TEN REASONS FOR PURSUING MULTI-COMMUTATIVE QUANTUM THEORIES

A consistent presentation of the multi-commutativity quantum project (with further references) can be found in
arxiv.org/abs/quant-ph/9711004
http://arxiv.org/PS_cache/quant-ph/pdf/9711/9711004v1.pdf

Here we focus on the motivation of the project, highlighting how it changes the quantum language and takes advantage of its newly gained flexibility. We stress its ultimate goal to overcome the conceptual limitations of the standard quantum theory and any other model that tends to identify quantum physics with non-commutative formalisms.

Ordered Lie Algebras Replace the Associative Algebras in Describing Dynamical Systems

In 1976, we learned something of primary importance for quantum theory (E. Alfsen, F. Shultz, Non-Commutative Spectral Theory for Affine Function Spaces on Convex Sets, Memoirs AMS, Number 172, July 1976): the standard operator (or algebraic) quantum language works because the cone of the positive quantum variables (observables) exhibits a specific, “spectral” geometry. We can forget the associative algebraic structures altogether and redefine the variables of a generic dynamical system as an invariantly ordered real Lie algebra in spectral duality with its dual space (briefly, C-invariant system, C for “cone”). The classical/quantum case corresponds to a lattice/anti-lattice geometry of the invariant cone. In the classical vector-lattice case, the commutative associative algebraic structure is implicitly present. In the general case, the language of ordered Lie algebras may lead us beyond the known algebraic models, implying that the standard language is too restrictive. Technical details about this formalism shift are given in the arxiv.org eprint.

After this language recasting, the transition from classical to quantum systems translates into dropping the lattice geometry of the positive cone while preserving its invariance, a procedure resulting in non-Boolean event spaces and non-commutative physical variables. That is what von Neumann’s achievement looks like half a century later. Now we can ask a question von Neumann couldn’t ask—what happens if we depart from the classical pattern in the opposite direction, if we drop the invariance of the cone and preserve its lattice geometry. That is, if we refuse to sacrifice the Boolean probabilities for purely pragmatic reasons, without clear theoretical motivation. The new path leads to a family of isomorphic lattice cones (with a common order unit), giving rise to a family of commutative associative products in the space of the variables, and a family of (partly overlapping) Boolean algebras as an event space, a structure referred to as multi-commutativity (or multi-Booleanness). Remember that the lattice order in the space of the variables, its commutative associative algebraic structure, and the Boolean structure in the probability event space are (roughly speaking) equivalent properties.

There is a natural way to arrive at multi-commutative entities that have a reasonable chance of being physically meaningful and we are going to make use of it. We build our physical models as a combination of two structures: a vector lattice, responsible for all probabilistic concepts, and a Lie product, introducing Hamiltonian dynamics. The two structures are independent and we have to secure a sufficiently large group of common Lie and order automorphisms (symmetries of the physical system). Let's make it a requirement that the group of the common automorphisms be maximal: either (C) the group of the order automorphisms contains the group of the Lie automorphisms or, conversely, (L) the Lie automorphisms form the larger group. Case (C) leads to the study of invariant lattice cones in Lie algebras and to classical C-invariant systems. The structures of case (L) can be appropriately called invariant Lie products in vector lattices or, briefly, L-invariant systems (L for "Lie product"). In the L-invariant systems the Lie automorphisms do not leave the original lattice cone invariant, and its rotation round the order unit gives rise to a family of commutative structures.

The C-invariant systems inherit their defining properties directly from the known classical and quantum Hamiltonian systems. Unlike that clear case, we do not know if the L-invariant systems admit interesting physical interpretation at all. In fact, no mathematical results about the invariant Lie products in vector lattices are known (beyond the basic definitions and simple examples). Nonetheless, we can assert that a hypothetical L-invariant physical system, with the usual relationship between variables and states, possesses a typically quantum behavior. It may be even "more quantum" than today's quantum theory.

Quantum Properties Rediscovered

The first quantum property is the uncertainty relation. It is inherent to the L-invariance since the set of states lies in the intersection of the bases of all the dual cones of the family. The extreme points of the "classical" probability measures fail to meet this strong positivity requirement and are excluded from the set of states, which is all we need to assert the existence of uncertainty relations.

