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Space-Time and Probability

Simon Saunders

Special relativity is most naturally formulated as a theory of spacetime geometry, but within the spacetime framework probability appears to be a purely epistemic notion. It is possible that progress can be made with rather different approaches - covariant stochastic equations, in particular - but the results to date are not encouraging. However, it seems a non-epistemic notion of probability can be made out in Minkowski space on Everett's terms.

I shall work throughout with the consistent histories formalism. I shall start with a conservative interpretation, and then go on to Everett's.

1 Probability in Consistent Histories

In the consistent histories approach histories are represented by ordered products of Heisenberg-picture projection operators, of the form

$$C(\underline{\alpha}(n,-m)) = P_{\alpha_n}(t_n)...P_{\alpha_1}(t_1)P_{\alpha_0}(t_0)P_{\alpha_{-1}}(t_{-1})...P_{\alpha_{-m}}(t_{-m}). \quad (1)$$

Here $\underline{\alpha}(n,-m)$ is the ordered sequence of variables $< \alpha_n, ...\alpha_{-m} >$; its values are *histories.* (As is usual, variables will on occasion stand for values as well. I shall also write $\underline{\alpha}$, $\underline{\alpha}'$, etc. where beginning and end points of histories do not need to be made explicit.) Each variable α_k, where $k \in \{n, ... -m\}$, ranges over subspaces of Hilbert space as defined by a resolution of the identity, i.e. a pairwise disjoint set of projections $\{P_{a_k}(t_k)\}$ which sum to the identity. Histories, therefore, are ordered sequences of sub-spaces of Hilbert space. The question of which set of projections is to be selected at each time is a version of *the preferred basis problem* of quantum mechanics. I shall come back to this presently; for the time being, for the sake of concreteness, suppose the spectrum of each projection is a subset of the configuration space of the system (call it an *outcome).* The probability of the history $\underline{\alpha}(n,-m)$ given state ρ (which may be pure or mixed) is:

$$\Pr(\underline{\alpha}(n,-m)) = Tr(C(\underline{\alpha}(n,-m))\rho C(\underline{\alpha}(n,-m))^*). \quad (2)$$

This quantity can be obtained by repeated application of Luders' rule, supposing the projections of Eq.(1) to be measured sequentially, conditionalizing the state on the outcome in each case.

The conditional probability of history $\underline{\alpha}(n,1)$, given history $\underline{\alpha}(0,-m)$ ("future outcomes conditional on the present and the past") - assuming the latter has non-zero probability - is:

$$\Pr(\underline{\alpha}(n,1)/\underline{\alpha}(0,-m)) = \frac{Tr(C(\underline{\alpha}(n,1))C(\underline{\alpha}(0,-m))\rho C(\underline{\alpha}(0,-m))^* C(\underline{\alpha}(n,1))^*)}{Tr(C(\underline{\alpha}(0,-m))\rho C(\underline{\alpha}(0,-m))^*)} \tag{3}$$

These quantities are real numbers in the interval $[0,1]$. The sum of the probabilities for all such histories (holding the conditions fixed) is equal to one. Neither property holds for the retrospective conditionals, of the form:

$$\Pr(\underline{\alpha}(-1,-m)/\underline{\alpha}(0)) = \frac{Tr(C(\underline{\alpha}(0))C(\underline{\alpha}(-1,-m))\rho C(\underline{\alpha}(-1,-m))^* C(\underline{\alpha}(0))^*)}{Tr(C(\underline{\alpha}(0))\rho C(\underline{\alpha}(0))^*)} \tag{4}$$

("past outcomes conditional on the present"). A condition that ensures that these too are correctly normalized and sum to one is the so called *consistency condition* (also called the *weak decoherence condition*) [Griffiths 1986], [Omnes 1988]:

$$\underline{\alpha} \neq \underline{\alpha}' \Rightarrow Tr(C(\underline{\alpha})\rho C(\underline{\alpha}')^*) + Tr(C(\underline{\alpha}')\rho C(\underline{\alpha})^*) = 0. \tag{5}$$

Consistency is a restriction on the preferred basis. As it stands it is a weak constraint [Kent and Dowker 1996]; it is more stringent if it is to hold on variation of the state. Certain variables, and associated spectral decompositions - integrals of densities that obey local conservation laws, for example - habitually decohere [Halliwell 1998].

