, if $f \in S(G_1)$ and $g \in S(G_2)$, then $\mathrm{Im}(f,g) = 0$. Indeed, if $u_0 \oplus u_1 \in S(G_1)$ and $v_0 \oplus v_1 \in S(G_2)$, then

$$\sigma(u_0 \oplus u_1, v_0 \oplus v_1) = \int_{\mathbb{R}^3} (u_0 v_1 - u_1 v_0)\, d^3\mathbf{x} = 0, \tag{4.17}$$

since the u_i and v_i have disjoint regions of support.

Now, we say that a Weyl operator $W(f)$ acting on $\mathcal{F}(\mathcal{H})$ is *classically localized* in G just in case $f \in S(G)$. ("Classically" here refers simply to the fact that our notion of localization is derived from the local structure of the classical phase space S.) Let $\mathbf{B}(\mathcal{F}(\mathcal{H}))$ denote the algebra of bounded operators on $\mathcal{F}(\mathcal{H})$. We then define the subalgebra $\mathcal{R}(G) \subseteq \mathbf{B}(\mathcal{F}(\mathcal{H}))$ of operators classically localized in G to be the "von Neumann algebra" generated by the Weyl operators classically localized in G. That is, $\mathcal{R}(G)$ consists of arbitrary linear combinations and "weak limits" of Weyl operators classically localized in G.[5]

If $\mathcal{R} \subseteq \mathbf{B}(\mathcal{F}(\mathcal{H}))$, we let $\mathcal{R}'$ denote all operators in $\mathbf{B}(\mathcal{F}(\mathcal{H}))$ that commute with every operator in $\mathcal{R}$. If $\mathcal{R}$ contains I and is closed under taking adjoints, then von Neumann's "double commutant theorem" entails that $(\mathcal{R}')'$ is the von Neumann algebra generated by $\mathcal{R}$. Thus, we have

$$\mathcal{R}(G) = \{W(f) : f \in S(G)\}''\,. \tag{4.18}$$

In order also to associate unbounded operators with local regions, we say that an unbounded operator A is *affiliated* with the local algebra $\mathcal{R}(G)$ just in case $U^{-1}AU = A$ for any unitary operator $U \in \mathcal{R}(G)'$. It then follows

[5] Since $f \mapsto W(f)$ is weakly continuous, $\mathcal{R}(G)$ contains $W(f)$ for all f in the closure of $S(G)$.

that $\Phi(f)$ is affiliated with $\mathcal{R}(G)$ just in case $W(f) \in \mathcal{R}(G)$.

The correspondence $G \mapsto \mathcal{R}(G)$ clearly satisfies isotony. That is, if $G_1 \subseteq G_2$ then $\mathcal{R}(G_1) \subseteq \mathcal{R}(G_2)$. Moreover, the local algebras also satisfy fixed-time microcausality. That is, if $G_1 \cap G_2 = \emptyset$ then all operators in $\mathcal{R}(G_1)$ commute with all operators in $\mathcal{R}(G_2)$. (This follows directly from Eq. (4.14) and the fact that $S(G_1)$ and $S(G_2)$ are symplectically orthogonal.)

4.3.1 Anti-locality and the Reeh-Schlieder theorem

Let $\mathcal{R}$ be some subalgebra of $\mathbf{B}(\mathcal{F}(\mathcal{H}))$. We say that a vector $\psi \in \mathcal{F}(\mathcal{H})$ is *cyclic* for $\mathcal{R}$ just in case $[\mathcal{R}\psi] = \mathcal{F}(\mathcal{H})$, where $[\mathcal{R}\psi]$ denotes the closed linear span of $\{A\psi : A \in \mathcal{R}\}$. Of course, every vector in $\mathcal{F}(\mathcal{H})$, including the vacuum vector Ω, is cyclic for the global algebra $\mathbf{B}(\mathcal{F}(\mathcal{H}))$ of all bounded operators on $\mathcal{F}(\mathcal{H})$. The Reeh-Schlieder theorem, however, tells us that the vacuum vector Ω is cyclic for any *local* algebra $\mathcal{R}(G)$.

The first version of the Reeh-Schlieder theorem I will present is a restricted version of the theorem—due to Segal and Goodman—applicable only to the free Bose field model. The key concept in this version of the theorem is the notion of an "anti-local" operator.

Definition. *An operator A on $L_2(\mathbb{R}^3)$ is said to be anti-local just in case: For any $f \in L_2(\mathbb{R}^3)$ and for any open subset G of $\mathbb{R}^3$, $\text{supp}(f) \cap G = \emptyset$ and $\text{supp}(Af) \cap G = \emptyset$ only if $f = 0$.*

Thus, in particular, an anti-local operator maps any wavefunction with support inside a bounded region to a wavefunction with infinite "tails."

The following lemma may be the most important lemma for understanding the local structure of the free Bose field model:

Lemma (Segal and Goodman 1965). *The operator $H = (-\nabla^2 + m^2)^{1/2}$ is anti-local.*

This lemma has the important consequence that for any non-empty open subset G of $\mathbb{R}^3$, the *complex*-linear span of $S(G)$ is dense in $\mathcal{H}$ (cf. Segal and Goodman 1965, Corollary 1). However, for any real-linear subspace E of $\mathcal{H}$, Ω is cyclic for the algebra generated by $\{W(f) : f \in E\}$ if and only if the complex-linear span of E is dense in $\mathcal{H}$ (cf. Petz 1990, Proposition 7.7). Thus, the anti-locality of H entails that Ω is cyclic for every local algebra.

Reeh-Schlieder Theorem. *Let G be any nonempty open subset of $\mathbb{R}^3$. Then, Ω is cyclic for $\mathcal{R}(G)$.*

What is the significance of this cyclicity result? Segal (1964, 140) claims that the theorem is "striking," since it entails that, "...the entire state vector space of the field could be obtained from measurements in an arbitrarily small region of space-time!" He then goes on to claim that the result is, "quite at variance with the spirit of relativistic causality" (143). Fleming also sees the cyclicity result as counterintuitive, apparently because it does not square well with our understanding of relativistic causality. For example (cf. Fleming 2000, 499), the Reeh-Schlieder theorem entails that for any state $\psi \in \mathcal{F}(\mathcal{H})$, and for any predetermined ϵ, there is an operator $A \in \mathcal{R}(G)$ such that $\|A\Omega - \psi\| < \epsilon$. In particular, ψ may be a state that differs from the vacuum only in some region G' that is disjoint (and hence spacelike separated) from G. If, then, A is interpreted as an "operation" that can be performed in the region G, it follows that operations performed in G can result in arbitrary changes of the state in the region G'. This, then, is taken by Fleming to show that, "the local fields allow the possibility of arbitrary space-like distant effects from arbitrary localized actions" (Fleming 2000, 513).

Fleming's use of "actions" and "effects" seems to construe a local operation—represented by an operator $A \in \mathcal{R}(G)$—as a purely *physical* disturbance of the system; i.e., the operation here is a *cause* with an *effect* at spacelike separation. If this were the only way to think of local operations, then I would grant that the Reeh-Schlieder theorem is counterintuitive, and indeed very contrary to the spirit of relativisitic causality. However, once one makes the crucial distinction between selective and nonselective local operations, local cyclicity does not obviously conflict with relativistic causality (see chapter 3, section 2). Rather than dwell on that here, however, I will proceed to spell out some of the further "counterintuitive" consequences of the Reeh-Schlieder theorem.

1. Let G_1 and G_2 be disjoint subsets of $\mathbb{R}^3$. Suppose that $W(f)$ is classically localized in G_1 and $W(g)$ is classically localized in G_2. Then, $\mathrm{Im}(f,g) = 0$ and therefore $W(f)W(g) = W(f+g)$. Thus,

$$\langle \Omega, W(f)W(g)\Omega \rangle = \exp(-\|f+g\|^2/4) \quad (4.19)$$

$$= \langle \Omega, W(f)\Omega \rangle \cdot \langle \Omega, W(g)\Omega \rangle \cdot e^{-\mathrm{Re}(f,g)/2}. \quad (4.20)$$

However, $S(G_1)$ and $S(G_2)$ are not orthogonal relative to the real part of the inner product $(\cdot,\cdot)$. Indeed, if $f = u_0 \oplus u_1$ and $g = v_0 \oplus v_1$, then

$$\begin{aligned} \mathrm{Re}(f,g) &= (u_0, Hv_0) + (u_1, H^{-1}v_1) && (4.21)\\ &= \int_{\mathbb{R}^3} u_0(Hv_0)\, d^3\mathbf{x} + \int_{\mathbb{R}^3} u_1(H^{-1}v_1)\, d^3\mathbf{x}. && (4.22)\end{aligned}$$

But since H and H^{-1} are anti-local, the two integrals in (4.22) will not generally vanish. Therefore, the vacuum state is not a product state across $\mathcal{R}(G_1)$ and $\mathcal{R}(G_2)$.

It should be noted, however, that the above argument does not entail that the vacuum state is "entangled"—since it could still be a *mixture* of product states across $\mathcal{R}(G_1)$ and $\mathcal{R}(G_2)$. However, it can be shown directly from the cyclicity of the vacuum vector Ω that the vacuum state is not even a mixture of product states across $\mathcal{R}(G_1)$ and $\mathcal{R}(G_2)$ (see chapter 2). Moreover, the vacuum predicts a maximal violation of Bell's inequality relative to the algebras $\mathcal{R}(G)$ and $\mathcal{R}(G')$, where $G' = \mathbb{R}^3\backslash G$ (Summers & Werner 1985). (Bell correlation, however, is not entailed by cyclicity; see Clifton, Halvorson, & Kent 2000.)

2. The cyclicity of the vacuum combined with (equal-time) microcausality entails that the vacuum vector is *separating* for any local algebra $\mathcal{R}(G)$, where G' has non-empty interior. That is, for any operator $A \in \mathcal{R}(G)$, if $A\Omega = 0$ then $A = 0$. In particular, for any local event—represented by projection operator $P \in \mathcal{R}(G)$—the probability that event will occur in the vacuum state is nonzero. Thus, the vacuum is "seething with activity" at the local level.

Since the vacuum is entangled across $\mathcal{R}(G)$ and $\mathcal{R}(G')$, it follows that the vacuum is a mixed state when restricted to the local algebra $\mathcal{R}(G)$. In fact, when we restrict the vacuum to $\mathcal{R}(G)$, it is *maximally mixed* in the sense that the vacuum may be written as a mixture with any one of a dense set of states of $\mathcal{R}(G)$ (see chapter 3, page 36). Intuitively speaking, then, the vacuum state provides minimal information about local states of affairs. This is quite similar to the singlet state, which restricts to the maximally mixed state $(1/2)I$ on either one-particle subsystem (cf. Redhead 1995a).

3. For any annihilation operator $a(f)$, we have $a(f)\Omega = 0$. Thus, $a(f)$ cannot be affiliated with the local algebra $\mathcal{R}(G)$. Since the family of operators affiliated with $\mathcal{R}(G)$ is closed under taking adjoints, it also follows that no creation operators are affiliated with $\mathcal{R}(G)$.

The concreteness of the model we are dealing with allows a more direct understanding of why, mathematically speaking, local algebras do not contain creation and annihilation operators. Inverting the relation in (4.13), and using the fact that $f \mapsto a^+(f)$ is linear and $f \mapsto a(f)$ is anti-linear, it follows that

$$a^+(f) = 2^{-1/2}(\Phi(f) - i\Phi(if)), \tag{4.23}$$

$$a(f) = 2^{-1/2}(\Phi(f) + i\Phi(if)). \tag{4.24}$$

Thus, an algebra generated by the operators $\{W(f) : f \in E\}$, will contain the creation and annihilation operators $\{a^+(f), a(f) : f \in E\}$ only if E is a *complex*-linear subspace of $\mathcal{H}$. This is *not* the case for a local algebra $\mathcal{R}(G)$ where $E = S(G)$ is a *real*-linear subspace of $\mathcal{H}$. In fact, referring to the concrete one-particle space $\mathcal{H}_\mu$ allows us to see clearly that $S(G)$ is not invariant under the complex structure J. If $u_0 \oplus u_1 \in S(G)$, then

$$J(u_0 \oplus u_1) = -H^{-1}u_1 \oplus Hu_0. \tag{4.25}$$

But since H and H^{-1} are anti-local, it is not the case that $Hu_0 \in C^\infty(G)$ or $-H^{-1}u_1 \in C^\infty(G)$. Thus, $Jf \notin S(G)$ when $f \in S(G)$. What is more, since the complex span of $S(G)$ is dense in $\mathcal{H}_\mu$, if $S(G)$ *were* a complex subspace, then it would follow that $\mathcal{R}(G) = \mathbf{B}(\mathcal{F}(\mathcal{H}))$.

Number operators also annihilate the vacuum. Since the vacuum is separating for local algebras, no number operator is affiliated with any local algebra. Thus, an observer in the region G cannot count the number of particles in G!

How should we understand the inability of local observers to count the number of particles in their vicinity? According to Redhead (1995b), a heuristic calculation shows that the local number density operator N_G does not commute with the density operator $N_{G'}$ (where G' is the complement of G). Thus, he claims that

> ...it is usual in axiomatic formulations of quantum field theory to impose a microcausality condition on physically significant local observables, *viz* that the associated operators *should* commute at space-like separation. The conclusion of this line of argument is that number densities are not physical observables, and hence we do not have to bother about trying to interpret them. (Redhead 1995b, 81)

While Redhead's conclusion is correct, it is instructive to note that his reasoning cannot be reproduced in a mathematically rigorous fashion. That is, there are *no* local number density operators—in particular, neither N_G nor $N_{G'}$ exist—and so it cannot be literally true that N_G and $N_{G'}$ fail to commute.

In order to see this, consider first the (single wavefunction) number operator $N_f = a^+(f)a(f)$, where f is "classically localized" in G, i.e., $f \in S(G)$. Since $f \mapsto a^+(f)$ is linear, and $f \mapsto a(f)$ is anti-linear, it follows that $N_f = N_{(e^{it}f)}$ for all $t \in \mathbb{R}$. That is, a single wavefunction number operator N_f is invariant under phase tranformations of f. However, classical localization of a wavefunction is *not* invariant under phase transformations. Thus, it is not possible to formulate a well-defined notion of classical localization for a single wavefunction number operator.

How, though, do we define a number density operator N_G? Heuristically, one sets

$$N_G = \int_G N(\mathbf{x}) d^3\mathbf{x}, \tag{4.26}$$

where $N(\mathbf{x}) = a^+(\mathbf{x})a(\mathbf{x})$. Since, however, $N(\mathbf{x})$ is not a well-defined mathematical object, Eq. (4.26) is a purely formal expression. Thus, we replace $N(\mathbf{x})$ with the single wavefunction number operator N_f and we set,

$$N_G = \sum_i N_{f_i} , \tag{4.27}$$

where $\{f_i\}$ is a basis of the real-linear subspace $S(G)$ of $\mathcal{H}$.[6] Using the fact that $N_f = N_{if}$ for each f, it follows then that $N_G = N_{[G]}$, where $N_{[G]}$ is the number operator for the closed complex-linear span $[S(G)]$ of $S(G)$ in $\mathcal{H}$; and the anti-locality of H entails that $[S(G)] = \mathcal{H}$. Therefore, the operator we defined in Eq. (4.27) turns out to be the *total* number operator N.

4. The Reeh-Schlieder theorem also has implications for the *internal* structure of the local algebra $\mathcal{R}(G)$. In particular, the local algebra $\mathcal{R}(G)$ is what is called a "type III" von Neumann algebra (Araki 1964). (The algebra $\mathbf{B}(\mathcal{F}(\mathcal{H}))$ of all bounded operators on $\mathcal{F}(\mathcal{H})$ is called a type I von Neumann algebra.) From a physical point of view, this is significant since type III algebras contain only infinite-dimensional projections—which entails that there are strict limits on our ability to "isolate" a local system from outside influences (see section 3.4). Type III algebras also have *no*

[6] Actually, this infinite sum is also a formal expression, since it sums unbounded operators. A technically correct definition would define N_G as an upper bound of quadratic forms (see Bratteli & Robinson 1996).

pure (normal) states.

4.4 Newton-Wigner localization

In the previous section, we saw that the standard localization scheme $G \mapsto \mathcal{R}(G)$ has a number of "counterintuitive" features, all of which follow from the Reeh-Schlieder theorem. These counterintuitive features prompted Segal (1964) and Fleming (2000) to suggest a reworking of the correspondence between spatial regions and subalgebras of observables. In this section, I give a mathematically rigorous rendering of the Segal-Fleming proposal, and I show how it avoids both the Reeh-Schlieder theorem and its consequences. (Here I deal only with Fleming's first proposal, prior to his generalization to "covariant fields.")

Recall that a localization scheme defines a correspondence between regions in space and real-linear subspaces of the one-particle space $\mathcal{H}$. The Newton-Wigner localization scheme defines this correspondence in precisely the way it is done in elementary quantum mechanics: A region G in $\mathbb{R}^3$ corresponds to the subspace $L_2(G) \subseteq L_2(\mathbb{R}^3)$ of wavefunctions with probability amplitude vanishing (almost everywhere) outside of G. We may then use the unitary mapping V between $\mathcal{H}_\mu$ and $L_2(\mathbb{R}^3)$ to identify the subspace $V^{-1}L_2(G)$ of Newton-Wigner localized wavefunctions in $\mathcal{H}_\mu$. Hereafter, I will suppress reference to V^{-1} and use $L_2(G)$ to denote the pertinent subspace in either concrete version of the one-particle space.

Note that the correspondence $G \mapsto L_2(G)$ is monotone; i.e., if $G_1 \subseteq G_2$ then $L_2(G_1) \subseteq L_2(G_2)$. Moreover, if $G_1 \cap G_2 = \emptyset$, then $L_2(G_1)$ and $L_2(G_2)$ are *fully* orthogonal—a key difference between NW localization and classical localization.

Now, we say that a Weyl operator $W(f)$ acting on $\mathcal{F}(\mathcal{H})$ is *NW-localized* in G just in case $f \in L_2(G)$. We then define the algebra $\mathcal{R}_{NW}(G)$ of NW-localized operators on $\mathcal{F}(\mathcal{H})$ as the von Neumann algebra generated by the Weyl operators NW-localized in G. That is,

$$\mathcal{R}_{NW}(G) = \{W(f) : f \in L_2(G)\}'' . \tag{4.28}$$

Clearly, the correspondence $G \mapsto \mathcal{R}_{NW}(G)$ satisfies isotony. Moreover, since $G_1 \cap G_2 = \emptyset$ entails that $L_2(G_1)$ and $L_2(G_2)$ are orthogonal subspaces of $\mathcal{H}$, the correspondence $G \mapsto \mathcal{R}_{NW}(G)$ satisfies fixed-time microcausality. Thus, at least in this fixed-time formulation, the NW localization scheme appears to have all the advantages of the classical localization scheme. I will now proceed to spell out some features of the NW localization scheme that may make it seem *more* attractive than the standard localization scheme.

If G is an open subset of $\mathbb{R}^3$ and $G' = \mathbb{R}^3 \backslash G$, then

$$L_2(\mathbb{R}^3) = L_2(G \cup G') = L_2(G) \oplus L_2(G'). \tag{4.29}$$

Accordingly, if we let $\mathcal{F}_G = \mathcal{F}(L_2(G))$ and $\mathcal{F}_{G'} = \mathcal{F}(L_2(G'))$ then it follows that

$$\mathcal{F}(\mathcal{H}) \quad = \quad \mathcal{F}_G \otimes \mathcal{F}_{G'}. \tag{4.30}$$

(Here the equality sign is intended to denote that there is a natural isomorphism between $\mathcal{F}(\mathcal{H})$ and $\mathcal{F}_G \otimes \mathcal{F}_{G'}$.) Moreover, the vacuum vector $\Omega \in \mathcal{F}(\mathcal{H})$ is the product $\Omega_G \otimes \Omega_{G'}$ of the respective vacuum vectors in $\mathcal{F}_G$ and $\mathcal{F}_{G'}$. By definition, $\Phi(f)$ is affiliated with $\mathcal{R}_{NW}(G)$ when $f \in L_2(G)$. Since $L_2(G)$ is a complex-linear subspace of $\mathcal{H}$, it follows that $\Phi(if)$ is also affiliated with $\mathcal{R}_{NW}(G)$, and hence that $a^+(f), a(f)$, and N_f are all affiliated with $\mathcal{R}_{NW}(G)$. If we let U denote the unitary operator that maps $\mathcal{F}_G \otimes \mathcal{F}_{G'}$ naturally onto $\mathcal{F}(\mathcal{H})$, then it is not difficult to see that

$$U^{-1} a^+(f) U = a_G^+(f) \otimes I, \tag{4.31}$$

where $a_G^+(f)$ is the creation operator on $\mathcal{F}_G$. Thus, we also have $U^{-1}a(f)U = a_G(f) \otimes I$, and since the creation and annihilation operators $\{a_G^{\pm}(f) : f \in L_2(G)\}$ form an irreducible set of operators on $\mathcal{F}_G$, it follows that

$$\mathcal{R}_{NW}(G) \quad = \quad \mathbf{B}(\mathcal{F}_G) \otimes I, \tag{4.32}$$

$$\mathcal{R}_{NW}(G') \quad = \quad I \otimes \mathbf{B}(\mathcal{F}_{G'}). \tag{4.33}$$

(Again, equality here means there is a natural isomorphism.)

It follows then that acting on $\Omega = \Omega_G \otimes \Omega_{G'}$ with elements from $\mathcal{R}_{NW}(G)$ results only in vectors of the form $\psi \otimes \Omega_{G'}$ for some $\psi \in \mathcal{F}_G$. Thus, the vacuum is *not* cyclic for the local algebra $\mathcal{R}_{NW}(G)$.

1. It is obvious from the preceding that the vacuum is a product state across $\mathcal{R}_{NW}(G)$ and its complement $\mathcal{R}_{NW}(G')$. This also follows directly from the fact that $L_2(G)$ and $L_2(G')$ are fully orthogonal subspaces of $\mathcal{H}$. Indeed, let $W(f) \in \mathcal{R}_{NW}(G)$ and $W(g) \in \mathcal{R}_{NW}(G')$. Then since $\|f+g\|^2 =$

$\|f\|^2 + \|g\|^2$, it follows that

$$\begin{aligned} \langle \Omega, W(f)W(g)\Omega \rangle &= \langle \Omega, W(f+g)\Omega \rangle & (4.34) \\ &= \exp(-\|f+g\|^2/4) & (4.35) \\ &= \langle \Omega, W(f)\Omega \rangle \cdot \langle \Omega, W(g)\Omega \rangle. & (4.36) \end{aligned}$$

2. Restricting the vacuum state Ω to $\mathcal{R}_{NW}(G)$ is equivalent to restricting the product state $\Omega_G \otimes \Omega_{G'}$ to $\mathbf{B}(\mathcal{F}_G) \otimes I$. Thus, the restriction of Ω to $\mathcal{R}_{NW}(G)$ is pure, and the global vacuum provides a "maximally specific" description of local states of affairs.

3. If $\{f_i\}$ is an orthonormal basis of $L_2(G)$, then the number operator $N_G = \sum_i N_{f_i}$ is affiliated with $\mathcal{R}_{NW}(G)$. Moreover, the number operator $N_{G'}$ is affiliated with $\mathcal{R}_{NW}(G')$, and by microcausality we have $[N_G, N_{G'}] = 0$. We may also see this by employing the correspondence between $\mathcal{F}(\mathcal{H})$ and $\mathcal{F}_G \otimes \mathcal{F}_{G'}$. The Fock space $\mathcal{F}_G$ has its own total number operator $\widetilde{N}_G$. Similarly, $\mathcal{F}_{G'}$ has its own total number operator $\widetilde{N}_{G'}$. Obviously then, $\widetilde{N}_G \otimes I$ is affiliated with $\mathbf{B}(\mathcal{F}_G) \otimes I$, and $I \otimes \widetilde{N}_{G'}$ is affiliated with $I \otimes \mathbf{B}(\mathcal{F}_{G'})$. Just as obviously, $\widetilde{N}_G \otimes I$ commutes with $I \otimes \widetilde{N}_{G'}$.

4. As can be seen from Eq. (4.32), the local algebra $\mathcal{R}_{NW}(G)$ is a type I von Neumann algebra. According to Segal (1964, 140), this is precisely the structure of local algebras that is "suggested by considerations of causality and empirical accessibility."

4.5 The full strength of Reeh-Schlieder

The results of the previous two sections speak for themselves: The Newton-Wigner localization scheme results in a mathematical structure that appears to be much more in accord with our a priori physical intuitions than the structure obtained from the standard localization scheme. In this section, however, I show that the NW localization scheme "avoids" the Reeh-Schlieder theorem in only a trivial sense, and I show that the NW localization scheme has its own counterintuitive features without parallel in the standard localization scheme.

First, while the NW-local algebras avoid cyclicity of the vacuum vector, they still have a dense set of cyclic vectors.[7]

[7]Cf. Fleming's claim that, "...it is remarkable that *any state* can have enough struc-

Theorem 4.1. *$\mathcal{R}_{NW}(G)$ has a dense set of cyclic vectors in $\mathcal{F}(\mathcal{H})$.*

Proof. Since the Hilbert spaces $\mathcal{F}_G$ and $\mathcal{F}_{G'}$ have the same (infinite) dimension, it follows from Theorem 4 of (Clifton et al. 1998) that $\mathcal{R}_{NW}(G) = \mathbf{B}(\mathcal{F}_G) \otimes I$ has a dense set of cyclic vectors in $\mathcal{F}(\mathcal{H}) = \mathcal{F}_G \otimes \mathcal{F}_{G'}$. □

Thus, if the worry about the Reeh-Schlieder theorem is about cyclicity in general, adopting the NW localization scheme does nothing to alleviate this worry.

Perhaps, however, the worry about the Reeh-Schlieder theorem is specifically a worry about cyclicity of the *vacuum* state. (One wonders, though, why this would be worse than cyclicity of any other state.) Even so, I argue now that the NW localization scheme does not avoid the "vacuum-specific" consequences of the full Reeh-Schlieder theorem.

Let $\mathcal{K}$ be an arbitrary Hilbert space, representing the state space of some quantum field theory. (For example, $\mathcal{K} = \mathcal{F}(\mathcal{H})$ in the case of the free Bose field.) Suppose also that there is a representation $\mathbf{a} \mapsto U(\mathbf{a})$ of the spacetime translation group in the group of unitary operators on $\mathcal{K}$. Given such a representation, there is a "four operator" $\mathbf{P}$ on $\mathcal{K}$ such that $U(\mathbf{a}) = e^{i\mathbf{a}\cdot\mathbf{P}}$. We say that the representation $\mathbf{a} \mapsto U(\mathbf{a})$ satisfies the *spectrum condition* just in case the spectrum of $\mathbf{P}$ is contained in the forward light cone. From a physical point of view, the spectrum condition corresponds to the assumption that (a) all physical effects propagate at velocities at most the speed of light, and (b) energy is positive. Note, consequently, that the spectrum condition is a purely global condition, and so is not likely to be a source of dispute between proponents of differing localization schemes.

A *net of local observable algebras* is an assignment $O \mapsto \mathcal{A}(O)$ of open regions in Minkowski spacetime to von Neumann subalgebras of $\mathbf{B}(\mathcal{K})$. (Note that this definition is not immediately pertinent to the localization schemes presented in sections 4.3 and 4.4, since they gave an assignment of algebras to open regions in space at a fixed time.) The full Reeh-Schlieder theorem will apply to this net if it satisfies the following postulates:

1. *Isotony:* If $O_1 \subseteq O_2$, then $\mathcal{A}(O_1) \subseteq \mathcal{A}(O_2)$.
2. *Translation Covariance:* $U(\mathbf{a})^{-1}\mathcal{A}(O)U(\mathbf{a}) = \mathcal{A}(O + \mathbf{a})$.

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ture within an arbitrarily small region, O, to enable even the mathematical reconstituting of essentially the whole state space" (Fleming 2000, 499).

3. *Weak Additivity:* For any open $O \subseteq M$, the set

$$\bigcup_{\mathbf{a}\in M} \mathcal{A}(O+\mathbf{a})$$

of operators is irreducible (i.e., leaves no subspace of $\mathcal{K}$ invariant).

In this general setting, a vacuum vector Ω can be taken to be any vector invariant under all spacetime translations $U(\mathbf{a})$.

Full Reeh-Schlieder Theorem. *Suppose that $\{\mathcal{A}(O)\}$ is a net of local observable algebras satisfying postulates 1–3. Then, for any open region O in Minkowski spacetime, Ω is cyclic for $\mathcal{A}(O)$.*

Note that the Reeh-Schlieder theorem does *not* require the postulate of microcausality (i.e., if $A \in \mathcal{A}(O_1)$ and $B \in \mathcal{A}(O_2)$, where O_1 and O_2 are spacelike separated, then $[A, B] = 0$).[8]

For the standard localization scheme, there is a straightforward connection between the full Reeh-Schlieder theorem and the fixed-time version given in section 4.3. In particular, there is an alternative method for describing the standard localization scheme that involves appeal to spacetime regions rather than space regions at a fixed time (see Horuzhy 1988, Chapter 4). It then follows that $\mathcal{R}(G) = \mathcal{A}(O_G)$, where O_G is the "domain of dependence" of the spatial region G. Thus, the fixed-time version of the Reeh-Schlieder theorem may be thought of as corollary of the full Reeh-Schlieder theorem in connection with the fact that $\mathcal{R}(G) = \mathcal{A}(O_G)$.

Segal and Fleming avoid the fully general version of the Reeh-Schlieder theorem only by remaining silent about how we ought to assign algebras of observables to open regions of *spacetime*.[9] Since, however, the typical quantum field theory cannot be expected to admit a fixed-time (3+1) formulation (cf. Haag 1992, 59), it is not at all clear that they have truly avoided the Reeh-Schlieder theorem in any interesting sense. It would certainly be interesting to see which, *if any*, of the full Reeh-Schlieder theorem's three premises would be rejected by a more general NW localization scheme.

However, we need not speculate about the possibility that the full Reeh-Schlieder theorem will apply to some generalization of NW localization

[8] To see that microcausality is logically independent from postulates 1–3, take the trivial localization scheme: $\mathcal{A}(O) = \mathbf{B}(\mathcal{K})$, for each O.

[9] It is essential for the proof of the full Reeh-Schlieder theorem that the region O has some "temporal extension": The theorem uses the fact that if $A \in \mathcal{A}(O_1)$ where $O_1 \subset O$, then $U(\mathbf{a})^{-1}AU(\mathbf{a}) \in \mathcal{A}(O)$ for sufficiently small $\mathbf{a}$ in four independent directions.

scheme: The Reeh-Schlieder theorem already has "counterintuitive" consequences for the fixed-time NW localization scheme. In particular, although the vacuum Ω is not cyclic under operations NW-localized in some spatial region G at a single time, Ω *is* cyclic under operators NW-localized in G *within an arbitrary short time interval.* Before I give the precise version of this result, I should clarify some matters concerning the relationship between the dynamics of the field and local algebras.

In the standard localization scheme, the dynamics of local algebras may be thought of two ways. On the one hand, we may think of the assignment $G \mapsto \mathcal{R}(G)$ as telling us, once and for all, which observables are associated with the region G, in which case the state of $\mathcal{R}(G)$ (i.e., the reduced state of the entire field) changes via the unitary evolution $U(t)$ (Schrödinger picture). On the other hand, we may think of the state of the field as fixed, in which case the algebra $\mathcal{R}(G)$ evolves over time to the algebra $U(t)^{-1}\mathcal{R}(G)U(t)$ (Heisenberg picture). Thus, $U(t)^{-1}\mathcal{R}(G)U(t)$ gives those operators classically localized in G at time t. The Schrödinger picture is particularly intuitive in this case, since it mimics the dynamics of a classical field where quantities associated with points in space change their values over time.

Now, neither Segal nor Fleming explain how we should think of the dynamics of the NW-local algebras. Presumably, however, we are to think of the dynamics of the NW-local algebras in precisely the same way as we think of the dynamics of the standard local algebras.[10] In particular, we may suppose that the state of the field is, at all times, the vacuum state Ω, and that $U(t)^{-1}\mathcal{R}_{NW}(G)U(t)$ gives those operators NW-localized in G at time t.

Now for any $\Delta \subseteq \mathbb{R}$ let

$$\mathbf{S}_{\Delta} = \{U(t)^{-1}AU(t) : A \in \mathcal{R}_{NW}(G),\, t \in \Delta\}\,. \tag{4.37}$$

That is, $\mathbf{S}_{\Delta}$ consists of those operators NW-localized in G at some time $t \in \Delta$.

Theorem 4.2. *For any interval (a,b) around 0, Ω is cyclic for $\mathbf{S}_{(a,b)}$.*

Sketch of proof: Let $[\mathbf{S}_{(a,b)}\Omega]$ denote the closed linear span of $\{A\Omega : A \in$

[10]It is conceivable that Segal or Fleming have some different idea concerning the relationship between NW-local algebras at different times. For example, perhaps even in the Schrödinger picture, the map $G \mapsto \mathcal{R}_{NW}(G)$ should be thought of as time-dependent. Although this is surely a formal possibility, it is exceedingly difficult to understand what it might mean, physically, to have a time-dependent association of physical magnitudes with regions in space.

$\mathbf{S}_{(a,b)}\}$. Since the infinitesimal generator $d\Gamma(H)$ of the group $U(t)$ is positive, Kadison's "little Reeh-Schlieder theorem" (1970) entails that $[\mathbf{S}_{(a,b)}\Omega] = [\mathbf{S}_{\mathbb{R}}\Omega]$. However, $[\mathbf{S}_{\mathbb{R}}\Omega] = \mathcal{F}(\mathcal{H})$; i.e., Ω is cyclic under operators NW-localized in G over all times (Segal 1964, 143). Therefore, Ω is cyclic for $\mathbf{S}_{(a,b)}$. □

In Fleming's language, then, the NW-local fields "allow the possibility of arbitrary space-like distant effects" from actions localized in an arbitrarily small region of space over an arbitrarily short period of time. Is this any less "counterintuitive" than the instantaneous version of the Reeh-Schlieder theorem for the standard localization scheme?[11]

Finally, we are in a position to see explicitly a "counterintuitive" feature of the NW localization scheme that is not shared by the standard localization scheme: NW-local operators fail to commute at spacelike separation. For this, choose mutually disjoint regions G_1 and G_2 in $\mathbb{R}^3$, and choose an interval (a,b) around 0 so that $O_1 := \bigcup_{t\in(a,b)}(G_1+t)$ and $O_2 := \bigcup_{t\in(a,b)}(G_2+t)$ are spacelike separated. Let $\mathcal{A}_{NW}(O_i)$ be the von Neumann algebra generated by

$$\bigcup_{t\in(a,b)} U(t)^{-1}\mathcal{R}_{NW}(G_i)U(t). \tag{4.38}$$

Then it follows from Theorem 2 that the vacuum is cyclic for $\mathcal{A}_{NW}(O_2)$. However, since $\mathcal{A}_{NW}(O_1) \supseteq \mathcal{R}_{NW}(G)$ contains annihilation operators and number operators, it follows that $\mathcal{A}_{NW}(O_1)$ and $\mathcal{A}_{NW}(O_2)$ do not satisfy microcausality. (Microcausality, in conjunction with cyclicity of the vacuum vector, would entail that the vacuum vector is separating.) More specifically, while the algebras $U(t)^{-1}\mathcal{R}_{NW}(G_1)U(t)$ and $U(t)^{-1}\mathcal{R}_{NW}(G_2)U(t)$ do satisfy microcausality for any fixed t, microcausality does not generally hold for the algebras $U(t)^{-1}\mathcal{R}_{NW}(G_1)U(t)$ and $U(s)^{-1}\mathcal{R}_{NW}(G_2)U(s)$ when $t \neq s$ (despite the fact that G_1+t and G_2+s are spacelike separated).

It would be naive at this stage to claim that failure of generalized microcausality provides a simple reductio on the NW localization scheme. As I will argue in the next section, however, the failure of generalized microcausality for the NW-local algebras leaves little room for making any physical sense of the NW localization scheme.

[11]One may, however, reject the interpretation of elements of $\mathcal{R}_{NW}(G)$ as operations that can be performed in G. I return to this point in the next section.