Another quantum feature immediately draws our attention. The L-invariant variables depend on an additional parameter, the choice of the ordering cone or, which is the same thing, the choice of a global Boolean algebra (among the many) containing the Boolean sub-algebra generated by the spectral resolution of a given variable. We encounter here a genuinely new phenomenon—the variables are not completely determined by their physical names or by their probabilistic characteristics in all states. There emerge classes of physically equivalent but theoretically distinguishable variables. Can this distinguishability be physically meaningful? It can, if and only if it makes sense to differentiate between well-defined classes of procedures measuring the same variable. Such a situation never arises in classical physics but we are now in the quantum world and some aspects of the measuring processes are expected to be part of the theory.

In the L-invariant family of lattice cones, it is reasonable to link the dependence of the variables on the measuring procedures to various sets of jointly measured basic variables (possibly incompatible), whose transformation law is given by the transformations between the cones. In this sense, we can speak of "covariantly ordered" Lie algebras. We seek to put the family of lattice cones in close relation to the common method of introducing basic variables (together with their physical names) as generators of a Lie group of symmetries. If that is possible to achieve, the L-invariant formalism becomes essentially superior to the standard theory. The standard theory, somewhat paradoxically, represents all conceivable measuring procedures associated with a given variable by a single mathematical object, exactly as in the classical theories. That is a conceptual deficiency of the standard quantum theory or, if you like, incompleteness or, actually, both.

Thus, the L-invariant multi-commutative structures are tailored to describe quantum systems. Perhaps an entirely new class of quantum systems? Or they provide a more complete description of the quantum world known to stand behind the standard non-commutative quantum theory? At this early stage, both developments are open, the second one being the first on our agenda. There is no guarantee, however, that the multi-valued multiplication of the variables, implied by the multi-commutativity, can always be given acceptable physical interpretation. That is Risk Number 1. Next, we have to find a reasonable bridge from the L-invariant systems back to the standard theory, or rather to the C-invariant quantum theory, its more economical counterpart. A blueprint for such a bridge is given below. Its viability is impossible to assess right now and we have to face Risk Number 2. We are in a position to localize the risk areas in our project and that leaves us enough space for safe maneuvering.

The Multi-Commutativity Begets Non-Commutativity

Seeking to link the L-invariance to the C-invariance, we can follow a natural guideline. The C-invariant quantum systems share with the L-invariant systems the same basic mathematical language, but the C-invariant variables do not depend on their measuring procedures. The appropriate role for them is to serve as a simplified description of the classes of equivalent L-invariant variables where their inner structure is erased and the class as a whole is treated as a single variable. Such a factorization will delete the multi-commutativity while retaining the local statistical properties.

The main conjecture of the multi-commutativity project postulates the simplest possible factorization mechanism: the Lie algebra in the L-invariant systems is supposed to admit an invariant spectral cone (preserving the set of states and necessarily non-lattice) that solves the factorization problem. That is, it generates a C-invariant system that reproduces the Boolean algebras and the spectral properties of all basic variables (but destroys the global Boolean structures). More generally, the C-invariant quantum systems—among them all standard systems—are assumed to be factorizations of the richer multi-commutative L-invariant systems. That is our hypothetical bridge connecting C- and L-invariant quantum systems. Of course, we do not exclude the existence of non-factorizable L-invariant systems.

The alternating (possibly infinite) chains of invariant Lie-algebraic and order structures (L, C, L … or C, L, C …) are interesting objects of study, not yet classified and waiting their turn. The factorization conjecture breaks the chain L, C, L … after the second term and is, perhaps, the first step towards more complex and subtle dynamical systems.

The factorization conjecture repeats, in a sense, von Neumann's feat—assigning physical responsibilities to non-commutative structures. However, our position now is much stronger, because we move in the opposite direction, eliminating not the classical commutativity but the quantum multi-commutativity. Equally strong is our motivation: deliberate factorization to get a simplified and handier quantum language.

In the process, we recapture the standard operator language but with the awareness that it is an approximation and any formal operations in it with incompatible variables are meaningless. In particular, in the lattice of the projection operators, only the Boolean sub-lattices are, strictly speaking, probability event spaces. The widespread attempts to interpret the quantum logic as a non-Boolean event space are pointless. Instead of non-Booleanness, we should speak of multi-Booleanness, taking into account the sensitivity of the quantum variables to the choice of the measuring procedures. We should adopt such a cautious attitude even before the multi-Boolean extension has proven its feasibility.

We are now in a position to identify the self-delusion of axiomatic quantum mechanics. The axiomatics normally begins with the requirement that the variables should be uniquely determined by their mean values in all states. That's a dangerous shortcut, a premature factorization excluding from the theory all traces of the measuring procedures. It must be the last—not the first—phase in any quantum axiomatics.

It is time to sum up what we can expect from the multi-commutativity adventure.

