Violation of the Locality condition in CFT_2

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Abstract

We present some recent results concerning the identification of the modular structure of von Neumann algebras with spacetime symmetries within the framework of the algebraic formulation of conformal quantum field theory in 2 dimensions. We discuss the localization properties of a new class of KMS-states invariant under representations of the Moebius group generated by higher modes of the Virasoro algebra and show that the usual formulation of locality within the algebraic approach fails in this setting. This can either be circumvented by modifying the notion of spacelike separation or by starting out with multilocalized von Neumann algebras. We argue that this violation of the locality condition is closely connected to the KMS-states not being faithful on the algebra.

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1 Introduction

The connection between the mathematical structure of modular theory on von Neumann algebras and quantum physics was established in the mid-seventies and has proven to be the basis of many new concepts in QFT, e.g. the thermal aspects of QFT and the formulation of equilibrium quantum statistical mechanics by Haag, Hugenholz and Winnink. Another important aspect was first realized by Bisognano and Wichmann. They were able to re-identify the modular operator as the generator of the Lorentz boosts for the algebra generated by fields restricted to the wedge region. Recent works of Schroer and Wiesbrock have shown that this is a rewarding ansatz especially in chiral conformal field theories in 2 dimensions. The modular group of chiral nets turns out to be the automorphism group of dilations, which is part of the Moebius group, the conformal extension of the Poincaré group in two dimensions.

On the other hand the notion of locality is at the conceptual and therefore mathematical basis of the algebraic approach to Quantum Field Theory, in which the results above are accessible. It implies a certain degree of independence to objects located at spacelike distances. This is usually encoded by the requirement that algebras that are spacelike separated must commute. But various formulations of this idea have been discussed, e.g. strict locality, Haag Duality or the Schlieder property. Although all of them are expressing more or less the same physical content, the relations between these formulations is (in some cases) far from being clear. In addition the analysis of the locality condition in algebraic quantum theory is further complicated since other assumptions like causality or relativistic covariance import more "local" structure into the theory.

In this report we shortly present the idea of the modular structure for the higher Virasoro modes generating the Moebius group. It appears that in the setting of a chiral theory invariant under transformations generated by the higher Virasoro modes the naive locality condition cannot be applied to this situation, simply because spacelike separated algebras do not necessarily commute. The physical interpretation of this situation is not an easy task. We argue that one is forced to adopt either a modified notion of spacelike separation or regard what we call multilocalized algebras, i.e. algebras with localization in different, disconnected regions of spacetime as the fundamental building blocks of the theory. While the first solution appears to be rather ad hoc and matched just to the geometrical situation at hand, the ansatz with multilocalized algebras has another important drawback. The states associated to the multilocalized algebras are no longer faithful states. What first appeared to be a technical detail turned out to be closely connected to the violation of locality, since the attempt to restore the faithfulness of the states by restricting the algebra leads to a situation where the locality problem is no longer present. This may also support the widespread belief that all physically "sensible" states are faithful states. But either way one is left with a puzzling "non-local" picture of this situation. We hope that the investigation of localization properties sheds new light on the interpretation of the locality condition of the algebraic approach to quantum field theory.

This report is organized as follows: In the next section we briefly review the algebraic approach to QFT, modular theory and the KMS property. We present the model of a chiral conformal net and discuss the identification of the dilation group as the modular group of the theory in section 3. In section 4 this ansatz is generalized to the higher modes of the Virasoro algebra in the compact geometric setting of the unit circle. We show that the usual locality condition is violated and discuss the two alternative solutions. We end this article by connecting the locality problem to the non-faithfullness of the states on the multilocalized algebras.

2 Algebraic QFT and Modular Structure

In this section we would like to remind the reader of the basic definitions and structure of the algebraic approach to quantum field theory[3]. We present briefly the connection between the modular theory of von Neumann algebras developed mainly by M. Takesaki and algebraic field theory given by the Kubo-Martin-Schwinger (KMS) condition.