4.6 Local properties and local measurements

Mathematically speaking, there is no limit to the number of ways we could associate operators with subsets of a spacetime manifold. But when does such an association have physical significance, or a natural physical interpretation? In other words, when does a mathematical relation, such as $A \in \mathcal{R}(G)$, correspond to some physical relation of "localization" between the corresponding observable and region of space? The standard localization scheme was originally introduced with the explicit intention that the mathematical relation $A \in \mathcal{R}(G)$ should denote that the observable represented by A is measurable in the region of space denoted by G. On the other hand, advocates of the NW localization scheme have not been uniformly clear concerning its intended physical significance. In this section, I will argue that advocates of the NW localization scheme are impaled on the horns of a dilemma: Either $A \in \mathcal{R}_{NW}(G)$ entails that A is measurable in G, in which case the NW localization scheme predicts act-outcome correlations at spacelike separation, or the NW localization scheme is a formal recipe without physical significance.

Note first that if $A \in \mathcal{R}_{NW}(G)$ entails that A is measurable in G, then the NW localization scheme is *empirically inequivalent* to the standard localization scheme. [For example, the vacuum displays Bell correlations relative to the algebras $\mathcal{R}(G)$ and $\mathcal{R}(G')$, while the vacuum is a product state across $\mathcal{R}_{NW}(G)$ and $\mathcal{R}_{NW}(G')$.] Segal is clear that he is willing to accept this consequence, and indeed, he believes the NW localization scheme gives a more accurate account of what is locally measurable. He says,

> From an operational viewpoint it is these variables [i.e., $\Phi(f)$ with $f \in L_2(G)$] ... that appear as the localized field variables, and the ring $\mathcal{R}_{NW}(G)$... appears as the appropriate ring of local field observables, rather than the ring $\mathcal{R}(G)$.... (Segal 1964, 142; notation adapted)

However, if A and B are two observables that do not commute, then a standard von Neumann measurement of A can alter the statistics for measurement outcomes of B. As a result, the failure of generalized microcausality (i.e., commutation at spacelike separation) for NW local algebras entails the possibility of act-outcome correlations at spacelike separation.[12] Thus, when

[12] Although I lack direct historical evidence, it appears that Segal eventually abandoned the NW localization scheme due to the conflict with relativistic causality (cf. Baez, Segal, & Zhou 1992, 173).

equipped with the local measurability interpretation, the NW localization scheme appears to be inconsistent with special relativity.

Although Fleming argues for the "physical significance" of the NW localization scheme, he does not put it forward as a replacement for, or competitor to, the standard localization scheme:

> How shall we choose between these perspectives? We need not choose and we should not. Rather, wisdom lies in exploring the implications and the subtler details of the interpretation of both perspectives. (Fleming 2000, 513)

Since the two localization schemes are empirically *in*equivalent, when both are interpreted in terms of local measurability, Fleming must eschew the claim that elements of NW local algebras are locally measurable. Indeed, Fleming notes elsewhere that

> . . . one naturally assumes that one can interpret the *association* of an operator with a spacetime region as implying that one can *measure it by performing operations confined* to that region,

but he goes on to "question [this] interpretive assumption" (Fleming & Butterfield 1999, 158–159). How then *does* Fleming interpret the association of an observable with a region in space? That is, what does he mean by saying that an observable is localized in a region of space?

In his explanation of NW-localization, Fleming refers to the NW position operator (which, in the case of the free Bose field, is identical to the center of energy position operator). He argues that,

> . . . HD [hyperplane dependent] position operators, such as the general CE [center of energy] and the general NW position operators, are more closely related than the local field coordinate to assessments of *where*, on hyperplanes and in space-time, objects, systems, their localizable properties and phenomena are located. (Fleming 2000, 514)

However, the NW position operator is not contained in any NW local algebra, and there is no natural correspondence between the spectral projections of the NW position operator and the NW local algebras.[13] Thus, even if we were to concede that the NW position operator has "unequivocal physical

[13] Suppose that G_1 and G_2 are disjoint. Then, no pair of non-trivial projections from $\mathcal{R}_{NW}(G_1)$ and $\mathcal{R}_{NW}(G_2)$ is orthogonal (cf. Eqs. (4.32) and (4.33)).

significance," this would not appear to clarify the physical significance of NW local algebras.

Perhaps, however, the physical significance of the NW local algebras can be derived from their relationship to the relevant number operators. In particular, the NW number operator N_G is affiliated with $\mathcal{R}_{NW}(G)$; and, as a result, the projection P onto the complement of the nullspace of N_G is contained in $\mathcal{R}_{NW}(G)$. Now, according to the advocate of NW localization, P represents that property possessed by the system iff. there are particles in G. Thus, it would seem reasonable to say that P represents a property that is localized in G, and, by extension, that any projection operator in $\mathcal{R}_{NW}(G)$ represents a property that is localized in G.

Despite the shift in emphasis to "properties," this interpretation of the NW localization scheme does not differ from the interpretation of the standard localization scheme. Indeed, the standard localization scheme also says that elements of $\mathcal{R}(G)$ correspond to properties that are localized in G. The only difference between the two cases is that the standard localization scheme defines the relation "is localized in" in terms of the (more fundamental) relation "is measurable in," whereas Fleming appears to take the localization relation to be primitive.

However, if localization is a primitive relation, it is not obvious why we should think it coincides with the assignments made by the NW localization scheme. In particular, let Σ be some spacelike hypersurface in Minkowski spacetime, and let h be a symmetry of Σ. Let $U(h)$ denote the unitary transformation of $\mathcal{F}(\mathcal{H})$ induced by h, and let $\widetilde{\mathcal{R}}_{NW}(G) = U(h)^{-1}\mathcal{R}_{NW}(G)U(h)$. Then $\mathcal{R}_{NW}(G)$ and $\widetilde{\mathcal{R}}_{NW}(G)$ are identical in their formal properties, and thus have, prima facie, an equal claim as descriptions of which properties are localized in G. Thus, it is incumbent upon Fleming to describe some relevant difference between the two algebras.

To clarify this point further, consider the analogous situation of a spatially extended, classical system (cf. Fleming 2000, 507). Let C denote the center of energy of the system. Then, in each state of the system (i.e., at each time) C may be identified with some point x in the hypersurface Σ. For each $x \in \Sigma$, let $P(x) = 1$ if $C = x$, and let $P(x) = 0$ otherwise. Then, $P(x)$ represents that property possessed by the system iff. the center of energy is x. Thus, we might wish to infer that $P(x)$ represents a property that is localized at x. Suppose, however, that we are given some symmetry h of Σ. Let $\widetilde{P}(x) = 1$ if $C = h^{-1}(x)$, and let $\widetilde{P}(x) = 0$ otherwise. Then, $\widetilde{P}(x)$ represents that property possessed by the system iff. the quantity $\widetilde{C} \equiv h(C)$ takes the value x. Applying the same reasoning we used to conclude that $P(x)$ is localized at x, it follows that $\widetilde{P}(x) = P(h^{-1}(x))$ is localized at x.

Since $h^{-1}(x)$ could be any point of Σ, the argument for the claim that $P(x)$ is localized at x is clearly invalid.

In fact, it is only in cases where we have locally measurable quantities that we can resolve the arbitrariness introduced by the possibility of shifting quantities from point to point (or from region to region). For example, let X denote the position observable of a classical point particle. Let $P(x)$ denote that property possessed by the system iff. $X = x$, and let $\widetilde{P}(x)$ denote that property possessed by the system iff. $X = h^{-1}(x)$. Then $P(x)$, but not $\widetilde{P}(x)$, is measurable at x. Thus, there is a significant difference between these two ways of associating quantities with points. On the other hand, in the center of energy example, neither $P(x)$ nor $\widetilde{P}(x)$ is measurable at x. Thus, there are no relevant grounds for favoring one of the two associations between quantities and points.

To sum up: In the absence of some other criterion for distinguishing NW local algebras, we must conclude that either the NW localization scheme is arbitrary, or $A \in \mathcal{R}_{NW}(G)$ entails that A is measurable in G. However, if $A \in \mathcal{R}_{NW}(G)$ entails that A is measurable in G, then the NW localization scheme predicts the possibility of act-outcome correlations at spacelike separation. Therefore, the NW localization scheme is either incurably arbitrary, or is inconsistent with special relativity.

4.7 Conclusion

Introduction of the NW localization scheme into quantum field theory was an ingenious move. By means of one deft transformation, it appears to thwart the Reeh-Schlieder theorem and to restore the "intuitive" picture of localization from non-relativistic quantum mechanics. However, there are many reasons to doubt that Newton-Wigner has truly spared us of the counterintuitive consequences of the Reeh-Schlieder theorem. First, NW-local algebras still have a dense set of cyclic vectors. Second, since general quantum field theories cannot be expected to admit a fixed-time formulation, it is not clear that the NW localization scheme has any interesting level of generality. Third, NW-local operations on the vacuum over an arbitrarily short period of time do generate the state space of the entire field. And, finally, the failure of generalized microcausality for the NW local algebras entails the possibility of act-outcome correlations at spacelike separation.

After showing that the Reeh-Schlieder theorem fails for NW-local algebras, Fleming (2000, 505) states that, "Now it is clear why it would be worthwhile to see the NW fields as covariant structures." While there may be very good reasons for seeing the NW fields as covariant structures, avoiding the Reeh-Schlieder theorem is *not* one of them.

Chapter 5

No place for particles in relativistic quantum theories?

5.1 Introduction

It is a widespread belief, at least within the physics community, that there is no particle mechanics that is simultaneously relativistic and quantum-theoretic; and, thus, that the only relativistic quantum theory is a *field* theory. This belief has received much support in recent years in the form of rigorous "no-go theorems" by Malament (1996) and Hegerfeldt (1998a, 1998b). In particular, Hegerfeldt shows that in a generic quantum theory (relativistic or non-relativistic), if there are states with localized particles, and if there is a lower bound on the system's energy, then superluminal spreading of the wavefunction must occur. Similarly, Malament shows the inconsistency of a few intuitive desiderata for a relativistic, quantum-mechanical theory of (localizable) particles. Thus, it appears that there is a fundamental conflict between the demands of relativistic causality and the requirements of a theory of localizable particles.

What is the philosophical lesson of this apparent conflict between relativistic causality and localizability? One the one hand, if we believe that the assumptions of Malament's theorem must hold for any theory that is descriptive of our world, then it follows that our world cannot be correctly described by a particle theory. On the other hand, if we believe that our world *can* be correctly described by a particle theory, then one (or more) of Malament's assumptions must be false. Malament clearly endorses the first response; that is, he argues that his theorem entails that there is no relativistic quantum mechanics of localizable particles (insofar as any relativistic theory precludes act-outcome correlations at spacelike separation). Others, however, have argued that the assumptions of Malament's theorem need not hold for any relativistic, quantum-mechanical theory (cf. Fleming & Butterfield 1999), or that we cannot judge the truth of the assumptions until we resolve the interpretive issues of elementary quantum mechanics (cf. Barrett 2001).

Although we do not think that these arguments against Malament's assumptions succeed, there are other reasons to doubt that Malament's theorem is sufficient to support a sound argument against the possibility of a relativistic quantum mechanics of localizable particles. First, Malament's theorem depends on a specific assumption about the structure of Minkowski spacetime—a "no preferred reference frame" assumption—that could be seen as having less than full empirical warrant. Second, Malament's theorem establishes only that there is no relativistic quantum mechanics in which particles can be completely localized in spatial regions with sharp boundaries; it leaves open the possibility that there might be a relativistic quantum mechanics of "unsharply" localized particles. In this paper, we present two new no-go theorems which, together, suffice to close these loopholes in the argument against relativistic quantum mechanics. First, we present a new no-go theorem that generalizes some of the aspects of Malament's and Hegerfeldt's theorems, and which does not depend on the "no preferred frame" assumption (Theorem 5.1). Second, we derive a generalized version of Malament's theorem that shows that there is no relativistic quantum mechanics of "unsharply" localized particles (Theorem 5.2).

However, it would be a mistake to think that these result show—or, are intended to show—that a field ontology, rather than a particle ontology, is appropriate for relativistic quantum theories. While these results show that there are no position observables that satisfy certain relativistic constraints, quantum field theories—both relativistic *and* non-relativistic—already reject the notion of position observables in favor of "localized" field observables. Thus, no-go results against relativistic position operators have nothing to say about the possibility that relativistic quantum field theory might permit a "particle interpretation," in which localized particles are supervenient on the underlying localized field observables. To exclude this latter possibility, we formulate (in section 5.6) a necessary condition for a generic quantum theory to permit a particle interpretation, and we then show that this condition fails in *any* relativistic theory (Theorem 5.3).

Since *our world* is presumably both relativistic and quantum-theoretic, these results show that there are no localizable particles. However, in section 5.7 we shall argue that relativistic quantum field theory itself warrants an approximate use of "particle talk" that is sufficient to save the phenomena.

5.2 Malament's theorem

Malament's theorem shows the inconsistency of a few intuitive desiderata for a relativistic quantum mechanics of (localizable) particles. It strengthens previous results (e.g., Schlieder 1971) by showing that the assumption of "no superluminal wavepacket spreading" can be replaced by the weaker assumption of "microcausality," and by making it clear that Lorentz invariance is not needed to derive a conflict between relativistic causality and localizability.

In order to present Malament's result, we assume that our background spacetime M is an affine space, with a foliation $\mathcal{S}$ into spatial hyperplanes. (For ease, we can think of an affine space as a vector space, so long as we do not assign any physical significance to the origin.) This will permit us to consider a wide range of relativistic (e.g., Minkowski) as well as non-relativistic (e.g., Galilean) spacetimes. The pure states of our quantum-mechanical system are given by rays in some Hilbert space $\mathcal{H}$. We assume that there is a mapping $\Delta \mapsto E_\Delta$ of *bounded* subsets of hyperplanes in M into projections on $\mathcal{H}$. We think of E_Δ as representing the proposition that the particle is localized in Δ; or, from a more operational point of view, E_Δ represents the proposition that a position measurement is certain to find the particle within Δ. We also assume that there is a strongly continuous representation $\mathbf{a} \mapsto U(\mathbf{a})$ of the translation group of M in the unitary operators on $\mathcal{H}$. Here strong continuity means that for any unit vector $\psi \in \mathcal{H}$, $\langle \psi, U(\mathbf{a})\psi \rangle \to 1$ as $\mathbf{a} \to 0$; and it is equivalent (via Stone's theorem) to the assumption that there are energy and momentum observables for the particle. If all of the preceding conditions hold, we say that the triple $(\mathcal{H}, \Delta \mapsto E_\Delta, \mathbf{a} \mapsto U(\mathbf{a}))$ is a *localization system* over M.

The following conditions should hold for any localization system—either relativistic or non-relativistic—that describes a single particle.

Localizability: If Δ and Δ' are disjoint subsets of a single hyperplane, then $E_\Delta E_{\Delta'} = 0$.

Translation covariance: For any Δ and for any translation $\mathbf{a}$ of M, $U(\mathbf{a}) E_\Delta U(\mathbf{a})^* = E_{\Delta + \mathbf{a}}$.

Energy bounded below: For any timelike translation $\mathbf{a}$ of M, the generator $H(\mathbf{a})$ of the one-parameter group $\{U(t\mathbf{a}) : t \in \mathbb{R}\}$ has a spectrum bounded from below.

We recall briefly the motivation for each of these conditions. "Localizability" says that the particle cannot be detected in two disjoint spatial sets at a given time. "Translation covariance" gives us a connection between the symmetries of the spacetime M and the symmetries of the quantum-mechanical system. In particular, if we displace the particle by a spatial translation $\mathbf{a}$, then the original wavefunction ψ will transform to some wavefunction $\psi_{\mathbf{a}}$. Since the statistics for the displaced detection experiment should be identical to the original statistics, we have $\langle \psi, E_\Delta \psi \rangle = \langle \psi_{\mathbf{a}}, E_{\Delta+\mathbf{a}} \psi_{\mathbf{a}} \rangle$. By Wigner's theorem, however, the symmetry is implemented by some unitary operator $U(\mathbf{a})$. Thus, $U(\mathbf{a})\psi = \psi_{\mathbf{a}}$, and $U(\mathbf{a}) E_\Delta U(\mathbf{a})^* = E_{\Delta+\mathbf{a}}$. In the case of time translations, the covariance condition entails that the particle has unitary dynamics. (This might seem to beg the question against a collapse interpretation of quantum mechanics; we dispell this worry at the end of this section.) Finally, the "energy bounded below" condition asserts that, relative to any free-falling observer, the particle has a lowest possible energy state. If it were to fail, we could extract an arbitrarily large amount of energy from the particle as it drops down through lower and lower states of energy.

We now turn to the "specifically relativistic" assumptions needed for Malament's theorem. The special theory of relativity entails that there is a finite upper bound on the speed at which (detectable) physical disturbances can propagate through space. Thus, if Δ and Δ' are distant regions of space, then there is a positive lower bound on the amount of time it should take for a particle localized in Δ to travel to Δ'. We can formulate this requirement precisely by saying that for any timelike translation $\mathbf{a}$, there is an $\epsilon > 0$ such that, for every state ψ, if $\langle \psi, E_\Delta \psi \rangle = 1$ then $\langle \psi, E_{\Delta'+t\mathbf{a}} \psi \rangle = 0$ whenever $0 \leq t < \epsilon$. This is equivalent to the following assumption.

Strong causality: If Δ and Δ' are disjoint subsets of a single hyperplane, and if the distance between Δ and Δ' is nonzero, then for any timelike translation $\mathbf{a}$, there is an $\epsilon > 0$ such that $E_\Delta E_{\Delta'+t\mathbf{a}} = 0$ whenever $0 \leq t < \epsilon$.

(Note that strong causality entails localizability.) Although strong causality is a reasonable condition for relativistic theories, Malament's theorem requires only the following weaker assumption (which he himself calls "locality").

Microcausality: If Δ and Δ' are disjoint subsets of a single hyperplane, and if the distance between Δ and Δ' is nonzero, then for any timelike translation $\mathbf{a}$, there is an $\epsilon > 0$ such that $[E_\Delta, E_{\Delta'+t\mathbf{a}}] = 0$ whenever $0 \leq t < \epsilon$.

If E_Δ can be measured within Δ, microcausality is equivalent to the assumption that a measurement within Δ cannot influence the statistics of measurements performed in regions that are spacelike to Δ (see Malament 1996, 5). Conversely, a failure of microcausality would entail the possibility of act-outcome correlations at spacelike separation. Note that both strong and weak causality make sense for non-relativistic spacetimes (as well as for relativistic spacetimes); though, of course, we should not expect either causality condition to hold in the non-relativistic case.

Theorem (Malament). *Let* $(\mathcal{H}, \Delta \mapsto E_\Delta, \mathbf{a} \mapsto U(\mathbf{a}))$ *be a localization system over Minkowski spacetime that satisfies:*

1. *Localizability*
2. *Translation covariance*
3. *Energy bounded below*
4. *Microcausality*

Then $E_\Delta = 0$ *for all* Δ*.*

Thus, in every state, there is no chance that the particle will be detected in any local region of space. As Malament claims, this serves as a *reductio ad absurdum* of any relativistic quantum mechanics of a single (localizable) particle.

5.2.1 Malament's critics

Several authors have claimed that Malament's theorem is not sufficient to rule out a relativistic quantum mechanics of localizable particles. In particular, these authors argue that it is not reasonable to expect the conditions of Malament's theorem to hold for any relativistic, quantum-mechanical theory of particles. For example, Dickson (1997) argues that a 'quantum' theory does not need a position *operator* (equivalently, a system of localizing projections) in order to treat position as a physical quantity; Barrett (2001) argues that time-translation covariance is suspect; and Fleming and Butterfield (1999) argue that the microcausality assumption is not warranted by special relativity. We now show, however, that none of these arguments is decisive against the assumptions of Malament's theorem.

Dickson (1997, 214) cites the Bohmian interpretation of the Dirac equation as a counterexample to the claim that any 'quantum' theory must represent position by an operator. In order to see what Dickson might mean by this, recall that the Dirac equation admits both positive and negative energy solutions. If $\mathcal{H}$ denotes the Hilbert space of all (both positive and negative energy) solutions, then we may define the 'standard position operator' Q by setting $Q\psi(\mathbf{x}) = \mathbf{x} \cdot \psi(\mathbf{x})$ (Thaller 1992, 7). If, however, we restrict to the Hilbert space $\mathcal{H}_{\text{pos}} \subset \mathcal{H}$ of positive energy solutions, then the probability density given by the Dirac wavefunction does not correspond to a self-adjoint position operator (Thaller 1992, 32). According to Holland (1993, 502), this lack of a position operator on $\mathcal{H}_{\text{pos}}$ precludes a Bohmian interpretation of $\psi(\mathbf{x})$ as a probability amplitude for finding the particle in an elementary volume $d^3\mathbf{x}$ around $\mathbf{x}$.

Since the Bohmian interpretation of the Dirac equation uses all states (both positive and negative energy), and the corresponding position observable Q, it is not clear what Dickson means by saying that the Bohmian interpretation of the Dirac equation dispenses with a position observable. Moreover, since the energy is not bounded below in $\mathcal{H}$, this would not in any case give us a counterexample to Malament's theorem. However, Dickson could have developed his argument by appealing to the positive energy subspace $\mathcal{H}_{\text{pos}}$. In this case, we *can* talk about positions despite the fact that we do not have a position observable in the usual sense. In particular, we shall show in section 5.5 that, for talk about positions, it suffices to have a family of "unsharp" localization observables. (And, yet, we shall show that relativistic quantum theories do not permit even this attenuated notion of localization.)

Barrett (2001) argues that the significance of Malament's theorem cannot be assessed until we have solved the measurement problem:

> If we might have to violate the apparently weak and obvious assumptions that go into proving Malament's theorem in order to get a satisfactory solution to the measurement problem, then all bets are off concerning the applicability of the theorem to the detectible entities that inhabit our world. (Barrett 2001, 16)

In particular, a solution to the measurement problem may require that we abandon unitary dynamics. But if we abandon unitary dynamics, then the translation covariance condition does not hold, and we need not accept the conclusion that there is no relativistic quantum mechanics of (localizable) particles.

Unfortunately, it is not clear that we could avoid the upshot of Malament's theorem by moving to a collapse theory. Existing (non-relativistic) collapse theories take the empirical predictions of quantum theory seriously. That is, the "statistical algorithm" of quantum mechanics is assumed to be at least approximately correct; and collapse is introduced only to ensure that we obtain determinate properties at the end of a measurement. However, in the present case, Malament's theorem shows that the statistical algorithm of any quantum theory predicts that if there are local particle detections, then act-outcome correlations are possible at spacelike separation. Thus, if a collapse theory is to stay close to these predictions, it too would face a conflict between localizability and relativistic causality.

Perhaps, then, Barrett is suggesting that the price of accomodating localizable particles might be a complete abandonment of unitary dynamics, *even at the level of a single particle*. In other words, we may be forced to adopt a collapse theory *without* having any underlying (unitary) quantum theory. But even if this is correct, it wouldn't count against Malament's theorem, which was intended to show that there is no relativistic *quantum* theory of localizable particles. Furthermore, noting that Malament's theorem requires unitary dynamics is one thing; it would be quite another thing to provide a model in which there *are* localizable particles—at the price of non-unitary dynamics—but which is also capable of reproducing the well-confirmed quantum interference effects at the micro-level. Until we have such a model, pinning our hopes for localizable particles on a failure of unitary dynamics is little more than wishful thinking.

Like Barrett, Fleming (Fleming & Butterfield 1999, 158ff) disagrees with the reasonableness of Malament's assumptions. Unlike Barrett, however, Fleming provides a concrete model in which there are localizable particles (viz., using the Newton-Wigner position operator as a localizing observable) and in which Malament's microcausality assumption fails. Nonetheless, Fleming argues that this failure of microcausality is perfectly consistent with relativistic causality.

According to Fleming, the property "localized in Δ" (represented by E_Δ) need not be detectable within Δ. As a result, $[E_\Delta, E_{\Delta'}] \neq 0$ does not entail that it is possible to send a signal from Δ to Δ'. However, by claiming that local *beables* need not be local *observables*, Fleming undercuts the primary utility of the notion of localization, which is to indicate those physical quantities that are operationally accessible in a given region of spacetime. Indeed, it is not clear what motivation there could be—aside from indicating what is locally measurable—for assigning observables to spatial regions. If E_Δ is *not* measurable in Δ, then why should we say that "E_Δ is localized in Δ"? Why not say instead that "E_Δ is localized in Δ'" (where $\Delta' \neq \Delta$)? Does either statement have any empirical consequences and, if so, how do their empirical consequences differ? Until these questions are answered, we maintain that local beables are always local observables; and a failure of microcausality *would* entail the possibility of act-outcome correlations at spacelike separation. Therefore, the microcausality assumption is an essential feature of any relativistic quantum theory with "localized" observables. (For a more detailed argument along these lines, see section 4.6.)

Thus, the arguments against the four (explicit) assumptions of Malament's theorem are unsuccessful; these assumptions are perfectly reasonable, and we should expect them to hold for any relativistic, quantum-mechanical theory. However, there is another difficulty with the argument against any relativistic quantum mechanics of (localizable) particles: Malament's theorem makes *tacit* use of specific features of Minkowski spacetime which—some might claim—have less than perfect empirical support. First, the following example shows that Malament's theorem fails if there is a preferred reference frame.

Example 1. Let $M = \mathbb{R}^1 \oplus \mathbb{R}^3$ be full Newtonian spacetime (with a distinguished timelike direction $\mathbf{a}$). To any set of the form $\{(t,x) : x \in \Delta\}$, with $t \in \mathbb{R}$, and Δ a bounded open subset of $\mathbb{R}^3$, we assign the spectral projection E_Δ of the position operator for a particle in three dimensions. Let $H(\mathbf{a}) = 0$ so that $U(t\mathbf{a}) = e^{it0} = I$ for all $t \in \mathbb{R}$. Since the energy in every state is zero, the energy condition is trivially satisfied.

Note, however, that if the background spacetime is *not* regarded as having a distinguished timelike direction, then this example violates the energy condition. Indeed, the generator of an arbitrary timelike translation has the form

$$H(\mathbf{b}) \;=\; \mathbf{b}\cdot\mathbf{P} \;=\; b_0 0 + b_1 P_1 + b_2 P_2 + b_3 P_3 \;=\; b_1 P_1 + b_2 P_2 + b_3 P_3, \quad (5.1)$$

where $\mathbf{b} = (b_0, b_1, b_2, b_3) \in \mathbb{R}^4$ is a timelike vector, and P_i are the three orthogonal components of the total momentum. But since each P_i has spectrum $\mathbb{R}$, the spectrum of $H(\mathbf{b})$ is *not* bounded from below when $\mathbf{b}$ is not a scalar multiple of $\mathbf{a}$. □

Malament's theorem does not require the full structure of Minkowski spacetime (e.g., the Lorentz group). Rather, it suffices to assume that the affine space M satisfies the following condition.

No absolute velocity: Let $\mathbf{a}$ be a spacelike translation of M. Then there is a pair $(\mathbf{b}, \mathbf{c})$ of timelike translations of M such that $\mathbf{a} = \mathbf{b} - \mathbf{c}$.

Despite the fact that "no absolute velocity" is a feature of all post-Galilean spacetimes, there are some who claim that the existence of a (undetectable) preferred reference frame is perfectly consistent with the empirical evidence on which relativistic theories are based (cf. Bell 1987, Chap. 9). What is more, the existence of a preferred frame is an absolutely essential feature of a number of "realistic" interpretations of quantum theory (cf. Maudlin 1994, Chap. 7). Thus, this tacit assumption of Malament's theorem has the potential to be a major source of contention for those wishing to maintain that there can be a relativistic quantum mechanics of localizable particles.

There is a further worry about the generality of Malament's theorem: It is not clear whether the result can be expected to hold for arbitrary *relativistic* spacetimes, or whether it is an artifact of peculiar features of Minkowski spacetime (e.g., that space is infinite). To see this, suppose that M is an arbitrary globally hyperbolic manifold. (That is, M is a manifold that permits at least one foliation $\mathcal{S}$ into spacelike hypersurfaces). Although M will not typically have a translation group, we suppose that M has a transitive Lie group G of diffeomorphisms. (Just as a manifold is locally isomorphic to $\mathbb{R}^n$, a Lie group is locally isomorphic to a group of translations.) We require that G has a representation $g \mapsto U(g)$ in the unitary operators on $\mathcal{H}$; and, the translation covariance condition now says that $E_{g(\Delta)} = U(g)E_\Delta U(g)^*$ for all $g \in G$.

The following example shows that Malament's theorem fails even for the very simple case where M is a two-dimensional cylinder.

Example 2. Let $M = \mathbb{R} \oplus S^1$, where S^1 is the one-dimensional unit circle, and let G denote the Lie group of timelike translations and rotations of M. It is not difficult to construct a unitary representation of G that satisfies the energy bounded below condition. (We can use the Hilbert space of square-integrable functions from S^1 into $\mathbb{C}$, and the procedure for constructing the unitary representation is directly analogous to the case of a single particle moving on a line.) Fix a spacelike hypersurface Σ, and let μ denote the normalized rotation-invariant measure on Σ. For each open subset Δ of Σ, let $E_\Delta = I$ if $\mu(\Delta) \geq 2/3$, and let $E_\Delta = 0$ if $\mu(\Delta) < 2/3$. Then localizability holds, since for any pair (Δ, Δ') of disjoint open subsets of Σ, either $\mu(\Delta) < 2/3$ or $\mu(\Delta') < 2/3$. □

Nonetheless, Examples 1 and 2 hardly serve as physically interesting counterexamples to a strengthened version of Malament's theorem. In particular, in Example 1 the energy is identically zero, and therefore the probability for finding the particle in a given region of space remains constant over time. In Example 2, the particle is localized in every region of space with volume greater than 2/3, and the particle is never localized in a region of space with volume less than 2/3. In the following two sections, then, we will formulate explicit conditions to rule out such pathologies, and we will use these conditions to derive a strengthened version of Malament's theorem that applies to generic spacetimes.

5.3 Hegerfeldt's theorem

Hegerfeldt's (1998a, 1998b) recent results on localization apply to arbitrary (globally hyperbolic) spacetimes, and they do not make use of the "no absolute velocity" condition. Thus, we will suppose henceforth that M is a globally hyperbolic spacetime, and we will fix a foliation $\mathcal{S}$ of M, as well as a unique isomorphism between any two hypersurfaces in this foliation. If $\Sigma \in \mathcal{S}$, we will write $\Sigma + t$ for the hypersurface that results from "moving Σ forward in time by t units"; and if Δ is a subset of Σ, we will use $\Delta + t$ to denote the corresponding subset of $\Sigma + t$. We assume that there is a representation $t \mapsto U_t$ of the time-translation group $\mathbb{R}$ in the unitary operators on $\mathcal{H}$, and we will say that the localization system $(\mathcal{H}, \Delta \mapsto E_\Delta, t \mapsto U_t)$ satisfies *time-translation covariance* just in case $U_t E_\Delta U_{-t} = E_{\Delta+t}$ for all Δ and all $t \in \mathbb{R}$.

Hegerfeldt's result is based on the following root lemma.

Lemma 5.1 (Hegerfeldt). *Suppose that $U_t = e^{itH}$, where H is a self-adjoint operator with spectrum bounded from below. Let A be a positive operator (e.g., a projection operator). Then for any state ψ, either*

$$\langle U_t\psi, AU_t\psi\rangle \neq 0\,, \qquad \text{for almost all } \ t \in \mathbb{R},$$

or

$$\langle U_t\psi, AU_t\psi\rangle = 0\,, \qquad \text{for all } \ t \in \mathbb{R}\,.$$

Hegerfeldt claims that this lemma has the following consequence for localization:

> If there exist particle states which are strictly localized in some finite region at $t = 0$ and later move towards infinity, then finite propagation speed cannot hold for localization of particles. (Hegerfeldt 1998a, 243)

Hegerfeldt's argument for this conclusion is as follows:

> Now, if the particle or system is strictly localized in Δ at $t = 0$ it is, a fortiori, also strictly localized in any larger region Δ' containing Δ. If the boundaries of Δ' and Δ have a finite distance and *if finite propagation speed holds* then the probability to find the system in Δ' must also be 1 for sufficiently small times, e.g. $0 \leq t < \epsilon$. But then [Lemma 5.1], with $A \equiv I - E_{\Delta'}$, states that the system stays in Δ' for *all* times. Now, we can make Δ' smaller and let it approach Δ. Thus we conclude that if a particle or system is at time $t = 0$ strictly localized in a region Δ, then finite propagation speed implies that it stays in Δ for all times and therefore prohibits motion to infinity. (Hegerfeldt 1998a, 242–243; notation adapted, but italics in original)

Let us attempt now to put this argument into a more precise form.

First, Hegerfeldt claims that the following is a consequence of "finite propagation speed": If $\Delta \subseteq \Delta'$, and if the boundaries of Δ and Δ' have a finite distance, then a state initially localized in Δ will continue to be localized in Δ' for some finite amount of time. We can capture this precisely by means of the following condition.

No instantaneous wavepacket spreading (NIWS): If $\Delta \subseteq \Delta'$, and the boundaries of Δ and Δ' have a finite distance, then there is an $\epsilon > 0$ such that $E_\Delta \leq E_{\Delta'+t}$ whenever $0 \leq t < \epsilon$.

(Note that NIWS plus localizability entails strong causality.) In the argument, Hegerfeldt also assumes that if a particle is localized in every one of a family of sets that "approaches" Δ, then it is localized in Δ. We can capture this assumption in the following condition.

Monotonicity: If $\{\Delta_n : n \in \mathbb{N}\}$ is a downward nested family of subsets of Σ such that $\bigcap_n \Delta_n = \Delta$, then $\bigwedge_n E_{\Delta_n} = E_\Delta$.

Using this assumption, Hegerfeldt argues that if NIWS holds, and if a particle is initially localized in some finite region Δ, then it will remain in Δ for all subsequent times. In other words, if $E_\Delta \psi = \psi$, then $E_\Delta U_t \psi = U_t \psi$ for all $t \geq 0$. We can now translate this into the following rigorous no-go theorem.

Theorem (Hegerfeldt). *Suppose that the localization system* $(\mathcal{H}, \Delta \mapsto E_\Delta, t \mapsto U_t)$ *satisfies:*

1. *Monotonicity*
2. *Time-translation covariance*
3. *Energy bounded below*
4. *No instantaneous wavepacket spreading*

Then $U_t E_\Delta U_{-t} = E_\Delta$ *for all* $\Delta \subset \Sigma$ *and all* $t \in \mathbb{R}$.

Thus, conditions 1–4 can be satisfied only if the particle has trivial dynamics. If M is an affine space, and if we add "no absolute velocity" as a fifth condition in this theorem, then we get the stronger conclusion that $E_\Delta = 0$ for all bounded Δ (see Lemma 5.2, section 5.9). Thus, there is an obvious similarity between Hegerfeldt's and Malament's theorems. However, NIWS is a stronger causality assumption than microcausality. In fact, while NIWS plus localizability entails strong causality (and hence microcausality), the following example shows that NIWS is not entailed by the conjunction of strong causality, monotonicity, time-translation covariance, and energy bounded below.

Example 3. Let Q, P denote the standard position and momentum operators on $\mathcal{H} = L_2(\mathbb{R})$, and let $H = P^2/2m$ for some $m > 0$. Let $\Delta \mapsto E^Q_\Delta$ denote the spectral measure for Q. Fix some bounded subset Δ_0 of $\mathbb{R}$, and let $E_\Delta = E^Q_\Delta \otimes E^Q_{\Delta_0}$ (a projection operator on $\mathcal{H} \otimes \mathcal{H}$) for all Borel subsets Δ of $\mathbb{R}$. Thus, $\Delta \mapsto E_\Delta$ is a (non-normalized) projection-valued measure. Let $U_t = I \otimes e^{itH}$, and let $E_{\Delta+t} = U_t E_\Delta U_{-t}$ for all $t \in \mathbb{R}$. It is clear that monotonicity, time-translation covariance, and energy bounded below hold. To see that strong causality holds, let Δ and Δ' be disjoint subsets of a single hyperplane Σ. Then,

$$E_\Delta U_t E_{\Delta'} U_{-t} \;=\; E^Q_\Delta E^Q_{\Delta'} \otimes E^Q_{\Delta_0} E^Q_{\Delta_0 + t} \;=\; 0 \otimes E^Q_\Delta E^Q_{\Delta_0+t} \;=\; 0\,, \tag{5.2}$$

for all $t \in \mathbb{R}$. On the other hand, $U_t E_\Delta U_{-t} \neq E_\Delta$ for any nonempty Δ and for any $t \neq 0$. Thus, it follows from Hegerfeldt's theorem that NIWS fails. □

Thus, we could not recapture the full strength of Malament's theorem simply by adding "no absolute velocity" to the conditions of Hegerfeldt's theorem.

