Local Primitive Causality and the Common Cause Principle in Quantum Field Theory

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Abstract

If $\{\mathcal{A}(V)\}$ is a net of local von Neumann algebras satisfying standard axioms of algebraic relativistic quantum field theory and V_1 and V_2 are spacelike separated spacetime regions, then the system $(\mathcal{A}(V_1), \mathcal{A}(V_2), \phi)$ is said to satisfy the Weak Reichenbach's Common Cause Principle iff for every pair of projections $A \in \mathcal{A}(V_1)$, $B \in \mathcal{A}(V_2)$ correlated in the normal state ϕ there exists a projection C belonging to a von Neumann algebra associated with a spacetime region V contained in the union of the backward light cones of V_1 and V_2 and disjoint from both V_1 and V_2, a projection having the properties of a Reichenbachian common cause of the correlation between A and B. It is shown that if the net has the local primitive causality property then every local system $(\mathcal{A}(V_1), \mathcal{A}(V_2), \phi)$ with a locally normal and locally faithful state ϕ and open bounded V_1 and V_2 satisfies the Weak Reichenbach's Common Cause Principle.

1 Introduction

An operationally motivated and mathematically powerful approach to quantum field theory is algebraic quantum field theory (AQFT) (cf. [7]). Although in its axiomatic nature

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it includes models which have no physical significance at all, it does subsume all physically interesting models, as well, for the assumptions made in AQFT are ordinarily of such a nature that they are the desiderata of any reasonable quantum field theory. The initial data in AQFT are a collection $\{\mathcal{A}(V)\}$ of algebras indexed by a suitable set of open subregions of the space–time of interest, $\mathcal{A}(V)$ understood as being generated by all the observables measurable in the spacetime region V, and a state ϕ on these algebras, understood as representing the preparation of the quantum system under investigation. It is remarkable that it is not necessary to make a specific choice of observables and state (*i.e.* to choose a particular model) in order to establish results of physical interest — many such results follow from quite general assumptions.

A characteristic feature of (relativistic) local algebraic quantum field theory is that it predicts correlations between projections A, B lying in von Neumann algebras $\mathcal{A}(V_1), \mathcal{A}(V_2)$ associated with spacelike separated spacetime regions V_1, V_2. Typically, if $\{\mathcal{A}(V)\}$ is a net of local algebras in a vacuum representation, then there exist many normal states ϕ on $\mathcal{A}(V_1 \cup V_2)$ such that $\phi(A \wedge B) > \phi(A)\phi(B)$ for suitable projections $A \in \mathcal{A}(V_1)$, $B \in \mathcal{A}(V_2)$ (this will be explained below).

According to a classical tradition in the philosophy of science, such probabilistic correlations are always signs of causal relations. More precisely, this position is typically formulated in the form of what has become known as *Reichenbach's Common Cause Principle*. This principle asserts (cf. [24]) that if two events A and B are correlated, then the correlation between A and B is either due to a direct causal influence connecting A and B, or there is a third event C which is a common cause of the correlation. The latter means that C satisfies four simple probabilistic conditions which together imply the correlation in question. (These conditions will be given below.)

The self–adjoint elements of the local von Neumann algebras $\mathcal{A}(V)$ associated with spacetime regions V are interpreted in AQFT as the mathematical representatives of the physical quantities observable in region V. Since, in particular, projections in the local von Neumann algebras are interpreted as 0-1–valued observables and the expectation values of the projections in ϕ as probabilities of the events that these two-valued observables take on the value 1 when the system has been prepared in state ϕ, then the above–mentioned correlations predicted by AQFT lead naturally to the question of the status of Reichenbach's Common Cause Principle within AQFT. If the correlated projections belong to algebras associated with spacelike separated regions, a direct causal influence between them is excluded by the theory of relativity. Consequently, compliance of AQFT with Reichenbach's Common Cause Principle would mean that for every correlation between projections A and B lying in von Neumann algebras associated with spacelike separated spacetime regions V_1, V_2 there must exist a projection C possessing the probabilistic properties which qualify it to be a Reichenbachian common cause of the correlation between A and B. However, since observables and hence also the projections in AQFT must be localized, in the case of the spacelike correlations predicted by AQFT, one also has to specify the spacetime region V with which the von Neumann algebra $\mathcal{A}(V)$ containing the common cause C is associated. Intuitively, the region V should be disjoint from both V_1 and V_2 but should not be causally disjoint from them, in order to leave room for a causal effect of C on the correlated events. Thus the natural requirement concerning the region V is that it be contained in the intersection of the backward light cones of V_1 and V_2.

This requirement and the resulting notion of Reichenbachian common cause in AQFT (Definition 4) was formulated in a previous paper [20] (see also Chapter 8 in [21]), together with the problem of whether AQFT complies with Reichenbach's Common Cause

Principle, as described. This problem is still open — the present paper does not settle the issue. What we can show here is less — we shall prove that if a net $\{\mathcal{A}(V)\}$ of local von Neumann algebras satisfies some standard, physically natural assumptions as well as the so–called *local primitive causality* condition (Definition 1), then every local system $(\mathcal{A}(V_1), \mathcal{A}(V_2), \phi)$ with a locally normal and locally faithful state ϕ and open, bounded spacelike separated spacetime regions V_1, V_2 satisfies the Weak Reichenbach's Common Cause Principle, where "weak" means that there exists a region V contained in the *union* of the backward light cones of V_1 and V_2 such that the local von Neumann algebra $\mathcal{A}(V)$ contains a common cause C of the correlation (Definition 5 and Proposition 3). We shall show that such states include the states of physical interest in vacuum representations for relativistic quantum field theories on Minkowski space. Hence, although the question of whether general local systems $(\mathcal{A}(V_1), \mathcal{A}(V_2), \phi)$ with non-faithful normal states ϕ satisfy the Weak Reichenbach's Common Cause Principle remains open, we will interpret Proposition 3 as a clear indication that AQFT is a causally rich enough theory to comply with the Weak Common Cause Principle – and possibly also with the strong one.