The consistency condition is not needed to ensure normalization and additivity of probability for histories in the predictive case. We can dispense with it in the retrodictive case as well; we need only replace the denominator in Eq.(4) by:

$$\sum_{<\alpha_{-1},...,\alpha_{-m}>} Tr(C(\underline{\alpha}(0,-m))\rho C(\underline{\alpha}(0,-m))^*). \tag{6}$$

But this strategy is seriously deficient when we consider the algebraic structure which histories inherit from their definition. For example - the one we have at the back of our minds - consider outcomes as subsets of configuration space. As such they inherit the structure of a Boolean algebra. One outcome can be contained in another by set inclusion, and this generalizes naturally to histories:

$$\alpha_n \subseteq \alpha'_n, ..., \alpha_{-m} \subseteq \alpha'_{-m} \Longleftrightarrow \underline{\alpha}(n,-m)) \subseteq \underline{\alpha}'(n,-m). \tag{7}$$

The Boolean operations of intersection and union extend to histories as well. If a probability measure on this space of histories is to respect this Boolean algebra, then:

$$\Pr(\underline{\alpha} \cup \underline{\alpha}') = \Pr(\underline{\alpha}) + \Pr(\underline{\alpha}') - \Pr(\underline{\alpha} \cap \underline{\alpha}'). \quad (8)$$

This condition is not in general satisfied when probabilities are defined by Eqs.(1), (2); nor does it follow on introducing normalization factors such as Eq.(6). Using (1) and (2), from (8) one obtains (5); Eq.(8) is in fact a form of the consistency condition. It ensures that the probability measure over histories respects the natural set-theoretic relations that follow from coarse-graining.

In physical terms, an outcome consists in values of certain variables (in our case, configuration space variables) having values in designated intervals of reals. On partitioning these intervals one obtains new, finer-grained outcomes, and new, finer-grained histories. The sums of probabilities for non-intersecting histories should equal the probability of their union; this is the consistency condition. The analogous condition is automatically satisfied by histories in the pilot-wave theory, where - because deterministic - it is equivalent to an additivity requirement for probabilities of disjoint subsets of configuration space at a single time.

Consider now the interpretation of probability. Although the notation for histories, including the time t_0, is suggestive, it does not in fact imply that "the present" is in any way preferred; it only reflects, let us say, our particular location in a given history, the one which is actual. We suppose it is a consistent history. We suppose, moreover, that although the available data is necessarily approximate, telling us that variables take values in certain intervals, there is all the same a unique, maximally fine-grained, history of events - the one which actually occurs. We proceed to assign probabilities for all such fine-grained histories, conditional on the coarse-grained history fixed by the available data. The conditional probabilities of Eq.(3) and (4) are special cases of them. But if the actual history is maximally fine-grained - no matter that we do not know what it is - then the probability of every possible outcome at every time, conditional on the actual history, is either zero or one. The only non-trivial notion of probability available can only concern incompletely specified histories, histories which can be further fine-grained (whilst still satisfying the consistency condition). Probability is epistemic.

This framework as it stands cannot be cast in relativistic form, but there is a near neighbour to it, where the preferred basis concerns subsets of the spectra of self-adjoint quantities built out of algebras of local fields. The various measures and the consistency condition can in turn be expressed in terms of path integrals [Hartle 1989]. There seems to be no fundamental obstacle to extending this notion of spacetime probability to relativistic quantum theory. But the probabilities arrived at in this way are all epistemic.

2 Probability in the Everett Interpretation.

There is no non-epistemic notion of probability consistent with relativity theory because relativity requires the tenseless spacetime perspective, and that in turn forces an epistemic notion of probability.

There has been plenty of debate in the philosophy literature about this argument [Putnam 1966, Maxwell 1986, Stein 1991]; here I wish only to show that there is certainly a loophole in it - if one is prepared to follow the logic of the treatment of tense, in moving to the spacetime perspective, to include the treatment of the determinate and indeterminate as well.

Suppose as before that the present is not in any way privileged - that the question of what is "now", like the question of what is "here", is simply a matter of where one happens to be located (in space-time). Now extend this analysis to "determinateness". We are to view these terms as what philosophers call *indexicals*, terms whose reference depends on the context of utterance. But in conformity with relativity, suppose that such contexts are ineluctably *local*; what is determinate is in the first instance is what is here and now - and, derivative on this, what has probability one relative to the here and now. If we are to calculate these probabilities using Eq.(3),(4) (and supposing our history space is in fact a quasiclassical domain [Gell-Mann and Hartle 1993]), we will obtain pretty much the same results as using quantum mechanics as standardly interpreted: events in the past which would have left a record in the present will be determinate; events of a similar kind, but in the future, will in general be indeterminate; events remote from the here and now will likewise, in general, be indeterminate This, typically, is our determinate vicinity. So one would expect on general physical grounds.

We arrive, given the preferred basis, not at a space of histories, but at a space of vicinities. They are all of them possibilities; the question is which of them are realized. But it would be quite impossible to suppose that only one of them is; that would be akin to solipsism, an egocentric or at best an anthropocentric view of reality. No more can there be only a single "here". One might suppose that there is only a single "now"; there are those who believe that the past and future are not real, or not in the sense that "the now" is real. But it had better be a global, universal now, or else there is some local *spatia*l vicinity which has this special status as well - and we are back to the conflict with special relativity (according to which there *is* no privileged global notion of "now"). In non-relativistic physics we have a half-way house, in which moments but not places can be singled out as real; it is not available in the relativistic case.