5.4 A new Hegerfeldt-Malament type theorem

Example 3 shows that Hegerfeldt's theorem fails if NIWS is replaced by strong causality (or by microcausality). On the other hand, Example 3 is hardly a physically interesting counterexample to a strengthened version of Hegerfeldt's theorem. In particular, if Σ is a fixed spatial hypersurface, and if $\{\Delta_n : n \in \mathbb{N}\}$ is a covering of Σ by bounded sets (i.e., $\bigcup_n \Delta_n = \Sigma$), then $\bigvee_n E_{\Delta_n} = I \otimes E_{\Delta_0} \neq I \otimes I$. Thus, it is not certain that the particle will be detected *somewhere or other* in space. In fact, if $\{\Delta_n : n \in \mathbb{N}\}$ is a covering of Σ and $\{\Pi_n : n \in \mathbb{N}\}$ is a covering of $\Sigma + t$, then

$$\bigvee_{n\in\mathbb{N}} E_{\Delta_n} \;=\; I \otimes E_{\Delta_0} \;\neq\; I \otimes E_{\Delta_0+t} \;=\; \bigvee_{n\in\mathbb{N}} E_{\Pi_n}. \tag{5.3}$$

Thus, the total probability for finding the particle somewhere or other in space can change over time.

It would be completely reasonable to require that $\bigvee_n E_{\Delta_n} = I$ whenever $\{\Delta_n : n \in \mathbb{N}\}$ is a covering of Σ. This would be the case, for example, if the mapping $\Delta \mapsto E_\Delta$ (restricted to subsets of Σ) were the spectral measure of some position operator. However, we propose that—at the very least—any physically interesting model should satisfy the following weaker condition.

Probability conservation: If $\{\Delta_n : n \in \mathbb{N}\}$ is a covering of Σ, and $\{\Pi_n : n \in \mathbb{N}\}$ is a covering of $\Sigma + t$, then $\bigvee_n E_{\Delta_n} = \bigvee_n E_{\Pi_n}$.

Probability conservation guarantees that there is a well-defined total probability for finding the particle somewhere or other in space, and this probability remains constant over time. In particular, if both $\{\Delta_n : n \in \mathbb{N}\}$ and $\{\Pi_n : n \in \mathbb{N}\}$ consist of pairwise disjoint sets, then the localizability condition entails that $\bigvee_n E_{\Delta_n} = \sum_n E_{\Delta_n}$ and $\bigvee_n E_{\Pi_n} = \sum_n E_{\Pi_n}$. In this case, probability conservation is equivalent to

$$\sum_{n\in\mathbb{N}} \mathrm{Prob}^{\psi}(E_{\Delta_n}) \;=\; \sum_{n\in\mathbb{N}} \mathrm{Prob}^{\psi}(E_{\Pi_n})\,, \tag{5.4}$$

for any state ψ. Note, finally, that probability conservation is neutral with respect to relativistic and non-relativistic models.[1]

[1]Probability conservation would fail if a particle could escape to infinity in a finite amount of time (cf. Earman 1986, 33). However, a particle can escape to infinity only if there is an infinite potential well, and this would violate the energy condition. Thus, given the energy condition, probability conservation should also hold for non-relativistic particle theories.

Theorem 5.1. *Suppose that the localization system* $(\mathcal{H}, \Delta \mapsto E_\Delta, t \mapsto U_t)$ *satisfies:*

1. *Localizability*
2. *Probability conservation*
3. *Time-translation covariance*
4. *Energy bounded below*
5. *Microcausality*

Then $U_t E_\Delta U_{-t} = E_\Delta$ *for all* Δ *and all* $t \in \mathbb{R}$*.*

If M is an affine space, and if we add "no absolute velocity" as a sixth condition in this theorem, then it follows that $E_\Delta = 0$ for all Δ (see Lemma 5.2). Thus, modulo the probability conservation condition, Theorem 5.1 recaptures the full strength of Malament's theorem. Moreover, we can now trace the difficulties with localization to microcausality *alone*: there are localizable particles only if it is possible to have act-outcome correlations at spacelike separation.

We now give examples to show that each condition in Theorem 5.1 is indispensable; that is, no four of the conditions suffices to entail the conclusion. (Example 1 shows that conditions 1–5 can be simultaneously satisfied.) Suppose for simplicity that M is two-dimensional. (All examples work in the four-dimensional case as well.) Let Q, P be the standard position and momentum operators on $L_2(\mathbb{R})$, and let $H = P^2/2m$. Let Σ be a spatial hypersurface in M, and suppose that a coordinatization of Σ has been fixed, so that there is a natural association between each bounded open subset Δ of Σ and a corresponding spectral projection E_Δ of Q.

(1+2+3+4) (a) Consider the standard localization system for a single non-relativistic particle. That is, let Σ be a fixed spatial hyperplane, and let $\Delta \mapsto E_\Delta$ (with domain the Borel subsets of Σ) be the spectral measure for Q. For $\Sigma + t$, set $E_{\Delta+t} = U_t E_\Delta U_{-t}$, where $U_t = e^{itH}$. (b) The Newton-Wigner approach to relativistic QM uses the standard localization system for a non-relativistic particle, only replacing the non-relativistic Hamiltonian $P^2/2m$ with the relativistic Hamiltonian $(P^2 + m^2 I)^{1/2}$, whose spectrum is also bounded from below.

(1+2+3+5) (a) For a mathematically simple (but physically uninteresting) example, take the first example above and replace the Hamiltonian

$P^2/2m$ with P. In this case, microcausality trivially holds, since $U_t E_\Delta U_{-t}$ is just a shifted spectral projection of Q. (b) For a physically interesting example, consider the relativistic quantum theory of a single spin-1/2 electron (see section 5.2.1). Due to the negative energy solutions of the Dirac equation, the spectrum of the Hamiltonian is not bounded from below.

(1+2+4+5) Consider the the standard localization system for a non-relativistic particle, but set $E_{\Delta+t} = E_\Delta$ for all $t \in \mathbb{R}$. Thus, we escape the conclusion of trivial dynamics, but only by disconnecting the (nontrivial) unitary dynamics from the (trivial) association of projections with spatial regions.

(1+3+4+5) (a) Let Δ_0 be some bounded open subset of Σ, and let E_{Δ_0} be the corresponding spectral projection of Q. When $\Delta \neq \Delta_0$, let $E_\Delta = 0$. Let $U_t = e^{itH}$, and let $E_{\Delta+t} = U_t E_\Delta U_{-t}$ for all Δ. This example is physically uninteresting, since the particle cannot be localized in any region besides Δ_0, including proper supersets of Δ_0. (b) See Example 3.

(2+3+4+5) Let Δ_0 be some bounded open subset of Σ, and let E_{Δ_0} be the corresponding spectral projection of Q. When $\Delta \neq \Delta_0$, let $E_\Delta = I$. Let $U_t = e^{itH}$, and let $E_{\Delta+t} = U_t E_\Delta U_{-t}$ for all Δ. Thus, the particle is always localized in every region other than Δ_0, and is sometimes localized in Δ_0 as well.

5.5 Are there unsharply localizable particles?

We have argued that attempts to undermine the four explicit assumptions of Malament's theorem are unsuccessful. We have also now shown that the tacit assumption of "no absolute velocity" is not necessary to derive Malament's conclusion. And, yet, there is one more loophole in the argument against a relativistic quantum mechanics of localizable particles. In particular, the basic assumption of a family $\{E_\Delta\}$ of localizing projections is unnecessary; it is possible to have a quantum-mechanical particle theory in the absence of localizing projections. What is more, one might object to the use of localizing projections on the grounds that they represent an unphysical idealization—viz., that a "particle" can be completely contained in a finite region of space with a sharp boundary, when in fact it would require an infinite amount of energy to prepare a particle in such a state. Thus, there remains a possibility that relativistic causality can be reconciled with "unsharp" localizability.

To see how we can define "particle talk" without having projection operators, consider the relativistic theory of a single spin-1/2 electron (where we now restrict to the subspace $\mathcal{H}_{\text{pos}}$ of positive energy solutions of the Dirac equation). In order to treat the '$\mathbf{x}$' of the Dirac wavefunction as an observable, we need only to define a probability amplitude and density for the particle to be found at $\mathbf{x}$; and these can be obtained from the Dirac wavefunction itself. That is, for a subset Δ of Σ, we set

$$\text{Prob}^{\psi}(\mathbf{x} \in \Delta) = \int_{\Delta} |\psi(\mathbf{x})|^2 d\mathbf{x}\,. \tag{5.5}$$

Now let $\Delta \mapsto E_\Delta$ be the spectral measure for the standard position operator on the Hilbert space $\mathcal{H}$ (which includes both positive and negative energy solutions). That is, E_Δ multiplies a wavefunction by the characteristic function of Δ. Let F denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_{\text{pos}}$. Then,

$$\int_{\Delta} |\psi(\mathbf{x})|^2 d\mathbf{x} = \langle \psi, E_\Delta \psi \rangle = \langle \psi, F E_\Delta \psi \rangle, \tag{5.6}$$

for any $\psi \in \mathcal{H}_{\text{pos}}$. Thus, we can apply the standard recipe to the operator FE_Δ (defined on $\mathcal{H}_{\text{pos}}$) to compute the probability that the particle will be found within Δ. However, FE_Δ does *not* define a projection operator on $\mathcal{H}_{\text{pos}}$. (In fact, it can be shown that FE_Δ does not have any eigenvectors with eigenvalue 1.) Thus, we do not need a family of *projection* operators in order to define probabilities for localization.

Now, in general, to define the probability that a particle will be found in Δ, we need only assume that there is an operator A_Δ such that $\langle \psi, A_\Delta \psi \rangle \in [0,1]$ for any unit vector ψ. Such operators are called *effects*, and include the projection operators as a proper subclass. Thus, we say that the triple $(\mathcal{H}, \Delta \mapsto A_\Delta, \mathbf{a} \mapsto U(\mathbf{a}))$ is an *unsharp localization system* over M just in case $\Delta \mapsto A_\Delta$ is a mapping from subsets of hyperplanes in M to effects on $\mathcal{H}$, and $\mathbf{a} \mapsto U(\mathbf{a})$ is a continuous representation of the translation group of M in unitary operators on $\mathcal{H}$. (We assume for the present that M is again an affine space.)

Most of the conditions from the previous sections can be applied, with minor changes, to unsharp localization systems. In particular, since the energy bounded below condition refers only to the unitary representation, it can be carried over intact; and translation covariance also generalizes straightforwardly. However, we will need to take more care with microcausality and with localizability.

If E and F are projection operators, $[E, F] = 0$ just in case for any state, the statistics of a measurement of F are not affected by a non-selective measurement of E and vice versa (cf. Malament 1996, 5). This fact, along with the assumption that E_Δ is measurable in Δ, motivates the microcausality assumption. For the case of an association of arbitrary effects with spatial regions, Busch (1999, Proposition 2) has shown that $[A_\Delta, A_{\Delta'}] = 0$ just in case for any state, the statistics for a measurement of A_Δ are not affected by a non-selective measurement of $A_{\Delta'}$ and vice versa. Thus, we may carry over the microcausality assumption intact, again seen as enforcing a prohibition against act-outcome correlations at spacelike separation.

The localizability condition is motivated by the idea that a particle cannot be simultaneously localized (with certainty) in two disjoint regions of space. In other words, if Δ and Δ' are disjoint subsets of a single hyperplane, then $\langle\psi, E_\Delta\psi\rangle = 1$ entails that $\langle\psi, E_{\Delta'}\psi\rangle = 0$. It is not difficult to see that this last condition is equivalent to the assumption that $E_\Delta + E_{\Delta'} \leq I$. That is,

$$\langle\psi, (E_\Delta + E_{\Delta'})\psi\rangle \leq \langle\psi, I\psi\rangle \,, \tag{5.7}$$

for any state ψ. Now, it is an accidental feature of projection operators (as opposed to arbitrary effects) that $E_\Delta + E_{\Delta'} \leq I$ is equivalent to $E_\Delta E_{\Delta'} = 0$. Thus, the appropriate generalization of localizability to unsharp localization systems is the following condition.

Localizability: If Δ and Δ' are disjoint subsets of a single hyperplane, then $A_\Delta + A_{\Delta'} \leq I$.

That is, the probability for finding the particle in Δ, plus the probability for finding the particle in some disjoint region Δ', never totals more than 1. It would, in fact, be reasonable to require a slightly stronger condition, viz., the probability of finding a particle in Δ plus the probability of finding a particle in Δ' equals the probability of finding a particle in $\Delta \cup \Delta'$. If this is true for all states ψ, we have:

Additivity: If Δ and Δ' are disjoint subsets of a single hyperplane, then $A_\Delta + A_{\Delta'} = A_{\Delta\cup\Delta'}$.

With just these mild constraints, Busch (1999) was able to derive the following no-go result.

Theorem (Busch). *Suppose that the unsharp localization system* $(\mathcal{H}, \Delta \mapsto A_\Delta, \mathbf{a} \mapsto U(\mathbf{a}))$ *satisfies localizability, translation covariance, energy bounded* *below, microcausality, and no absolute velocity. Then, for all* Δ, A_Δ *has no eigenvector with eigenvalue* 1.

Thus, it is not possible for a particle to be localized with certainty in any bounded region Δ. Busch's theorem, however, leaves it open question whether there are (nontrivial) "strongly unsharp" localization systems that satisfy microcausality. The following result shows that there are not.

Theorem 5.2. *Suppose that the unsharp localization system* $(\mathcal{H}, \Delta \mapsto A_\Delta, \mathbf{a} \mapsto U(\mathbf{a}))$ *satisfies:*

1. *Additivity*
2. *Translation covariance*
3. *Energy bounded below*
4. *Microcausality*
5. *No absolute velocity*

Then $A_\Delta = 0$ *for all* Δ.

Theorem 5.2 shows that invoking the notion of unsharp localization does nothing to resolve the tension between relativistic causality and localizability. For example, we can now show that the (positive energy) Dirac theory—in which there are localizable particles—violates relativistic causality. Indeed, it is clear that the conclusion of Theorem 5.2 fails.[2] On the other hand, additivity, translation covariance, energy bounded below, and no absolute velocity hold. Thus, microcausality fails, and the (positive energy) Dirac theory permits superluminal signalling.

Unfortunately, Theorem 5.2 does not generalize to arbitrary globally hyperbolic spacetimes, as the following example shows.

Example 4. Let M be the cylinder spacetime from Example 2. Let G denote the group of timelike translations and rotations of M, and let $g \mapsto U(g)$ be a positive energy representation of G in the unitary operators on a Hilbert space $\mathcal{H}$. For any $\Sigma \in \mathcal{S}$, let μ denote the normalized rotation-invariant measure on Σ, and let $A_\Delta = \mu(\Delta)I$. Then, conditions 1–5 of Theorem 5.2 are satisfied, but the conclusion of the theorem is false. □

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[2]For any unit vector $\psi \in \mathcal{H}_{\text{pos}}$, there is a bounded set Δ such that $\int_\Delta |\psi|^2 d\mathbf{x} \neq 0$. Thus, $A_\Delta \neq 0$.

The previous counterexample can be excluded if we require there to be a fixed positive constant δ such that, for each Δ, there is a state ψ with $\langle \psi, A_\Delta \psi \rangle \geq \delta$. In fact, with this condition added, Theorem 5.2 holds for any globally hyperbolic spacetime. (The proof is an easy modification of the proof we give in Section 5.9.) However, it is not clear what physical motivation there could be for requiring this further condition. Note also that Example 4 has trivial dynamics; i.e., $U_t A_\Delta U_{-t} = A_\Delta$ for all Δ. We conjecture that every counterexample to a generalized version of Theorem 5.2 will have trivial dynamics.

Theorem 5.2 strongly supports the conclusion that there is no relativistic quantum mechanics of a single (localizable) particle; and that the only consistent combination of special relativity and quantum mechanics is in the context of quantum field theory. However, neither Theorem 5.1 nor Theorem 5.2 says anything about the ontology of relativistic quantum field theory itself; they leave open the possibility that relativistic quantum field theory might permit an ontology of localizable particles. To eliminate this latter possibility, we will now proceed to present a more general result which shows that there are no localizable particles in *any* relativistic quantum theory.

5.6 Are there localizable particles in RQFT?

The localizability assumption is motivated by the idea that a "particle" cannot be detected in two disjoint spatial regions at once. However, in the case of a many-particle system, it is certainly possible for there to be particles in disjoint spatial regions. Thus, the localizability condition does not apply to many-particle systems; and Theorems 5.1 and 5.2 cannot be used to rule out a relativistic quantum mechanics of $n > 1$ localizable particles.

Still, one might argue that we could use E_Δ to represent the proposition that a measurement is certain to find that *all* n particles lie within Δ, in which case localizability should hold. Note, however, that when we alter the interpretation of the localization operators $\{E_\Delta\}$, we must alter our interpretation of the conclusion. In particular, the conclusion now shows only that it is not possible for all n particles to be localized in a bounded region of space. This leaves open the possibility that there are localizable particles, but that they are governed by some sort of "exclusion principle" that prohibits them all from clustering in a bounded spacetime region.

Furthermore, Theorems 5.1 and 5.2 only show that it is impossible to define *position operators* that obey appropriate relativistic constraints. But it does not immediately follow from this that we lack any notion of localization in relativistic quantum theories. Indeed,

> ...a position operator is inconsistent with relativity. This compels us to find another way of modeling localization of events. In field theory, we model localization by making the observables dependent on position in spacetime. (Ticiatti 1999, 11)

However, it is not a peculiar feature of *relativistic* quantum field theory that it lacks a position operator: Any quantum field theory (either relativistic or non-relativistic) will model localization by making the observables dependent on position in spacetime. Moreover, in the case of non-relativistic QFT, these "localized" observables suffice to provide us with a concept of localizable particles. In particular, for each spatial region Δ, there is a "number operator" N_Δ whose eigenvalues give the number of particles within the region Δ. Thus, we have no difficultly in talking about the particle content in a given region of space despite the absence of any position operator.

Abstractly, a number operator N on $\mathcal{H}$ is any operator with eigenvalues contained in $\{0, 1, 2, \dots\}$. In order to describe the number of particles locally, we require an association $\Delta \mapsto N_\Delta$ of subsets of spatial hyperplanes in M to number operators on $\mathcal{H}$, where N_Δ represents the number of particles in the spatial region Δ. If $\mathbf{a} \mapsto U(\mathbf{a})$ is a unitary representation of the translation group, we say that the triple $(\mathcal{H}, \Delta \mapsto N_\Delta, \mathbf{a} \mapsto U(\mathbf{a}))$ is a *system of local number operators* over M. Note that a localization system $(\mathcal{H}, \Delta \mapsto E_\Delta, \mathbf{a} \mapsto U(\mathbf{a}))$ is a special case of a system of local number operators where the eigenvalues of each N_Δ are restricted to $\{0, 1\}$. Furthermore, if we loosen our assumption that number operators have a discrete spectrum, and instead require only that they have spectrum contained in $[0, \infty)$, then we can also include unsharp localization systems within the general category of systems of local number operators. Thus, a system of local number operators is the *minimal* requirement for a concept of localizable particles in any quantum theory.

In addition to the natural analogues of the energy bounded below condition, translation covariance, and microcausality, we will be interested in the following two requirements on a system of local number operators:[3]

[3] Due to the unboundedness of number operators, we would need to take some care in giving technically correct versions of the following conditions. In particular, the additivity condition should technically include the clause that N_Δ and $N_{\Delta'}$ have a common dense domain, and the operator $N_{\Delta \cup \Delta'}$ should be thought of as the self-adjoint closure of $N_\Delta + N_{\Delta'}$. In the number conservation condition, the sum $N = \sum_n N_{\Delta_n}$ can be made rigorous by exploiting the correspondence between self-adjoint operators and "quadratic forms" on

Additivity: If Δ and Δ' are disjoint subsets of a single hyperplane, then $N_{\Delta} + N_{\Delta'} = N_{\Delta \cup \Delta'}$.

Number conservation: If $\{\Delta_n : n \in \mathbb{N}\}$ is a disjoint covering of Σ, then the sum $\sum_n N_{\Delta_n}$ converges to a densely defined, self-adjoint operator N on $\mathcal{H}$ (independent of the chosen covering), and $U(\mathbf{a})NU(\mathbf{a})^* = N$ for any timelike translation $\mathbf{a}$ of M.

Additivity asserts that, when Δ and Δ' are disjoint, the expectation value (in any state ψ) for the number of particles in $\Delta \cup \Delta'$ is the sum of the expectations for the number of particles in Δ and the number of particles in Δ'. In the pure case, it asserts that the number of particles in $\Delta \cup \Delta'$ is the sum of the number of particles in Δ and the number of particles in Δ'. The "number conservation" condition tells us that there is a well-defined total number of particles (at a given time), and that the total number of particles does not change over time. This condition holds for any non-interacting model of QFT.

It is a well-known consequence of the Reeh-Schlieder theorem that relativistic quantum field theories do not admit systems of local number operators (cf. Redhead 1995b). We will now derive the same conclusion from strictly weaker assumptions. In particular, we show that microcausality is the *only* specifically relativistic assumption needed for this result. The relativistic spectrum condition—which requires that the spectrum of the four-momentum lie in the forward light cone, and which is used in the proof of the Reeh-Schlieder theorem—plays no role in our proof.[4]

Theorem 5.3. *Suppose that the system* $(\mathcal{H}, \Delta \mapsto N_{\Delta}, \mathbf{a} \mapsto U(\mathbf{a}))$ *of local number operators satisfies:*

1. *Additivity*
2. *Translation covariance*
3. *Energy bounded below*
4. *Number conservation*

$\mathcal{H}$. In particular, we can think of N as deriving from the upper bound of quadratic forms corresponding to finite sums.

[4]Microcausality is not only sufficient, but also necessary for the proof that there are no local number operators. The Reeh-Schlieder theorem entails the cyclicity of the vacuum state. But the cyclicity of the vacuum state alone does not entail that there are no local number operators; we must also assume microcausality (cf. Requardt 1982 and section 4.5 of this dissertation).

5. *Microcausality*

6. *No absolute velocity*

Then $N_\Delta = 0$ *for all* Δ.

Thus, in every state, there are no particles in any local region. This serves as a reductio ad absurdum for any notion of localizable particles in a relativistic quantum theory.

Unfortunately, Theorem 5.3 is not the strongest result we could hope for, since "number conservation" can only be expected to hold in the (trivial) case of non-interacting fields. However, we would need a more general approach in order to deal with interacting relativistic quantum fields, because (due to Haag's theorem; cf. Streater & Wightman 2000, 163) their dynamics are not unitarily implementable on a fixed Hilbert space. On the other hand it would be wrong to think of this as indicating a limitation on the generality of our conclusion: Haag's theorem also entails that interacting models of RQFT have no number operators—either global or local.[5] Still, it would be interesting to recover this conclusion (perhaps working in a more general algebraic setting) without using the full strength of Haag's assumptions.

5.7 Particle talk without particle ontology

The results of the previous sections show that, insofar as we can expect any relativistic quantum theory theory to satisfy a few basic conditions, these theories do not admit (localizable) particles into their ontology. We also considered and rejected several arguments which attempt to show that one (or more) of these conditions can be jettisoned without doing violence to the theory of relativity or to quantum mechanics. Thus, we have yet to find a good reason to reject one of the premises on which our argument against localizable particles is based. However, Segal (1964) and Barrett (2001) claim that we have independent grounds for rejecting the conclusion; that is, we have good reasons for believing that there *are* localizable particles.

The argument for localizable particles appears to be very simple: Our experience shows us that objects (particles) occupy finite regions of space. But the reply to this argument is just as simple: These experiences are

[5]If a total number operator exists in a representation of the canonical commutation relations, then that representation is quasi-equivalent to a free-field (Fock) representation (Chaiken 1968). However, Haag's theorem entails that in relativistic theories, representations with nontrivial interactions are *not* quasi-equivalent to a free-field representation.

illusory! Although no object is strictly localized in a bounded region of space, an object can be well-enough localized to give the appearance to us (finite observers) that it is strictly localized. In fact, relativistic quantum field theory *itself* shows how the "illusion" of localizable particles can arise, and how talk about localizable particles can be a useful fiction.

In order to assess the possibility of "approximately localized" objects in relativistic quantum field theory, we shall now pursue the investigation in the framework of algebraic quantum field theory.[6] Here, one assumes that there is a correspondence $O \mapsto \mathcal{R}(O)$ between bounded open subsets of M and subalgebras of observables on some Hilbert space $\mathcal{H}$. Observables in $\mathcal{R}(O)$ are considered to be "localized" (i.e., measurable) in O. Thus, if O and O' are spacelike separated, we require that $[A, B] = 0$ for any $A \in \mathcal{R}(O)$ and $B \in \mathcal{R}(O')$. Furthermore, we assume that there is a continuous representation $\mathbf{a} \mapsto U(\mathbf{a})$ of the translation group of M in unitary operators on $\mathcal{H}$, and that there is a unique "vacuum" state $\Omega \in \mathcal{H}$ such that $U(\mathbf{a})\Omega = \Omega$ for all $\mathbf{a}$. This latter condition entails that the vacuum appears the same to all observers, and that it is the unique state of lowest energy.

In this context, a particle detector can be represented by an effect C such that $\langle \Omega, C\Omega \rangle = 0$. That is, C should register no particles in the vacuum state. However, the Reeh-Schlieder theorem entails that no positive local observable can have zero expectation value in the vacuum state. Thus, we again see that (strictly speaking) it is impossible to detect particles by means of local measurements; instead, we will have to think of particle detections as "approximately local" measurements.

If we think of an observable as representing a measurement procedure (or, more precisely, an equivalence class of measurement procedures), then the norm distance $\|C - C'\|$ between two observables gives a quantitative measure of the physical similarity between the corresponding procedures. (In particular, if $\|C - C'\| < \delta$, then the expectation values of C and C' never differ by more than δ.)[7] Moreover, in the case of real-world measurements, the existence of measurement errors and environmental noise make it impossible for us to determine precisely which measurement procedure we have performed. Thus, practically speaking, we can at best determine a neighborhood of observables corresponding to a concrete measurement

[6]For general information on algebraic quantum field theory, see (Haag 1992) and (Buchholz 2000). For specific information on particle detectors and "almost local" observables, see Chapter 6 of (Haag 1992) and Section 4 of (Buchholz 2000).

[7]Recall that $\|C - C'\| = \sup\{\|(C - C')\psi\| : \psi \in \mathcal{H}, \|\psi\| = 1\}$. It follows then from the Cauchy-Schwarz inequality that $|\langle \psi, (C - C')\psi \rangle| \leq \|C - C'\|$ for any unit vector ψ.

procedure.

In the case of present interest, what we actually measure is always a local observable—i.e., an element of $\mathcal{R}(O)$, where O is bounded. However, given a fixed error bound δ, if an observable C is within norm distance δ from some local observable $C' \in \mathcal{R}(O)$, then a measurement of C' will be practically indistinguishable from a measurement of C. Thus, if we let

$$\mathcal{R}_\delta(O) = \{C : \exists C' \in \mathcal{R}(O) \text{ such that } \|C - C'\| < \delta\}, \tag{5.8}$$

denote the family of observables "almost localized" in O, then 'FAPP' (i.e., 'for all practical purposes') we can locally measure any observable from $\mathcal{R}_\delta(O)$. That is, measurement of an element from $\mathcal{R}_\delta(O)$ can be simulated to a high degree of accuracy by local measurement of an element from $\mathcal{R}(O)$. However, for any local region O, and for any $\delta > 0$, $\mathcal{R}_\delta(O)$ *does* contain (nontrivial) effects that annihilate the vacuum.[8] Thus, particle detections can always be simulated by purely local measurements; and the appearance of (fairly-well) localized objects can be explained without the supposition that there are localizable particles in the strict sense.

However, it may not be easy to pacify Segal and Barrett with a FAPP solution to the problem of localization. Both appear to think that the absence of localizable particles (in the strict sense) is not simply contrary to our manifest experience, but would undermine the very possibility of objective empirical science. For example, Segal claims that,

> ...it is an elementary fact, *without which experimentation of the usual sort would not be possible*, that particles are indeed localized in space at a given time. (Segal 1964, 145; my italics)

Furthermore, "particles would not be observable without their localization in space at a particular time" (1964, 139). In other words, experimentation involves observations of particles, and these observations can occur only if particles are localized in space. Unfortunately, Segal does not give any argument for these claims. It seems to us, however, that the moral we should draw from the no-go theorems is that Segal's account of observation is false. In particular, it is not (strictly speaking) true that we observe particles. Rather, there are 'observation events', and these observation events are consistent (to a good degree of accuracy) with the supposition that they are brought about by (localizable) particles.

[8] Suppose that $A \in \mathcal{R}(O)$, and let $A(\mathbf{x}) = U(\mathbf{x})AU(\mathbf{x})^*$. If f is a test function on M whose Fourier transform is supported in the complement of the forward light cone, then $L = \int f(\mathbf{x})A(\mathbf{x})d\mathbf{x}$ is almost localized in O and $\langle\Omega, L\Omega\rangle = 0$ (cf. Buchholz 2000, 7).

Like Segal, Barrett (2001) claims that we will have trouble explaining how empirical science can work if there are no localizable particles. In particular, Barrett claims that empirical science requires that we be able to keep an account of our measurement results so that we can compare these results with the predictions of our theories. Furthermore, we identify measurement records by means of their location in space. Thus, if there were no localized objects, then there would be no identifiable measurement records, and "…it would be difficult to account for the possibility of empirical science at all" (Barrett 2001, 3).

However, it's not clear what the difficulty here is supposed to be. On the one hand, we have seen that relativistic quantum field theory does predict that the appearances will be FAPP consistent with the supposition that there are localized objects. So, for example, we could distinguish two record tokens at a given time if there were two disjoint regions O and O' and particle detector observables $C \in \mathcal{R}_\delta(O)$ and $C' \in \mathcal{R}_\delta(O')$ (approximated by observables *strictly* localized in O and O respectively) such that $\langle \psi, C\psi \rangle \approx 1$ and $\langle \psi, C'\psi \rangle \approx 1$. Now, it may be that Barrett is also worried about how, given a field ontology, we could assign any sort of trans-temporal identity to our record tokens. But this problem, however important philosophically, is distinct from the problem of localization. Indeed, it also arises in the context of non-relativistic quantum field theory, where there is *no* problem with describing localizable particles. Finally, Barrett might object that once we supply a quantum-theoretical model of a particle detector itself, then the superposition principle will prevent the field and detector from getting into a state where there is a fact of the matter as to whether, "a particle has been detected in the region O." But this is simply a restatement of the standard quantum measurement problem that infects *all* quantum theories—and we have made no pretense of solving that here.

5.8 Conclusion

Malament claims that his theorem justifies the belief that,

> …in the attempt to reconcile quantum mechanics with relativity theory…one is driven to a field theory; all talk about "particles" has to be understood, at least in principle, as talk about the properties of, and interactions among, quantized fields. (Malament 1996, 1)

We have argued that the first claim is correct—quantum mechanics and relativity can be reconciled only in the context of quantum field theory.

In order, however, to close a couple of loopholes in Malament's argument for this conclusion, we provided two further results (Theorems 5.1 and 5.2) which show that the conclusion continues to hold for generic spacetimes, as well as for "unsharp" localization observables. We then went on to show that relativistic quantum field theory also does not permit an ontology of localizable particles; and so, strictly speaking, our talk about localizable particles is a fiction. Nonetheless, relativistic quantum field theory does permit *talk* about particles—albeit, if we understand this talk as really being about the properties of, and interactions among, quantized fields. Indeed, modulo the standard quantum measurement problem, relativistic quantum field theory has no trouble explaining the appearance of macroscopically well-localized objects, and shows that our talk of particles, though a *façon de parler*, has a legitimate role to play in empirically testing the theory.

5.9 Appendix: Proofs of theorems

Theorem (Hegerfeldt). *Suppose that the localization system* $(\mathcal{H}, \Delta \mapsto E_\Delta, t \mapsto U_t)$ *satisfies monotonicity, time-translation covariance, energy bounded below, and NIWS. Then* $U_t E_\Delta U_{-t} = E_\Delta$ *for all* $\Delta \subset \Sigma$ *and all* $t \in \mathbb{R}$.

Proof. The formal proof corresponds directly to Hegerfeldt's informal proof. Thus, let Δ be a subset of some spatial hypersurface Σ. If $E_\Delta = 0$ then obviously $U_t E_\Delta U_{-t} = E_\Delta$ for all $t \in \mathbb{R}$. So, suppose that $E_\Delta \neq 0$, and let ψ be a unit vector such that $E_\Delta \psi = \psi$. Since Σ is a manifold, and since $\Delta \neq \Sigma$, there is a family $\{\Delta_n : n \in \mathbb{N}\}$ of subsets of Σ such that, for each $n \in \mathbb{N}$, the distance between the boundaries of Δ_n and Δ is nonzero, and such that $\bigcap_n \Delta_n = \Delta$. Fix $n \in \mathbb{N}$. By NIWS and time-translation covariance, there is an $\epsilon_n > 0$ such that $E_{\Delta_n} U_t \psi = U_t \psi$ whenever $0 \leq t < \epsilon_n$. That is, $\langle U_t \psi, E_{\Delta_n} U_t \psi \rangle = 1$ whenever $0 \leq t < \epsilon_n$. Since energy is bounded from below, we may apply Lemma 5.1 with $A = I - E_{\Delta_n}$ to conclude that $\langle U_t \psi, E_{\Delta_n} U_t \psi \rangle = 1$ for all $t \in \mathbb{R}$. That is, $E_{\Delta_n} U_t \psi = U_t \psi$ for all $t \in \mathbb{R}$. Since this holds for all $n \in \mathbb{N}$, and since (by monotonicity) $E_\Delta = \bigwedge_n E_{\Delta_n}$, it follows that $E_\Delta U_t \psi = U_t \psi$ for all $t \in \mathbb{R}$. Thus, $U_t E_\Delta U_{-t} = E_\Delta$ for all $t \in \mathbb{R}$. □

Lemma 5.2. *Suppose that the localization system* $(\mathcal{H}, \Delta \mapsto E_\Delta, \mathbf{a} \mapsto U(\mathbf{a}))$ *satisfies localizability, time-translation covariance, and no absolute velocity. Let* Δ *be a bounded spatial set. If* $U(\mathbf{a}) E_\Delta U(\mathbf{a})^* = E_\Delta$ *for all timelike translations* $\mathbf{a}$ *of* M*, then* $E_\Delta = 0$.

Proof. By no absolute velocity, there is a pair $(\mathbf{a}, \mathbf{b})$ of timelike translations such that $\Delta + (\mathbf{a} - \mathbf{b})$ is in Σ and is disjoint from Δ. By time-translation covariance, we have,

$$E_{\Delta+(\mathbf{a}-\mathbf{b})} \;=\; U(\mathbf{a})U(\mathbf{b})^* E_\Delta U(\mathbf{b})U(\mathbf{a})^* \;=\; E_\Delta. \tag{5.9}$$

Thus, localizability entails that E_Δ is orthogonal to itself, and so $E_\Delta = 0$. □

Lemma 5.3. *Let $\{\Delta_n : n = 0, 1, 2, \dots\}$ be a covering of Σ, and let $E = \bigvee_{n=0}^{\infty} E_{\Delta_n}$. If probability conservation and time-translation covariance hold, then $U_t E U_{-t} = E$ for all $t \in \mathbb{R}$.*

Proof. Since $\{\Delta_n + t : n \in \mathbb{N}\}$ is a covering of $\Sigma + t$, probability conservation entails that $\bigvee_n E_{\Delta_n + t} = E$. Thus,

$$U_t E U_{-t} \;=\; U_t \Big[\bigvee_{n=0}^{\infty} E_{\Delta_n} \Big] U_{-t} \;=\; \bigvee_{n=0}^{\infty} \Big[U_t E_{\Delta_n} U_{-t} \Big] \tag{5.10}$$

$$\;=\; \bigvee_{n=0}^{\infty} E_{\Delta_n + t} \;=\; E, \tag{5.11}$$

where the third equality follows from time-translation covariance. □

In order to prove the next result, we will need to invoke the following lemma from Borchers (1967).

Lemma (Borchers). *Let $U_t = e^{itH}$, where H is a self-adjoint operator with spectrum bounded from below. Let E and F be projection operators such that $EF = 0$. If there is an $\epsilon > 0$ such that*

$$[E, U_t F U_{-t}] = 0\,, \qquad 0 \le t < \epsilon\,,$$

then $E U_t F U_{-t} = 0$ for all $t \in \mathbb{R}$.