2 Spacelike Correlations in Quantum Field Theory

In light of the interpretation given to the net $\{\mathcal{A}(V)\}$ of C^*–algebras $\mathcal{A}(V)$ (with common identity element) indexed by the open, bounded subsets V of Minkowski space M,[1] the following properties are physically natural requirements (for further discussion of these axioms, see [7] and [15]):

(i) Isotony: if V_1 is contained in V_2, then $\mathcal{A}(V_1)$ is a subalgebra of $\mathcal{A}(V_2)$;

(ii) Einstein causality: if V_1 is spacelike separated from V_2, then every element of $\mathcal{A}(V_1)$ commutes with every element of $\mathcal{A}(V_2)$; letting $\mathcal{A}(V)'$ denote the commutant of $\mathcal{A}(V)$ in $\mathcal{A}$, this can be succinctly expressed by $\mathcal{A}(V_1) \subset \mathcal{A}(V_2)'$;

(iii) Relativistic covariance: there is a representation α of the identity-connected component $\mathcal{P}$ of the Poincaré group by automorphisms on $\mathcal{A}$ such that $\alpha_g(\mathcal{A}(V)) = \mathcal{A}(gV)$ for all V and $g \in \mathcal{P}$.

The smallest C^*–algebra $\mathcal{A}$ containing all the local algebras $\mathcal{A}(V))$ is called the quasilocal algebra. This is the algebra on which ϕ is defined as a state. In this paper we shall be interested in irreducible vacuum representations, so we shall also assume:

(iv) Vacuum representation: for each V, $\mathcal{A}(V)$ is a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$ in which $\mathcal{A} = \mathcal{B}(\mathcal{H})$ (the set of all bounded operators on $\mathcal{H}$) and there is a distinguished unit vector Ω, and on which there is a strongly continuous unitary representation $U(\mathcal{P})$ such that $U(g)\Omega = \Omega$, for all $g \in \mathcal{P}$, and

$$\alpha_g(A) = U(g)AU(g)^{-1} \quad , \quad \text{for all} \quad A \in \mathcal{A} \quad ,$$

as well as the spectrum condition — the spectrum of the self–adjoint generators of the strongly continuous unitary representation $U(\mathbb{R}^4)$ of the translation subgroup of $\mathcal{P}$ (which has the physical interpretation of the global energy–momentum spectrum of the theory) must lie in the closed forward light cone.

[1] The results presented in this paper can be generalized to suitable nets and states on stationary, globally hyperbolic space–times, but we shall content ourselves here with illustrating the ideas in Minkowski space.

The vacuum state ϕ on $\mathcal{A}$ is defined by $\phi(A) = \langle\Omega, A\Omega\rangle$, for all $A \in \mathcal{A}$. This state is thus Poincaré invariant: $\phi(\alpha_g(A)) = \phi(A)$, for all $g \in \mathcal{P}$ and $A \in \mathcal{A}$. We shall further assume that there are sufficiently many observables localized in arbitrarily small spacetime regions:

(v) Weak additivity: for any nonempty open region V, the set of operators $\cup_{g\in\mathbb{R}^4}\mathcal{A}(gV)$ is dense in $\mathcal{A}$ (in the weak operator topology).

An immediate consequence of assumptions (i)–(v) is the Reeh–Schlieder Theorem: for any nonempty open region V, the set of vectors $\mathcal{A}(V)\Omega = \{A\Omega \mid A \in \mathcal{A}(V)\}$ is dense in $\mathcal{H}$. In fact, one has a stronger result, which will be useful for us. First, we must recall some definitions. For a spacetime region V, let V' denote the (interior of the) causal complement of V, *i.e.* the set of all points $x \in M$ which are spacelike separated from every point in V. In addition, one says that a vector Φ in $\mathcal{H}$ is analytic for the energy if the function $\mathbb{C} \ni z \mapsto e^{zH}\Phi \in \mathcal{H}$ is analytic, where H is the generator of the one-parameter group $U(t) \in U(\mathbb{R}^4)$, $t \in \mathbb{R}$, implementing the time translations. By the spectrum condition, H is a positive operator. Note that any vector Φ with finite energy content — *i.e.* for which there exists a compact set in the spectrum of H such that the corresponding spectral projection P leaves Φ fixed, *i.e.* $P\Phi = \Phi$ — is a vector analytic for the energy. In particular, the vacuum vector Ω is analytic for the energy. And since no preparation of a quantum system which can be carried out by man can require infinite energy, it is evident that (convex combinations of) states induced by such analytic vectors include all of the physically interesting states in this representation.