We might link up these possible local vicinities, in spacelike and timelike directions, and thereby arrive at the set of all possible histories. We would be led back, that is to say, to the framework previously considered, the consistent histories formalism - and to the epistemic notion of probability already considered. The only remaining alternative is to suppose that *all* possible local states of affairs exist. This I take to be Everett's approach. All that there is is the set of vicinities, with all the conditional probabilities defined between them. In physics we do not only take up an *atemporal* perspective, situated at no particular moment in time, we take up an *acontingent* one, independent of any contingent event too. But it is a *local* perspective. It says nothing about which history a particular vicinity is part of; indeed, one and the same vicinity will be part of many distinct histories - depending on the probabilities. There is no hint of the epistemic notion of probability that we had before.

We should be clear on the difference between this and a *many histories i*nterpretation. Reifying all the histories of a given history space does not materially change the situation regarding the interpretation of probability. Probability remains as before, epistemic. The actual history is the one in which one happens to be; probability, as before, concerns the measure of histories consistent with a given coarse-grained description. It makes no difference if they are fictitious or real.

What is so different about the Everett approach? It is that there is no univocal criterion for identity over time. There is no fact of the matter as to what a given actuality will later turn into. Transition probabilities can be defined as before (Eqs.(3),(4), where the outcomes now concern local events), but there is no further criterion of identity over time. This criterion is determinate - one has a unique future event - only insofar as the probability for that event is one (in which case it is already part of the vicinity, according to our earlier definition).

It is has been objected by many that the notion of probability is incoherent in the Everett interpretation [Loewer 1996], but one can hardly insist that for the notion of probability to make any sense the future must already be settled. The approach is moreover much less extravagant - and stays within the bounds of what can be locally defined - than the many histories approach, according to which there are vast numbers of qualitatively identical local vicinities, a different one for every distinct history that can accommodate it (a point well-known in the philosophy literature [Albert and Loewer 1988]).

But here I am concerned with the intelligibility of the approach, not its plausibility. Does probability make any sense on its terms? In every other solution to the problem of measurement, including many-histories, the future is univocal. There is a unique fact of the matter as what will actually be. It is by no means clear that this statement can be coherently denied, or that any meaningful notion of probability can be made out in its absence.

It is not a conceptual necessity, however. It is obviously unnecessary in the case of ordinary objects. Whether the Ship of Theseus, parts of which are steadily replaced over time, remains really the same, or whether it is the ship built from its parts that is really the original, has been a worry for philosophers for millennia. No one else has been much concerned by it. If we have any very pressing *a priori* convictions on this, it seems they come into play at the level of *personal* identify; that there must be a unequivocal notion of identity when we come to *human* beings. But this too has been the subject of a long and inconclusive debate in philosophy.

The problem can be starkly posed [Saunders 1998]. The human brain has remarkable bilateral symmetry. It can indeed be divided - an operation known as a commissurotomy - without obvious deleterious effects (although not as far as the brain stem and the rectilinear formation). But as shown by a series of classic experiments [Sperry 1982] the two hemispheres can be made to operate as distinct and non-communicating centres of consciousness. It is a stretch to take the further step - and to consider the two hemispheres as physically separated altogether, so that they may function as separate persons - but it is hard to see why objections on the grounds of medical feasibility should be germane to the conceptual difficulty. So let us take this further step. In the process, suppose too that there is rather greater functional symmetry between the two hemispheres than is in fact typical. It follows that after the process of fision there are two perfectly normal persons each with equal right - and claiming that right - to count themselves the same as the person prior to the division. Prior to division, it seems there can be no fact of the matter as to which of the two one is going to be.

How are we to understand this? One can hardly expect to have some sort of shared identity, that the two persons that result will somehow think in tandem. It is unreasonable to expect death - each of the two that result will certainly *say* that they have survived; one would be inclined to call it survival if only a single hemisphere were to remain. So what does one expect? On any behavioral criterion, the situation is exactly the same as a probabilistic event with two possible outcomes. We can speak of probability, even though there is no fact of the matter as to what one will be.

This is a distinctively philosophical thought experiment, but I do not think it differs fundamentally from physical thought experiments. Unlike many of the latter, it may even turn out to be practically possible. And I do not see why the concept of unequivocal identity over time should be retained as sacrosanct, when so many other basic concepts of space and time have been so radically revised in physical science.

There remain of course other difficulties with Everett's ideas - the preferred basis problem among them. This problem is particularly severe for a one-history interpretation of history space, where the ontology - the question of what exists - depends on the answer to it. On reflection, for the same reason, it is just as bad in a many-histories interpretation. It is ameliorated but not removed in the Everett approach, where the preferred basis does not determine what ultimately exists, but only the constitution of particular observers - of which vicinities in the universal state are habitable [Zurek 1994, Saunders 1993]; but this is a much larger question than the one that I have been concerned with.

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