Lemma 5.4. *Let $U_t = e^{itH}$, where H is a self-adjoint operator with spectrum bounded from below. Let $\{E_n : n = 0, 1, 2, \dots\}$ be a family of projection operators such that $E_0 E_n = 0$ for all $n \ge 1$, and let $E = \bigvee_{n=0}^{\infty} E_n$. If $U_t E U_{-t} = E$ for all $t \in \mathbb{R}$, and if for each $n \ge 1$ there is an $\epsilon_n > 0$ such that*

$$[E_0, U_t E_n U_{-t}] = 0, \qquad 0 \le t < \epsilon_n, \tag{5.12}$$

then $U_t E_0 U_{-t} = E_0$ for all $t \in \mathbb{R}$.

Proof. If $E_0 = 0$ then the conclusion obviously holds. Suppose then that $E_0 \neq 0$, and let ψ be a unit vector in the range of E_0. Fix $n \geq 1$. Using (5.12) and Borchers' lemma, it follows that $E_0 U_t E_n U_{-t} = 0$ for all $t \in \mathbb{R}$. Then,

$$\begin{aligned} \|E_n U_{-t}\psi\|^2 &= \langle U_{-t}\psi, E_n U_{-t}\psi\rangle = \langle \psi, U_t E_n U_{-t}\psi\rangle & (5.13)\\ &= \langle E_0\psi, U_t E_n U_{-t}\psi\rangle = \langle \psi, E_0 U_t E_n U_{-t}\psi\rangle = 0\,, & (5.14)\end{aligned}$$

for all $t \in \mathbb{R}$. Thus, $E_n U_{-t}\psi = 0$ for all $n \geq 1$, and consequently, $[\bigvee_{n\geq 1} E_n]U_{-t}\psi = 0$. Since $E_0 = E - [\bigvee_{n\geq 1} E_n]$, and since (by assumption) $EU_{-t} = U_{-t}E$, it follows that

$$E_0 U_{-t}\psi = EU_{-t}\psi = U_{-t}E\psi = U_{-t}\psi\,, \tag{5.15}$$

for all $t \in \mathbb{R}$. □

Theorem 5.1. *Suppose that the localization system $(\mathcal{H}, \Delta \mapsto E_\Delta, t \mapsto U_t)$ satisfies localizability, probability conservation, time-translation covariance, energy bounded below, and microcausality. Then $U_t E_\Delta U_{-t} = E_\Delta$ for all Δ and all $t \in \mathbb{R}$.*

Proof. Let Δ be an open subset of Σ. If $\Delta = \Sigma$ then probability conservation and time-translation covariance entail that $E_\Delta = E_{\Delta+t} = U_t E_\Delta U_{-t}$ for all $t \in \mathbb{R}$. If $\Delta \neq \Sigma$ then, since Σ is a manifold, there is a covering $\{\Delta_n : n \in \mathbb{N}\}$ of $\Sigma\backslash\Delta$ such that the distance between Δ_n and Δ is nonzero for all n. Let $E_0 = E_\Delta$, and let $E_n = E_{\Delta_n}$ for $n \geq 1$. Then 1 entails that $E_0 E_n = 0$ when $n \geq 1$. If we let $E = \bigvee_{n=0}^{\infty} E_n$ then probability conservation entails that $U_t E U_{-t} = E$ for all $t \in \mathbb{R}$ (see Lemma 5.3). By time-translation covariance and microcausality, for each $n \geq 1$ there is an $\epsilon_n > 0$ such that

$$[E_0, U_t E_n U_{-t}] = 0, \qquad 0 \leq t < \epsilon_n. \tag{5.16}$$

Since the energy is bounded from below, Lemma 5.4 entails that $U_t E_0 U_{-t} = E_0$ for all $t \in \mathbb{R}$. That is, $U_t E_\Delta U_{-t} = E_\Delta$ for all $t \in \mathbb{R}$. □

Theorem 5.2. *Suppose that the unsharp localization system $(\mathcal{H}, \Delta \mapsto A_\Delta, \mathbf{a} \mapsto U(\mathbf{a}))$ satisfies additivity, translation covariance, energy bounded below, microcausality, and no absolute velocity. Then $A_\Delta = 0$ for all Δ.*

Proof. We prove by induction that $\|A_\Delta\| \leq (2/3)^m$, for each $m \in \mathbb{N}$, and for each bounded Δ. For this, let F_Δ denote the spectral measure for A_Δ.

(Base case: $m = 1$) Let $E_\Delta = F_\Delta(2/3, 1)$. We verify that $(\mathcal{H}, \Delta \mapsto E_\Delta, \mathbf{a} \mapsto U(\mathbf{a}))$ satisfies the conditions of Malament's theorem. Clearly, no absolute velocity and energy bounded below hold. Moreover, since unitary transformations preserve spectral decompositions, translation covariance holds; and since spectral projections of compatible operators are also compatible, microcausality holds. To see that localizability holds, let Δ and Δ' be disjoint bounded subsets of a single hyperplane. Then microcausality entails that $[A_\Delta, A_{\Delta'}] = 0$, and therefore $E_\Delta E_{\Delta'}$ is a projection operator. Suppose for reductio ad absurdum that ψ is a unit vector in the range of $E_\Delta E_{\Delta'}$. By additivity, $A_{\Delta\cup\Delta'} = A_\Delta + A_{\Delta'}$, and we therefore obtain the contradiction:

$$1 \;\geq\; \langle \psi, A_{\Delta\cup\Delta'}\psi\rangle \;=\; \langle \psi, A_\Delta \psi\rangle + \langle \psi, A_{\Delta'}\psi\rangle \;\geq\; 2/3 + 2/3\,. \tag{5.17}$$

Thus, $E_\Delta E_{\Delta'} = 0$, and Malament's theorem entails that $E_\Delta = 0$ for all Δ. Therefore, $A_\Delta = A_\Delta F_\Delta(0, 2/3)$ has spectrum lying in $[0, 2/3]$, and $\|A_\Delta\| \leq 2/3$ for all bounded Δ.

(Inductive step) Suppose that $\|A_\Delta\| \leq (2/3)^{m-1}$ for all bounded Δ. Let $E_\Delta = F_\Delta((2/3)^m, (2/3)^{m-1})$. In order to see that Malament's theorem applies to $(\mathcal{H}, \Delta \mapsto E_\Delta, \mathbf{a} \mapsto U(\mathbf{a}))$, we need only check that localizability holds. For this, suppose that Δ and Δ' are disjoint subsets of a single hyperplane. By microcausality, $[A_\Delta, A_{\Delta'}] = 0$, and therefore $E_\Delta E_{\Delta'}$ is a projection operator. Suppose for reductio ad absurdum that ψ is a unit vector in the range of $E_\Delta E_{\Delta'}$. Since $\Delta \cup \Delta'$ is bounded, the induction hypothesis entails that $\|A_{\Delta\cup\Delta'}\| \leq (2/3)^{m-1}$. By additivity, $A_{\Delta\cup\Delta'} = A_\Delta + A_{\Delta'}$, and therefore we obtain the contradiction:

$$(2/3)^{m-1} \;\geq\; \langle \psi, A_{\Delta\cup\Delta'}\psi\rangle \;=\; \langle \psi, A_\Delta \psi\rangle + \langle \psi, A_{\Delta'}\psi\rangle \;\geq\; (2/3)^m + (2/3)^m\,. \tag{5.18}$$

Thus, $E_\Delta E_{\Delta'} = 0$, and Malament's theorem entails that $E_\Delta = 0$ for all Δ. Therefore, $\|A_\Delta\| \leq (2/3)^m$ for all bounded Δ. □

Theorem 5.3. *Suppose that the system $(\mathcal{H}, \Delta \mapsto N_\Delta, \mathbf{a} \mapsto U(\mathbf{a}))$ of local number operators satisfies additivity, translation covariance, energy bounded below, number conservation, microcausality, and no absolute velocity. Then, $N_\Delta = 0$ for all bounded Δ.*

Proof. Let N be the unique total number operator obtained from taking the sum $\sum_n N_{\Delta_n}$ where $\{\Delta_n : n \in \mathbb{N}\}$ is a disjoint covering of Σ. Note that for any $\Delta \subseteq \Sigma$, we can choose a covering containing Δ, and hence, $N = N_\Delta + A$, where A is a positive operator. By microcausality, $[N_\Delta, A] = 0$, and therefore

$[N_\Delta, N] = [N_\Delta, N_\Delta + A] = 0$. Furthermore, for any vector ψ in the domain of N, $\langle\psi, N_\Delta\psi\rangle \leq \langle\psi, N\psi\rangle$.

Let E be the spectral measure for N, and let $E_n = E(0,n)$. Then, NE_n is a bounded operator with norm at most n. Since $[E_n, N_\Delta] = 0$, it follows that

$$\langle\psi, N_\Delta E_n\psi\rangle = \langle E_n\psi, N_\Delta E_n\psi\rangle \leq \langle E_n\psi, NE_n\psi\rangle \leq n\,, \tag{5.19}$$

for any unit vector ψ. Thus, $\|N_\Delta E_n\| \leq n$. Since $\bigcup_{n=1}^{\infty} E_n(\mathcal{H})$ is dense in $\mathcal{H}$, and since $E_n(\mathcal{H})$ is in the domain of N_Δ (for all n), it follows that if $N_\Delta E_n = 0$, for all n, then $N_\Delta = 0$. We now concentrate on proving the antecedent.

For each Δ, let $A_\Delta = (1/n)N_\Delta E_n$. We show that the structure $(\mathcal{H}, \Delta \mapsto A_\Delta, \mathbf{a} \mapsto U(\mathbf{a}))$ satisfies the conditions of Theorem 5.2. Clearly, energy bounded below and no absolute velocity hold. It is also straightforward to verify that additivity and microcausality hold. To check translation covariance, we compute:

$$\begin{aligned} U(\mathbf{a})A_\Delta U(\mathbf{a})^* &= U(\mathbf{a})N_\Delta E_n U(\mathbf{a})^* && (5.20)\\ &= U(\mathbf{a})N_\Delta U(\mathbf{a})^* U(\mathbf{a})E_n U(\mathbf{a})^* && (5.21)\\ &= U(\mathbf{a})N_\Delta U(\mathbf{a})^* E_n \;=\; N_{\Delta+\mathbf{a}}E_n \;=\; A_{\Delta+\mathbf{a}}. && (5.22) \end{aligned}$$

The third equality follows from number conservation, and the fourth equality follows from translation covariance. Thus, $N_\Delta E_n = A_\Delta = 0$ for all Δ. Since this holds for all $n \in \mathbb{N}$, $N_\Delta = 0$ for all Δ. □

Part III

Inequivalent Particle Concepts

Chapter 6

Inequivalent representations of the canonical commutation relations in quantum field theory

6.1 Introduction

Philosophical reflection on quantum field theory has tended to focus on how it revises our conception of what a particle is. For instance, though there is a self-adjoint operator in the theory representing the total number of particles of a field, the standard "Fock space" formalism does not individuate particles from one another. Thus, Teller (1995, Chapter 2) suggests that we speak of *quanta* that can be "aggregated", instead of (enumerable) *particles*—which implies that they can be distinguished and labelled. Moreover, because the theory *does* contain a total number of quanta observable (which, therefore, has eigenstates corresponding to different values of this number), a field state can be a nontrivial superposition of number eigenstates that fails to predict any particular number of quanta with certainty. Teller (1995, 105–106) counsels that we think of these superpositions as not actually containing any quanta, but only propensities to *display* various numbers of quanta when the field interacts with a "particle detector".

The particle concept seems so thoroughly denuded by quantum field theory that is hard to see how it could possibly underwrite the particulate nature of laboratory experience. Those for whom fields are the fundamental objects of the theory are especially aware of this explanatory burden:

> . . . quantum field theory is the quantum theory of a field, not a theory of "particles". However, when we consider the manner in which a quantum field interacts with other systems to which it is coupled, an interpretation of the states in [Fock space] in terms of "particles" naturally arises. It is, of course, essential that this be the case if quantum field theory is to describe observed phenomena, since "particle-like" behavior is commonly observed. (Wald 1994, 46–47)

These remarks occur in the context of Wald's discussion of yet another threat to the "reality" of quanta.

The threat arises from the possibility of inequivalent representations of the algebra of observables of a field in terms of operators on a Hilbert space. Inequivalent representations are required in a variety of situations; for example, interacting field theories in which the scattering matrix does not exist ("Haag's theorem"), free fields whose dynamics cannot be unitarily implemented (Arageorgis et al. 2001), and states in quantum statistical mechanics corresponding to different temperatures (Emch 1972). The catch is that each representation carries with it a distinct notion of "particle". Our main goal in this chapter and the next is to clarify the subtle relationship between inequivalent representations of a field theory and their associated particle concepts.

Most of our discussion will apply to any case in which inequivalent representations of a field are available. However, we have a particular interest in the case of the Minkowski versus Rindler representations of a free Boson field. What makes this case intriguing is that it involves two radically different descriptions of the particle content of the field in the *very same* spacetime region. The questions we aim to answer are:

- Are the Minkowski and Rindler descriptions nevertheless, in some sense, *physically* equivalent?
- Or, are they incompatible, even theoretically *incommensurable*?
- Can they be thought of as *complementary* descriptions in the same way that the concepts of position and momentum are?
- Or, can at most one description, the "inertial" story in terms Minkowski quanta, be the *correct* one?

Few discussions of Minkowski versus Rindler quanta broaching these questions can be found in the philosophical literature, and what discussion there is has not been sufficiently grounded in a rigorous mathematical treatment to deliver cogent answers (as we shall see). We do not intend to survey the vast physics literature about Minkowski versus Rindler quanta, nor all physical aspects of the problem. Yet a proper appreciation of what is at stake, and of which answers to the above questions are sustainable, requires that we lay out the basics of the relevant formalism. We have striven for a self-contained treatment, in the hopes of opening up the discussion to philosophers of physics already familiar with elementary non-relativistic quantum theory. (We are inclined to agree with Torretti's recent diagnosis that most philosophers of physics tend to neglect quantum field theory because they are "sickened by untidy math" (1999, 397).)

We begin in section 6.2 with a general introduction to the problem of quantizing a classical field theory. This is followed by a detailed discussion of the conceptual relationship between inequivalent representations in which we reach conclusions at variance with some of the extant literature. In section 6.3, we explain how the state of motion of an observer is taken into account when constructing a Fock space representation of a field.

6.2 Inequivalent field quantizations

In section 6.2.1 we discuss the Weyl algebra, which in the case of infinitely many degrees of freedom circumscribes the basic kinematical structure of a free Boson field. After introducing in section 6.2.2 some important concepts concerning representations of the Weyl algebra in terms of operators on Hilbert space, we shall be in a position to draw firm conclusions about the conceptual relation between inequivalent representations in section 6.2.3.

6.2.1 The Weyl algebra

Consider how one constructs the quantum-mechanical analogue of a classical system with a finite number of degrees of freedom, described by a $2n$-dimensional phase space S. Each point of S is determined by a pair of vectors $\vec{a}, \vec{b} \in \mathbb{R}^n$ whose components $\{a_j\}$ and $\{b_j\}$ encode all the position and momentum components of the system

$$x(\vec{a}) = \sum_{j=1}^{n} a_j x_j, \qquad p(\vec{b}) = \sum_{j=1}^{n} b_j p_j. \tag{6.1}$$

To quantize the system, we impose the *canonical commutation relations* (CCRs)

$$[x(\vec{a}), x(\vec{a}')] = [p(\vec{b}), p(\vec{b}')] = 0, \qquad [x(\vec{a}), p(\vec{b})] = i(\vec{a} \cdot \vec{b})I, \tag{6.2}$$

and, then, seek a representation of these relations in terms of operators on a Hilbert space $\mathcal{H}$. In the standard *Schrödinger representation*, $\mathcal{H}$ is the space of square-integrable wavefunctions $L_2(\mathbb{R}^n)$, $x(\vec{a})$ becomes the operator that multiplies a wavefunction $\psi(x_1, \ldots, x_n)$ by $\sum_{j=1}^{n} a_j x_j$, and $p(\vec{b})$ is the partial differential operator $-i \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j}$.

Note the action of $x(\vec{a})$ is not defined on an element $\psi \in L_2(\mathbb{R}^n)$ unless $x(\vec{a})\psi$ is again square-integrable, and $p(\vec{b})$ is not defined on ψ unless it is suitably differentiable. This is not simply a peculiarity of the Schrödinger representation. Regardless of the Hilbert space on which they act, two self-adjoint operators whose commutator is a nonzero scalar multiple of the identity, as in (6.2), cannot both be everywhere defined (KR 1997, Remark 3.2.9). To avoid the technical inconvenience of dealing with domains of definition, it is standard to reformulate the representation problem in terms of unitary operators which are bounded, and hence everywhere defined.

Introducing the two n-parameter families of unitary operators

$$U(\vec{a}) := e^{ix(\vec{a})}, \quad V(\vec{b}) := e^{ip(\vec{b})}, \qquad \vec{a}, \vec{b} \in \mathbb{R}^n, \tag{6.3}$$

it can be shown, at least formally, that the CCRs are equivalent to

$$U(\vec{a})U(\vec{a}') = U(\vec{a} + \vec{a}'), \qquad V(\vec{b})V(\vec{b}') = V(\vec{b} + \vec{b}'), \tag{6.4}$$

$$U(\vec{a})V(\vec{b}) = e^{i(\vec{a} \cdot \vec{b})} V(\vec{b})U(\vec{a}), \tag{6.5}$$

called the *Weyl form* of the CCRs. This equivalence holds rigorously in the Schrödinger representation, however there are "irregular" representations in which it fails (see Segal 1967, Sec. 1; Emch 1972, 228; Summers 2001). Thus, one reconstrues the goal as that of finding a representation of the Weyl form of the CCRs in terms of two concrete families of unitary operators $\{U(\vec{a}), V(\vec{b}) : \vec{a}, \vec{b} \in \mathbb{R}^n\}$ acting on a Hilbert space $\mathcal{H}$ that *can* be related, via (6.3), to canonical position and momentum operators on $\mathcal{H}$ satisfying the CCRs. We shall return to this latter "regularity" requirement later in this section.

Though the position and momentum degrees of freedom have so far been treated on a different footing, we can simplify things further by introducing the composite *Weyl operators*

$$W(\vec{a},\vec{b}) := e^{i(\vec{a}\cdot\vec{b})/2}V(\vec{b})U(\vec{a}), \qquad \vec{a},\vec{b} \in \mathbb{R}^n. \tag{6.6}$$

Combining this definition with Eqns. (6.4) and (6.5) yields the multiplication rule

$$W(\vec{a},\vec{b})W(\vec{a}',\vec{b}') = e^{-i\sigma((\vec{a},\vec{b}),(\vec{a}',\vec{b}'))/2}W(\vec{a}+\vec{a}',\vec{b}+\vec{b}'), \tag{6.7}$$

where

$$\sigma((\vec{a},\vec{b}),(\vec{a}',\vec{b}')) := (\vec{a}'\cdot\vec{b}) - (\vec{a}\cdot\vec{b}'). \tag{6.8}$$

Observe that $\sigma(\cdot,\cdot)$ is a symplectic form (i.e., an anti-symmetric, bilinear functional) on S. (Note, also, that σ is nondegenerate; i.e., if for any $f \in S$, $\sigma(f,g) = 0$ for all $g \in S$, then $f = 0$.) We set

$$W(\vec{a},\vec{b})^* := e^{-i(\vec{a}\cdot\vec{b})/2}U(-\vec{a})V(-\vec{b}) = W(-\vec{a},-\vec{b}). \tag{6.9}$$

Clearly, then, any representation of the Weyl operators $W(\vec{a},\vec{b})$ on a Hilbert space $\mathcal{H}$ gives rise to a representation of the Weyl form of the CCRs, and vice-versa.

Now, more generally, we allow our classical phase space S to be a vector space of arbitrary dimension; e.g., S may be an infinite-dimensional space constructed out of solutions to some relativistic wave equation. We assume S comes equipped with a (nondegenerate) symplectic form σ, and we say that a family $\{W_\pi(f) : f \in S\}$ of unitary operators acting on some Hilbert space $\mathcal{H}_\pi$ satisfies *the Weyl relations* just in case (cf. (6.7), (6.9))

$$W_\pi(f)W_\pi(g) = e^{-i\sigma(f,g)/2}W_\pi(f+g), \qquad f,g \in S, \tag{6.10}$$

$$W_\pi(f)^* = W_\pi(-f), \qquad f \in S. \tag{6.11}$$

We may go on to form arbitrary linear combinations of the Weyl operators, and thus obtain (at least some of) the self-adjoint operators that will serve as observables of the system.

Let $\mathcal{F}$ be the complex linear span of the set of Weyl operators $\{W_\pi(f) : f \in S\}$ acting on $\mathcal{H}_\pi$. (It follows from (6.10) that $\mathcal{F}$ is closed under taking operator products.) We say that a bounded operator A on $\mathcal{H}_\pi$ is *uniformly* approximated by operators in $\mathcal{F}$ just in case for every $\epsilon > 0$, there is an operator $\tilde{A} \in \mathcal{F}$ such that

$$\|(A-\tilde{A})x\| < \epsilon, \qquad \text{for all unit vectors } x \in \mathcal{H}_\pi. \tag{6.12}$$

If we let $\mathcal{A}_\pi$ denote the set of all bounded operators on $\mathcal{H}_\pi$ that can be uniformly approximated by elements in $\mathcal{F}$, then $\mathcal{A}_\pi$ is the C^*-algebra generated by the Weyl operators $\{W_\pi(f) : f \in S\}$. In particular, $\mathcal{A}_\pi$ is a C^*-subalgebra of the algebra $\mathbf{B}(\mathcal{H}_\pi)$ of *all* bounded operators on $\mathcal{H}_\pi$, which is itself uniformly closed and closed under taking adjoints $A \mapsto A^*$.

Suppose now that $\{W_\pi(f) : f \in S\}$ and $\{W_\phi(f) : f \in S\}$ are two systems of Weyl operators representing the same classical system but acting, respectively, on two different Hilbert spaces $\mathcal{H}_\pi$ and $\mathcal{H}_\phi$. Let $\mathcal{A}_\pi, \mathcal{A}_\phi$ denote the corresponding C^*-algebras. A bijective mapping $\alpha : \mathcal{A}_\pi \mapsto \mathcal{A}_\phi$ is called a $*$-*isomorphism* just in case α is linear, multiplicative, and commutes with the adjoint operation. We then have the following uniqueness result for the C^*-algebra generated by Weyl operators (see Bratteli & Robinson (henceforth, BR) 1996, Thm. 5.2.8).

Proposition 6.1. *There is a $*$-isomorphism α from $\mathcal{A}_\pi$ onto $\mathcal{A}_\phi$ such that $\alpha(W_\pi(f)) = W_\phi(f)$ for all $f \in S$.*

This Proposition establishes that the C^*-algebra constructed from any representation of the Weyl relations is, in fact, a unique object, independent of the representation in which we chose to construct it. We shall denote this abstract algebra, called the *Weyl algebra over* (S, σ), by $\mathcal{A}[S, \sigma]$ (and, when no confusion can result, simply say "Weyl algebra" and write $\mathcal{A}$ for $\mathcal{A}[S, \sigma]$). So our problem boils down to choosing a *representation* $(\pi, \mathcal{H}_\pi)$ of $\mathcal{A}[S, \sigma]$ given by a mapping $\pi : \mathcal{A}[S, \sigma] \mapsto \mathbf{B}(\mathcal{H}_\pi)$ preserving all algebraic relations. Note, also, that since the image $\pi(\mathcal{A}[S, \sigma])$ will always be an isomorphic copy of $\mathcal{A}[S, \sigma]$, π will always be one-to-one, and hence provide a *faithful* representation of $\mathcal{A}[S, \sigma]$.

With the representation-independent character of $\mathcal{A}[S, \sigma]$, why should we care any longer to choose a representation? After all, there is no technical obstacle to proceeding abstractly. We can take the self-adjoint elements of $\mathcal{A}[S, \sigma]$ to be the quantum-mechanical observables of our system. A linear functional ω on $\mathcal{A}[S, \sigma]$ is called a *state* just in case ω is positive (i.e., $\omega(A^*A) \geq 0$) and normalized (i.e., $\omega(I) = 1$). As usual, a state ω is said to be *pure* (and *mixed* otherwise) just in case it is not a nontrivial convex combination of other states of $\mathcal{A}$. The dynamics of the system can be represented by a one-parameter group α_t of automorphisms of $\mathcal{A}$ (i.e., each α_t is just a map of $\mathcal{A}$ onto itself that preserves all algebraic relations). Hence, if we have some initial state ω_0, the final state will be given by $\omega_t = \omega_0 \circ \alpha_t$. We can even supply definitions for the probability in the state ω_t that a self-adjoint element $A \in \mathcal{A}$ takes a value lying in some Borel subset of its spectrum (Wald 1994, 79–80), and for transition probabilities between, and superpositions of, pure states of $\mathcal{A}$ (Roberts & Roepstorff 1969). At no stage, it seems, need we ever introduce a Hilbert space as an essential element of the formalism. In fact, Haag and Kastler (1964, 852) and Robinson (1966, 488) maintain that the choice of a representation is largely a matter of analytical convenience without physical implications.

Nonetheless, the abstract Weyl algebra does not contain unbounded operators, many of which are naturally taken as corresponding to important physical quantities. For instance, the total energy of the system, the canonically conjugate position and momentum observables—which in field theory play the role of the local field observables—and the total number of particles are all represented by unbounded operators. Also, we shall see later that not even any *bounded* function of the total number of particles (apart from zero and the identity) lies in the Weyl algebra. Surprisingly, Irving Segal (one of the founders of the mathematically rigorous approach to quantum field theory) has written that this:

> . . . has the simple if quite rough and somewhat oversimplified interpretation that the total number of "bare" particles is devoid of physical meaning. (Segal 1963, 56; see also Segal 1959, 12)

We shall return to this issue of physical meaning shortly. First, let us see how a representation can be used to expand the observables of a system beyond the abstract Weyl algebra.

Let $\mathcal{F}$ be a family of bounded operators acting on a representation space $\mathcal{H}_\pi$. We say that a bounded operator A on $\mathcal{H}_\pi$ can be *weakly* approximated by elements of $\mathcal{F}$ just in case for any vector $x \in \mathcal{H}$, and any $\epsilon > 0$, there is some $\tilde{A} \in \mathcal{F}$ such that

$$|\langle x, Ax\rangle - \langle x, \tilde{A}x\rangle| < \epsilon. \tag{6.13}$$

(Note the important quantifier change between the definitions of uniform and weak approximation, and that weak approximation has no abstract representation-independent counterpart.) Consider the family $\pi(\mathcal{A})^-$ of bounded operators that can be weakly approximated by elements of $\pi(\mathcal{A})$, i.e., $\pi(\mathcal{A})^-$ is the weak closure of $\pi(\mathcal{A})$. By von Neumann's double commutant theorem, $\pi(\mathcal{A})^- = \pi(\mathcal{A})''$, where the prime operation on a family of operators (here applied twice) denotes the set of all bounded operators on $\mathcal{H}_\pi$ commuting with each member of that family. $\pi(\mathcal{A})''$ is called the *von Neumann algebra* generated by $\pi(\mathcal{A})$. Clearly $\pi(\mathcal{A}) \subseteq \pi(\mathcal{A})''$, however we can hardly expect that $\pi(\mathcal{A}) = \pi(\mathcal{A})''$ when $\mathcal{H}_\pi$ is infinite-dimensional (which it *must* be, since there is no finite-dimensional representation of the

Weyl algebra for even a single degree of freedom). Nor should we generally expect that $\pi(\mathcal{A})'' = \mathbf{B}(\mathcal{H}_\pi)$, though this does hold in "irreducible" representations, as we explain in the next subsection.

We may now expand our observables to include all self-adjoint operators in $\pi(\mathcal{A})''$. And, although $\pi(\mathcal{A})''$ still contains only bounded operators, it is easy to associate (potentially physically significant) unbounded observables with this algebra as well. We say that a (possibly unbounded) self-adjoint operator A on $\mathcal{H}_\pi$ is *affiliated* with $\pi(\mathcal{A})''$ just in case all A's spectral projections lie in $\pi(\mathcal{A})''$. Of course, we could have adopted the same definition for self-adjoint operators "affiliated to" $\pi(\mathcal{A})$ itself, but C^*-algebras do not generally contain nontrivial projections (or, if they do, will not generally contain even the spectral projections of their self-adjoint members).

As an example, suppose we now demand to have a (so-called) *regular* representation π, in which, for each fixed $f \in S$, the mapping $\mathbb{R} \ni t \mapsto \pi(W(tf))$ is weakly continuous. Then Stone's theorem will guarantee the existence of unbounded self-adjoint operators $\{\Phi(f) : f \in S\}$ on $\mathcal{H}_\pi$ satisfying $\pi(W(tf)) = e^{it\,\Phi(f)}$, and it can be shown that all these operators are affiliated to $\pi(\mathcal{A}[S,\sigma])''$ (KR 1997, Ex. 5.7.53(ii)). In this way, we can recover as observables our original canonically conjugate positions and momenta (cf. Eqn. (6.3)), which the Weyl relations ensure will satisfy the original unbounded form of the CCRs.

It is important to recognize, however, that by enlarging the set of observables to include those affiliated to $\pi(\mathcal{A}[S,\sigma])''$, we have now left ourselves open to arbitrariness. In contrast to Proposition 6.1, we now have:

Proposition 6.2. *If S is infinite-dimensional, then there are regular representations π, ϕ of $\mathcal{A}[S,\sigma]$ for which there is no $*$-isomorphism α from $\pi(\mathcal{A}[S,\sigma])''$ onto $\phi(\mathcal{A}[S,\sigma])''$ such that $\alpha(\pi(W(f))) = \phi(W(f))$ for all $f \in S$.*

This occurs when the representations are "disjoint," which we discuss in the next subsection.

Proposition 6.2 is what motivates Segal to argue that observables affiliated to the weak closure $\pi(\mathcal{A}[S,\sigma])''$ in a representation of the Weyl algebra are "somewhat unphysical" and "have only analytical significance" (1963, 11–14, 134).[1] Segal is explicit that by "physical" he means "empirically

[1] Actually, Segal consistently finds it convenient to work with a strictly larger algebra than our (minimal) Weyl algebra, sometimes called the *mode finite* or *tame* Weyl algebra. However, both Proposition 1 (see Baez et al. 1992, Thm. 5.1) and Proposition 2 continue to hold for the tame Weyl algebra (also cf. Segal 1967, 128–129).

measurable in principle" (1963, 11). We should not be confused by the fact that he often calls observables that fail this test "conceptual" (suggesting they are more than mere analytical crutches). For in (Baez et al. 1992, 145), Segal gives as an example the bounded self-adjoint operator $\cos(P) + (Q^2 + 1)^{-1}$ on $L_2(\mathbb{R})$ "for which no known 'Gedanken experiment' will actually directly determine the spectrum, and so [it] represents an observable in a purely conceptual sense." Thus, the most obvious reading of Segal's position is that he subscribes to an operationalist view about the physical significance of theoretical quantities. Indeed, since good reasons *can* be given for the impossibility of exact ("sharp") measurements of observables in the von Neumann algebra generated by a C^*-algebra (see Wald 1994, 79ff; Halvorson 2001a), operationalism explains Segal's dismissal of the physical (as opposed to analytical) significance of observables not in the Weyl algebra *per se*. (And it is worth recalling that Bridgman himself was similarly unphased by having to relegate much of the mathematical structure of a physical theory to "a ghostly domain with no physical relevance" (1936, 116).)

Of course, insofar as operationalism is philosophically defensible at all, it does not compel assent. And, in this instance, Segal's operationalism has not dissuaded others from taking the more liberal view advocated by Wald:

> . . . one should not view [the Weyl algebra] as encompassing *all* observables of the theory; rather, one should view [it] as encompassing a "minimal" collection of observables, which is sufficiently large to enable the theory to be formulated. One may later wish to enlarge [the algebra] and/or further restrict the notion of "state" in order to accommodate the existence of additional observables. (Wald 1994, 75)

The conservative and liberal views entail quite different commitments about the physical equivalence of representations—or so we shall argue.

6.2.2 Equivalence and disjointness of representations

It is essential that precise mathematical definitions of equivalence be clearly distinguished from the, often dubious, arguments that have been offered for their conceptual significance. We confine this section to discussing the definitions.

Since our ultimate goal is to discuss the Minkowski and Rindler quantizations of the Weyl algebra, we only need to consider the case where one of the two representations at issue, say π, is "irreducible" and the other, ϕ, is "factorial". A representation π of $\mathcal{A}$ is called *irreducible* just in case no non-trivial subspace of the Hilbert space $\mathcal{H}_\pi$ is invariant under the action of all operators in $\pi(\mathcal{A})$. It is not difficult to see that this is equivalent to $\pi(\mathcal{A})'' = \mathbf{B}(\mathcal{H}_\pi)$ (using the fact that a subspace is left invariant by $\pi(\mathcal{A})$ just in case the projection onto that subspace commutes with all operators in $\pi(\mathcal{A})$). A representation ϕ of $\mathcal{A}$ is called *factorial* just in case the von Neumann algebra $\phi(\mathcal{A})''$ is a *factor*, i.e., it has trivial center (the only operators in $\phi(\mathcal{A})''$ that commute with all other operators in $\phi(\mathcal{A})''$ are multiples of the identity). Since $\mathbf{B}(\mathcal{H}_\pi)$ is a factor, π is irreducible only if it is factorial. Thus, the Schrödinger representation of the Weyl algebra is both irreducible and factorial.

The strongest form of equivalence between representations is unitary equivalence: ϕ and π are said to be *unitarily equivalent* just in case there is a unitary operator U mapping $\mathcal{H}_\phi$ onto $\mathcal{H}_\pi$, and such that

$$U\phi(A)U^{-1} = \pi(A) \qquad \forall A \in \mathcal{A}. \tag{6.14}$$

There are two other weaker definitions of equivalence.

Given a family π_i of irreducible representations of the Weyl algebra on Hilbert spaces $\mathcal{H}_i$, we can construct another (reducible) representation ϕ of the Weyl algebra on the direct sum Hilbert space $\sum_i \oplus \mathcal{H}_i$, by setting

$$\phi(A) = \sum_i \oplus \, \pi_i(A), \qquad A \in \mathcal{A}. \tag{6.15}$$

If each representation $(\pi_i, \mathcal{H}_i)$ is unitarily equivalent to some fixed representation $(\pi, \mathcal{H})$, we say that $\phi = \sum \oplus \pi_i$ is a *multiple* of the representation π. Furthermore, we say that two representations of the Weyl algebra, ϕ (factorial) and π (irreducible), are *quasi-equivalent* just in case ϕ is a multiple of π. It should be obvious from this characterization that quasi-equivalence weakens unitary equivalence. Another way to see this is to use the fact (KR 1997, Def. 10.3.1, Cor. 10.3.4) that quasi-equivalence of ϕ and π is equivalent to the existence of a $*$-isomorphism α from $\phi(\mathcal{A})''$ onto $\pi(\mathcal{A})''$ such that $\alpha(\phi(A)) = \pi(A)$ for all $A \in \mathcal{A}$. Unitary equivalence is then just the special case where the $*$-isomorphism α can be implemented by a unitary operator.

If ϕ is not even quasi-equivalent to π, then we say that ϕ and π are *disjoint* representations of $\mathcal{A}$.[2] Note, then, that if both π and ϕ are irreducible,

[2] In general, disjointness is not defined as the negation of quasi-equivalence, but by the more cumbersome formulation: Two representations π, ϕ are disjoint just in case π has no "subrepresentation" quasi-equivalent to ϕ, and ϕ has no subrepresentation quasi-

they are either unitarily equivalent or disjoint.

We can now state the following pivotal result (von Neumann 1931).

Stone-von Neumann Uniqueness Theorem. *When S is finite-dimensional, every regular representation of $\mathcal{A}[S,\sigma]$ is quasi-equivalent to the Schrödinger representation.*

This theorem is usually interpreted as saying that there is a unique quantum theory corresponding to a classical theory with finitely-many degrees of freedom. (But see section 7.5, where we question this interpretation of the theorem.) The theorem *fails* in field theory—where S is infinite-dimensional—opening the door to disjoint representations and Proposition 6.2.