Proposition 1 [1] *Under the assumptions (i)–(v), for any nonempty open region V, the set of vectors $\mathcal{A}(V)\Phi$ is dense in $\mathcal{H}$, for all vectors Φ which are analytic for the energy.*

Note that assumption (ii) entails that such vectors are also separating (*i.e.* $X \in \mathcal{A}(V)$ and $X\Phi = 0$ imply $X = 0$) for all algebras $\mathcal{A}(V)$ such that V' is nonempty. Hence, (convex combinations of) the states ϕ induced by analytic vectors are faithful (*i.e.* $X \in \mathcal{A}(V)$ and $\phi(XX^*) = 0$ imply $X = 0$) on each such algebra $\mathcal{A}(V)$. Such states are said to be locally faithful. We emphasize: given assumptions (i)–(v), all physically interesting states will be locally faithful.

We shall be interested in special bounded regions V, called double cones. Let the point $x \in \mathcal{M}$ be in the interior of the forward light cone with apex at $y \in \mathcal{M}$. Any two such points determine a double cone as the interior of the intersection of the backward light cone with apex at x with forward light cone with apex at $y \in \mathcal{M}$. The set of double cones forms a basis for the topology on $\mathcal{M}$. Double cones are nonempty, as are their causal complements. We state a further assumption.

(vi) Type of double cone algebras: for every double cone V, the von Neumann algebra $\mathcal{A}(V)$ is type III. [2]

This assumption appears to be of a different sort than the previous ones, but it has been verified in many models by direct computation, and it has been derived from general principles (for example, the existence of a scaling limit [4]). So, typically, double cone algebras in vacuum representations are, in fact, type III, and condition (vi) is not anticipated to be physically restrictive.

We formulate our final assumption. For a convex spacetime region V, let $V'' = (V')'$ denote the *causal completion* (also called causal closure and causal hull in the literature)

[2] cf. [16] for a description of types of von Neumann algebras

of V. One notes that every light ray running through any given point in V'' must intersect V. One should also note that $V = V''$ for every double cone V. There are many ways of formulating a causality condition in AQFT short of specifying a particular time propagation (cf. [8]). The following is suitable for our purposes and postulates a hyperbolic propagation within lightlike characteristics.

Definition 1 *The net $\{\mathcal{A}(V)\}$ is said to satisfy the* local primitive causality *condition if $\mathcal{A}(V'') = \mathcal{A}(V)$ for every nonempty convex region V.*

This is an additional assumption, which does not follow from assumptions (i)–(vi). Indeed, there exist nets associated with certain generalized free fields which satisfy conditions (i)–(vi) but which violate local primitive causality [5]. However, this condition has been verified in many concrete models. For further insight into the content of local primitive causality, see the discussion directly after the proof of Prop. 3 below.

For later purposes we collect some definitions concerning the independence of local algebras.[3] A pair $(\mathcal{A}_1, \mathcal{A}_2)$ of C^*–subalgebras of the C^*–algebra $\mathcal{C}$ has the Schlieder property if $XY \neq 0$ for any $0 \neq X \in \mathcal{A}_1$ and $0 \neq Y \in \mathcal{A}_2$. We note that given assumptions (i)–(v), $(\mathcal{A}(V_1), \mathcal{A}(V_2))$ has the Schlieder property for all spacelike separated, regular shaped[4] V_1, V_2, in particular for double cones.

A pair $(\mathcal{A}_1, \mathcal{A}_2)$ of such algebras is called C^*–independent if for any state ϕ_1 on $\mathcal{A}_1$ and for any state ϕ_2 on $\mathcal{A}_2$ there exists a state ϕ on $\mathcal{C}$ such that $\phi(X) = \phi_1(X)$ and $\phi(Y) = \phi_2(Y)$, for any $X \in \mathcal{A}_1, Y \in \mathcal{A}_2$. Under assumptions (i)–(v), algebras associated with spacelike separated double cones are C^*–independent, since, as pointed out above, they are a mutually commuting pair of algebras satisfying the Schlieder property, which in this context is equivalent with C^*–independence [23].

Two von Neumann subalgebras $\mathcal{N}_1, \mathcal{N}_2$ of the von Neumann algebra $\mathcal{N}$ are called logically independent [18] [19] if $A \wedge B \neq 0$ for any projections $0 \neq A \in \mathcal{N}_1$, $0 \neq B \in \mathcal{N}_2$. If $\mathcal{N}_1, \mathcal{N}_2$ is a mutually commuting pair of von Neumann algebras, then C^*–independence and logical independence are equivalent [21].[5] In light of our preceding remarks, we conclude:

Lemma 1 *Assumptions (i)–(v) entail that the pair $(\mathcal{A}(V_1), \mathcal{A}(V_2))$ is logically independent for any spacelike separated double cones V_1, V_2.*

Let V_1 and V_2 be two spacelike separated spacetime regions and $A \in \mathcal{A}(V_1)$ and $B \in \mathcal{A}(V_2)$ be two projections. If ϕ is a state on $\mathcal{A}(V_1 \cup V_2)$, then it can happen that

$$\phi(A \wedge B) > \phi(A)\phi(B) \quad . \tag{1}$$

If (1) is the case, then we say that there is *superluminal (or spacelike) correlation* between A and B in the state ϕ. We now explain why such correlations are common when assumptions (i)–(v) hold.

The ubiquitous presence of superluminal correlations is one of the consequences of the generic violation of Bell's inequalities in AQFT. To make this clear, we need to establish some further notions. Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be two commuting von Neumann subalgebras of the

[3] For the origin and a detailed analysis of the interrelation of these and other notions of statistical independence, see the review [25] and Chapter 11 in [21] — for more recent results, see [3] [10].