Why the L-Invariant Multi-Commutativity Quantum Project Is Worth Pursuing

The multi-commutativity project offers a remarkable set of potential benefits.

- The L-invariant theory removes a conceptual deficiency in the standard theory. The dependence of the quantum variables on the measuring procedures is tacitly assumed in the standard theory, but it never materializes, never leaves the gray interpretation area.
- The L-invariant theory can settle the old controversy over the completeness of the standard theory. The solution is a real surprise—we detect incompleteness, and eventually can remedy it, but it is not where one has been looking for it. What the standard theory is lacking turns out to be quantum-ness.
- The sudden appearance of the standard non-commutativity finds a natural explanation. The non-commutativity points to a hidden factorization, takes the place of absent multi-commutativity. This perspective can help break the vicious circle that has been troubling the axiomatic quantum mechanics for decades, in particular its operational models.
- The joint measurement of incompatible variables is clearly confirmed. The L-invariant theory shows that the uncertainty relation and the impossibility of joint measurement are, as common sense suggests, logically independent properties. Their relationship in the

standard theory has always been contentious, because of the dubious status of the non-Boolean probability spaces.

- The L-invariant theory makes visible the origin of the uncertainty relations. They arise immediately from the positivity of the states. One can hardly imagine a more convincing restriction on states.
- The L-invariant systems generate (through factorization) C-invariant systems that are equally complete but mathematically more economical than their standard counterparts. Less restrictive formalism gives us a chance to expand the collection of quantum systems in a logically transparent way.
- We get a new approach to the relationship between classical and quantum physics, all their differences now encoded in the inclusion relation between two automorphism groups. The classical and the quantum worlds appear as mirror images of each other, the standard theory fixed at the middle point, as it reveals two-sided LC-invariance (the two automorphism groups tend to coincide).
- The language of L- or C-invariance is more flexible than the monolithic standard theory. It works with two independent structures (Lie product and order relation), their choice and interrelations being largely under our control. The scheme outlined above rests on this flexibility without exhausting it.
- Quite apart from the problems of quantum physics, the L-invariant structures can be regarded as a class of abstract multi-Boolean probabilities, with all sorts of possible applications.
- The L-invariant project stimulates new mathematical research. The theory of invariant Lie products in vector lattices can be as rich and elegant as the classification of the invariant cones in Lie algebras. A question at the highest mathematical level—do we know how to define the independence of two mathematical structures? There is certainly more than one reasonable definition.

Let's recap. The incompleteness and over-completeness coexisting in the standard theory are both attributable to the historically accidental shortcut commutativity ➟ non-commutativity that misses a major stage in the more consistent route commutativity ➟ multi-commutativity ➟ non-commutativity. Our quantum language needs correction and it is time to invest in multi-commutativity as a new quantum paradigm. The stakes are high enough.

A Case for Science Policy

We have to come to terms with the conclusion that fundamental physical theories can be informed by historical contingencies. The 20th century offers some key examples, the advent of quantum mechanics being the most interesting among them. The lack of synchronization between mathematics and physics, always a source of headaches, claims our attention today as a top priority on the physicists' science policy agenda.

We need autonomous science policy to help us handle—qualify—our sudden 20th-century freedom in defining the language of physics. The multi-commutativity quantum project is predicated by this line of reasoning—it is an attempt to avoid arbitrary definitions, to reduce the excessive freedom to relative necessity. It begins with a call for a specific type of mathematical results—geometric spectral theory in the context of ordered Lie algebras. Most significantly, the L-invariant quantum project conflicts with the mainstream of axiomatic quantum mechanics. They both hope to reproduce the standard theory, but they have little in common in their philosophy and technical tools. Their conflict testifies that we have to venture deep into the huge, exacting (and exciting) area lying between physics and mathematics. The reliable science policy is a project of the future.

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The task of rethinking the non-commutative quantum language arises almost a century after von Neumann's breakthrough and should not be delayed much longer. The multi-commutative L-invariant project relates to significant themes in quantum physics, probability theory and Lie theory. Its ultimate success or failure depends on results we'll assemble during decades to come. In contrast, its embryonic cell, the C-invariant quantum model, involves foolproof premises and is in itself a legitimate successor to von Neumann's quantum program. The C-invariant quantum theory should not wait for the multi-commutative projects to supersede the current standard theory. There is a direct access to the C-invariant quantum models, independent of their role as factorizations of a richer quantum theory.

No doubt, however, the full picture of interacting commutativity, multi-commutativity and non-commutativity, culminating in the factorizable multi-commutative L-invariant quantum systems, is the big, inspiring goal where our endeavors should converge.