There is another way to think of the relations between representations, in terms of states. Recall the abstract definition of a state of a C^*-algebra, as simply a positive normalized linear functional on the algebra. Since, for any representation π, $\pi(\mathcal{A})$ is isomorphic to $\mathcal{A}$, π induces a one-to-one correspondence between the abstract states of $\mathcal{A}$ and the abstract states of $\pi(\mathcal{A})$. Note now that *some* of the abstract states on $\pi(\mathcal{A})$ are the garden-variety density operator states that we are familiar with from elementary quantum mechanics. In particular, define ω_D on $\pi(\mathcal{A})$ by picking a density operator D on $\mathcal{H}_\pi$ and setting

$$\omega_D(A) := \mathrm{Tr}(DA), \qquad A \in \pi(\mathcal{A}). \tag{6.16}$$

In general, however, there are abstract states of $\pi(\mathcal{A})$ that are not given by density operators via Eqn. (6.16).[3] We say then that an abstract state ω of $\pi(\mathcal{A})$ is *normal* just in case it is given (via Eqn. (6.16)) by some density operator D on $\mathcal{H}_\pi$. We let $\mathfrak{F}(\pi)$ denote the subset of the abstract state space of $\mathcal{A}$ consisting of those states that correspond to normal states in the representation π, and we call $\mathfrak{F}(\pi)$ the *folium* of the representation π. That is, $\omega \in \mathfrak{F}(\pi)$ just in case there is a density operator D on $\mathcal{H}_\pi$ such that

$$\omega(A) = \mathrm{Tr}(D\pi(A)), \qquad A \in \mathcal{A}. \tag{6.17}$$

equivalent to π. Since we are only interested, however, in the special case where π is irreducible (and hence has no non-trivial subrepresentations) and ϕ is "factorial" (and hence is quasi-equivalent to each of its subrepresentations), the cumbersome formulation reduces to our definition.

[3] Gleason's theorem does not rule out these states because it is not part of the definition of an abstract state that it be countably additive over mutually orthogonal projections. Indeed, such additivity does not even make sense abstractly, because an infinite sum of orthogonal projections can never converge uniformly, only weakly (in a representation).

We then have the following equivalences (KR 1997, Prop. 10.3.13):

$$\begin{aligned} \pi \text{ and } \phi \text{ are quasi-equivalent} &\iff \mathfrak{F}(\pi) = \mathfrak{F}(\phi), \\ \pi \text{ and } \phi \text{ are disjoint} &\iff \mathfrak{F}(\pi) \cap \mathfrak{F}(\phi) = \emptyset. \end{aligned}$$

In other words, π and ϕ are quasi-equivalent just in case they share the same normal states; and π and ϕ are disjoint just in case they have *no* normal states in common.

In fact, if π is disjoint from ϕ, then all normal states in the representation π are "orthogonal" to all normal states in the representation ϕ. We may think of this situation intuitively as follows. Define a third representation ψ of $\mathcal{A}$ on $\mathcal{H}_\pi \oplus \mathcal{H}_\phi$ by setting

$$\psi(A) = \pi(A) \oplus \phi(A), \qquad A \in \mathcal{A}. \tag{6.18}$$

Then, every normal state of the representation π is orthogonal to every normal state of the representation ϕ.[4] This makes sense of the oft-repeated phrase (see e.g. Gerlach 1989) that "The Rindler vacuum is orthogonal to all states in the Minkowski vacuum representation".

While not every abstract state of $\mathcal{A}$ will be in the folium of a given representation, there is always *some* representation of $\mathcal{A}$ in which the state *is* normal, as a consequence of the following (see KR 1997, Thms. 4.5.2 and 10.2.3).

Gelfand-Naimark-Segal Theorem. *Any abstract state ω of a C^*-algebra $\mathcal{A}$ gives rise to a unique (up to unitary equivalence) representation $(\pi_\omega, \mathcal{H}_\omega)$ of $\mathcal{A}$ and vector $\Omega_\omega \in \mathcal{H}_\omega$ such that*

$$\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle, \qquad A \in \mathcal{A}, \tag{6.19}$$

and such that the set $\{\pi_\omega(A)\Omega_\omega : A \in \mathcal{A}\}$ is dense in $\mathcal{H}_\omega$. Moreover, π_ω is irreducible just in case ω is pure.

The triple $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ is called the *GNS representation* of $\mathcal{A}$ induced by the state ω, and Ω_ω is called a *cyclic* vector for the representation. We shall see in the next main section how the Minkowski and Rindler vacuum states of $\mathcal{A}$ induce disjoint GNS representations.

There is a third notion of equivalence of representations, still weaker than quasi-equivalence. Let π be a representation of $\mathcal{A}$, and let $\mathfrak{F}(\pi)$ be the

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[4] This intuitive picture may be justified by making use of the "universal representation" of $\mathcal{A}$ (KR 1997, Thm. 10.3.5).

folium of π. We say that an abstract state ω of $\mathcal{A}$ can be *weak* approximated* by states in $\mathfrak{F}(\pi)$ just in case for each $\epsilon > 0$, and for each finite collection $\{A_i : i = 1, \ldots, n\}$ of operators in $\mathcal{A}$, there is a state $\omega' \in \mathfrak{F}(\pi)$ such that

$$\left|\omega(A_i) - \omega'(A_i)\right| < \epsilon, \qquad i = 1, \ldots, n\,. \tag{6.20}$$

Two representations π, ϕ are then said to be *weakly equivalent* just in case all states in $\mathfrak{F}(\pi)$ may be weak* approximated by states in $\mathfrak{F}(\phi)$ and vice versa. In other words, the weak* closure of $\mathfrak{F}(\pi)$ contains $\mathfrak{F}(\phi)$ and vice versa. We then have the following fundamental result (Fell 1960).

Fell's Theorem. *Let π be a faithful representation of a C^*-algebra $\mathcal{A}$. Then, every abstract state of $\mathcal{A}$ is weak* approximated by states in $\mathfrak{F}(\pi)$.*

In particular, then, it follows that *all* representations of $\mathcal{A}[S, \sigma]$ are weakly equivalent.

In summary, we have the following implications for any two representations π, ϕ:

Unitarily equivalent $\Longrightarrow$ Quasi-equivalent $\Longrightarrow$ Weakly equivalent.

If π and ϕ are both irreducible, then the first arrow is reversible.

6.2.3 Physical equivalence of representations

Do disjoint representations yield *physically* inequivalent theories? It depends on what one takes to be the physical content of a theory, and what one means by "equivalent theories"—subjects about which philosophers of science have had plenty to say.

Recall that Reichenbach (1938) deemed two theories "the same" just in case they are empirically equivalent, i.e., they are confirmed equally under all possible evidence. Obviously this criterion, were we to adopt it here, would beg the question against those who (while agreeing that, strictly speaking, only self-adjoint elements of the Weyl algebra can actually be measured) attribute physical significance to "global" quantities only definable in a representation, like the total number of particles.

A stronger notion of equivalence, formulated by Glymour (1971) (who proposed it only as a *necessary* condition), is that two theories are equivalent only if they are "intertranslatable". This is often cashed out in logical terms as the possibility of defining the primitives of one theory in terms of those of the other so that the theorems of the first appear as logical consequences of those of the second, and vice-versa. Prima facie, this criterion is ill-suited to the present context, because the different "theories" are not presented to us as syntactic structures or formalized logical systems, but rather two competing algebras of observables whose states represent physical predictions. In addition, intertranslatability *per se* has nothing to say about what portions of the mathematical formalism of the two physical theories being compared ought to be intertranslatable, and what should be regarded as "surplus mathematical structure" not required to be translated.

Nevertheless, we believe the intertranslatability thesis can be naturally expressed in the present context and rendered neutral as between the conservative and liberal approaches to physical observables discussed earlier. Think of the Weyl operators $\{\phi(W(f)) : f \in S\}$ and $\{\pi(W(f)) : f \in S\}$ as the primitives of our two "theories", in analogy with the way the natural numbers can be regarded as the primitives of a "theory" of real numbers. Just as we may define rational numbers as ratios of natural numbers, and then construct real numbers as the limits of Cauchy sequences of rationals, we construct the Weyl algebras $\phi(\mathcal{A})$ and $\pi(\mathcal{A})$ by taking linear combinations of the Weyl operators and then closing in the uniform topology. We then close in the weak topology of the two representations to obtain the von Neumann algebras $\phi(\mathcal{A})''$ and $\pi(\mathcal{A})''$. Whether the observables affiliated with this second closure have physical significance is controversial, as is whether we should be conservative and take only normal states in the given representation to be physical, or be more liberal and admit a broader class of algebraic states. The analogue of the "theorems" of the theory are then statements about the expectation values dictated by the physical states for the self-adjoint elements in the physically relevant algebra of the theory.

We therefore propose the following formal rendering of Glymour's intertranslatability thesis adapted to the present context: Representations ϕ and π are *physically equivalent* only if there exists a bijective mapping α from the physical observables of the representation ϕ to the physical observables of the representation π, and another bijective mapping β from the physical states of the representation ϕ to the physical states of the representation π, such that

$$\alpha(\phi(W(f))) = \pi(W(f)), \qquad \forall f \in S, \tag{6.21}$$
$$\text{("primitives")}$$

$$\beta(\omega)(\alpha(A)) = \omega(A), \qquad \forall \text{ states } \omega,\ \forall \text{ observables } A. \tag{6.22}$$
$$\text{("theorems")}$$

Of course, the notion of equivalence we obtain depends on how we construe the domain of the universal quantifiers in (6.22). According to a conservative rendering of observables, only the self-adjoint elements of the Weyl algebra $\pi(\mathcal{A})$ are genuine physical observables of the representation π. (More generally, an unbounded self-adjoint operator on $\mathcal{H}_\pi$ is a physical observable only if all of its bounded functions lie in $\pi(\mathcal{A})$.) On the other hand, a liberal rendering of observables considers all self-adjoint operators in the weak closure $\pi(\mathcal{A})^-$ of $\pi(\mathcal{A})$ as genuine physical observables. (More generally, those unbounded self-adjoint operators whose bounded functions lie in $\pi(\mathcal{A})^-$, i.e., all such operators affiliated with $\pi(\mathcal{A})^-$, should be considered genuine physical observables.) A conservative with respect to states claims that only the density operator states (i.e., normal states) of the algebra $\pi(\mathcal{A})$ are genuine physical states. On the other hand, a liberal with respect to states claims that all algebraic states of $\pi(\mathcal{A})$ should be thought of as genuine physical states. We thereby obtain *four distinct* necessary conditions for physical equivalence, according to whether one is conservative or liberal about observables, and conservative or liberal about states.[5]

Arageorgis (1995, 302) and Arageorgis et al. (2001, 3) also take the correct notion of physical equivalence in this context to be intertranslatability. On the basis of informal discussions (with rather less supporting argument than one would have liked), they claim that physical equivalence of representations requires that they be unitarily equivalent. (They do not discuss quasi-equivalence.) We disagree with this conclusion, but there is still substantial overlap between us. For instance, with our precise necessary condition for physical equivalence above, we may establish the following elementary result.

Proposition 6.3. *Under the conservative approach to states, ϕ and π are physically equivalent representations of $\mathcal{A}$ only if they are quasi-equivalent.*

With somewhat more work, the following result may also be established.[6]

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[5]The distinction between the conservative and liberal positions about observables could be further ramified by taking into account the distinction—which is suppressed throughout this chapter—between local and global observables. In particular, if all (and only) locally measurable observables have genuine physical status, then physical equivalence of π and ϕ would require a bijection α between *local* observables in $\pi(\mathcal{A}[S,\sigma])^-$ and *local* observables in $\phi(\mathcal{A}[S,\sigma])^-$. Similarly, the distinction between the conservative and liberal positions about states could be further ramified by taking into account the distinction between normal states and "locally normal" states.

[6]Our proof in the appendix makes rigorous Arageorgis' brief (and insufficient) reference to Wigner's symmetry representation theorem in his (1995, 302, footnote).

Proposition 6.4. *Under the liberal approach to observables, ϕ and π are physically equivalent representations of $\mathcal{A}$ only if they are quasi-equivalent.*

The above results leave only the position of the "conservative about observables/liberal about states" undecided. However, we claim, *pace* Arageorgis et al., that a proponent of this position can satisfy conditions (6.21), (6.22) *without* committing himself to quasi-equivalence of the representations. Since he is conservative about observables, Proposition 6.1 already guarantees the existence of a bijective mapping α—in fact, a $*$-isomorphism from the whole of $\phi(\mathcal{A})$ to the whole of $\pi(\mathcal{A})$—satisfying (6.21). And if he is liberal about states, the state mapping β need not map any normal state of $\phi(\mathcal{A})$ into a normal state of $\pi(\mathcal{A})$, bypassing the argument for Proposition 6.3. Indeed, since the liberal takes *all* algebraic states of $\phi(\mathcal{A})$ and $\pi(\mathcal{A})$ to be physically significant, for any algebraic state ω of $\phi(\mathcal{A})$, the bijective mapping β that sends ω to the state $\omega \circ \alpha^{-1}$ on $\pi(\mathcal{A})$ trivially satisfies condition (6.22) even when ϕ and π are disjoint.

Though we have argued that Segal was conservative about observables, we are not claiming he was liberal about states. In fact, Segal consistently maintained that only the "regular states" of the Weyl algebra have physical relevance (1961, 7; 1967, 120, 132). A state ω of $\mathcal{A}[S, \sigma]$ is called *regular* just in case the map $f \mapsto \omega(W(f))$ is continuous on all finite-dimensional subspaces of S; or, equivalently, just in case the GNS representation of $\mathcal{A}[S, \sigma]$ determined by ω is regular (Segal 1967, 134). However, note that, unlike normality of a state, regularity is representation-independent. Taking the set of all regular states of the Weyl algebra to be physical is therefore still liberal enough to permit satisfaction of condition (6.22). For the mapping β of the previous paragraph trivially preserves regularity, insofar as both ω and $\omega \circ \alpha^{-1}$ induce the same abstract regular state of $\mathcal{A}[S, \sigma]$.

Our verdict, then, is that Segal, for one, is not committed to saying that physically equivalent representations are quasi-equivalent. And this explains why he sees fit to *define* physical equivalence of representations in such a way that Proposition 6.1 secures the physical equivalence of all representations (see Segal 1961, Defn. 1(c)). (Indeed, Segal regards Proposition 6.1 as the appropriate generalization of the Stone-von Neumann uniqueness theorem to infinite-dimensional S.) One might still ask what the point of passing to a concrete Hilbert space representation of $\mathcal{A}[S, \sigma]$ is if one is going to allow as physically possible regular states not in the folium of the chosen representation. The point, we take it, is that if we are interested in drawing out the predictions of some particular regular state, such as the Minkowski vacuum or the Rindler vacuum, then passing to a particular representation will put at our disposal all the standard analytical techniques of Hilbert space quantum mechanics to facilitate calculations in that particular state.[7]

Haag and Kastler (1964, 852) and Robinson (1966, 488) have argued that *by itself* the *weak* equivalence of all representations of the Weyl algebra entails their physical equivalence.[8] Their argument starts from the fact that, by measuring the expectations of a finite number of observables $\{A_i\}$ in the Weyl algebra, each to a finite degree of accuracy ϵ, we can determine the state of the system only to within a weak* neighborhood. But by Fell's density theorem, states from the folium of *any* representation lie in this neighborhood. So we can never determine in practice which representation is the physically "correct" one and they all, in some (as yet, unarticulated!) sense, carry the same physical content. And as a corollary, choosing a representation is simply a matter of convention.

Clearly the necessary conditions for physical equivalence we have proposed constitute very different notions of equivalence than weak equivalence, so we are not disposed to agree with this argument. Evidently it presupposes that only the observables in the Weyl algebra itself are physically significant, which we have granted *could* be grounded in operationalism. However, there is an additional layer of operationalism that the argument must presuppose: scepticism about the physical meaning of postulating an *absolutely precise* state for the system. If we follow this scepticism to its logical conclusion, we should instead think of physical states of the Weyl algebra as represented by weak* neighborhoods of algebraic states. What it would then mean to falsify a state, so understood, is that some finite number of expectation values measured to within finite accuracy are found to be incompatible with all the algebraic states in some proposed weak* neighborhood. Unfortunately, no particular "state" in this sense can ever be fully empirically adequate, for any hypothesized state (= weak* neighborhood) will be subject to constant revision as the accuracy and number of our experiments increase. We agree with Summers (2001) that this would do irreparable damage to the predictive power of the theory—damage that can only be avoided by maintaining

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[7]In support of not limiting the physical states of the Weyl algebra to any one representation's folium, one can also cite the cases of non-unitarily implementable dynamics discussed by Arageorgis et al. (2001) in which dynamical evolution occurs between regular states that induce disjoint GNS representations. In such cases, it would hardly be coherent to maintain that regular states *dynamically accessible to one another* are not physically co-possible.

[8]Indeed, the term "physical equivalence" is often used synonomously with weak equivalence; for example, by Emch (1972, 108), who, however, issues the warning that "we should be seriously wary of semantic extrapolations" from this usage. Indeed!

that there is a correct algebraic state.

We do not, however, agree with Summers' presumption (tacitly endorsed by Arageorgis et al. 2001) that we not only need the correct algebraic state, but "...the correct state *in the correct representation*" (2001, 13; italics ours). This added remark of Summers' is directed against the conventionalist corollary to Fell's theorem. Yet we see nothing in the point about predictive power that privileges any particular representation, not even the GNS representation of the predicted state. We might well have good reason to deliberately choose a representation in which the *precise* algebraic state predicted is not normal. (For example, Kay (1985) does exactly this, by "constructing" the Minkowski vacuum as a thermal state in the Rindler quantization.) The role Fell's theorem plays is then, at best, methodological. All it guarantees is that when we calculate with density operators in our chosen represention, we can always get a reasonably good indication of the predictions of *whatever* precise algebraic state we have postulated for the system.

6.2.4 The liberal stance on observables

So much for the conservative stance on observables. An interpreter of quantum field theory is not likely to find it attractive, if only because none of the observables that have any chance of underwriting the particle concept lie in the Weyl algebra. But suppose, as interpreters, we adopt the liberal approach to observables. Does the physical inequivalence of disjoint representations entail their incompatibility, or even incommensurability? By this, we do not mean to conjure up Kuhnian thoughts about incommensurable "paradigms", whose proponents share no methods to resolve their disputes. Rather, we are pointing to the (more Feyerabendian?) possibility of an unanalyzable shift in meaning between disjoint representations as a consequence of the fact that the concepts (observables and/or states) of one representation are not wholly definable or translatable in terms of those of the other.

One might think of neutralizing this threat by viewing disjoint representations as sub-theories or models of a more general theory built upon the Weyl algebra. Consider the analogy of *two different* classical systems, modelled, say, by phase spaces of different dimension. Though not physically equivalent, these models hardly define incommensurable theories insofar as they share the characteristic kinematical and dynamical features that warrant the term "classical." Surely the same could be said of disjoint representations of the Weyl algebra?

There is, however, a crucial disanalogy that needs to be taken into account. In the case of the Minkowski and Rindler representations, physicists freely switch between them to describe the quantum state of the *very same* "system"—in this case, the quantum field in a fixed region of spacetime (see, e.g., Unruh & Wald 1984; Wald 1994, Sec. 5.1). And, as we shall see later, the weak closures of these representations provide physically inequivalent descriptions of the particle content in the region. So it is tempting to view this switching back and forth between disjoint representations as conceptually incoherent (Arageorgis 1995, 268), and to see the particle concepts associated to the representations as not just different, but outright incommensurable (Arageorgis et al. 2001).

We shall argue that this view, tempting as it is, goes too far. For suppose we *do* take the view that the observables affiliated to the von Neumann algebras generated by two disjoint representations ϕ and π simply represent different physical aspects of the same physical system. If we are also liberal about states (not restricting ourselves to any one representation's folium), then it is natural to ask what implications a state ω of our system, that happens to be in the folium of ϕ, has for the observables in $\pi(\mathcal{A})''$. In many cases, it is possible to extract a definite answer.

In particular, any abstract state ω of $\mathcal{A}$ gives rise to a state on $\pi(\mathcal{A})$, which may be extended to a state on the weak closure $\pi(\mathcal{A})^-$ (KR 1997, Thm. 4.3.13). The only catch is that unless $\omega \in \mathfrak{F}(\pi)$, this extension will not be unique. For, only normal states of $\pi(\mathcal{A})$ possess sufficiently nice continuity properties to ensure that their values on $\pi(\mathcal{A})$ uniquely fix their values on the weak closure $\pi(\mathcal{A})^-$ (see KR 1997, Thm. 7.1.12). *However*, it may happen that all extensions of ω agree on the expectation value they assign to a *particular observable* affiliated to $\pi(\mathcal{A})^-$. This is the strategy we shall use to make sense of assertions such as "The Minkowski vacuum in a (Rindler) spacetime wedge is full of Rindler quanta" (cf., e.g., DeWitt 1979a). The very fact that such assertions can be made sense of *at all* takes the steam out of claims that disjoint representations are necessarily incommensurable. Indeed, we shall ultimately argue that this shows disjoint representations should not be treated as *competing* "theories" in the first place. Rather, they are better viewed as supplying physically different, "complementary" perspectives on the same quantum system from within a broader theoretical framework that does not privilege a particular representation.

6.3 Constructing representations

We now explain how to construct "Fock representations" of the CCRs. In sections 6.3.1 and 6.3.2 we show how this construction depends on one's choice of preferred timelike motion in Minkowski spacetime. In section 6.4, we show that alternative choices of preferred timelike motion can result in unitarily inequivalent—indeed, disjoint—representations.

6.3.1 First quantization ("Splitting the frequencies")

The first step in the quantization scheme consists in turning the classical phase space (S, σ) into a quantum-mechanical "one particle space"—i.e., a Hilbert space. *The non-uniqueness of the quantization scheme comes in at this very first step.*

Depending on our choice of preferred timelike motion, we will have a one-parameter group T_t of linear mappings from S onto S representing the evolution of the classical system in time. The flow $t \mapsto T_t$ should also preserve the symplectic form. A bijective real-linear mapping $T : S \mapsto S$ is called a *symplectomorphism* just in case T preserves the symplectic form; i.e., $\sigma(Tf, Tg) = \sigma(f, g)$ for all $f, g \in S$.

We say that J is a *complex structure* for (S, σ) just in case

1. J is a symplectomorphism,
2. $J^2 = -I$,
3. $\sigma(f, Jf) > 0, \qquad 0 \neq f \in S$.

Relative to a complex structure J, we can extend the scalar multiplication on S to complex numbers; viz., define multiplication by $a + ib$ by setting $(a + ib)f := af + bJf \in S$. We can also define an inner product $(\cdot, \cdot)_J$ on the resulting complex vector space by setting

$$(f, g)_J := \sigma(f, Jg) + i\sigma(f, g), \qquad f, g \in S. \tag{6.23}$$

We let $\mathcal{S}_J$ denote the Hilbert space that results when we equip (S, σ) with the extended scalar multiplication and inner product $(\cdot, \cdot)_J$ and then take the completion in the usual way.

A symplectomorphism T is (by assumption) a real-linear operator on S. However, it does not automatically follow that T is a *complex*-linear operator on $\mathcal{S}_J$, since $T(if) = i(Tf)$ may fail. If, however, T commutes with J, then T will be a complex-linear operator on $\mathcal{S}_J$, and it is easy to see that $(Tf, Tg)_J = (f, g)_J$ for all $f, g \in \mathcal{S}_J$, so T would in fact be unitary. Accordingly, we say that a group T_t of symplectomorphisms on (S, σ) is *unitarizable* relative to J just in case $[J, T_t] = 0$ for all $t \in \mathbb{R}$.

If T_t is unitarizable and $t \mapsto T_t$ is weakly continuous, so that we have $T_t = e^{itH}$ (by Stone's theorem), we say that T_t has *positive energy* just in case H is a positive operator. In general, we say that $(\mathcal{H}, U_t)$ is a *quantum one particle system* just in case $\mathcal{H}$ is a Hilbert space and U_t is a weakly continuous one-parameter unitary group on $\mathcal{H}$ with positive energy. Kay (1979) proved:

Theorem (Kay). *Let T_t be a one-parameter group of symplectomorphisms of (S, σ). If there is a complex structure J on (S, σ) such that $(\mathcal{S}_J, T_t)$ is a quantum one particle system, then J is unique.*

Physically, the time translation group T_t determines a natural decomposition (or "splitting") of the solutions of the relativistic wave equation we are quantizing into those that oscillate with purely positive and with purely negative frequency with respect to the motion. This has the effect of uniquely fixing a choice of J, and the Hilbert space $\mathcal{S}_J$ then provides a representation of the positive frequency solutions alone.[9]

We shall see in the next section how the representation space of a 'Fock' representation of the Weyl algebra is constructed directly from the Hilbert space $\mathcal{S}_J$. Thus, as we claimed, the nonuniqueness of the resulting representation stems entirely from the *arbitrary* choice of the time translation group T_t in Minkowski spacetime and the complex structure J on S it determines.

6.3.2 Second quantization (Fock space)

Once we have used some time translation group T_t to fix the Hilbert space $\mathcal{S}_J$, the "second quantization" procedure yields a unique representation $(\pi, \mathcal{H}_\pi)$ of the Weyl algebra $\mathcal{A}[S, \sigma]$.

Let $\mathcal{H}^n$ denote the n-fold symmetric tensor product of $\mathcal{S}_J$ with itself. That is, using $\mathcal{S}_J^n$ to denote $\mathcal{S}_J \otimes \cdots \otimes \mathcal{S}_J$ (n times), $\mathcal{H}^n = P_+(\mathcal{S}_J^n)$ where P_+ is the projection onto the symmetric subspace. Then we define a Hilbert space

$$\mathcal{F}(\mathcal{S}_J) := \mathbb{C} \oplus \mathcal{H}^1 \oplus \mathcal{H}^2 \oplus \mathcal{H}^3 \oplus \cdots , \tag{6.24}$$

[9] For more physical details, see Fulling (1972, Secs. VIII.3,4) and Wald (1994, 41–42, 63, 111).

called the *bosonic Fock space over* $\mathcal{S}_J$. Let

$$\Omega := 1 \oplus 0 \oplus 0 \oplus \cdots , \tag{6.25}$$

denote the privileged "Fock vacuum" state in $\mathcal{F}(\mathcal{S}_J)$.

Now, we define creation and annihilation operators on $\mathcal{F}(\mathcal{S}_J)$ in the usual way. For any fixed $f \in S$, we first consider the unique bounded linear extensions of the mappings $a_n^*(f) : \mathcal{S}_J^{n-1} \to \mathcal{S}_J^n$ and $a_n(f) : \mathcal{S}_J^n \to \mathcal{S}_J^{n-1}$ defined by the following actions on product vectors

$$a_n^*(f)(f_1 \otimes \cdots \otimes f_{n-1}) = f \otimes f_1 \otimes \cdots \otimes f_{n-1}, \tag{6.26}$$

$$a_n(f)(f_1 \otimes \cdots \otimes f_n) = (f, f_1)_J \; f_2 \otimes \cdots \otimes f_n. \tag{6.27}$$

We then define the *unbounded* creation and annihilation operators on $\mathcal{F}(\mathcal{S}_J)$ by

$$a^*(f) := a_1^*(f) \oplus \sqrt{2}P_+ a_2^*(f) \oplus \sqrt{3}P_+ a_3^*(f) \oplus \cdots , \tag{6.28}$$

$$a(f) := 0 \oplus a_1(f) \oplus \sqrt{2}a_2(f) \oplus \sqrt{3}a_3(f) \oplus \cdots . \tag{6.29}$$

(Note that the mapping $f \mapsto a^*(f)$ is linear while $f \mapsto a(f)$ is anti-linear.)

As the definitions and notation suggest, $a^*(f)$ and $a(f)$ are each other's adjoint, $a^*(f)$ is the creation operator for a particle with wavefunction f, and $a(f)$ the corresponding annihilation operator. The unbounded self-adjoint operator $N(f) = a^*(f)a(f)$ represents the number of particles in the field with wavefunction f (unbounded, because we are describing bosons to which no exclusion principle applies). Summing $N(f)$ over any J-orthonormal basis of wavefunctions in $\mathcal{S}_J$, we obtain the *total* number operator N on $\mathcal{F}(\mathcal{S}_J)$, which has the form

$$N = 0 \oplus 1 \oplus 2 \oplus 3 \oplus \cdots . \tag{6.30}$$

Next, we define the self-adjoint "field operators"

$$\Phi(f) := 2^{-1/2}(a^*(f) + a(f)), \qquad f \in S. \tag{6.31}$$

(In heuristic discussions of free quantum field theory, these are normally encountered as "operator-valued solutions" $\Phi(x)$ to a relativistic field equation at some fixed time. However, if we want to associate a properly defined self-adjoint field operator with the spatial point x, we must consider a neighborhood of x, and an operator of form $\Phi(f)$, where the "test-function" $f \in S$

has support in the neighborhood.[10]) Defining the unitary operators

$$\pi(W(tf)) := \exp(it\Phi(f)), \qquad t \in \mathbb{R},\ f \in S, \tag{6.32}$$

it can then be verified (though it is not trivial) that the $\pi(W(f))$ satisfy the Weyl form of the CCRs. In fact, the mapping $W(f) \mapsto \pi(W(f))$ gives an irreducible regular representation π of $\mathcal{A}[S,\sigma]$ on $\mathcal{F}(\mathcal{S}_J)$.

We also have

$$\langle \Omega, \pi(W(f))\Omega \rangle = e^{-(f,f)_J/4}, \qquad f \in S. \tag{6.33}$$

(We use angle brackets to distinguish the inner product of $\mathcal{F}(\mathcal{S}_J)$ from that of $\mathcal{S}_J$.) The vacuum vector $\Omega \in \mathcal{F}(\mathcal{S}_J)$ defines an abstract regular state ω_J of $\mathcal{A}[S,\sigma]$ via $\omega_J(A) := \langle \Omega, \pi(A)\Omega \rangle$ for all $A \in \mathcal{A}[S,\sigma]$. Since the action of $\pi(\mathcal{A}[S,\sigma])$ on $\mathcal{F}(\mathcal{S}_J)$ is irreducible, $\{\pi(A)\Omega : A \in \mathcal{A}[S,\sigma]\}$ is dense in $\mathcal{F}(\mathcal{S}_J)$ (else its closure would be a non-trivial subspace invariant under all operators in $\pi(\mathcal{A}[S,\sigma])$). Thus, the Fock representation of $\mathcal{A}[S,\sigma]$ on $\mathcal{F}(\mathcal{S}_J)$ is unitarily equivalent to the GNS representation of $\mathcal{A}[S,\sigma]$ determined by the pure state ω_J.

In sum, a complex structure J on (S,σ) gives rise to an abstract vacuum state ω_J on $\mathcal{A}[S,\sigma]$ whose GNS representation $(\pi_{\omega_J}, \mathcal{H}_{\omega_J}, \Omega_{\omega_J})$ is just the standard Fock vacuum representation $(\pi, \mathcal{F}(\mathcal{S}_J), \Omega)$. Note also that inverting Eqn. (6.31) yields

$$\begin{aligned} a^*(f) &= 2^{-1/2}(\Phi(f) - i\Phi(if))\,, && (6.34)\\ a(f) &= 2^{-1/2}(\Phi(f) + i\Phi(if))\,. && (6.35) \end{aligned}$$

Thus, we could just as well have arrived at the Fock representation of $\mathcal{A}[S,\sigma]$ "abstractly" by *starting* with the pure regular state ω_J on $\mathcal{A}[S,\sigma]$ as our proposed vacuum, exploiting its regularity to guarantee the existence of field operators $\{\Phi(f) : f \in S\}$ acting on $\mathcal{H}_{\omega_J}$, and then using Eqns. (6.34) to define $a^*(f)$ and $a(f)$ (and, from thence, the number operators $N(f)$ and N).

There is a natural way to construct operators on $\mathcal{F}(\mathcal{S}_J)$ out of operators on the one-particle space $\mathcal{S}_J$, using the *second quantization map* Γ and its "derivative" $d\Gamma$. Unlike the representation map π, the operators on $\mathcal{F}(\mathcal{S}_J)$ in the range of Γ and $d\Gamma$ do not "come from" $\mathcal{A}[S,\sigma]$, but rather $\mathbf{B}(\mathcal{S}_J)$. Since

[10]The picture of a quantum field as an operator-valued *field*—or, as Teller (1995, Ch. 5) aptly puts it, a field of "determinables"—unfortunately, has no mathematically rigorous foundation.

the latter depends on how S was complexified, we cannot expect second quantized observables to be representation-independent.

To define $d\Gamma$, first let H be a self-adjoint (possibly unbounded) operator on $\mathcal{S}_J$. We define H_n on $\mathcal{H}^n$ by setting $H_0 = 0$ and

$$H_n(P_+(f_1 \otimes \cdots \otimes f_n)) = P_+ \left(\sum_{i=1}^{n} f_1 \otimes f_2 \otimes \cdots \otimes Hf_i \otimes \cdots \otimes f_n \right), \quad (6.36)$$

for all f_i in the domain of H, and then extending by continuity. It then follows that $\oplus_{n \geq 0} H_n$ is an "essentially selfadjoint" operator on $\mathcal{F}(\mathcal{S}_J)$ (see BR 1996, 8). We let

$$d\Gamma(H) := \overline{\bigoplus_{n \geq 0} H_n}, \quad (6.37)$$

denote the resulting (closed) self-adjoint operator. The simplest example occurs when we take $H = I$, in which case it is easy to see that $d\Gamma(H) = N$. However, the total number operator N is not affiliated with the Weyl algebra.[11]

Proposition 6.5. *Let $(\pi, \mathcal{F}(\mathcal{H}))$ be a Fock representation of $\mathcal{A}[S, \sigma]$. If S is infinite-dimensional, then $\pi(\mathcal{A}[S, \sigma])$ contains no non-trivial bounded functions of the total number operator on $\mathcal{F}(\mathcal{H})$.*

In particular, $\pi(\mathcal{A}[S, \sigma])$ does not contain any of the spectral projections of N. Thus, while the *conservative* about observables is free to refer to the abstract state ω_J of $\mathcal{A}[S, \sigma]$ as a "vacuum" state, he cannot use that language to underwrite the claim that ω_J is a state of "no particles"!

To define Γ, let U be a unitary operator on $\mathcal{S}_J$. Then $U_n = P_+(U \otimes \cdots \otimes U)$ is a unitary operator on $\mathcal{H}^n$. We define the unitary operator $\Gamma(U)$ on $\mathcal{F}(\mathcal{S}_J)$ by

$$\Gamma(U) := \bigoplus_{n \geq 0} U_n. \quad (6.38)$$

If $U_t = e^{itH}$ is a weakly continuous unitary group on $\mathcal{S}_J$, then $\Gamma(U_t)$ is a weakly continuous group on $\mathcal{F}(\mathcal{S}_J)$, and we have

$$\Gamma(U_t) = e^{itd\Gamma(H)}. \quad (6.39)$$

In particular, the one-particle evolution $T_t = e^{itH}$ that was used to fix J

[11] Our proof in the appendix reconstructs the argument briefly sketched by Segal (1959, 12).

lifts to a field evolution given by $\Gamma(T_t)$, where $d\Gamma(H)$ represents the energy of the field and has the vacuum Ω as a ground state.

The maps π and Γ satisfy the following relation:

$$\pi(W(Uf)) = \Gamma(U)^* \pi(W(f))\Gamma(U), \tag{6.40}$$

for any unitary operator U on $\mathcal{S}_J$. Taking the phase transformation $U = e^{it}I$, it follows that

$$\pi(W(e^{it}f)) = e^{-itN}\pi(W(f))e^{itN}, \qquad t \in \mathbb{R}. \tag{6.41}$$

Using Eqn. (6.33), it also follows that

$$\langle \Gamma(U)\Omega, \pi(W(f))\Gamma(U)\Omega\rangle = \langle \Omega, \pi(W(Uf))\Omega\rangle = \langle \Omega, \pi(W(f))\Omega\rangle. \tag{6.42}$$

Thus, Ω and $\Gamma(U)\Omega$ determine the same state of $\pi(\mathcal{A}[S,\sigma])^-$. If $\pi(\mathcal{A}[S,\sigma])^- = \mathbf{B}(\mathcal{F}(\mathcal{S}_J))$, then $\Gamma(U)\Omega = c\Omega$ for some complex number c of unit modulus. In particular, the vacuum is invariant under the group $\Gamma(T_t)$, and is therefore time-translation invariant.

6.4 Disjointness of the Minkowski and Rindler representations

We omit the details of the construction of the classical phase space (S,σ), since they are largely irrelevant to our concerns. The only information we need is that the space S may be taken (roughly) to be solutions to some relativistic wave equation, such as the Klein-Gordon equation. More particularly, S may be taken to consist of pairs of smooth, compactly supported functions on $\mathbb{R}^3$: one function specifies the values of the field at each point in space at some initial time (say $t = 0$), and the other function is the time-derivative of the field (evaluated at $t = 0$). If we then choose a "timelike flow" in Minkowski spacetime, we will get a corresponding flow in the solution space S; and, in particular, this flow will be given by a one-parameter group T_t of symplectomorphisms on (S,σ).