[4] see precise statements in [25]

[5] If $\mathcal{N}_1, \mathcal{N}_2$ do not mutually commute, then C^*–independence is strictly weaker than logical independence [10].

von Neumann algebra $\mathcal{N}$ and let ϕ be a state on $\mathcal{N}$. Following [26], the Bell correlation $\beta(\phi, \mathcal{N}_1, \mathcal{N}_2)$ between the algebras $\mathcal{N}_1, \mathcal{N}_2$ in state ϕ is defined by

$$\beta(\phi, \mathcal{N}_1, \mathcal{N}_2) \equiv \sup \frac{1}{2}\phi(X_1(Y_1 + Y_2) + X_2(Y_1 - Y_2)) \quad , \tag{2}$$

where the supremum in (2) is taken over all self–adjoint $X_i \in \mathcal{N}_1, Y_j \in \mathcal{N}_2$ with norm less than or equal to 1. It can be shown [2] [27] that $\beta(\phi, \mathcal{N}_1, \mathcal{N}_2) \leq \sqrt{2}$. The Clauser–Holt–Shimony–Horne version of Bell's inequality in this notation reads:

$$\beta(\phi, \mathcal{N}_1, \mathcal{N}_2) \leq 1 \quad , \tag{3}$$

and a state ϕ for which $\beta(\phi, \mathcal{N}_1, \mathcal{N}_2) > 1$ is called *Bell correlated.* It is known [27] that if $\mathcal{N}_1$ or $\mathcal{N}_2$ is abelian, or if ϕ is a product state across the algebras $\mathcal{N}_1, \mathcal{N}_2$ (*i.e.*, if $\phi(XY) = \phi(X)\phi(Y)$, for all $X \in \mathcal{N}_1$ and $Y \in \mathcal{N}_2$), then $\beta(\phi, \mathcal{N}_1, \mathcal{N}_2) = 1$. Hence, if ϕ is Bell correlated then ϕ cannot be a product state across the algebras $\mathcal{N}_1, \mathcal{N}_2$. But, as we shall next see, non–product states yield correlations (1) for suitable choices of projections.

Lemma 2 *Let $\mathcal{N}_1$ and $\mathcal{N}_1$ be subalgebras of the von Neumann algebra $\mathcal{N}$ such that $\mathcal{N}_1 \subset \mathcal{N}_2'$, and let ϕ be a normal state on $\mathcal{N}$ which is not a product state across the algebras $\mathcal{N}_1, \mathcal{N}_2$. Then there exist projections $A \in \mathcal{N}_1$ and $B \in \mathcal{N}_2$ such that $\phi(A \wedge B) > \phi(A)\phi(B)$.*

Proof. By the spectral theorem, if $\phi(A \wedge B) = \phi(A)\phi(B)$, for all projections $A \in \mathcal{N}_1$ and $B \in \mathcal{N}_2$, then ϕ is a product state across $\mathcal{N}_1, \mathcal{N}_2$. Hence, there must exist projections $A \in \mathcal{N}_1$ and $B \in \mathcal{N}_2$ such that $\phi(A \wedge B) \neq \phi(A)\phi(B)$. If $\phi(A \wedge B) < \phi(A)\phi(B)$, then

$$\begin{aligned} \phi((1 - A)B) &= \phi(B) - \phi(AB) > \phi(B) - \phi(A)\phi(B) \\ &= \phi(1 - A)\phi(B) \quad , \end{aligned}$$

and $1 - A \in \mathcal{N}_1$ is a projector. Similarly, $\phi(A(1 - B)) > \phi(A)\phi(1 - B)$. On the other hand, if $\phi(A \wedge B) > \phi(A)\phi(B)$, then also $\phi((1 - A) \wedge (1 - B)) > \phi(1 - A)\phi(1 - B)$. □

There are many situations in which $\beta(\phi, \mathcal{A}(V_1), \mathcal{A}(V_2)) = \sqrt{2}$, in other words, where there is maximal violation of Bell's inequalities (cf. [27] [28] [29]). Indeed, if V_1 and V_2 are tangent spacelike separated wedges or double cones, there is maximal violation of Bell's inequalities in *every* normal state [29]. We refer the interested reader to those papers. To keep the discussion as simple as possible, we shall only discuss the recent results established by Halvorson and Clifton.[6] The symbol $\mathcal{N}_1 \vee \mathcal{N}_2$ denotes the smallest von Neumann algebra containing both $\mathcal{N}_1$ and $\mathcal{N}_2$.

Proposition 2 [9] *If $(\mathcal{N}_1, \mathcal{N}_2)$ is a pair of commuting type III von Neumann algebras acting on the Hilbert space $\mathcal{H}$ and having the Schlieder property, then the set of unit vectors which induce Bell correlated states on $\mathcal{N}_1, \mathcal{N}_2$ is open and dense in the unit sphere of $\mathcal{H}$. Indeed, the set of normal states on $\mathcal{N}_1 \vee \mathcal{N}_2$ which are Bell correlated on $\mathcal{N}_1, \mathcal{N}_2$ is norm dense in the normal state space of $\mathcal{N}_1 \vee \mathcal{N}_2$.*

From the above remarks, given the assumptions (i)–(vi), we see that for any spacelike separated double cones V_1, V_2, the pair $(\mathcal{A}(V_1), \mathcal{A}(V_2))$ satisfies the hypothesis of Prop. 2. So, "most" normal states on such pairs of algebras manifest superluminal correlations (1).[7] Hence, superluminal correlations abound in AQFT, and the question posed in the introduction is not vacuous. In the next section, we address this question.

[6] In point of fact, Halvorson and Clifton required of $\mathcal{N}_1$ and $\mathcal{N}_2$ that they be of infinite type, not the more restrictive type III, but this is the version which we shall employ below.