2.1 The algebraic approach to QFT

The basic entities in the algebraic approach[4] are C^*-algebras $\mathcal{A}$ or von Neumann algebras $\mathcal{N}$ and states ω i.e. linear, positive, normalized functionals on the algebras[5]. These algebras are associated to spacetime regions $\mathcal{O}$, which localizes the algebras and enables a physical interpretation in terms of observables. Heuristically, the algebra $\mathcal{N}(\mathcal{O})$ contains the bounded fields $\phi(f)$ and $\phi^*(f)$ with f being a test function with support in $\mathcal{O}$ with the consequence that multiplication of fields is well defined. Hence the following mapping is of fundamental interest:

$$\mathcal{O} \longrightarrow \mathcal{N}(\mathcal{O})$$

with $\mathcal{O}$ being open, bounded regions of Minkowski spacetime $\mathcal{M}$. The causal complement of a spacetime region $\mathcal{O}$ is the set of all points that are spacelike separated from $\mathcal{O}$ and is denoted by $\mathcal{O}'$. The commutant $\mathcal{N}'$ of an algebra $\mathcal{N}$ is defined as the set of all elements that commute with all other elements of the algebra[6]. The basic assumptions of the algebraic approach can be summarized as follows[7]:

- Isotony: $\mathcal{O}_1 \subseteq \mathcal{O}_2 \Rightarrow \mathcal{N}(\mathcal{O}_1) \subseteq \mathcal{N}(\mathcal{O}_2)$
  This condition defines the algebraic structure of a net on $\mathcal{N}(\mathcal{O}_i)$. An unbounded region may also be taken into account, since the inductive limit of a sequence of monotonously expanding, bounded regions exists. The von

[3] For a very recent review of the current state of algebraic QFT see [8].

[4] Cf. for detailed exposition [20, 21, 22].

[5] For the sake of convenience we will proceed with von Neumann algebras only.

[6] $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, $\mathcal{A}' := \{B \in \mathcal{B}(\mathcal{H}) \mid AB = BA \; \forall A \in \mathcal{A}\}$, $\mathcal{B}(\mathcal{H})$ is the set of all bounded operators on the Hilbert space $\mathcal{H}$.

[7] For a discussion of various formulations of the axioms of AQFT see [1].

Neumann algebra $\mathcal{N}(\mathcal{M})$ associated to Minkowski spacetime is called the *global algebra*.

- Locality: $\mathcal{O}_1 \subset \mathcal{O}_2{}' \Rightarrow [\mathcal{N}(\mathcal{O}_1), \mathcal{N}(\mathcal{O}_2)] = 0$
This in the first place enables compatibility of observables measured at a spacelike distance.

- Causality: For any family of regions $\mathcal{O}_i$ with $(\bigcup_i \mathcal{O}_i)' = \emptyset$, one has

$$\mathcal{N}(\bigcup_i \mathcal{O}_i) = \mathcal{N}(\mathcal{M}).$$

This condition states that the quantities describing the system propagate causally and their values at any initial time moment determine the values at any other time.

- Covariance: There is a representation of the symmetry group $\mathcal{P}$(e.g. Poincaré group, conformal group) with automorphisms α_g on the net such that

$$\forall g \in \mathcal{P} : \alpha_g(\mathcal{N}\left(\mathcal{O}\right)) = \mathcal{N}\left(g\mathcal{O}\right).$$

Note that these conditions of the algebraic approach in contrast to the Wightman axioms mention neither a vacuum vector nor a Hilbert space. The Hilbert space is reconstructed from the algebra and a state on this algebra via the famous Gelfand-Naimark-Segal (GNS) construction.
Yet, generally one extends for physical reasons the purely algebraic assumptions above by an analytic stability condition, i.e. a so-called spectrum condition. It is assumed that there exists a so-called particle representation, i.e. a non-degenerate representation π of the algebra $\mathcal{N}\left(\mathcal{O}\right)$ on a Hilbert space, with an implementation of the automorphism α_g by continuous unitary operators:

$$\mathcal{U}_\pi(g)\pi(A)\mathcal{U}_\pi(g)^* = \pi(\alpha_g(A)).$$

The stability of the theory is now guaranteed by the

- Spectrum condition: The spectrum of the generator P_μ of the translations $\mathcal{U}_\pi(g_x)$ is contained in the closed forward light cone $\overline{V}_+$.