First, consider the group T_t of symplectomorphisms of (S,σ) induced by the standard inertial timelike flow. (See Figure 6.1, which suppresses two spatial dimensions. Note that it is irrelevant which *inertial* frame's flow we pick, since they all determine the same representation of $\mathcal{A}[S,\sigma]$ up to unitary equivalence; see Wald 1994, 106.) It is well-known that there is a complex structure M on (S,σ) such that $(\mathcal{S}_M, T_t)$ is a quantum one-particle

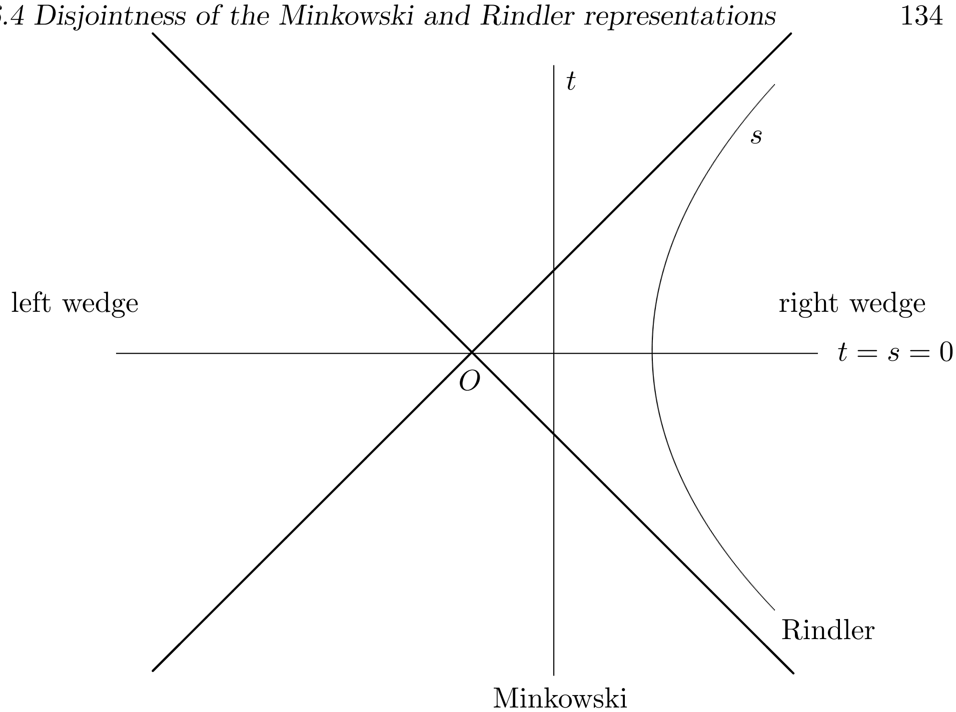

Figure 6.1: Minkowski and Rindler Motions.

system (see Kay 1985; Horuzhy 1988, Ch. 4). We call the associated pure regular state ω_M of $\mathcal{A}[S, \sigma]$ the *Minkowski vacuum state.* As we have seen, it gives rise via the GNS construction to a unique Fock vacuum representation π_{ω_M} on the Hilbert space $\mathcal{H}_{\omega_M} = \mathcal{F}(\mathcal{S}_M)$.

Next, consider the group of Lorentz boosts about a given center point O in spacetime. This also gives rise to a one-parameter group T_s of symplectomorphisms of (S, σ) (cf. Figure 6.1). Let $S(\triangleleft)$ be the subspace of S consisting of Cauchy data with support in the right Rindler wedge $(x_1 > 0)$; i.e., at $s = 0$, both the field and its first derivative vanish when $x_1 \leq 0$. Let $\mathcal{A}_\triangleleft := \mathcal{A}[S(\triangleleft), \sigma]$ be the Weyl algebra over the symplectic space $(S(\triangleleft), \sigma)$. Then, T_s leaves $S(\triangleleft)$ invariant, and hence gives rise to a one-parameter group of symplectomorphisms of $(S(\triangleleft), \sigma)$. Kay (1985) has shown rigorously that there is indeed a complex structure R on $(S(\triangleleft), \sigma)$ such that $(\mathcal{S}(\triangleleft)_R, T_s)$ is a quantum one particle system. We call the resulting state $\omega_R^\triangleleft$ of $\mathcal{A}_\triangleleft$ the *(right) Rindler vacuum state.* It gives rise to a unique GNS-Fock representation $\pi_{\omega_R^\triangleleft}$ of $\mathcal{A}_\triangleleft$ on $\mathcal{H}_{\omega_R^\triangleleft} = \mathcal{F}(\mathcal{S}(\triangleleft)_R)$ and, hence, a quantum field theory for the spacetime consisting of the right wedge *alone.*

The Minkowski vacuum state ω_M of $\mathcal{A}[S, \sigma]$ also determines, by restriction, a state $\omega_M^{\triangleleft}$ of $\mathcal{A}_{\triangleleft}$ (i.e., $\omega_M^{\triangleleft} := \omega_M|_{\mathcal{A}_{\triangleleft}}$). Thus, we may apply the GNS construction to obtain the Minkowski representation $(\pi_{\omega_M^{\triangleleft}}, \mathcal{H}_{\omega_M^{\triangleleft}})$ of $\mathcal{A}_{\triangleleft}$. It can be shown that $\omega_M^{\triangleleft}$, unlike $\omega_R^{\triangleleft}$, is a highly mixed state (cf. section 3.3). Therefore, $\pi_{\omega_M^{\triangleleft}}$ is reducible.

To obtain a concrete picture of this representation, note that (again, as a consequence of the Reeh-Schlieder theorem) Ω_{ω_M} is a cyclic vector for the subalgebra $\pi_{\omega_M}(\mathcal{A}_{\triangleleft})$ acting on the "global" Fock space $\mathcal{F}(\mathcal{S}_M)$. Thus, by the uniqueness of the GNS representation $(\pi_{\omega_M^{\triangleleft}}, \mathcal{H}_{\omega_M^{\triangleleft}})$, it is unitarily equivalent to the representation $(\pi_{\omega_M}|_{\mathcal{A}_{\triangleleft}}, \mathcal{F}(\mathcal{S}_M))$. It can be shown that $\pi_{\omega_M}(\mathcal{A}_{\triangleleft})''$ is a factor (Horuzhy 1988, Thm 3.3.4). Thus, while reducible, $\pi_{\omega_M^{\triangleleft}}$ is still factorial.

Under the liberal approach to observables, the representations $\pi_{\omega_M^{\triangleleft}}$ (factorial) and $\pi_{\omega_R^{\triangleleft}}$ (irreducible) provide physically inequivalent descriptions of the physics in the right wedge.[12]

Proposition 6.6. *The Minkowski and Rindler representations of $\mathcal{A}_{\triangleleft}$ are disjoint.*

Now let $\triangleright$ denote the left Rindler wedge, and define the subspace $S(\triangleright)$ of S as $S(\triangleleft)$ was defined above. (Of course, by symmetry, Proposition 6.6 holds for $\mathcal{A}_{\triangleright}$ as well.) Let $\mathcal{A}_{\bowtie} := \mathcal{A}[S(\triangleright) \oplus S(\triangleleft), \sigma]$ denote the Weyl algebra over the symplectic space $(S(\triangleright) \oplus S(\triangleleft), \sigma)$. Then $\mathcal{A}_{\bowtie} = \mathcal{A}_{\triangleright} \otimes \mathcal{A}_{\triangleleft}$, and $\omega_M^{\bowtie} := \omega_M|_{\mathcal{A}_{\bowtie}}$ is pure (Kay 1985, Defn., Thm. 1.3(iii)).[13] Thus, the GNS representation induced by $\omega_M^{\bowtie}$ is irreducible, and (again invoking the uniqueness of the GNS representation) it is equivalent to $(\pi_{\omega_M}|_{\mathcal{A}_{\bowtie}}, \mathcal{F}(\mathcal{S}_M))$ (since $\Omega_{\omega_M} \in \mathcal{F}(\mathcal{S}_M)$ is a cyclic vector for the subalgebra $\pi_{\omega_M}(\mathcal{A}_{\bowtie})$ as well).

The tensor product of the pure left and right Rindler vacua $\omega_R^{\bowtie} := \omega_R^{\triangleright} \otimes$

[12]If only locally measurable observables have genuine physical significance (see note 5), then the Minkowski and Rindler representations *are* physically equivalent. Indeed, since both $\omega_M^{\triangleleft}$ and $\omega_R^{\triangleleft}$ are "of Hadamard form," it follows that $\pi_{\omega_M^{\triangleleft}}$ and $\pi_{\omega_R^{\triangleleft}}$ are "locally quasi-equivalent" (cf. Verch 1994, Theorem 3.9). That is, for each algebra $\mathcal{A}(O)$ of local observables in $\mathcal{A}_{\triangleleft}$, $\pi_{\omega_M^{\triangleleft}}|_{\mathcal{A}(O)}$ is quasi-equivalent to $\pi_{\omega_R^{\triangleleft}}|_{\mathcal{A}(O)}$. Clearly, this fact only strengthens our case against the claim that the Minkowski and Rindler representations correspond to incommensurable theories of the quantum field.

[13]The restriction of ω_M to $\mathcal{A}_{\bowtie}$ is a pure "quasifree" state. Thus, there is a complex structure M' on $S(\triangleright) \oplus S(\triangleleft)$ such that

$$\omega_M(W(f)) = \exp(-\sigma(f, M'f)/4) = \exp(-\sigma(f, Mf)/4), \tag{6.43}$$

for all $f \in S(\triangleright) \oplus S(\triangleleft)$ (Petz 1990, Prop. 3.9). It is not difficult to see then that $M|_{S(\triangleright)\oplus S(\triangleleft)} = M'$ and therefore that M leaves $S(\triangleright) \oplus S(\triangleleft)$ invariant. Hereafter, we will use M to denote the complex structure on S as well as its restriction to $S(\triangleright) \oplus S(\triangleleft)$.

$\omega_R^{\triangleleft}$ is of course also a pure state of $\mathcal{A}_{\bowtie}$.[14] It induces a GNS representation of the latter on the Hilbert space $\mathcal{H}_{\omega_R^{\bowtie}}$ given by $\mathcal{F}(\mathcal{S}_R) \equiv \mathcal{F}(\mathcal{S}(\triangleright)_R) \otimes \mathcal{F}(\mathcal{S}(\triangleleft)_R)$. It is not difficult to show that $\omega_R^{\bowtie}$ and $\omega_M^{\bowtie}$, both now irreducible, are also disjoint.[15]

Proposition 6.7. *The Minkowski and Rindler representations of $\mathcal{A}_{\bowtie}$ are disjoint.*

In the next chapter we shall discuss the conceptually problematic implications that the *M-vacuum* states $\omega_M^{\bowtie}$ and $\omega_M^{\triangleleft}$ have for the *presence* of R-quanta in the double and right wedge spacetime regions. However, we note here an important difference between Rindler and Minkowski observers.

The total number of R-quanta, according to a Rindler observer confined to the left (resp., right) wedge, is represented by the number operator $N_{\triangleright}$ (resp., $N_{\triangleleft}$) on $\mathcal{F}(\mathcal{S}(\triangleright)_R)$ (resp., $\mathcal{F}(\mathcal{S}(\triangleleft)_R)$). However, because of the spacelike separation of the wedges, no single Rindler observer has access, even in principle, to the expectation value of the "overall" total Rindler number operator $N_R = N_{\triangleright} \otimes I + I \otimes N_{\triangleleft}$ acting on $\mathcal{F}(\mathcal{S}(\triangleright)_R) \otimes \mathcal{F}(\mathcal{S}(\triangleleft)_R)$.

The reverse is true for a Minkowski observer. While she has access, at least in principle, to the total number of M-quanta operator N_M acting on $\mathcal{F}(\mathcal{S}_M)$, N_M is a purely global observable that does not split into the sum of two separate number operators associated with the left and right wedges (as a general consequence of the Reeh-Schlieder theorem—see Redhead 1995b). In fact, since the Minkowski complex structure M is an "anti-local" operator (Segal & Goodman, 1965), it fails to leave either of the subspaces $S(\triangleright)$ or $S(\triangleleft)$ invariant, and it follows that no M-quanta number operator is affiliated with $\pi_{\omega_M^{\triangleleft}}(\mathcal{A}_{\triangleleft})''$.[16] Thus, even a liberal about observables must say that a Minkowski observer with access only to the right wedge does not have the capability of counting M-quanta.

So, while it might be sensible to ask for the probability in state $\omega_M^{\triangleleft}$ that a Rindler observer detects particles in the right wedge, it is *not* sensible to ask, conversely, for the probability in state $\omega_R^{\triangleleft}$ that a Minkowski observer will detect particles in the right wedge. Note also that since N_M is

[14] More precisely, $\omega_R^{\triangleleft}$ arises from a complex structure $R_{\triangleleft}$ on $S(\triangleleft)$, $\omega_R^{\triangleright}$ arises from a complex structure $R_{\triangleright}$ on $S(\triangleright)$, and $\omega_R^{\bowtie}$ arises from the complex structure $R_{\triangleright} \oplus R_{\triangleleft}$ of $S(\triangleright) \oplus S(\triangleleft)$. When no confusion can result, we will use R to denote the complex structure on $S(\triangleright) \oplus S(\triangleleft)$ and its restriction to $S(\triangleleft)$.

[15] We give proofs of Propositions 6.6 and 6.7 in the appendix. For another proof, employing quite different methods, see the appendix of (Beyer 1991).

[16] See chapter 4 for further details and a critical analysis of different approaches to the problem of particle localization in quantum field theory.

a purely global observable (i.e., there is no sense to be made of "the number of Minkowski quanta in a bounded spatial or spacetime region"), what a Minkowski observer might *locally* detect with a "particle detector" (over an extended, but finite, interval of time) can at best give an approximate indication of the global Minkowski particle content of the field.

6.5 Appendix: Proofs of theorems

Proposition 6.3. *Under the conservative approach to states, ϕ (factorial) and π (irreducible) are physically equivalent representations of $\mathcal{A}$ only if they are quasi-equivalent.*

Proof. Let ω be a normal state of $\phi(\mathcal{A})$. Then, by hypothesis, $\beta(\omega)$ is a normal state of $\pi(\mathcal{A})$. Define a state ρ on $\mathcal{A}$ by

$$\rho(A) = \omega(\phi(A)), \qquad A \in \mathcal{A}. \tag{6.44}$$

Since ω is normal, $\rho \in \mathfrak{F}(\phi)$. Define a state ρ' on $\mathcal{A}$ by

$$\rho'(A) = \beta(\omega)(\pi(A)), \qquad A \in \mathcal{A}. \tag{6.45}$$

Since $\beta(\omega)$ is normal, $\rho' \in \mathfrak{F}(\pi)$. Now, conditions (6.21) and (6.22) entail that

$$\omega(\phi(A)) = \beta(\omega)(\alpha(\phi(A))) = \beta(\omega)(\pi(A)), \tag{6.46}$$

for any $A = W(f) \in \mathcal{A}$, and thus $\rho(W(f)) = \rho'(W(f))$ for any $f \in S$. However, a state of the Weyl algebra is uniquely determined (via linearity and uniform continuity) by its action on the generators $\{W(f) : f \in S\}$. Thus, $\rho = \rho'$ and since $\rho \in \mathfrak{F}(\phi) \cap \mathfrak{F}(\pi)$, it follows that ϕ and π are quasi-equivalent. □

Proposition 6.4. *Under the liberal approach to observables, ϕ (factorial) and π (irreducible) are physically equivalent representations of $\mathcal{A}$ only if they are quasi-equivalent.*

Proof. By hypothesis, the bijective mapping α must map the self-adjoint part of $\phi(\mathcal{A})''$ onto that of $\pi(\mathcal{A})''$. Extend α to *all* of $\phi(\mathcal{A})''$ by defining

$$\alpha(X) := \alpha(\mathrm{Re}(X)) + i\alpha(\mathrm{Im}(X)), \qquad X \in \phi(\mathcal{A})''. \tag{6.47}$$

Clearly, then, α preserves adjoints.

Recall that a family of states S_0 on a C^*-algebra is called *full* just in case S_0 is convex, and for any $A \in \mathcal{A}$, $\rho(A) \geq 0$ for all $\rho \in S_0$ only if $A \geq 0$. By hypothesis, there is a bijective mapping β from the "physical" states of $\phi(\mathcal{A})''$ onto the "physical" states of $\pi(\mathcal{A})''$. According to both the conservative and liberal construals of physical states, the set of physical states includes normal states. Since the normal states are full, the domain and range of β contain full sets of states of the respective C^*-algebras.

By condition (6.22) and the fact that the domain and range of β are full sets of states, α arises from a *symmetry* between the C^*-algebras $\phi(\mathcal{A})''$ and $\pi(\mathcal{A})''$ in the sense of Roberts & Roepstorff (1969, Sec. 3).[17] Their Propositions 3.1 and 6.3 then apply to guarantee that α must be linear and preserve Jordan structure (i.e., anti-commutator brackets). Thus α is a Jordan $*$-isomorphism.

Now both $\phi(\mathcal{A})''$ and $\pi(\mathcal{A})'' = \mathbf{B}(\mathcal{H}_\pi)$ are von Neumann algebras, and the latter has a trivial commutant. Thus Exercise 10.5.26 of (KR 1997) applies, and α is either a $*$-isomorphism or a $*$-anti-isomorphism, that reverses the order of products. However, such reversal is ruled out, otherwise we would have, using the Weyl relations (6.10),

$$\alpha(\phi(W(f))\phi(W(g))) = e^{-i\sigma(f,g)/2}\alpha(\phi(W(f+g))), \tag{6.48}$$
$$\Longrightarrow \quad \alpha(\phi(W(g)))\alpha(\phi(W(f))) = e^{-i\sigma(f,g)/2}\alpha(\phi(W(f+g))), \tag{6.49}$$
$$\Longrightarrow \quad \pi(W(g))\pi(W(f)) = e^{-i\sigma(f,g)/2}\pi(W(f+g)), \tag{6.50}$$
$$\Longrightarrow \quad e^{i\sigma(f,g)/2}\pi(W(f+g)) = e^{-i\sigma(f,g)/2}\pi(W(f+g)), \tag{6.51}$$

for all $f, g \in S$. This entails that the value of σ on any pair of vectors is always is a multiple of 2π which, since σ is bilinear, cannot happen unless $\sigma = 0$ identically (and hence $S = \{0\}$). It follows that α is in fact a $*$-isomorphism. And, by condition (6.21), α must map $\phi(A)$ to $\pi(A)$ for all $A \in \mathcal{A}$. Thus ϕ is quasi-equivalent to π. □

Proposition 6.5. *Let $(\pi, \mathcal{F}(\mathcal{H}))$ be a Fock representation of $\mathcal{A}[S,\sigma]$. If S is infinite-dimensional, then $\pi(\mathcal{A}[S,\sigma])$ contains no non-trivial bounded functions of the total number operator on $\mathcal{F}(\mathcal{H})$.*

Proof. For clarity, we suppress reference to the representation map π. Suppose that $F : \mathbb{N} \mapsto \mathbb{C}$ is a bounded function. We show that if $F(N) \in \mathcal{A}[S,\sigma]$, then $F(n) = F(n+1)$ for all $n \in \mathbb{N}$.

[17] Actually, they consider only symmetries of a C^*-algebra onto *itself*, but their results remain valid for our case.

The Weyl operators on $\mathcal{F}(\mathcal{H})$ and their generators satisfy the commutation relation (BR 1996, Prop. 5.2.4):

$$W(g)\Phi(f)W(g)^* = \Phi(f) - \mathrm{Im}(g,f)I. \tag{6.52}$$

Using the equation $a^*(f) = 2^{-1/2}(\Phi(f) - i\Phi(if))$, we find

$$W(g)a^*(f)W(g)^* = a^*(f) + 2^{-1/2}i(g,f)I, \tag{6.53}$$

and from this, $[W(g), a^*(f)] = 2^{-1/2}i(g,f)W(g)$. Now let $\psi \in \mathcal{F}(\mathcal{H})$ be in the domain of $a^*(f)$. Then a straightforward calculation shows that

$$\left\langle a^*(f)\psi, W(g)a^*(f)\psi\right\rangle = 2^{-1/2}i(g,f)\left\langle a^*(f)\psi, W(g)\psi\right\rangle + \left\langle a(f)a^*(f)\psi, W(g)\psi\right\rangle. \tag{6.54}$$

Let $\{f_k\}$ be an infinite orthonormal basis for $\mathcal{H}$, and let $\psi \in \mathcal{F}(\mathcal{H})$ be the vector whose n-th component is $P_+(f_1 \otimes \cdots \otimes f_n)$ and whose other components are zero. Now, for any $k > n$, we have $a(f_k)a^*(f_k)\psi = (n+1)\psi$. Thus, Eqn. (6.54) gives

$$\left\langle a^*(f_k)\psi, W(g)a^*(f_k)\psi\right\rangle = 2^{-1/2}i(g,f_k)\left\langle a^*(f_k)\psi, W(g)\psi\right\rangle + (n+1)\left\langle \psi, W(g)\psi\right\rangle. \tag{6.55}$$

Hence,

$$\lim_{k\to\infty}\left\langle a^*(f_k)\psi, W(g)a^*(f_k)\psi\right\rangle = (n+1)\left\langle \psi, W(g)\psi\right\rangle. \tag{6.56}$$

Since $\mathcal{A}[S,\sigma]$ is generated by $\{W(g) : g \in \mathcal{H}\}$, Eqn. (6.56) holds when $W(g)$ is replaced with any element in $\mathcal{A}[S,\sigma]$. On the other hand, ψ is an eigenvector with eigenvalue n for N while $a^*(f_k)\psi$ is an eigenvector with eigenvalue $n+1$ for N. Thus, $\langle \psi, F(N)\psi\rangle = F(n)\|\psi\|^2$ while

$$\begin{aligned} \langle a^*(f_k)\psi, F(N)a^*(f_k)\psi\rangle &= F(n+1)\,\|a^*(f_k)\psi\|^2 && (6.57)\\ &= (n+1)F(n+1)\,\|\psi\|^2, && (6.58)\end{aligned}$$

for all $k > n$. Thus, the assumption that $F(N)$ is in $\mathcal{A}[S,\sigma]$ (and hence satisfies (6.56)) entails that $F(n+1) = F(n)$. □

Proposition 6.6. *The Minkowski and Rindler representations of* $\mathcal{A}_\triangleleft$ *are disjoint.*

Proof. By Theorem 3.3.4 of (Horuzhy 1988), $\pi_{\omega_M^\triangleleft}(\mathcal{A}_\triangleleft)''$ is a type III von Neumann algebra, and therefore contains no atomic projections. Since $\pi_{\omega_R^\triangleleft}$ is irreducible and $\pi_{\omega_M^\triangleleft}$ factorial, either $\pi_{\omega_R^\triangleleft}$ and $\pi_{\omega_M^\triangleleft}$ are disjoint, or they are quasi-equivalent. However, since $\pi_{\omega_R^\triangleleft}(\mathcal{A}_\triangleleft)'' = \mathbf{B}(\mathcal{F}(\mathcal{S}(\triangleleft)_R))$, the weak closure of the Rindler representation clearly contains atomic projections. Moreover, $*$-isomorphisms preserve the ordering of projection operators. Thus there can be no $*$-isomorphism of $\pi_{\omega_M^{\triangleleft}}(\mathcal{A}_{\triangleleft})''$ onto $\pi_{\omega_R^{\triangleleft}}(\mathcal{A}_{\triangleleft})''$, and the Minkowski and Rindler representations of $\mathcal{A}_{\triangleleft}$ are disjoint. □

Proposition 6.7. *The Minkowski and Rindler representations of $\mathcal{A}_{\bowtie}$ are disjoint.*

Proof. Again, we use the fact that $\pi_{\omega_M^{\bowtie}}(\mathcal{A}_{\triangleleft})''$ ($\equiv \pi_{\omega_M^{\triangleleft}}(\mathcal{A}_{\triangleleft})''$) does not contain atomic projections, whereas $\pi_{\omega_R^{\bowtie}}(\mathcal{A}_{\triangleleft})''$ ($\equiv \pi_{\omega_R^{\triangleleft}}(\mathcal{A}_{\triangleleft})''$) does. Suppose, for reductio ad absurdum, that $\pi_{\omega_R^{\bowtie}}$ and $\pi_{\omega_M^{\bowtie}}$ are quasi-equivalent. Since the states $\omega_M^{\bowtie}$ and $\omega_R^{\bowtie}$ are pure, the representations $\pi_{\omega_M^{\bowtie}}$ and $\pi_{\omega_R^{\bowtie}}$ are irreducible and therefore unitarily equivalent. Thus, there is a *weakly continuous* $*$-isomorphism α from $\pi_{\omega_M^{\bowtie}}(\mathcal{A}_{\bowtie})''$ onto $\pi_{\omega_R^{\bowtie}}(\mathcal{A}_{\bowtie})''$ such that $\alpha(\pi_{\omega_M^{\bowtie}}(A)) = \pi_{\omega_R^{\bowtie}}(A)$ for each $A \in \mathcal{A}_{\bowtie}$. In particular, α maps $\pi_{\omega_M^{\bowtie}}(\mathcal{A}_{\triangleleft})$ onto $\pi_{\omega_R^{\bowtie}}(\mathcal{A}_{\triangleleft})$; and, since α is weakly continuous, it maps $\pi_{\omega_M^{\bowtie}}(\mathcal{A}_{\triangleleft})''$ onto $\pi_{\omega_R^{\bowtie}}(\mathcal{A}_{\triangleleft})''$. Consequently, $\pi_{\omega_M^{\bowtie}}(\mathcal{A}_{\triangleleft})''$ contains an atomic projection, in contradiction with the fact that $\pi_{\omega_M^{\bowtie}}(\mathcal{A}_{\triangleleft})''$ is a type III von Neumann algebra. □

Chapter 7

Minkowski versus Rindler quanta

Sagredo: Do we not see here another example of that all-pervading principle of complementarity which excludes the simultaneous applicability of concepts to the real objects of our world? Is it not so that, rather than being frustrated by this limitation of our conceptual grasp of the reality, we see in this unification of opposites the deepest and most satisfactory result of the dialectical process in our struggle for understanding?
— Josef Jauch, *Are Quanta Real? A Galilean Dialogue* (1973)

7.1 Introduction

We have seen that a Rindler observer will construct "his quantum field theory" of the right wedge spacetime region differently from a Minkowski observer. He will use the complex structure R picked out uniquely by the boost group about the origin, and build up a representation of $\mathcal{A}_{\triangleleft}$ on the Fock space $\mathcal{F}(\mathcal{S}(\triangleleft)_R)$. However, suppose that the state of $\mathcal{A}_{\triangleleft}$ is the state $\omega_M^{\triangleleft}$ of *no* particles (globally) according to a Minkowski observer. What, if anything, will our Rindler observer say about the particle content in the right wedge? And does *this* question even make sense?

We will argue that this question does make sense, notwithstanding the disjointness of the Minkowski and Rindler representations. And the answer is surprising: Not only does a Rindler observer have a nonzero chance of detecting the presence of R-quanta, but if a Rindler observer were to measure the *total* number of R-quanta in the right wedge, he would always find (as we show in section 7.3) that the probability of an *infinite* total number is *one*!

We begin in section 7.2 by discussing the paradox of observer-dependence of particles to which such results lead. In particular, we criticize Teller's (1995, 1996) resolution of this paradox. Later, in section 7.4, we will also criticize the arguments of Arageorgis (1995) and Arageorgis et al. (2001) for the incommensurability of inequivalent particle concepts, and argue, instead, for their complementarity (in *support* of Teller).

7.2 The paradox of the observer-dependence of particles

Not surprisingly, physicists initially found a Rindler observer's ability to detect particles in the Minkowski vacuum paradoxical (see Rüger 1989, 571; Teller 1995, 110). After all, particles are the sorts of things that are either there or not there, so how could their presence depend on an observer's state of motion?

One way to resist this paradox is to reject from the outset the physicality of the Rindler representation, thereby withholding bona fide particle status from Rindler quanta. For instance, one could be bothered by the fact the Rindler representation cannot be globally defined over the whole of Minkowski spacetime, or that the one-particle Rindler Hamiltonian lacks a mass gap, allowing an arbitrarily large number of R-quanta to have a fixed finite amount of energy ("infrared divergence"). Arageorgis (1995, Ch. 6) gives a thorough discussion of these and other "pathologies" of the Rindler representation.[1] In consequence, he argues that the phenomenology associated with a Rindler observer's "particle detections" in the Minkowski vacuum ought to be explained entirely in terms of observables affiliated to the Minkowski representation (such as garden-variety Minkowski vacuum fluctuations of the local field observables).

This is not the usual response to the paradox of observer-dependence. Rüger (1989) has characterized the majority of physicists' responses in terms of the *field approach* and the *detector approach*. Proponents of the field approach emphasize the need to forfeit particle talk at the fundamental level, and to focus the discussion on measurement of local field quantities. Those of the detector approach emphasize the need to relativize particle talk to the behavior of concrete detectors following specified world-lines. Despite their differing emphases, and the technical difficulties in unifying these pro-

[1] See also, more recently, Belinskiĭ 1997, Fedotov, Mur, Narozhny, Belinskiĭ, & Karnakov 1999, and Nikolić 2000.

grams (well-documented by Arageorgis 1995), neither eschews the Rindler representation as unphysical, presumably because of its deep connections with quantum statistical mechanics and blackhole thermodynamics (Sciama et al. 1981). Moreover, pathological or not, it remains of philosophical interest to examine the consequences of taking the Rindler representation seriously—just as the possibility of time travel in general relativity admitted by certain "pathological" solutions to Einstein's field equations is of interest. And it is remarkable that there should be *any* region of Minkowski spacetime that admits two physically inequivalent quantum field descriptions.

Teller (1995, 1996) has recently offered his own resolution of the paradox. We reproduce below the relevant portions of his discussion in (Teller 1995, 111). However, note that he does not distinguish between left and right Rindler observers, $|0;M\rangle$ refers, in our notation, to the Minkowski vacuum vector $\Omega_{\omega_M} \in \mathcal{F}(\mathcal{S}_M)$, and $|1,0,0,\ldots\rangle_M$ (respectively, $|1,0,0,\ldots\rangle_R$) is a one-particle state $0 \oplus f \oplus 0 \oplus 0 \oplus \cdots \in \mathcal{F}(\mathcal{S}_M)$ (respectively, $\in \mathcal{F}(\mathcal{S}_R)$).

> ... Rindler raising and lowering operators are expressible as superpositions of the Minkowski raising and lowering operators, and states with a definite number of Minkowski quanta are superpositions of states with different numbers of Rindler quanta. In particular, $|0;M\rangle$ is a superposition of Rindler quanta states, including states for arbitrarily large numbers of Rindler quanta. In other words, $|0;M\rangle$ has an exact value of zero for the Minkowski number operator, and is simultaneously highly indefinite for the Rindler number operator.
>
> ... In $|0;M\rangle$ there is *no* definite number of Rindler quanta. There is only a propensity for detection of one or another number of Rindler quanta by an accelerating detector. A state in which a quantity has no exact value is one in which no values for that quantity are definitely, and so actually, exemplified. Thus in $|0;M\rangle$ no Rindler quanta actually occur, so the status of $|0;M\rangle$ as a state completely devoid of quanta is not impugned.
>
> To be sure, this interpretive state of affairs is surprising. To spell it out one step further, in $|1,0,0,\ldots\rangle_M$ there is one actual Minkowski quantum, no actual Rindler quanta, and all sorts of propensities for manifestation of Rindler quanta, among other things. In $|1,0,0,\ldots\rangle_R$ the same comment applies with the role of Minkowski and Rindler reversed. It turns out that there are various kinds of quanta, and a state in which one kind of quanta

> actually occurs is a state in which there are only propensities for complementary kinds of quanta. Surprising, but perfectly consistent and coherent.

Teller's point is that R-quanta only exist (so to speak) potentially in the M-vacuum, not actually. Thus it is still an invariant observer-independent fact that there are no *actual quanta* in the field, and the paradox evaporates. Similarly for Minkowski states of one or more particles as seen by Rindler observers. There is the same definite number of *actual quanta* for all observers. Thus, since actual particles are the "real stuff", the real stuff *is* invariant!

Notice, however, that there is something self-defeating in Teller's final concession, urged by advocates of the field and detector approaches, that different kinds of quanta need to be distinguished. For if we do draw the distinction sharply, it is no longer clear why even the actual presence of R-quanta in the M-vacuum should bother us. Teller seems to want to have it both ways: while there are different kinds of quanta, there is still only one kind of *actual* quanta, and it better be invariant.

Does this invariance really hold? In one sense, Yes. Disjointness does not prevent us from building Rindler creation and annihilation operators on the Minkowski representation space $\mathcal{F}(\mathcal{S}_M)$. We simply need to define Rindler analogues, $a^*_R(f)$ and $a_R(f)$, of the Minkowski creation and annihilation operators via Eqns. (6.34) with $\Phi(Rf)$ in place of $\Phi(if)$ $(= \Phi(Mf))$ (noting that $f \mapsto a_R(f)$ will now be anti-linear with respect to the *Rindler* conjugation R). It is then easy to see, using (6.31), that

$$a_R(f) = (1/2)[a^*_M((I + MR)f) + a_M((I - MR)f)]. \tag{7.1}$$

This linear combination would be trivial if $R = \pm M$. However, we know $R \neq M$, and $R = -M$ is ruled out because it is inconsistent with both complex structures being positive definite. Consequently, $\Omega_{\omega^{\bowtie}_M}$ is a nontrivial superposition of eigenstates of the Rindler number operator $N_R(f) := a^*_R(f)a_R(f)$; for, an easy calculation, using (7.1), reveals that

$$N_R(f)\Omega_{\omega^{\bowtie}_M} \quad = \quad (1/4)\big[\,\Omega_{\omega^{\bowtie}_M} + a^*_M((I - MR)f)a^*_M((I + MR)f)\Omega_{\omega^{\bowtie}_M}\big], \tag{7.2}$$

which (the presence of the nonzero second term guarantees) is not a simply a multiple of $\Omega_{\omega^{\bowtie}_M}$. Thus, Teller would be correct to conclude that the Minkowski vacuum implies dispersion in the number operator $N_R(f)$. And the same conclusion would follow if, instead, we considered the Minkowski creation and annihilation operators as acting on the Rindler representation space $\mathcal{F}(\mathcal{S}_R)$. Since only finitely many degrees of freedom are involved, this is guaranteed by the Stone-von Neumann theorem.

However, therein lies the rub. $N_R(f)$ merely represents the number of R-quanta with a specified wavefunction f. What about the *total* number of R-quanta in the M-vacuum (which involves *all* degrees of freedom)? If Teller cannot assure us that this too has dispersion, his case for the invariance of "actual quanta" is left in tatters. In his discussion, Teller fails to distinguish $N_R(f)$ from the total number operator N_R, but the distinction is crucial. It is a well-known consequence of the disjointness of $\pi_{\omega_R^{\bowtie}}$ and $\pi_{\omega_M^{\bowtie}}$ that neither representation's total number operator is definable on the Hilbert space of the other (BR 1996, Thm. 5.2.14). Therefore, it is literally *nonsense* to speak of $\Omega_{\omega_M^{\bowtie}}$ as a superposition of eigenstates of N_R![2] If $x_n, x_m \in \mathcal{F}(\mathcal{S}_R)$ are eigenstates of N_R with eigenvalues n, m respectively, then $x_n + x_m$ again lies in $\mathcal{F}(\mathcal{S}_R)$, and so is "orthogonal" to all eigenstates of the Minkowski number operator N_M acting on $\mathcal{F}(\mathcal{S}_M)$. And, indeed, taking infinite sums of Rindler number eigenstates will again leave us in the folium of the Rindler representation. As Arageorgis (1995, 303) has also noted: "The Minkowski vacuum state is not a superposition of Rindler quanta states, despite 'appearances' ".[3]

Yet this point, by itself, does not tell us that Teller's discussion cannot be salvaged. Recall that a state ρ is said to be *dispersion-free* on a (bounded)

[2] In their review of Teller's (1995) book, Huggett and Weingard (1996) question whether Teller's "quanta interpretation" of quantum field theory can be implemented in the context of inequivalent representations. However, when they discuss Teller's resolution of the observer-dependence paradox, in terms of mere *propensities to display* R-quanta in the M-vacuum, they write "This seems all well and good" (1996, 309)! Their only criticism is the obvious one: legitimizing such propensity talk ultimately requires a solution to the measurement problem. Teller's response to their review is equally unsatisfactory. Though he pays lip-service to the possibility of inequivalent representations (1998, 156–157), he fails to notice how inequivalence undercuts his discussion of the paradox.