[7] If V_1 and V_2 are tangent spacelike separated wedges or double cones, *all* normal states on $(\mathcal{A}(V_1), \mathcal{A}(V_2))$ manifest superluminal correlations.

3 The Notion of Reichenbachian Common Cause in AQFT

Let (Ω, p) be a classical probability measure space with Boolean algebra Ω and probability measure p. If $A, B \in \Omega$ are such that

$$p(A \wedge B) > p(A)p(B) \quad , \tag{4}$$

then the events A and B are said to be (positively) *correlated.*

Definition 2 *$C \in \Omega$ is a common cause of the correlation (4) if the following (independent) conditions hold:*

$$\begin{aligned}
p(A \wedge B|C) &= p(A|C)p(B|C) \quad , &(5)\\
p(A \wedge B|C^{\perp}) &= p(A|C^{\perp})p(B|C^{\perp}) \quad , &(6)\\
p(A|C) &> p(A|C^{\perp}) \quad , &(7)\\
p(B|C) &> p(B|C^{\perp}) \quad , &(8)
\end{aligned}$$

where $p(X|Y)$ denotes here the conditional probability of X on condition Y, and it is assumed that none of the probabilities $p(X)$, $(X = A, B, C)$ is equal to zero.

The above definition is due to Reichenbach ([22], Section 19). We wish to extend this definition to the setting of AQFT. To do this, we first define a notion of common cause of a correlation in a noncommutative measure space $(\mathcal{P}(\mathcal{N}), \phi)$ with a non-distributive von Neumann lattice $\mathcal{P}(\mathcal{N})$ in place of the Boolean algebra Ω and a normal state ϕ playing the role of p.[8] Here, $\mathcal{N}$ is a von Neumann algebra and $\mathcal{P}(\mathcal{N})$ denotes the set of projections in $\mathcal{N}$.

Definition 3 *Let $A, B \in \mathcal{P}(\mathcal{N})$ be two commuting projections which are correlated in ϕ:*

$$\phi(A \wedge B) > \phi(A)\phi(B) \quad . \tag{9}$$

$C \in \mathcal{P}(\mathcal{N})$ is a common cause *of the correlation (9) if C commutes with both A and B and the following conditions (completely analogous to (5)-(8)) hold:*

$$\begin{aligned}
\frac{\phi(A \wedge B \wedge C)}{\phi(C)} &= \frac{\phi(A \wedge C)}{\phi(C)} \frac{\phi(B \wedge C)}{\phi(C)} \quad , &(10)\\
\frac{\phi(A \wedge B \wedge C^{\perp})}{\phi(C^{\perp})} &= \frac{\phi(A \wedge C^{\perp})}{\phi(C^{\perp})} \frac{\phi(B \wedge C^{\perp})}{\phi(C^{\perp})} \quad , &(11)\\
\frac{\phi(A \wedge C)}{\phi(C)} &> \frac{\phi(A \wedge C^{\perp})}{\phi(C^{\perp})} \quad , &(12)\\
\frac{\phi(B \wedge C)}{\phi(C)} &> \frac{\phi(B \wedge C^{\perp})}{\phi(C^{\perp})} \quad . &(13)
\end{aligned}$$

[8] Only normal states are considered, since they have the continuity property which generalizes the property of σ–additivity in the classical context [16].

Definition 3 is a natural specification in a noncommutative probability space $(\mathcal{P}(\mathcal{N}), \phi)$ of the classical notion of common cause. The only deviation from Definition 2 is that the commutativity of the events involved is now required explicitly. One could, in principle, also define a common cause C which does not commute with A or B, but then one would have to expand Reichenbach's original scheme by allowing noncommutative conditionalization, which we do not wish to consider here. (See the papers [11], [12] for an analysis of some technical and conceptual difficulties concerning the generalization of Reichenbach's scheme to non-distributive event structures.)

The following formulation of Reichenbach's Common Cause Principle for local relativistic nets was proposed in [20].

Definition 4 *Let $\{\mathcal{A}(V)\}$ be a net of local von Neumann algebras over Minkowski space. Let V_1 and V_2 be two spacelike separated spacetime regions, $BLC(V_1)$ and $BLC(V_2)$ be their backward light cones, and let ϕ be a locally normal state on the quasilocal algebra $\mathcal{A}$.*[9] *We say that the local system $(\mathcal{A}(V_1), \mathcal{A}(V_2), \phi)$* satisfies Reichenbach's Common Cause Principle *if and only if for any pair of projections $A \in \mathcal{A}(V_1)$ $B \in \mathcal{A}(V_2)$ we have the following: if*

$$\phi(A \wedge B) > \phi(A)\phi(B) \quad , \tag{14}$$

then there exists a projection C in the von Neumann algebra $\mathcal{A}(V)$ associated with a region V such that

$$V \subseteq (BLC(V_1) \setminus V_1) \cap (BLC(V_2) \setminus V_2)$$

and such that C is a common cause of the correlation (14) in the sense of Definition 3. We say that Reichenbach's Common Cause Principle holds for the net iff for every pair of spacelike separated spacetime regions V_1, V_2 and every normal state ϕ, Reichenbach's Common Cause Principle holds for the local system $(\mathcal{A}(V_1), \mathcal{A}(V_2), \phi)$.

As we indicated in the Introduction, it is still an open problem whether this condition holds for the nets of local von Neumann algebras occurring in AQFT. The next definition formulates a weaker form of the Common Cause Principle for a net $\{\mathcal{A}(V)\}$ — it is for this weaker notion that we can prove something in this paper.