[3] Arageorgis presumes Teller's discussion is based upon the appearance of the following purely formal (i.e., non-normalizable) expression for $\Omega_{\omega_M^{\bowtie}}$ as a superposition in $\mathcal{F}(\mathcal{S}_R) \equiv \mathcal{F}(\mathcal{S}(\triangleright)_R) \otimes \mathcal{F}(\mathcal{S}(\triangleleft)_R)$ over left ("I") and right ("II") Rindler modes (Wald 1994, Eqn. 5.1.27):

$$\prod_i \left\{ \sum_{n=0}^{\infty} \exp(-n\pi\omega_i/a) \, |n_{iI}\rangle \otimes |n_{iII}\rangle \right\} \ . \tag{7.3}$$

However, it bears mentioning that, as this expression suggests: (a) the restriction of $\omega_M^{\bowtie}$ to either $\mathcal{A}_\triangleright$ or $\mathcal{A}_\triangleleft$ is indeed mixed; (b) $\omega_M^{\bowtie}$ can be shown rigorously to be an entangled state of $\mathcal{A}_\triangleright \otimes \mathcal{A}_\triangleleft$ (see chapter 2); and (c) the thermal properties of the "reduced density matrix" for either wedge obtained from this formal expression can be derived rigorously (Kay 1985). In addition, see Propositions 7.1 and 7.2 below!

observable X just in case $\rho(X^2) = \rho(X)^2$. Suppose, now, that Y is a possibly *un*bounded observable that is definable in some representation π of $\mathcal{A}[S,\sigma]$. We can then rightly say that an algebraic state ρ of $\mathcal{A}[S,\sigma]$ *predicts dispersion in* Y just in case, for *every* extension $\hat{\rho}$ of ρ to $\pi(\mathcal{A}[S,\sigma])''$, $\hat{\rho}$ is *not* dispersion-free on all bounded functions of Y. We then have the following result.

Proposition 7.1. *If* J_1, J_2 *are distinct complex structures on* (S,σ)*, then* ω_{J_1} *(respectively,* ω_{J_2}*) predicts dispersion in* N_{J_2} *(respectively,* N_{J_1}*).*

As a consequence, the Minkowski vacuum $\omega_M^{\bowtie}$ indeed predicts dispersion in the Rindler total number operator N_R (and in both $N_{\triangleright} \otimes I$ and $I \otimes N_{\triangleleft}$, invoking the symmetry between the wedges).

Teller also writes of the Minkowski vacuum as being a superposition of eigenstates of the Rindler number operator with *arbitrarily large* eigenvalues. Eschewing the language of superposition, the idea that there is no finite number of R-quanta to which the M-vacuum assigns probability one can also be rendered sensible. The relevant result was first obtained (heuristically) by Fulling (1972, Appendix F; 1989, 145):

Fulling's "Theorem". *Two Fock vacuum representations* $(\pi, \mathcal{F}(\mathcal{H}), \Omega)$ *and* $(\pi', \mathcal{F}(\mathcal{H}'), \Omega')$ *of* $\mathcal{A}[S,\sigma]$ *are unitarily equivalent if and only if* $\langle \Omega, N'\Omega \rangle < \infty$ *(or, equivalently,* $\langle \Omega', N\Omega' \rangle < \infty$*).*

As stated, this "theorem" fails to make sense, because it is only in the case where the representations are *already* equivalent that the primed total number operator is definable on the unprimed representation space and an expression like "$\langle \Omega, N'\Omega \rangle$" is well-defined. (We say more about why this is so in the next section.) However, there *is* a way to understand the expression "$\langle \Omega, N'\Omega \rangle < \infty$" (respectively, "$\langle \Omega, N'\Omega \rangle = \infty$") in a rigorous, non-question-begging way. We can take it to be the claim that all extensions $\hat{\rho}$ of the abstract unprimed vacuum state of $\mathcal{A}[S,\sigma]$ to $\mathbf{B}(\mathcal{F}(\mathcal{H}'))$ assign (respectively, do *not* assign) N' a finite value; i.e., for any such extension, $\sum_{n'=1}^{\infty} \hat{\rho}(P_{n'})n'$ converges (respectively, does not converge), where $\{P_{n'}\}$ are the spectral projections of N'. With this understanding, the following rigorization of Fulling's "theorem" can then be proved.

Proposition 7.2. *A pair of Fock representations* $\pi_{\omega_{J_1}}$, $\pi_{\omega_{J_2}}$ *are unitarily equivalent if and only if* ω_{J_1} *assigns* N_{J_2} *a finite value (equivalently,* ω_{J_2} *assigns* N_{J_1} *a finite value).*

It follows that $\omega_M^{\bowtie}$ cannot assign probability one to any finite number of R-quanta (and vice versa, with R and M interchanged).

Unfortunately, neither Proposition 7.1 or 7.2 is sufficient to rescue Teller's "actual quanta" invariance argument, for these propositions give no further information about the shape of the probability distribution that $\omega_M^{\bowtie}$ prescribes for N_R's eigenvalues. In particular, both propositions are compatible with there being a probability of *one* that *at least* $n > 0$ R-quanta obtain in the M-vacuum, for any $n \in \mathbb{N}$. If that were the case, Teller would then be forced to withdraw and concede that at least *some*, and perhaps many, Rindler quanta *actually* occur in a state with no actual Minkowski quanta. In the next section, we shall show that this—Teller's worst nightmare—is in fact the case.

7.3 Minkowski probabilities for Rindler number operators

We now defend the claim that a Rindler observer will say that there are actually *infinitely many* quanta while the field is in the Minkowski vacuum state (or, indeed, in any other state of the Minkowski folium).[4] This result applies more generally to any pair of disjoint regular representations, at least one of which is the GNS representation of an abstract Fock vacuum state. We shall specialize back down to the Minkowski/Rindler case later on.

Let ρ be a regular state of $\mathcal{A}[S, \sigma]$ inducing the GNS representation $(\pi_\rho, \mathcal{H}_\rho)$, and let ω_J be the abstract vacuum state determined by a complex structure J on (S, σ). The case we are interested in is, of course, when π_ρ, π_{ω_J} are disjoint. We first want to show how to define representation-independent probabilities in the state ρ for any J-quanta number operator that "counts" the number of quanta with wavefunctions in a fixed *finite*-dimensional subspace $F \subseteq \mathcal{S}_J$. (Parts of our exposition below follow BR (1996, 26–30), which may be consulted for further details.)

We know that, for any $f \in S$, there exists a self-adjoint operator $\Phi_\rho(f)$ on $\mathcal{H}_\rho$ such that

$$\pi_\rho(W(tf)) = \exp\left(it\Phi_\rho(f)\right), \qquad t \in \mathbb{R}. \tag{7.4}$$

[4]In fact, this was first proved, in effect, by Chaiken (1967). However his lengthy analysis focussed on comparing Fock with non-Fock (so-called "strange") representations of the Weyl algebra, and the implications of his result for disjoint Fock representations based on inequivalent one-particle structures seem not to have been carried down into the textbook tradition of the subject. (The closest result we have found is Theorem 5.2.14 of (BR 1996) which we are able to employ as a lemma to recover Chaiken's result for disjoint Fock representations—see section 6.5.)

We can also define unbounded annihilation and creation operators on $\mathcal{H}_\rho$ for J-quanta by

$$\begin{aligned} a_\rho(f) &:= 2^{-1/2}(\Phi_\rho(f) + i\Phi_\rho(Jf)), &(7.5)\\ a^*_\rho(f) &:= 2^{-1/2}(\Phi_\rho(f) - i\Phi_\rho(Jf)). &(7.6)\end{aligned}$$

Earlier, we denoted these operators by $a_J(f)$ and $a^*_J(f)$. However, we now want to emphasize the representation space upon which they act; and only the single complex structure J shall concern us in our general discussion, so there is no possibility of confusion with others.

Next, define a "quadratic form" $n_\rho(F) : \mathcal{H}_\rho \mapsto \mathbb{R}^+$. The domain of $n_\rho(F)$ is

$$D(n_\rho(F)) := \bigcap_{f\in F} D(a_\rho(f)), \tag{7.7}$$

where $D(a_\rho(f))$ is the domain of $a_\rho(f)$. Now let $\{f_k : k = 1, \ldots, m\}$ be some J-orthonormal basis for F, and define

$$[n_\rho(F)](\psi) := \sum_{k=1}^{m} \|a_\rho(f_k)\psi\|^2, \tag{7.8}$$

for any $\psi \in D(n_\rho(F))$. It can be shown that the sum in (7.8) is independent of the chosen orthonormal basis for F, and that $D(n_\rho(F))$ lies dense in $\mathcal{H}_\rho$. Given any densely defined, positive, closed quadratic form t on $\mathcal{H}_\rho$, there exists a unique positive self-adjoint operator T on $\mathcal{H}_\rho$ such that $D(t) = D(T^{1/2})$ and

$$t(\psi) = \langle T^{1/2}\psi, T^{1/2}\psi\rangle, \qquad \psi \in D(t). \tag{7.9}$$

We let $N_\rho(F)$ denote the finite-subspace J-quanta number operator on $\mathcal{H}_\rho$ arising from the quadratic form $n_\rho(F)$.

We seek a representation-*independent* value for "$\text{Prob}^\rho(N(F) \in \Delta)$", where $\Delta \subseteq \mathbb{N}$. So let τ be *any* regular state of $\mathcal{A}[S,\sigma]$, and let $N_\tau(F)$ be the corresponding number operator on $\mathcal{H}_\tau$. Let $\mathcal{A}[F]$ be the Weyl algebra over $(F, \sigma|_F)$, and let $E_\tau(F)$ denote the spectral measure for $N_\tau(F)$ acting on $\mathcal{H}_\tau$. Then, $[E_\tau(F)](\Delta)$ (the spectral projection representing the proposition "$N_\tau(F) \in \Delta$") is in the weak closure of $\pi_\tau(\mathcal{A}[F])$, by the Stone-von Neumann uniqueness theorem. In particular, there is a net $\{A_i\} \subseteq \mathcal{A}[F]$ such that $\pi_\tau(A_i)$ converges weakly to $[E_\tau(F)](\Delta)$. Now, the Stone-von Neumann uniqueness theorem also entails that there is a density operator D_ρ on $\mathcal{H}_\tau$ such that

$$\rho(A) = \text{Tr}(D_\rho \pi_\tau(A)), \qquad A \in \mathcal{A}[F]. \tag{7.10}$$

We therefore define

$$\begin{aligned}\text{Prob}^{\rho}(N(F) \in \Delta) &:= \lim_{i} \rho(A_i) && (7.11)\\ &= \lim_{i} \text{Tr}(D_{\rho}\pi_{\tau}(A_i)) && (7.12)\\ &= \text{Tr}(D_{\rho}[E_{\tau}(F)](\Delta)). && (7.13)\end{aligned}$$

The final equality displays that this definition is independent of the chosen approximating net $\{\pi_{\tau}(A_i)\}$, and the penultimate equality displays that this definition is independent of the (regular) representation π_{τ}. In particular, since we may take $\tau = \rho$, it follows that

$$\text{Prob}^{\rho}(N(F) \in \Delta) = \langle \Omega_{\rho}, [E_{\rho}(F)](\Delta)\Omega_{\rho}\rangle, \qquad (7.14)$$

exactly as expected.

We can also define a positive, closed quadratic form on $\mathcal{H}_{\rho}$ corresponding to the *total* J-quanta number operator by:

$$\begin{aligned} n_{\rho}(\psi) &= \sup_{F\in\mathbb{F}} [n_{\rho}(F)](\psi)\,, && (7.15)\\ D(n_{\rho}) &= \Big\{\psi \in \mathcal{H}_{\rho} : \psi \in \bigcap_{f\in S} D(a_{\rho}(f)),\, n_{\rho}(\psi) < \infty\Big\}, && (7.16)\end{aligned}$$

where $\mathbb{F}$ denotes the collection of all finite-dimensional subspaces of $\mathcal{S}_J$. If $D(n_{\rho})$ is dense in $\mathcal{H}_{\rho}$, then it makes sense to say that the total J-quanta number operator N_{ρ} exists on the Hilbert space $\mathcal{H}_{\rho}$. In general, however, $D(n_{\rho})$ will not be dense, and may contain only the 0 vector. Accordingly, we cannot use a direct analogue to Eqn. (7.13) to define the probability, in the state ρ, that there are, say, n or fewer J-quanta.

However, we can still proceed as follows. Fix $n \in \mathbb{N}$, and suppose $F \subseteq F'$ with both $F, F' \in \mathbb{F}$. Since any state with n or fewer J-quanta with wavefunctions in F' cannot have *more* than n J-quanta with wavefunctions in the (smaller) subspace F,

$$\text{Prob}^{\rho}(N(F) \in [0,n]) \;\geq\; \text{Prob}^{\rho}(N(F') \in [0,n]). \qquad (7.17)$$

Thus, whatever value we obtain for "$\text{Prob}^{\rho}(N \in [0,n])$", it should satisfy the inequality

$$\text{Prob}^{\rho}(N(F) \in [0,n]) \;\geq\; \text{Prob}^{\rho}(N \in [0,n]), \qquad (7.18)$$

for any finite-dimensional subspace $F \subseteq \mathcal{S}_J$. However, the following result holds.

Proposition 7.3. *If ρ is a regular state of $\mathcal{A}[S,\sigma]$ disjoint from the Fock state ω_J, then* $\inf_{F\in\mathbb{F}}\left\{\text{Prob}^{\,\rho}(N(F)\in[0,n])\right\}=0$ *for every $n\in\mathbb{N}$.*

Thus ρ must assign every finite number of J-quanta probability zero; i.e., ρ predicts an infinite number of J-quanta with probability 1!

Let us tighten this up some more. Suppose that we are in any regular representation $(\pi_\omega,\mathcal{H}_\omega)$ in which the total J-quanta number operator N_ω exists and is affiliated to $\pi_\omega(\mathcal{A}[S,\sigma])''$. (For example, we may take the Fock representation where $\omega=\omega_J$.) Let E_ω denote the spectral measure of N_ω on $\mathcal{H}_\omega$. Considering ρ as a state of $\pi_\omega(\mathcal{A}[S,\sigma])$, it is then reasonable to define

$$\text{Prob}^\rho(N\in[0,n]):=\hat{\rho}(E_\omega([0,n])), \tag{7.19}$$

where $\hat{\rho}$ is any extension of ρ to $\pi_\omega(\mathcal{A}[S,\sigma])''$, provided the right-hand side takes the same value for all extensions. (And, of course, it will when $\rho\in\mathfrak{F}(\pi_\omega)$, where (7.19) reduces to the standard definition.) Now clearly

$$[E_\omega(F)]([0,n])\;\geq\;E_\omega([0,n]),\qquad F\in\mathbb{F}. \tag{7.20}$$

("If there are at most n J-quanta in total, then there are at most n J-quanta whose wavefunctions lie in any finite-dimensional subspace of $\mathcal{S}_J$".) Since states preserve order relations between projections, every extension $\hat{\rho}$ must therefore satisfy

$$\text{Prob}^\rho(N(F)\in[0,n])\;=\;\hat{\rho}([E_\omega(F)]([0,n]))\;\geq\;\hat{\rho}(E_\omega([0,n])). \tag{7.21}$$

Thus, if ρ is disjoint from ω, Proposition 7.3 entails that $\text{Prob}^\rho(N\in[0,n])=0$ for all finite n.[5]

As an immediate consequence of this and the disjointness of the Minkowski and Rindler representations, we have (reverting to our earlier number operator notation):

$$\text{Prob}^{\omega_M^{\bowtie}}(N_R\in[0,n])=0=\text{Prob}^{\omega_R^{\bowtie}}(N_M\in[0,n]),\quad\forall n\in\mathbb{N}, \tag{7.22}$$

$$\text{Prob}^{\omega_M^{\triangleright}}(N_\triangleright\in[0,n])=0=\text{Prob}^{\omega_M^{\triangleleft}}(N_\triangleleft\in[0,n]),\quad\forall n\in\mathbb{N}. \tag{7.23}$$

The same probabilities obtain when the Minkowski vacuum is replaced with

[5]Notice that such a prediction could never be made by a state in the folium of π_ω, since density operator states are countably additive (see note 3).

any other state in the Minkowski folium.[6] So it could not be farther from the truth to say that there is merely the potential for Rindler quanta in the Minkowski vacuum, or in any other state in the folium of the Minkowski vacuum.

One must be careful, however, with an informal statement like "The M-vacuum contains infinitely many R-quanta with probability 1". Since Rindler wedges are unbounded, there is nothing unphysical, or otherwise metaphysically incoherent, about thinking of wedges as containing an infinite number of Rindler quanta. But we must not equate this with the quite different *empirical* claim "A Rindler observer's particle detector has the sure-fire disposition to register the value '∞' ". There is no such value! Rather, the empirical content of equations (7.22) and (7.23) is simply that an idealized "two-state" measuring apparatus designed to register whether there are more than n Rindler quanta in the Minkowski vacuum will always return the answer 'Yes'. This is a perfectly sensible physical disposition for a measuring device to have. Of course, we are not pretending to have in hand a specification of the physical details of such a device. Indeed, when physicists model particle detectors, these are usually assumed to couple to specific "modes" of the field, represented by finite-subspace, not total, number operators (cf., e.g., Wald 1994, Sec. 3.3). But, as regards our dispute with Teller, this is really beside the point, since Teller advertises his resolution of the paradox as a way to *avoid* a "retreat to instrumentalism" about the particle concept (1995, 110).

On Teller's behalf, one might object that there are still no grounds for saying any R-quanta obtain in the M-vacuum, since for any particular number n of R-quanta you care to name, equations (7.22) and (7.23) entail that n is *not* the number of R-quanta in the M-vacuum. But remember that the same is true for $n = 0$, and that, therefore, $n \geq 1$ R-quanta has probability 1! A further tack might be to deny that probability 0 for $n = 0$, or any other n, entails impossibility or non-actuality of that number of R-quanta. This would be similar to a common move made in response to the lottery paradox, in the hypothetical case where there are an infinite number of ticket holders. Since *someone* has to win, each ticket holder must still have the

[6]This underscores the utter bankruptcy, from the standpoint of the liberal about observables, in taking the weak equivalence of the Minkowski and Rindler representations to be sufficient for their physical equivalence. Yes, every Rindler state of the Weyl algebra is a weak* limit of Minkowski states. But the former all predict a finite number of Rindler quanta with probability 1, while the latter all predict an *infinite* number with probability 1. (Wald (1994, 82–83) makes the exact same point with respect to states that do and do not satisfy the "Hadamard" property.)

potential to win, even though his or her probability of winning is zero. The difficulty with this response is that in the Rindler case, we have no independent reason to think that some particular finite number of R-quanta *has* to be detected at all. Moreover, if we were to go soft on taking probability 0 to be sufficient for "not actual", we should equally deny that probability 1 is sufficient for "actual", and by Teller's lights the paradox would go away at a stroke (because there could never be actual Rindler *or* Minkowski quanta in *any* field state).

We conclude that Teller's resolution of the paradox of observer-dependence of particles fails. And so be it, since it was ill-motivated in the first place. We already indicated in the previous subsection that it should be enough of a resolution to recognize that there are different kinds of quanta. We believe the physicists of the field and detector approaches are correct to bite the bullet hard on this, even though it means abandoning naïve realism about particles (though not, of course, about detection events). We turn, next, to arguing that a coherent story can still be told about the relationship between the different kinds of particle talk used by different observers.

7.4 Incommensurable or complementary?

At the beginning of this chapter, we reproduced a passage from Jauch's amusing Galilean dialogue on the question "Are Quanta Real?". In that passage, Sagredo is glorying in the prospect that complementarity may be applicable even in classical physics; and, more generally, to solving the philosophical problem of the specificity of individual events versus the generality of scientific description. It is well-known that Bohr himself sought to extend the idea of complementarity to all different walks of life, beyond its originally intended application in quantum theory. And even within the confines of quantum theory, it is often the case that when the going gets tough, tough quantum theorists cloak themselves in the mystical profundity of complementarity, sometimes just to get philosophers off their backs.

So it seems with the following notorious comments of a well-known advocate of the detector approach that have received a predictably cool reception from philosophers:

> Bohr taught us that quantum mechanics is an algorithm for computing the results of measurements. Any discussion about what is a "real, physical vacuum", must therefore be related to the behavior of real, physical measuring devices, in this case particle-number detectors. Armed with such heuristic devices, we may

> then assert the following. There are quantum states and there are particle detectors. Quantum field theory enables us to predict probabilistically how a particular detector will respond to that state. That is all. That is all there can ever be in physics, because physics is about the observations and measurements that we can make in the world. We can't talk meaningfully about whether such-and-such a state contains particles except in the context of a specified particle detector measurement. To claim (as some authors occasionally do!) that when a detector responds (registers particles) in somebody's cherished vacuum state that the particles concerned are "fictitious" or "quasi-particles", or that the detector is being "misled" or "distorted", is an empty statement. (Davies 1984, 69)

We shall argue that, cleansed of Davies' purely operationalist reading of Bohr, complementarity does, after all, shed light on the relation between inequivalent particle concepts in quantum field theory.

Rüger (1989) balks at this idea. He writes:

> The "real problem"—how to understand how there might be particles for one observer, but none at all for another observer in a different state of motion—is not readily solved by an appeal to Copenhagenism.... Though quantum mechanics can tell us that the *properties* of micro-objects (like momentum or energy) depend in a sense on observers measuring them, the standard interpretation of the theory still does not tell us that whether there is a micro-object or not depends on observers. At least the common form of this interpretation is not of immediate help here. (Rüger 1989, 575–576)

Well, let us consider the "common form" of the Copenhagen interpretation. Whatever one's preferred embellishment of the interpretation, it must at least imply that observables represented by noncommuting "complementary" self-adjoint operators cannot have simultaneously determinate values in all states. Since field quantizations are built upon an abstract noncommutative algebra, the Weyl algebra, complementarity retains its application to quantum field theory. In particular, in any *single* Fock space representation—setting aside inequivalent representations for the moment—there will be a total number operator and nontrivial superpositions of its eigenstates. For these superpositions, which are eigenstates of observables failing to commute with the number operator, it is therefore perfectly in line with complementarity that we say they contain no actual particles in any substantive sense.[7] In addition, there are different number operators on Fock space that count the number of quanta with wavefunctions lying in different subspaces of the one-particle space, and they commute only if the corresponding subspaces are compatible. So even before we consider inequivalent particle concepts, we must already accept that there are different *complementary* "kinds" of quanta, according to what their wavefunctions are.

Does complementarity extend to the particle concepts associated with inequivalent Fock representations? *Contra* Rüger, we claim that it does. We saw earlier that one can build finite-subspace J-quanta number operators in *any* regular representation of $\mathcal{A}[S,\sigma]$, provided only that J defines a proper complex structure on S that leaves it invariant. In particular, using the canonical commutation relation $[\Phi(f),\Phi(g)] = i\sigma(f,g)I$, a tedious but elementary calculation reveals that, for any $f, g \in S$,

$$[N_{J_1}(f), N_{J_2}(g)] = \frac{i}{2}\Big\{\sigma(f,g)[\Phi(f),\Phi(g)]_+ + \sigma(f,J_2g)[\Phi(f),\Phi(J_2g)]_+ + \sigma(J_1f,g)[\Phi(J_1f),\Phi(g)]_+ + \sigma(J_1f,J_2g)[\Phi(J_1f),\Phi(J_2g)]_+\Big\}, \quad (7.24)$$

in any regular representation.[8] Thus, there are well-defined and, in general, *nontrivial* commutation relations between finite-subspace number operators, even when the associated particle concepts are inequivalent. We also saw in Eqn. (7.2) that when $J_2 \neq J_1$, no $N_{J_2}(f)$, for any $f \in \mathcal{S}_{J_2}$, will leave the zero-particle subspace of N_{J_1} invariant. Since it is a necessary condition that this nondegenerate eigenspace be left invariant by any self-adjoint operator commuting with N_{J_1}, it follows that $[N_{J_2}(f), N_{J_1}] \neq 0$ for all $f \in \mathcal{S}_{J_2}$. Thus finite-subspace number operators for one kind of quanta are complementary to the total number operators of inequivalent kinds of quanta.

Of course, we cannot give the same argument for complementarity between the *total* number operators N_{J_1} and N_{J_2} pertaining to inequivalent kinds of quanta, because, as we know, they cannot even be defined as operators on the same Hilbert space. However, we disagree with Arageorgis

[7] As Rüger notes earlier (1989, 571), in ordinary non-field-theoretic quantum theory, complementarity only undermined a naïve substance-properties ontology. However, this was only because there was no "number of quanta" observable in the theory.

[8] As a check on expression (7.24), note that it is invariant under the one-particle space phase transformations $f \to (\cos t + J_1 \sin t)f$ and $g \to (\cos t + J_2 \sin t)g$, and when $J_1 = J_2 = J$, reduces to zero just in case the rays generated by f and g are compatible subspaces of $\mathcal{S}_J$.

(1995, 303–304) that this means Teller's "complementarity talk" in relation to the Minkowski and Rindler total number operators is wholly inapplicable. We have two reasons for the disagreement.

First, since it is a necessary condition that a (possibly unbounded) self-adjoint observable Y on $\mathcal{H}_{\omega_{J_1}}$ commuting with N_{J_1} have $\Omega_{\omega_{J_1}}$ as an eigenvector, it is also necessary that the abstract vacuum state ω_{J_1} be dispersion-free on Y. But this latter condition is purely algebraic and makes sense even when Y does *not* act on $\mathcal{H}_{\omega_{J_1}}$. Moreover, as Proposition 7.1 shows, this condition fails when Y is the total number operator of any Fock representation inequivalent to $\pi_{\omega_{J_1}}$. So it is entirely natural to treat Proposition 7.1 as a vindication of the idea that inequivalent pairs of total number operators are complementary.

Secondly, we have seen that any state in the folium of a representation associated with one kind of quanta assigns probability zero to any finite number of an inequivalent kind of quanta. This has a direct analogue in the most famous instance of complementarity: that which obtains between the concepts of position and momentum.

Consider the unbounded position and momentum operators, Q and P ($= -i\frac{\partial}{\partial x}$), acting on $L_2(\mathbb{R})$. Let E and F be their respective spectral measures. We say that a state ρ of $\mathbf{B}(L_2(\mathbb{R}))$ assigns Q a *finite* dispersion-free value just in case ρ is dispersion-free on Q and there is a $\lambda \in \mathbb{R}$ such that $\rho(E(a,b)) = 1$ if and only if $\lambda \in (a,b)$. (Similarly for P.) Then the following is a direct consequence of the canonical commutation relation $[Q,P] \subseteq iI$ (see Halvorson & Clifton 1999, Prop. 3.7).

Proposition 7.4. *If ρ is a state of $\mathbf{B}(L_2(\mathbb{R}))$ that assigns Q (respectively, P) a finite dispersion-free value, then $\rho(F(a,b)) = 0$ (respectively, $\rho(E(a,b)) = 0$) for any $a, b \in \mathbb{R}$.*

This result makes rigorous the fact, suggested by Fourier analysis, that if either of Q or P has a sharp finite value in any state, the other is "maximally indeterminate". But the same goes for pairs of inequivalent number operators (N_{J_1}, N_{J_2}): if a regular state ρ assigns N_{J_1} a finite dispersion-free value, then $\rho \in \mathfrak{F}(\pi_{\omega_{J_1}})$ which, in turn, entails that ρ assigns probability zero to any finite set of eigenvalues for N_{J_2}. Thus, (N_{J_1}, N_{J_2}) are, in a natural sense, *maximally* complementary, despite the fact that they have no well-defined commutator.

One might object that our analogy is completely superficial; after all, Q and P still act on the *same* Hilbert space, $L_2(\mathbb{R})$! In the next section, however, we will show that the analogy between position-momentum complementarity and Minkowski-Rindler complementarity is exact.

7.5 Rethinking position-momentum complementarity

According to Bohr's notion of complementarity, a particle can have a sharp position, and it can have a sharp momentum, but it cannot have both simultaneously. In section 7.5.1, I argue that it is impossible to make sense of this idea in the "standard" Hilbert space formalism of quantum mechanics; in this case, a particle can have *neither* a sharp position *nor* a sharp momentum. In section 7.5.2, I argue that the proper way to make sense of position-momentum complementarity is in terms of inequivalent representations of the Weyl algebra. (I also argue that the Stone-von Neumann uniqueness theorem is an interpretive red herring.)

7.5.1 A problem with position-momentum complementarity

Complementarity is sometimes mistakenly equated with the the uncertainty relation

$$\Delta_\psi Q \cdot \Delta_\psi P \geq \hbar/2\,, \tag{7.25}$$

where $\Delta_\psi Q$ is the dispersion of Q in ψ, and $\Delta_\psi P$ is the dispersion of P in ψ. But the uncertainty relation says nothing about when Q and P can *possess* values; at best, it only tells us that there is a reciprocal relation between our *knowledge* of the value of Q and our *knowledge* of the value of P. To infer from this that Q and P do not simultaneously possess values would be to lapse into positivism.

A more promising analysis of complementarity is suggested by Bub and Clifton's (1996) classification of modal interpretations (see also Bub 1997; Halvorson & Clifton 1999). According to this analysis, we can think of Bohr's complementarity interpretation as a modal interpretation in which the measured observable R and a state e determines a unique maximal sublattice $\mathcal{L}(R,e)$ of the lattice $\mathcal{L}$ of all subspaces of the relevant Hilbert space. $\mathcal{L}(R,e)$ should be thought of as containing the propositions that have a definite truth value in the state e. In particular, $\mathcal{L}(R,e)$ always contains all propositions ascribing a value to R, and e can be decomposed into a mixture of pure states (i.e., truth valuations) of $\mathcal{L}(R,e)$. Thus, we can think of e as representing our *ignorance* of the possessed value of R. We would say that another observable R' is complementary to R just in case propositions attributing a value to R' are never contained in $\mathcal{L}(e,R)$.

But there is a serious problem in the case of position and momentum. In order to see this, recall that $E(S)\psi = \chi_S \cdot \psi$, for any $\psi \in L_2(\mathbb{R})$. Since two functions ψ and ϕ in $L_2(\mathbb{R})$ are identified when they agree except on a measure zero set, it follows that $E(S_1) = E(S_2)$ whenever $(S_1 - S_2) \cup (S_2 - S_1)$ has Lebesgue measure zero. This enables us to formulate a very simple "proof" that particles never have sharp positions.

> *Mathematical Fact:* $E(\{\lambda\}) = 0$.
>
> *Interpretive Assumption:* $E(\{\lambda\})$ means "the particle is located at the point λ."
>
> *Conclusion:* It is always false that the particle is located at λ.

What is more, the interpretive assumption entails that "being located in S_1" is literally the *same property* as "being located in S_2", whenever S_1 and S_2 agree almost everywhere. Thus, any attempt to attribute a position to the particle would force us to revise the classical notion of location in space.

Halvorson (2001a) argues that we can solve these difficulties by reinterpreting elements of $\mathcal{L}$ as "experimental propositions" rather than as "property ascriptions", and by introducing non-countably additive (i.e., non-vector) states on $\mathcal{L}$. In particular, suppose that we interpret $E(S)$ as the proposition: "A measurement of the position of the particle is certain to yield a value in S." Then, $E(\{\lambda\}) = 0$ does *not* entail that a particle cannot be located at λ, but only that no position measurement can be certain to yield the value λ. Moreover, there is a non-countably additive state h on $\mathcal{L}$ such that $h(E(S)) = 1$ for *all* open neighborhoods S of λ. Thus, we could think of h as representing a state in which the particle is located at λ.

However, there is still no proposition in the "object language" $\mathcal{L}$ that expresses the claim that the particle is located at λ.[9] Thus, if we think that particles really can have precise positions (or momenta), then the standard language $\mathcal{L}$ of quantum mechanics is *descriptively incomplete*.

7.5.2 The solution: Inequivalent representations

Let $\mathcal{A}[\mathbb{R}^2]$ be the Weyl algebra for a system with one degree of freedom. Recall that a representation $(\pi, \mathcal{H})$ of $\mathcal{A}[\mathbb{R}^2]$ is said to be *regular* just in

[9] The problem can be traced back to dropping the countably additivity condition. In particular, if a pure state h on $\mathcal{L}$ is countably additive, then there is a unique minimal element $E \in \mathcal{L}$ such that $h(E) = 1$. In other words, E represents the proposition that asserts that the state of affairs represented by h obtains. Therefore, by dropping the assumption of countable additivity, we allow for there to be more states than can be described in the language $\mathcal{L}$ of the theory.

case $a \mapsto \pi(U(a))$ and $b \mapsto \pi(V(b))$ are strongly continuous. According to the Stone-von Neumann uniqueness theorem, every regular representation of $\mathcal{A}[\mathbb{R}^2]$ is quasiequivalent to the "standard" representation on $L_2(\mathbb{R})$. However, I argue now that the Stone-von Neumann theorem has *absolutely no significance* from an interpretational perspective. *The "problem" of inequivalent representations arises already in elementary quantum mechanics.*

The Stone-von Neumann uniqueness theorem has no interpretive significance because the regularity assumption begs the question against position-momentum complementarity. In particular, a representation $(\pi, \mathcal{H})$ of $\mathcal{A}[\mathbb{R}^2]$ is regular just in case the self-adjoint generators Q of $\{\pi(U(a))\}$ and P of $\{\pi(V(b))\}$ exist on $\mathcal{H}$. However, if complementarity is correct, then Q and P cannot both possess sharp values. Why, then, do we need to assume that both operators exist in one representation space? What is more, we saw in the previous section that if *both* operators do exist, then *neither* can possess sharp values. Does this not give us a reason to rethink the regularity assumption?

It is only by employing nonregular representations of $\mathcal{A}[\mathbb{R}^2]$ that we can make sense of a particle's having sharp position (or momentum) values. I will now describe the "position representation" of $\mathcal{A}[\mathbb{R}^2]$, in which the position observable has a full (uncountably infinite) set of eigenstates. (The construction of the "momentum representation" proceeds along analogous lines. See Beaume et al. 1974; Fannes et al. 1974).

Let $l_2(\mathbb{R})$ denote the (nonseparable) Hilbert space of square-summable functions from $\mathbb{R}$ into $\mathbb{C}$. That is, an element f of $l_2(\mathbb{R})$ is supported on a countable subset S_f of $\mathbb{R}$ and $\|f\| := \sum_{x \in S_f} |f(x)|^2 < \infty$. The inner product on $l_2(\mathbb{R})$ is given by

$$\langle f, g \rangle = \sum_{x \in S_f \cap S_g} \overline{f(x)} g(x) \,. \tag{7.26}$$

For each $\lambda \in \mathbb{R}$, let φ_λ denote the characteristic function of $\{\lambda\}$. Thus, the set $\{\varphi_\lambda : \lambda \in \mathbb{R}\}$ is an orthonormal basis for $l_2(\mathbb{R})$. For each $a \in \mathbb{R}$, define $\pi(U(a))$ on the set $\{\varphi_\lambda : \lambda \in \mathbb{R}\}$ by

$$\pi(U(a))\varphi_\lambda = e^{ia\lambda}\varphi_\lambda \,. \tag{7.27}$$

Since $\pi(U(a))$ maps $\{\varphi_\lambda : \lambda \in \mathbb{R}\}$ onto an orthonormal basis for $\mathcal{H}$, $\pi(U(a))$ extends uniquely to a unitary operator on $\mathcal{H}$. Similarly, define $\pi(V(b))$ on $\{\varphi_\lambda : \lambda \in \mathbb{R}\}$ by

$$\pi(V(b))\varphi_\lambda = \varphi_{\lambda - b} \,. \tag{7.28}$$

Then $\pi(V(b))$ extends uniquely to a unitary operator on $\mathcal{H}$. Now, a straightforward calculation shows that,

$$\pi(U(a))\pi(V(b))\varphi_\lambda \quad = \quad e^{-iab}\pi(V(b))\pi(U(a))\varphi_\lambda\,, \qquad (7.29)$$

for any $a, b \in \mathbb{R}$. Thus, the operators $\{\pi(U(a)) : a \in \mathbb{R}\}$ and $\{\pi(V(b)) : b \in \mathbb{R}\}$ give a representation of $\mathcal{A}[\mathbb{R}^2]$ on $l_2(\mathbb{R})$. Moreover,

$$\lim_{a\to 0} \langle \varphi_\lambda, \pi(U(a))\varphi_\lambda\rangle = \lim_{a\to 0} e^{ia\lambda} = 1\,, \qquad (7.30)$$

for any $\lambda \in \mathbb{R}$. Thus, $a \mapsto \pi(U(a))$ is weakly continuous, and Stone's theorem entails that there is a self-adjoint operator Q on $\mathcal{H}$ such that $\pi(U(a)) = e^{iaQ}$. It is not difficult to see that $Q\varphi_\lambda = \lambda\varphi_\lambda$ for all $\lambda \in \mathbb{R}$.