Definition 5 *We say that the local system $(\mathcal{A}(V_1), \mathcal{A}(V_2), \phi)$ satisfies the* Weak Reichenbach's Common Cause Principle *if and only if for any pair of projections $A \in \mathcal{A}(V_1)$ $B \in \mathcal{A}(V_2)$ we have the following: if*

$$\phi(A \wedge B) > \phi(A)\phi(B) \quad , \tag{15}$$

then there exists a projection C in the von Neumann algebra $\mathcal{A}(V)$ associated with a region V such that

$$V \subseteq (BLC(V_1) \setminus V_1) \cup (BLC(V_2) \setminus V_2)$$

and such that C is a common cause of the correlation (15) in the sense of Definition 3.

Proposition 3 *If the net $\{\mathcal{A}(V)\}$ satisfies conditions (i)–(vi) and local primitive causality, then every local system $(\mathcal{A}(V_1), \mathcal{A}(V_2), \phi)$ with V_1, V_2 contained in a pair of spacelike separated double cones and with a locally normal and locally faithful state ϕ satisfies Weak Reichenbach's Common Cause Principle.*

[9] A state ϕ on $\mathcal{A}$ is locally normal if its restriction to $\mathcal{A}(V)$ is normal for every double cone V.

Note that because logical independence is hereditary, Lemma 1 entails that the pair of algebras $\mathcal{A}(V_1), \mathcal{A}(V_2)$ in Prop. 3 is logically independent. The proof of Prop. 3 is based on the following two Lemmas:

Lemma 3 *Let ϕ be a faithful state on a von Neumann algebra $\mathcal{N}$ containing two mutually commuting subalgebras $\mathcal{N}_1, \mathcal{N}_2$ which are logically independent. Let $A \in \mathcal{N}_1$ and $B \in \mathcal{N}_2$ be projections satisfying (15). Then a sufficient condition for C to satisfy (10)-(13) is that the following two conditions hold:*

$$C \quad < \quad A \wedge B \quad , \tag{16}$$

$$\phi(C) \quad = \quad \frac{\phi(A \wedge B) - \phi(A)\phi(B)}{1 - \phi(A \vee B)} \quad . \tag{17}$$

Lemma 4 *Let $\mathcal{N}$ be a type III von Neumann algebra on a separable Hilbert space $\mathcal{H}$, and let ϕ be a faithful normal state on $\mathcal{N}$. Then for every projection $A \in \mathcal{P}(\mathcal{N})$ and every positive real number $0 < r < \phi(A)$ there exists a projection $P \in \mathcal{P}(\mathcal{N})$ such that $P < A$ and $\phi(P) = r$.*

Proof of Lemma 3: Note first that if A and B are correlated, then one must have $1 - \phi(A \vee B) > 0$. If, on the contrary, one had $\phi(1) = 1 = \phi(A \vee B)$, then since $1 = A \vee B + A^\perp \wedge B^\perp$, one would have

$$\phi(A^\perp \wedge B^\perp) = 0 \quad . \tag{18}$$

On the other hand, since A and B are correlated, it follows that

$$\phi(A^\perp \wedge B^\perp) > \phi(A^\perp)\phi(B^\perp) \quad ,$$

contradicting (18). Hence, the right hand side of (17) is well defined and greater than zero. Further note that since $A \geq A \wedge B$ and $B \geq A \wedge B$, one must have $\phi(A) \geq \phi(A \wedge B)$ and $\phi(B) \geq \phi(A \wedge B)$. But equality in both of these cases is excluded by hypothesis, since if $\phi(A) = \phi(A \wedge B)$ or $\phi(B) = \phi(A \wedge B)$, then by the faithfulness of ϕ one has $A = A \wedge B$ or $B = A \wedge B$. This would imply $B \geq A$ or $A \geq B$, and so $A \wedge B^\perp = 0$ or $B \wedge A^\perp = 0$, which contradicts the logical independence of the pair $\mathcal{N}_1, \mathcal{N}_2$. Hence, one has $\phi(A) - \phi(A \wedge B) \equiv a > 0$ and $\phi(B) - \phi(A \wedge B) \equiv b > 0$.

Elementary algebraic calculation shows that the inequality

$$\frac{\phi(A \wedge B) - \phi(A)\phi(B)}{1 - \phi(A \vee B)} < \phi(A \wedge B)$$

is equivalent to the inequality

$$\phi(A)\phi(B) > [\phi(A) + \phi(B) - \phi(A \wedge B)]\, \phi(A \wedge B) \quad . \tag{19}$$

But

$$\phi(A)\phi(B) = \phi(A)[b + \phi(A \wedge B)]$$

and

$$[\phi(A) + \phi(B) - \phi(A \wedge B)]\phi(A \wedge B) = [\phi(A) + b]\, \phi(A \wedge B) \quad .$$

Since $\phi(A)b > b\phi(A \wedge B)$, inequality (19) follows. Therefore, the right hand side of (17) — and hence the value of $\phi(C)$ — is smaller than $\phi(A \wedge B)$, and so the conditions (16) and (17) are compatible.

It remains to see that conditions (10)-(13) hold. Conditions (10), (12) and (13) hold trivially, and since $C < A \wedge B$ commutes with both A and B, one can write

$$\phi(X) = \phi(X \wedge C) + \phi(X \wedge C^\perp) \quad , \quad X = A, B, A \wedge B \quad ,$$

implying

$$\phi(X \wedge C^\perp) = \phi(X) - \phi(X \wedge C) = \phi(X) - \phi(C) \quad , \quad X = A, B, A \wedge B \quad .$$

Hence, (11) is equivalent to

$$\frac{\phi(A \wedge B) - \phi(C)}{\phi(C^\perp)} = \frac{\phi(A) - \phi(C)}{\phi(C^\perp)} \frac{\phi(B) - \phi(C)}{\phi(C^\perp)} \quad ,$$

which, in turn, is equivalent to

$$\phi(C^\perp)[\phi(A \wedge B) - \phi(C)] = [\phi(A) - \phi(C)][\phi(B) - \phi(C)] \quad .$$

This latter equation can be seen after elementary algebra to be equivalent to (17). $\square$

Proof of Lemma 4: Consider the set S of projections defined by

$$S \equiv \{X \in \mathcal{P}(\mathcal{N}) \mid X < A \text{ and } \phi(X) \leq r\} \quad .$$

S is partially ordered with respect to the partial ordering inherited from the standard lattice ordering in $\mathcal{P}(\mathcal{N})$. Let S' be a linearly ordered subset of S. Since $\mathcal{P}(\mathcal{N})$ is a complete lattice, the least upper bound Z of S' exists. Since Z is the least upper bound of S', one has $Z \leq A$, and since Z is the strong operator limit of the net S' and ϕ is normal, one also has $\phi(Z) \leq r$. This implies that $Z < A$ holds, as well (because $r < \phi(A)$ and ϕ is faithful). So every linearly ordered subset of S has a maximal element in S; therefore by Zorn's Lemma, S has a maximal element P.

It shall be shown that $\phi(P) = r$. Assume for the purpose of contradiction that $\phi(P) < r$ and consider $Q \equiv A \wedge P^\perp \in \mathcal{P}(\mathcal{N})$. Since $P < A$, it follows that $Q \neq 0$. Since $\mathcal{N}$ is a type III algebra and $\mathcal{H}$ is separable, there exist a countably infinite number of mutually orthogonal projections $Q_i \in \mathcal{P}(\mathcal{N})$, each equivalent to Q in the Murray-von Neumann sense (cf. [16]), such that $Q = \vee_i Q_i$. So one has[10]

$$\phi(Q) = \phi(\vee_i Q_i) = \sum_i \phi(Q_i) \quad .$$

Since ϕ is faithful, one must have $\phi(Q_i) > 0$ for all i, and so $\phi(Q_i) \to 0$ $(i \to \infty)$. Consequently, there exists a sufficiently large k such that for the projection $P' \equiv P + Q_k$ one has

$$r > \phi(P') = \phi(P + Q_k) = \phi(P) + \phi(Q_k) > \phi(P) \quad ,$$

and $P < P' < A$, which contradicts the maximality of P in S. $\square$

[10]The second equality in the display is precisely the generalization of σ–additivity to which we have previously alluded and which obtains for normal states, but not for general states.

Proof of Proposition 3: Let ϕ be a locally normal and locally faithful state on $\mathcal{A}$ and assume that the projections $A \in \mathcal{A}(V_1), B \in \mathcal{A}(V_2)$ are correlated in ϕ:

$$\phi(A \wedge B) > \phi(A)\phi(B) \quad .$$

Since V_1 and V_2 are bounded, there exists a bounded (hyper)rectangular spacetime region V such that

$$V \subseteq (BLC(V_1) \setminus V_1) \cup (BLC(V_2) \setminus V_2)$$

and such that the causal completion V'' (which is a double cone) of V contains $V_1 \cup V_2$. Hence, by isotony and local primitive causality one has

$$A, B \in \mathcal{A}(V_1 \cup V_2) \subseteq \mathcal{A}(V'') = \mathcal{A}(V) \quad .$$

Let

$$r \equiv \frac{\phi(A \wedge B) - \phi(A)\phi(B)}{1 - \phi(A \vee B)} > 0 \quad .$$

It has already been established in the proof of Lemma 3 that $r < \phi(A \wedge B)$. Since double cone algebras are type III, by Lemma 4 there exists a projection $C \in \mathcal{A}(V)$ such that $C < A \wedge B$ and $\phi(C) = r$, and by Lemma 3 this C satisfies conditions (10)-(13). $\square$

The observant reader may be disturbed to note that in the proof just given the projections A, B were initially localized in the algebra $\mathcal{A}(V_1 \cup V_2)$ and then were located in the algebra $\mathcal{A}(V)$, even though, in fact, we have seen that $V \cap (V_1 \cup V_2) = \emptyset$. What kind of sleight of hand has taken place here? It is not widely understood that an "observable" A does not represent a unique measuring apparatus in some fixed laboratory, but rather represents an equivalence class of such apparata (cf. [17]). Consider two such idealized apparata X, Y such that $\phi(X) = \phi(Y)$ for all (idealized) states ϕ admitted in the theory (the set of such states contains as a subset — at least in principle — all states preparable in the laboratory). These two apparata are then identified to be in the same equivalence class and are thus represented by a single operator $A \in \mathcal{A}$. Hence, each of the projections A, B above, which are localized simultaneously in V and $V_1 \cup V_2$, represents two distinct events — one taking place in V and the other taking place in $V_1 \cup V_2$. The fact that it is possible, for every event in $V_1 \cup V_2$, to find an event in V which is equivalent to the first in the stated sense is part of the content of the local primitive causality condition. With this in mind, the reader is in the position to better appreciate the significance of the fact that the primitive causality condition can, in fact, be deduced for nets locally associated with, for example, free quantum fields satisfying hyperbolic equations of motion, as well as some interacting quantum field models whose construction has been carried out with mathematical rigor.