We are now in a position to see that the relationship between the position and momentum representations is *exactly the same* as the relation between Minkowski vacuum representation and the Rindler vacuum representation.

Proposition 7.5. *The position and momentum representations of $\mathcal{A}[\mathbb{R}^2]$ are disjoint.*

Proof. We actually prove the stronger claim that if $(\pi, \mathcal{H})$ is a representation in which there is a common eigenvector φ for $\{\pi(V(b)) : b \in \mathbb{R}\}$, then $a \mapsto \pi(U(a))$ is not weakly continuous. Indeed, if φ is an eigenvector for the family $\{\pi(V(b)) : b \in \mathbb{R}\}$, then,

$$e^{iab}\langle\varphi, \pi(U(a))\varphi\rangle = \langle\varphi, \pi(V(-b)U(a)V(b))\psi\rangle = \langle\varphi, \pi(U(a))\varphi\rangle\,, \qquad (7.31)$$

for any $a, b \in \mathbb{R}$. But this is possible only if $\langle\varphi, \pi(U(a))\varphi\rangle = 0$ when $a \neq 0$. Since $\langle\varphi, \pi(U(a))\varphi\rangle = 1$ when $a = 0$, it follows that $a \mapsto \pi(U(a))$ is not weakly continuous. □

We maintain, therefore, that there are compelling formal reasons for thinking of Minkowski and Rindler quanta as complementary. What's more, when a Minkowski observer sets out to detect particles, her state of motion determines that her detector will be sensitive to the presence of Minkowski quanta. Similarly for a Rindler observer and his detector. This is borne out by the analysis of Unruh and Wald (1984) in which they show how his detector will *itself* "define" (in a "nonstandard" way) what solutions of the relativistic wave equation are counted as having positive frequency, via the way the detector couples to the field. So we may think of the choice of an observer to follow an inertial or Rindler trajectory through spacetime as analogous to the choice between measuring the position or momentum of a particle. Each choice requires a distinct kind of coupling to the system, and both measurements cannot be executed on the field simultaneously and with arbitrarily high precision.[10] Moreover, execution of one type of measurement precludes meaningful discourse about the values of the observable that the observer did not choose to measure. All this is the essence of "Copenhagenism."

And it should *not* be equated with operationalism! The goal of the detector approach to the paradox of observer-dependence was to achieve clarity on the problem by reverting back to operational definitions of the word "particle" with respect to the concrete behaviour of particular kinds of detectors (cf., e.g., DeWitt 1979b, 692). But, as with early days of special relativity and quantum theory, operationalism can serve its purpose and then be jettisoned. Rindler quanta get their status as such not because they are, *by definition*, the sort of thing that accelerated detectors detect. This gets things backwards. Rindler detectors display Rindler quanta in the Minkowski vacuum *because* they couple to *Rindler* observables of the field that are distinct from, and indeed complementary to, Minkowski observables.

7.6 Against incommensurability

Arageorgis (1995) himself, together with his collaborators (Arageorgis et al. 2001), prefer to characterize inequivalent particle concepts, not as complementary, but *incommensurable*. At first glance, this looks like a trivial semantic dispute between us. For instance Glymour, in a recent introductory text on the philosophy of science, summarizes complementarity using the language of incommensurability:

> Changing the experiments we conduct is like changing conceptual schemes or paradigms: we experience a different world. Just

[10] Why can't *both* a Minkowski and a Rindler observer set off in different spacetime directions and *simultaneously* measure their respective (finite-subspace or total) number operators? Would it not, then, be a violation of relativistic causality when the Minkowski observer's measurement disturbs the statistics of the Rindler observer's measurement outcomes? No. We must remember that the Minkowski particle concept is global, so our Minkowski observer cannot make a precise measurement of any of her number operators unless it is executed throughout the whole of spacetime, which would necessarily destroy her spacelike separation from the Rindler observer. On the other hand, if she is content with only an approximate measurement of one of her number operators in a bounded spacetime region, it is well-known that simultaneous, nondisturbing "unsharp" measurements of incompatible observables *are* possible. For an analysis of the case of simultaneous measurements of unsharp position and momentum, see Busch et al. (1995).

> as no world of experience combines different conceptual schemes, no reality we can experience (even indirectly through our experiments) combines precise position and precise momentum. (Glymour 1992, 128)

However, philosophers of science usually think of incommensurability as a relation between theories *in toto*, not different parts of the same physical theory. Arageorgis et al. maintain that inequivalent quantizations define incommensurable *theories*.

Arageorgis (1995) makes the claim that "the degrees of freedom of the field in the Rindler model *simply cannot be described* in terms of the ground state and the elementary excitations of the degrees of freedom of the field in the Minkowski model" (1995, 268; our italics). Yet so much of our earlier discussion proves the contrary. Disjoint representations *are* commensurable, via the abstract Weyl algebra they share. The result is that the ground state of one Fock representation makes definite, if sometimes counterintuitive, predictions for the "differently complexified" degrees of freedom of other Fock representations.

Arageorgis et al. (2001) offer an *argument* for incommensurability—based on Fulling's "theorem". They begin by discussing the case where the primed and unprimed representations are unitarily equivalent. (Notice that they speak of two different "theorists", rather than two different observers.)

> ... while *different*, these particle concepts can nevertheless be deemed to be *commensurable*. The two theorists are just labelling the particle states in different ways, since each defines particles of a given type by mixing the creation and annihilation operators of the other theorist. Insofar as the primed and unprimed theorists disagree, they disagree over which of two intertranslatable descriptions of the same physical situation to use.
>
> The gulf of disagreement between two theorists using unitarily inequivalent Fock space representations is much deeper. If in this case the primed-particle theorist can speak sensibly of the unprimed-particle theorist's vacuum at all, he will say that its primed-particle content is infinite (or more properly, undefined), and the unprimed-theorist will say the same of the unprimed-particle content of the primed vacuum. Such disagreement is profound enough that we deem the particle concepts affiliated with unitarily inequivalent Fock representations *incommensurable*. (Arageorgis et al. 2001, 26)

The logic of this argument is curious. In order to make Fulling's "theorem" do the work for incommensurability that Arageorgis et al. want it to, one must first have in hand a rigorous version of the theorem (otherwise their argument would be built on sand). But any rigorous version, like our Proposition 7.2, has to presuppose that there is sense to be made of using a vector state from one Fock representation to generate a prediction for the expectation value of the total number operator in another inequivalent representation. Thus, one cannot even *entertain* the philosophical implications of Fulling's result if one has not first granted a certain level of commensurability between inequivalent representations.

Moreover, while it may be tempting to *define* what one means by "incommensurable representations" in terms of Fulling's characterization of inequivalent representations, it is difficult to see the exact motivation for such a definition. Even vector states *in the folium* of the unprimed "theorist's" Fock representation can fail to assign his total number operator a finite expectation value (just consider any vector not in the operator's domain). Yet it would be alarmist to claim that, were the field in such a state, the unprimed "theorist" would lose his conceptual grasp on, or his ability to talk about, his *own* unprimed kind of quanta! So long as a state prescribes a well-defined probability measure over the spectral projections of the unprimed "theorist's" total number operator—and all states in his *and* the folium of any primed "theorist's" representation *will*—we fail to see the difficulty.

7.7 Conclusion

Let us return to answer the questions we raised in the introduction to chapter 6.

We have argued that a conservative about physical observables is not committed to the physical inequivalence of disjoint representations, so long as he has no attachment to states in a particular folium being the only physical ones. On the other hand, a liberal about physical observables, no matter what his view on states, *must* say that disjoint representations yield physically inequivalent descriptions of a field. However, we steadfastly resisted the idea that this means an interpreter of quantum field theory must say disjoint representations are incommensurable, or even different, *theories*.

Distinguishing "potential" from "actual" quanta won't do to resolve the paradox of observer-dependence. Rather, the paradox forces us to thoroughly abandon the idea that Minkowski and Rindler observers moving through the same field are both trying to detect the presence of particles *simpliciter*. Their motions cause their detectors to couple to *different* incompatible particle observables of the field, making their perspectives on the field necessarily complementary. Furthermore, taking this complementarity seriously means saying that neither the Minkowski nor Rindler perspective yields the uniquely "correct" story about the particle content of the field, and that *both* are necessary to provide a complete picture.

So, "Are Rindler Quanta Real?" This is a loaded question that can be understood in two different ways.

First, we could be asking "Are *Any* Quanta Real?" without regard to inequivalent notions of quanta. Certainly particle detection events, modulo a resolution of the measurement problem, are real. But it should be obvious by now that detection events do not generally license naïve talk of individuatable, localizable, particles that come in determinate numbers in the *absence* of being detected.

A fuller response would be that quantum field theory is "fundamentally" a theory of a field, not particles. This is a reasonable response given that: (i) the field operators $\{\Phi(f) : f \in S\}$ exist in every regular representation; (ii) they can be used to construct creation, annihilation, and number operators; and (iii) their expectation values evolve in significant respects like the values of the counterpart classical field, modulo non-local Bell-type correlations. This "field approach" response might seem to leave the ontology of the theory somewhat opaque. The field operators, being subject to the canonical commutation relations, do not all commute; so we cannot speak sensibly of them all simultaneously having determinate values. However, the right way to think of the field approach, compatible with complementary, is to see it as viewing a quantum field as a collection of correlated "objective propensities" to display values of the field operators in more or less localized regions of spacetime, relative to various measurement contexts. This view makes room for the reality of quanta, but only as a kind of epiphenomenon of the field associated with certain functions of the field operators.

Second, we could be specifically interested in knowing whether it is sensible to say that *Rindler*, as opposed to just Minkowski, quanta are real. An uninteresting answer would be 'No'—on the grounds that quantum field theory on flat spacetime is not a serious candidate for describing our actual universe, or that the Rindler representation is too "pathological". But, as philosophers, we are content to leave to the physicists the task of deciding the question "Are Rindler Quanta *Empirically Verified*?". All we have tried to determine (to echo words of van Fraassen) is how the world *could possibly be* if both the Rindler and Minkowski representations were "true". We have argued that the antecedent of this counterfactual makes perfect sense, and that it forces us to view Rindler and Minkowski quanta as complementary. Thus, Rindler and Minkowski would be equally amenable to achieving "reality status" provided the appropriate measurement context were in place. As Wald has put it:

> Rindler particles are "real" to accelerating observers! This shows that different notions of "particle" are useful for different purposes. (1994, 116)

7.8 Appendix: Proofs of theorems

Proposition 7.1. *If J_1, J_2 are distinct complex structures on (S, σ), then ω_{J_1} (respectively, ω_{J_2}) predicts dispersion in N_{J_2} (respectively, N_{J_1}).*

Proof. We shall prove the contrapositive. Suppose, then, that there is some extension $\hat{\omega}_{J_1}$ of ω_{J_1} to $\mathbf{B}(\mathcal{F}(\mathcal{S}_{J_2}))$ that is dispersion-free on all bounded functions of N_{J_2}. Then $\hat{\omega}_{J_1}$ is multiplicative for the product of the bounded operator $e^{\pm itN_{J_2}}$ with any other element of $\mathbf{B}(\mathcal{F}(\mathcal{S}_{J_2}))$ (KR 1997, Ex. 4.6.16). Hence, by Eqn. (6.41),

$$\begin{aligned} \omega_{J_1}(W(\cos t + \sin t J_2 f)) &= \hat{\omega}_{J_1}\left(e^{-itN_{J_2}}\pi_{\omega_{J_2}}(W(f))e^{itN_{J_2}}\right) & (7.32)\\ &= \hat{\omega}_{J_1}(e^{-itN_{J_2}})\,\omega_{J_1}(W(f))\,\hat{\omega}_{J_1}(e^{itN_{J_2}}) & (7.33)\\ &= \omega_{J_1}(W(f)), & (7.34) \end{aligned}$$

for all $f \in S$ and $t \in \mathbb{R}$. In particular, we may set $t = \pi/2$, and it follows that $\omega_{J_1}(W(J_2 f)) = \omega_{J_1}(W(f))$ for all $f \in S$. Since e^{-x} is a one-to-one function of $x \in \mathbb{R}$, it follows from (6.33) that

$$(f, f)_{J_1} = (J_2 f, J_2 f)_{J_1}, \qquad f \in S, \tag{7.35}$$

and J_2 is a real-linear isometry of the Hilbert space $\mathcal{S}_{J_1}$. We next show that J_2 is in fact a unitary operator on $\mathcal{S}_{J_1}$.

Since J_2 is a symplectomorphism, $\mathrm{Im}(J_2 f, J_2 g)_{J_1} = \mathrm{Im}(f, g)_{J_1}$ for any two elements $f, g \in S$. We also have

$$|f+g|^2_{J_1} = |f|^2_{J_1} + |g|^2_{J_1} + 2\mathrm{Re}(f, g)_{J_1}, \tag{7.36}$$

$$
\begin{aligned}
|J_2 f + J_2 g|_{J_1}^2 &= |J_2 f|_{J_1}^2 + |J_2 g|_{J_1}^2 + 2\mathrm{Re}(J_2 f, J_2 g)_{J_1} &(7.37)\\
&= |f|_{J_1}^2 + |g|_{J_1}^2 + 2\mathrm{Re}(J_2 f, J_2 g)_{J_1}, &(7.38)
\end{aligned}
$$

using the fact that J_2 is isometric. But $J_2(f+g) = J_2 f + J_2 g$, since J_2 is real-linear. Thus,

$$|J_2 f + J_2 g|_{J_1}^2 = |J_2(f+g)|_{J_1}^2 = |f+g|_{J_1}^2, \tag{7.39}$$

using again the fact that J_2 is isometric. Cancellation with Eqns. (7.36) and (7.38) then gives $\mathrm{Re}(f,g)_{J_1} = \mathrm{Re}(J_2 f, J_2 g)_{J_1}$. Thus, J_2 preserves the inner product between any two vectors in $\mathcal{S}_{J_1}$. All that remains to show is that J_2 is complex-linear. So let $f \in \mathcal{S}_{J_1}$. Then,

$$(J_2(if), J_2 g)_{J_1} = (if, g)_{J_1} = -i(f,g)_{J_1} = -i(J_2 f, J_2 g)_{J_1} = (iJ_2 f, J_2 g)_{J_1}, \tag{7.40}$$

for all $g \in \mathcal{H}$. Since J_2 is onto, it follows that $(J_2(if), g)_{J_1} = (iJ_2 f, g)_{J_1}$ for all $g \in \mathcal{H}$ and therefore $J_2(if) = iJ_2 f$.

Finally, since J_2 is unitary and $J_2^2 = -I$, it follows that $J_2 = \pm iI = \pm J_1$. However, if $J_2 = -J_1$, then

$$-\sigma(f, J_1 f) = \sigma(f, J_2 f) \geq 0, \qquad f \in S, \tag{7.41}$$

since J_2 is a complex structure. Since J_1 is also a complex structure, it follows that $\sigma(f, J_1 f) = 0$ for all $f \in S$ and $S = \{0\}$. Therefore, $J_2 = J_1$. □

Proposition 7.2. *A pair of Fock representations $\pi_{\omega_{J_1}}$, $\pi_{\omega_{J_2}}$ are unitarily equivalent if and only if ω_{J_1} assigns N_{J_2} a finite value (equivalently, ω_{J_2} assigns N_{J_1} a finite value).*

Proof. S may be thought of as a *real* Hilbert space relative to either of the inner products μ_1, μ_2 defined by

$$\mu_{1,2}(\cdot,\cdot) := \mathrm{Re}(\cdot,\cdot)_{J_{1,2}} = \sigma(\cdot, J_{1,2}\cdot). \tag{7.42}$$

We shall use Theorem 2 of (van Daele & Verbeure 1971): $\pi_{\omega_{J_1}}, \pi_{\omega_{J_2}}$ are unitarily equivalent if and only if the positive operator $-[J_1, J_2]_+ - 2I$ on S is trace-class relative to μ_2. (Since unitary equivalence is symmetric, the same "if and only if" holds with $1 \leftrightarrow 2$.)

As we know, we can build any number operator $N_{J_2}(f)$ ($f \in S$) on $\mathcal{H}_{\omega_{J_1}}$ by using the complex structure J_2 in Eqns. (6.34). In terms of field operators, the result is

$$N_{J_2}(f) = 2^{-1}(\Phi(f)^2 + \Phi(J_2 f)^2 + i[\Phi(f), \Phi(J_2 f)]). \tag{7.43}$$

Observe that $N_{J_2}(J_2 f) = N_{J_2}(f)$, which had better be the case, since $N_{J_2}(f)$ represents the number of J_2-quanta with wavefunction in the *subspace* of $\mathcal{S}_{J_2}$ generated by f. The expectation value of an arbitrary "two-point function" in the J_1-vacuum state is given by

$$\begin{aligned}
&\langle \Omega_{\omega_{J_1}}, \phi(f_1)\phi(f_2)\Omega_{\omega_{J_1}}\rangle \\
&\quad= (-i)^2 \frac{\partial^2}{\partial t_1 \partial t_2}\, \omega_{J_1}(W(t_1 f_1)W(t_2 f_2))\Big|_{t_1=t_2=0} \\
&\quad= -\frac{\partial^2}{\partial t_1 \partial t_2} \exp\Big(-\frac{1}{2}t_1 t_2 (f_1, f_2)_{J_1} - \frac{1}{4}t_1^2 (f_1, f_1)_{J_1} - \frac{1}{4}t_2^2 (f_2, f_2)_{J_1}\Big)\Big|_{t_1=t_2=0} \\
&\quad= \frac{1}{2}(f_1, f_2)_{J_1},
\end{aligned} \tag{7.44}$$

invoking (6.32) in the first equality, and the Weyl relations (6.10) together with Eqns. (6.23), (6.33) to obtain the second. Plugging Eqn. (7.44) back into (7.43) and using (7.42) eventually yields

$$\langle \Omega_{\omega_{J_1}}, N_{J_2}(f)\Omega_{\omega_{J_1}}\rangle = 2^{-2}\mu_2(f, (-[J_1, J_2]_+ - 2I)f). \tag{7.45}$$

Next, recall that on the Hilbert space $\mathcal{H}_{\omega_{J_2}}$, $N_{J_2} = \sum_{k=1}^{\infty} N_{J_2}(f_k)$, where $\{f_k\} \subseteq \mathcal{S}_{J_2}$ is any orthonormal basis. Let $\hat{\omega}_{J_1}$ be any extension of ω_{J_1} to $\mathbf{B}(\mathcal{H}_{\omega_{J_2}})$. The calculation that resulted in expression (7.45) was done in $\mathcal{H}_{\omega_{J_1}}$, however, only finitely many-degrees of freedom were involved. Thus the Stone-von Neumann uniqueness theorem ensures that (7.45) gives the value of each individual $\hat{\omega}_{J_1}(N_{J_2}(f_k))$. Since for any finite m, $\sum_{k=1}^{m} N_{J_2}(f_k) \leq N_{J_2}$ as positive operators, we must also have

$$\sum_{k=1}^{m} \hat{\omega}_{J_1}(N_{J_2}(f_k)) = \hat{\omega}_{J_1}\left(\sum_{k=1}^{m} N_{J_2}(f_k)\right) \leq \hat{\omega}_{J_1}(N_{J_2}). \tag{7.46}$$

Thus, $\hat{\omega}_{J_1}(N_{J_2})$ will be defined only if the sum

$$\sum_{k=1}^{\infty} \hat{\omega}_{J_1}(N_{J_2}(f_k)) = \sum_{k=1}^{\infty} \hat{\omega}_{J_1}(N_{J_2}(J_2 f_k)) \tag{7.47}$$

converges. Using (7.45), this is, in turn, equivalent to

$$\sum_{k=1}^{\infty} \mu_2(f_k, (-[J_1, J_2]_+ - 2I)f_k) + \sum_{k=1}^{\infty} \mu_2(J_2 f_k, (-[J_1, J_2]_+ - 2I)J_2 f_k) < \infty. \tag{7.48}$$

However, it is easy to see that $\{f_k\}$ is a J_2-orthonormal basis just in case $\{f_k, J_2 f_k\}$ forms an orthonormal basis in S relative to the inner product μ_2. Thus, Eqn. (7.48) is none other than the statement that the operator $-[J_1, J_2]_+ - 2I$ on S is trace-class relative to μ_2, which is equivalent to the unitary equivalence of $\pi_{\omega_{J_1}}, \pi_{\omega_{J_2}}$. (The same argument, of course, applies with $1 \leftrightarrow 2$ throughout.) □

Proposition 7.3. *If ρ is a regular state of $\mathcal{A}[S, \sigma]$ disjoint from the Fock state ω_J, then* $\inf_{F \in \mathbb{F}} \Big\{ \mathrm{Prob}^{\rho}(N(F) \in [0, n]) \Big\} = 0$ *for every $n \in \mathbb{N}$.*

Proof. Suppose that ω_J and ρ are disjoint; i.e., $\mathfrak{F}(\omega_J) \cap \mathfrak{F}(\rho) = \emptyset$. First, we show that $D(n_\rho) = \{0\}$, where n_ρ is the quadratic form on $\mathcal{H}_\rho$ which, if densely defined, would correspond to the total J-quanta number operator.

Suppose, for reductio ad absurdum, that $D(n_\rho)$ contains some unit vector ψ. Let ω be the state of $\mathcal{A}[S, \sigma]$ defined by

$$\omega(A) = \langle \psi, \pi_\rho(A)\psi \rangle \,, \qquad A \in \mathcal{A}[S, \sigma] \,. \tag{7.49}$$

Since $\omega \in \mathfrak{F}(\rho)$, it follows that ω is a regular state of $\mathcal{A}[S, \sigma]$ (since ρ itself is regular), and that $\omega \notin \mathfrak{F}(\omega_J)$. Let P be the projection onto the closed subspace in $\mathcal{H}_\rho$ generated by the set $\pi_\rho(\mathcal{A}[S, \sigma])\psi$. If we let $P\pi_\rho$ denote the subrepresentation of π_ρ on $P\mathcal{H}_\rho$, then $(P\pi_\rho, P\mathcal{H}_\rho)$ is a representation of $\mathcal{A}[S, \sigma]$ with cyclic vector ψ. By the uniqueness of the GNS representation, it follows that $(P\pi_\rho, P\mathcal{H}_\rho)$ is unitarily equivalent to $(\pi_\omega, \mathcal{H}_\omega)$. In particular, since Ω_ω is the image in $\mathcal{H}_\omega$ of $\psi \in P\mathcal{H}_\rho$, $D(n_\omega)$ contains a vector cyclic for $\pi_\omega(\mathcal{A}[S, \sigma])$ in $\mathcal{H}_\omega$. However, by Theorem 4.2.12 of (BR 1996), this implies that $\omega \in \mathfrak{F}(\omega_J)$, which is a contradiction. Therefore, $D(n_\rho) = \{0\}$.

Now suppose, again for reductio ad absurdum, that

$$\inf_{F \in \mathbb{F}} \Big\{ \mathrm{Prob}^{\rho}(N(F) \in [0, n]) \Big\} \neq 0. \tag{7.50}$$

Let $E_F := [E_\rho(F)]([0, n])$ and let $E := \bigwedge_{F \in \mathbb{F}} E_F$. Since the family $\{E_F\}$ of projections is downward directed (i.e., $F \subseteq F'$ implies $E_F \geq E_{F'}$), we have

$$0 \;\neq\; \inf_{F \in \mathbb{F}} \{\langle \Omega_\rho, E_F \Omega_\rho \rangle\} \;=\; \langle \Omega_\rho, E\Omega_\rho \rangle \;=\; \|E\Omega_\rho\|^2. \tag{7.51}$$

Now since $E_F E\Omega_\rho = E\Omega_\rho$, it follows that

$$[n_\rho(F)](E\Omega_\rho) \leq n, \tag{7.52}$$

for all $F \in \mathbb{F}$. Thus, $E\Omega_\rho \in D(n_\rho)$ and $D(n_\rho) \neq \{0\}$, in contradiction with the conclusion of the previous paragraph. □

Chapter 8

Summary and outlook

8.1 Summary

In this dissertation, we considered three issues at the heart of the foundations of quantum field theory: Nonlocality, localizable particles, and inequivalent particle concepts.

Part I investigated the issue of nonlocality in quantum field theory. Here we saw that the generic state of any pair of spacelike separated regions violates Bell's inequalities, and thus predicts correlations that cannot be explained by any local hidden variable model. This first result depends only on the fact that local algebras of observables are infinite (more precisely, "of infinite type"), and so it holds not only for relativistic QFT, but also for non-relativistic QFT, as well as for a pair of particles (taking into account their position and momentum degrees of freedom). We also saw that any "cyclic" state is entangled across spacelike separated regions. Since the Reeh-Schlieder theorem entails that any field state with bounded energy is cyclic for each local algebra, this result shows that a number of physically interesting states—including the Minkowski vacuum state—are entangled.

In chapter 3, we considered a type of nonlocality that is novel to *relativistic* QFT. In particular, it is impossible to perform "isolating" operations that would remove the entanglement between any local system and its environment. From a structural point of view, this feature of RQFT can be traced to the fact that the algebras of local observables are type III von Neumann algebras, and so have no atomic projections. This fact can, in turn, be traced to the microcausality assumption (see Horuzhy 1988, Prop. 1.3.13), which is intended to enforce the restriction on no superluminal propagation. Thus, we have a curious fact in that a locality requirement actually enforces, rather than conflicts with, a form of quantum nonlocality.

Part II investigates the concept of localizable particles in relativistic quantum theories. In chapter 4, we considered a concrete proposal for localizing particles in relativistic QFT, viz., the Newton-Wigner localization scheme. Structurally, the Newton-Wigner localization scheme is very attractive; in fact, it is structurally identical to particle localization in nonrelativistic QFT. However, I argued that despite its structural simplicity, there is no cogent physical interpretation of the Newton-Wigner localization scheme. In particular, the Newton-Wigner scheme "assigns" observables to regions in spacetime, but this assignment does not satisfy microcausality—i.e., there are observables assigned to spacelike separated regions that are not compatible. Thus, the advocate of the Newton-Wigner scheme is impaled on the horns of a dilemma: If observables assigned to a region are measurable in that region, then superluminal signalling is possible. If observables assigned to a region are *not* measurable in that region, then the assignment has no empirical consequences, and is completely arbitrary.

In chapter 5, we considered the issue of particle localization from a more abstract perspective. Here we found that there *is* a fundamental conflict between the requirements of localizability and the constraints of relativistic causality. In particular, relativistic causality (expressed by means of the microcausality assumption) entails that there are no localizable particles. This claim holds no matter what sort of localizing observables we make use of—whether they be projection operators, positive operators, or number operators. Thus, in relativistic quantum theories, the concept of a localized object is at best an approximation that works fairly well at the macroscopic level.

Part III considered the issue of inequivalent particle concepts in RQFT. Whereas in Part II the difficulty was that there is no concept of localizable particles, the difficulty here is that there are *too many* particle concepts. In particular, we can construct two observables for a quantum field, both of which purport to be counting the total number of particles, but whose values cannot possibly be reconciled in one coherent story about the ontology of the field.

Each of these particle concepts corresponds to a representation of the canonical commutation relations. Thus, in chapter 6, we considered in general the status of inequivalent representations of the canonical commutation relations. Here we saw that one's position on inequivalent representations will depend both on one's attitude towards states that lie in different folia of the Weyl algebra, and on one's attitude towards "ideal" observables that can only be weakly approximated by elements of the abstract observable algebra.

Finally, in chapter 7, we considered the specific case of Minkowski versus Rindler quanta. Here I argued that the descriptions given by the Minkowski (inertial) and Rindler (accelerated) observers should not be thought of as deriving from incommensurable theories about the quantum field. Rather, these apparently conflicting accounts are simply another instance of the complementarity that is familiar from elementary quantum mechanics.

8.2 Open questions and directions for future research

There are a number of open questions that were not answered in this dissertation, and there are several further topics that received no attention in this dissertation.

1. Malament's conjecture. As we noted at the end of chapter 2, none of the results we obtained there settles the following conjecture made by David Malament.

Conjecture. *Let O_1, O_2 be a pair of spacelike separated regions of Minkowski spacetime. Then the vacuum state is Bell correlated across $\mathcal{R}(O_1), \mathcal{R}(O_2)$.*

This conjecture is philosophically important for a number of reasons. First, as Redhead (1995a) notes, the vacuum state in RQFT—unlike the vacuum state in non-relativistic QFT—should *not* be thought of as a state in which "nothing is happening" locally. If Malament's conjecture could be verified, it would further underscore the drastic difference between the ontology of non-relativistic vacuum state and the relativistic vacuum state. Second, the vacuum state is, by definition, the unique state that is *invariant* under all Lorentz transformations. Thus, if it could be shown that the vacuum state predicts Bell correlations for any two spacelike separated regions, then we would have good evidence for the fact that quantum nonlocality does not entail any sort of violation of the principles of relativity. Third, the Minkowski vacuum contains no interaction and is the state of lowest energy. Thus, if the vacuum state sustains nonlocal correlations at all distances, we would have a striking example of the fact that nonlocal correlations do not involve any transfer of energy momentum—which, it would be reasonble to assume, is a necessary condition of causality.

2. Theory of measurement. Unlike elementary quantum mechanics, quantum field theory does not yet have a rigorous theory of measurement.

There is every reason to expect that, when a measurement theory is formulated, QFT will also be plagued by a "measurement problem." However, it is not yet clear what form the problem will take, or what sorts of solutions it might admit. Thus, one of the most important problems in the foundations of QFT is to formulate a rigorous measurement theory. Furthermore, such a theory should take into explicit account the fact that measurements are *local*, or somehow result from localized interactions between a measuring apparatus and the field.

Even in the absence of a rigorous theory of measurement, certain portions of this dissertation favored a no-collapse interpretation of quantum field theory. For example, in making the distinction between selective and non-selective measurements (chapter 3), I argued that the change in statistics of distant systems induced by selective measurements is not a result of any physical disturbance; rather, the "collapse" results from the mental operation of choosing a subensemble.

Furthermore, I argued in chapter 7 that the choice of a representation should be thought of a analogous to the choice of a "privileged" observable—in the sense of Bub and Clifton (1996)—for a modal interpretation of quantum theory.[1] In particular, we can use this perspective on inequivalent representations to elaborate a "modal" version of Bohr's complementarity interpretation, in which the privileged observable can vary depending on the measurement context (cf. Bub 1997, Sec. 7.1). For example, in a position measurement context, we would treat the pure states in the position folium as the (modal) "value states" for the position observable. Furthermore, according to this interpretation, the issue of inequivalent representations is a conceptual problem for *all* quantum theories (except for toy models on finite-dimensional Hilbert spaces, where all representations are equivalent), and not just for quantum field theory.

3. Interacting quantum fields. For the most part, this dissertation has proceeded from an abstract point of view, considering model-independent features of generic quantum field theories. The only concrete model we have considered is the (trivial) free Bose field (see chapters 4, 6, and 7). Thus, we have omitted from discussion a number of the "most interesting" quantum field theories, such as quantum electrodynamics. (Of course, some of these interesting quantum field theories have not yet been given a mathematically

[1] The only disanalogy is that choosing a representation is a necessary, but not sufficient, condition for assigning values to observables. Each representation will be subject to the Kochen-Specker argument, and so we would need to go on to single out, within a representation, a sublattice of definite properties.

rigorous formulation.) There are, however, a number of interesting conceptual issues that arise only in the context of interacting quantum fields.

For example, Haag's theorem (Streater & Wightman 2000, 165) appears to show that the standard Hilbert space approach to quantum theory is not adequate for treating interacting quantum fields. But this raises interesting questions both of a methodological and of a metaphysical nature.

First, Emch has claimed that Haag's theorem shows, "the inability of conventional field theories to describe scattering situations, using the interaction picture, in which the S-matrix is different than the identity" (Emch 1972, 249). Thus, Haag's theorem poses a challenge for our understanding of the methodology of QFT. In particular, "heuristic" QFT proceeds to make predictions as if it were possible to use the standard Hilbert space approach to describe interacting quantum fields, and these predictions turn out to be quite accurate. How, then, is it possible to use an inconsistent mathematical formalism to derive these predictions?

Second, Haag's theorem presents us with an example of a result that is peculiar to *relativistic* quantum field theories. In particular, it *is* possible to construct interacting models of non-relativistic (Galilei-invariant) QFT within the confines of a single Hilbert space representation (cf. Lévy-Leblond 1967). (This can be contrasted with the case of the Reeh-Schlieder theorem which does remain valid in most reasonable models of non-relativistic QFT (Requardt 1982), since the theorem does not actually require the full strength of the relativistic spectrum condition.) Thus, in order to gain a better understanding of the relationship between the concepts of quantum theory and those of relativity, it would be desirable to isolate those features of relativistic theories that cause difficulties for the interaction picture.

4. Fields, observables, and ontology. In Part II of this dissertation, we gave some support for the dogma that relativistic QFT is not a theory of (localizable) particles. This dogma about particles goes hand in hand with another dogma—viz., that relativistic QFT *is* a theory of fields. However, it is far from clear what is meant by this second dogma, and whether it can be justified.

For example, in the case of the free Bose field, the local field observables are given by the elements $\{\Phi(f)\}$ where f runs through the compactly supported test-functions on Minkowski spacetime. However, in general $\Phi(f)$ and $\Phi(g)$ are not compatible, and so we cannot think of all elements of $\{\Phi(f)\}$ as possessing a value in an arbitrary state.

It is possible—at least in the case of the free Bose field—to split $\Phi(f)$ into two parts, the field configuration $\phi(h_f)$ and the canonically conjugate momentum $\pi(h_f)$. (For the definition of the test-function h_f, and other details, see Horuzhy 1988, 241.) In this case, we have

$$[\phi(h_1), \phi(h_2)] \;=\; 0 \;=\; [\pi(h_1), \pi(h_2)]\,, \tag{8.1}$$

$$[\phi(h_1), \pi(h_2)] \;=\; (h_1, h_2)I\,. \tag{8.2}$$

Thus, we could think of the field configuration observables $\{\phi(h)\}$ as possessing simultaneously definite values (since they are pairwise compatible), but only (as in the Bohm theory) at the expense of treating the momentum observables $\{\pi(h)\}$ as contextual. However, we should be just as suspicious of the claim that the field configuration observables are the "beables" of quantum field theory as we are of the claim that the position observable is the "beable" of elementary quantum mechanics.

The situation becomes more complicated in the general case. In general we have to distinguish between the net $\{\mathcal{F}(O)\}$ of local *field algebras* and the net $\{\mathcal{A}(O)\}$ local *observable algebras.* (The free Bose field model is peculiar in that the two nets are identical.) Since elements of the algebra $\mathcal{F}(O)$ are not typically observable, the net $\{\mathcal{F}(O)\}$ is not required to satisfy microcausality; i.e., elements of $\mathcal{F}(O)$ and $\mathcal{F}(O')$ may fail to commute even when O and O' are spacelike separated. However, this entails that the typical elements $A \in \mathcal{F}(O)$ and $B \in \mathcal{F}(O)$ cannot both possess a value in a given state. But what could explain their incompatibility?

A typical response from the algebraic field theorists is that "the physical content" of QFT is carried by the observable algebra, and that the description in terms of fields is mathematical "surplus structure." For example, it is sometimes claimed that the choice of field description is analogous to the choice in General Relativity of a coordinate chart for the manifold. From a philosophical perspective, however, this claim about quantum fields seems unjustifiably instrumentalistic. It would be interesting to see whether it is possible to make sense of this understanding of quantum fields without lapsing into instrumentalism.

5. Causality and Lorentz invariance. According to Graham Nerlich,

> Special relativity is based on the principle of Lorentz invariance, not on causality. The limit principle (all causal and signal connections are slower than light) is not a basic thesis of special relativity. . . . (Nerlich 1982)

We have seen, however, that RQFT stipulates from the outset, via the spectrum condition, that the limit principle holds. On the other hand, there are models of RQFT in which Lorentz invariance is violated (e.g., in charged sectors in quantum electrodynamics). Such features of RQFT have led the renowned physicist H.-J. Borchers (1985, 1991) to claim that Lorentz invariance is only "approximately" true (in the sense that it would be very difficult to design an experiment to show it false), and this as a consequence of the more basic limit principle. It would be particularly interesting, then, to investigate Borchers' claim from a more explicitly philosophical perspective, and to draw out its implications for our understanding of the conceptual foundations of special relativity.

6. Quantum statistical mechanics. The algebraic approach to quantum theory has proved useful not only for quantum field theory but also for quantum statistical mechanics. It would be interesting to consider, then, whether conclusions drawn in this dissertation—in particular, concerning the relationship between inequivalent representations—can be extended to that case.

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