To better understand the nature of the causal requirement being made here, we recall another, less frequently used algebraic causality condition. We say that V_1 is in the causal shadow of V_2 if every backward light ray from any point of V_1 passes through V_2. Then a more explicit algebraic formulation of the physical requirement that there is some kind of hyperbolic propagation of physical effects in action is that $\mathcal{A}(V_1) \subset \mathcal{A}(V_2)$, whenever V_1 is in the causal shadow of V_2. It is easy to see that the local primitive causality condition implies the latter condition, and it is clear that this latter causality condition would suffice to entail the conclusion of Prop. 3.

4 Concluding Remarks

Proposition 3 does not give an answer to the question of whether AQFT satisfies Reichenbach's Common Cause Principle interpreted in the sense of Definition 4, because it locates the common cause C only within the union of the backward light cones of V_1 and V_2 rather than in the intersection of these light cones. A bit more, however, can be said of the location of the specific common cause C displayed in the proof of Proposition 3. Define $\tilde{V}_1$ and $\tilde{V}_2$ by

$$\tilde{V}_1 \equiv (BLC(V_1) \cap V) \setminus (BLC(V_1) \cap BLC(V_2)) \tag{20}$$

$$\tilde{V}_2 \equiv (BLC(V_2) \cap V) \setminus (BLC(V_1) \cap BLC(V_2)) \tag{21}$$

(i.e. $\tilde{V}_1$ and $\tilde{V}_2$ are the parts of V that are in the backward light cones of V_1 and V_2, respectively, but do not intersect with the part of V which is in the common causal past of V_1 and V_2). Since $(\tilde{V}_1 \cup V_1)$ and $(\tilde{V}_2 \cup V_2)$ are contained in spacelike separated double cones, the algebras $\mathcal{N}(\tilde{V}_1 \cup V_1)$ and $\mathcal{N}(\tilde{V}_2 \cup V_2)$ are logically independent, hence the common cause $C < A \wedge B$ cannot belong to $\mathcal{N}(\tilde{V}_1)$ or to $\mathcal{N}(\tilde{V}_2)$ only, so neither $V \subseteq \tilde{V}_1$ nor $V \subseteq \tilde{V}_2$ is possible.

The common cause $C < A \wedge B$ is a very specific one — it *implies* both A and B. Such a common cause was called in [20] a *strong* common cause, whereas a common cause C is called *genuinely probabilistic* if neither $C \leq A$ nor $C \leq B$ is the case. We conjecture that the Weak Reichenbach's Common Cause Principle holds also with a genuinely probabilistic common cause. Indeed, we expect there to be an extraordinary richness of common causes of this type for any pair of correlated projections. After all, this is already evident for the strong common causes whose existence we have already established. A little thought will make clear that there is an infinite number of mutually disjoint (hyper)rectangular regions $V \subset M$ which are contained in $BLC(V_1) \cup BLC(V_2)$, disjoint from $V_1 \cup V_2$ and which satisfy $V_1 \cup V_2 \subset V''$. Hence, for given correlated projections A, B as described, there are infinitely many different strong common causes.

In [13] the probability space (Ω, p) was called *common cause incomplete* if it contains a pair of events (A, B) which are correlated in p but for which Ω does not contain a common cause C of the correlation between A and B. It was shown in [13] that every common cause incomplete probability space can be extended in such a way that the extension contains a common cause of the given correlated pair. Common cause incompleteness of a noncommutative space $(\mathcal{P}(\mathcal{N}), \phi)$ can be defined in complete analogy with the classical case (taking Definition 3 as the definition of common cause), and it can be shown [13], [14] that common cause incomplete noncommutative spaces also can be extended in such a manner that the extension contains a common cause of a given correlation. Because of conditions (7)–(8) (respectively (12)–(13)) required of the common cause, extending a given (Ω, p) (respectively $(\mathcal{P}(\mathcal{N}), \phi)$) by "adding" a common cause to it entails that the extension contains correlations which are not present in the original structure. It is therefore a nontrivial matter whether a given common cause incomplete space, classical or quantum, can be made *common cause closed*, or whether common cause closed spaces exist at all, where "common cause closedness" of a probability space means that the space contains a common cause of *every* correlation in it. It can be shown [6] that while common cause closed classical probability spaces exist, they cannot be small — no (Ω, p) with a finite Boolean algebra Ω can be common cause closed, with the exception of the Boolean algebra Ω generated by 5 atoms.

But the requirement of common cause closedness is too restrictive on intuitive grounds as well. One does not expect to have a common cause explanation of probabilistic correlations which arise as a consequence of a direct physical influence between the correlated events, or which are due to some logical relations between the correlated events. Thus it is a more reasonable to demand a space $(\mathcal{P}(\mathcal{N}), \phi)$ to be common cause closed with respect to two logically independent von Neumann sub-lattices $\mathcal{P}(\mathcal{N}_1)$ and $\mathcal{P}(\mathcal{N}_1)$ in $\mathcal{P}(\mathcal{N})$. Clearly, Lemma 3 and Lemma 4 imply that $(\mathcal{P}(\mathcal{N}), \phi)$ is common cause closed with respect to any two logically independent von Neumann sublattices, if $\mathcal{N}$ is a type III algebra and ϕ is a faithful normal state on $\mathcal{N}$.

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