Furthermore the physical vacuum enters this approach by the assumption of the existence of a state on the net which is invariant under the symmetry group $\mathcal{P}$. Via the GNS-construction a vacuum representation, i.e. a particle representation containing a vector state invariant under the unitary representation of the symmetry group can be constructed.
We would like to add the following definitions that will become useful later on in the text [22]:
A state is called faithful if and only if (iff) it is strictly positive on the (closure of the) convex cone of positive elements of the Algebra.

$$\forall A \in \overline{\mathcal{A}}_+ : \omega(A) > 0$$

A vector $|\psi\rangle \in \mathcal{H}$ is called separating iff

$$\forall A \in \mathcal{A} : A|\psi\rangle = 0 \Rightarrow A = 0.$$

The vector representation of a faithful state is separating and vice versa.
Given a C^*-algebra $\mathcal{A}$, one is able to construct a von Neumann algebra $\mathcal{N}$ by taking the double commutant:

$$\mathcal{N} = ((\mathcal{A})')' =: \mathcal{A}''.$$

2.2 Modular Theory and the KMS-property

Consider a von Neumann algebra $\mathcal{N}$ in standard form which means that $\mathcal{N}$ acts on a Hilbert space $\mathcal{H}$ with a cyclic and separating vector $|\Omega\rangle$. Then due to the famous Tomita-Takesaki-Theorem[8] one can associate a unique *modular structure* (Δ, J) with the tuple $(\mathcal{N}, |\Omega\rangle)$.

$$(\mathcal{N}, |\Omega\rangle) \longleftrightarrow (\Delta, J)$$

With Δ being a self adjoint, positive operator and J a usually antiunitary conjugation. The operators Δ and J have the following properties:

$$\begin{aligned} \Delta|\Omega\rangle &= |\Omega\rangle \quad J|\Omega\rangle = |\Omega\rangle \\ J\mathcal{N}J &= \mathcal{N}' \\ \Delta^{it}\mathcal{N}\Delta^{-it} &= \mathcal{N} \\ \Delta^{it}\mathcal{N}'\Delta^{-it} &= \mathcal{N}', \quad t \in \mathbb{R}. \end{aligned} \tag{1}$$

Δ^{it} defines an one-parameter unitary group on $\mathcal{N}$, the modular group which maps the von Neumann algebra $\mathcal{N}$ and its commutant $\mathcal{N}'$ onto itself, while the Tomita-conjugation J maps the algebra onto its commutant. Since the present work is solely dedicated to the geometrical interpretation of Δ^{it} we will not investigate the properties of the Tomita-conjugation J any further. So to put it in short terms, the main result is that given a von Neumann algebra in standard form one is able to identify an automorphism group Δ^{it} on this algebra. In the context of algebraic field theory the question naturally arises if there is a physical interpretation of this mathematical structure in terms of symmetry operations [7]. A first partial result in this direction was derived by Bisognano and Wichmann [5]. They were able to show that under certain circumstances[9] the modular structure of the von Neumann algebra of Wightman fields[10] localized in the (left or right-)wedge of Minkowski spacetime $\mathcal{M}$ can roughly be identified with the Lorentz boost in the 1-direction and the CPT operator:

$$\begin{aligned} \Delta^{it} &= U(\Lambda_{\mathcal{W}}(-2\pi t)) \\ J &= U(R^1_\pi)\Theta_{CPT}. \end{aligned} \tag{2}$$

[8]For a detailed exposition see e.g. [21, 22].

[9]See e.g. in [6] for a nice exposition of this point.

[10]$\mathcal{N}(\mathcal{W}) = \mathcal{A}(\mathcal{W})''$, $\mathcal{A}$ is the algebra of Wightman fields localized in the wedge $\mathcal{W} := \{x \in \mathcal{M} |\, x^1 > |x^0|\}$.

Starting with a von Neumann algebra and a state on it, another step in this enterprise was to realize that a symmetry operation by a one-parameter automorphism group on the von Neumann algebra could be re-identified as the *unique* modular structure essentially by showing that the state fulfills the Kubo-Martin-Schwinger (KMS) condition for this automorphism group [20]. Hence, the KMS condition works as a "hinge" between the modular group and the symmetry group of the theory. The KMS condition is a generalization of the characterization of Gibbs equilibrium states for systems with infinite degrees of freedom [16, 21]. Given a von Neumann algebra $\mathcal{N}$ and a one-parameter automorphism group $(\tau_t, t \in \mathbb{R})$ acting on this algebra, a state ω is a τ-KMS state for $\beta \in \mathbb{R}$, iff the following is true:

$$\omega(A\tau_{i\beta}(B)) = \omega(BA) \quad \forall A, B \in \mathcal{N}. \tag{3}$$

If this condition holds for $\beta = 1$ and ω is faithful and normal on $\mathcal{N}$, then the automorphism group τ_t is the unique modular group of $(\mathcal{N}, \omega)$ [22].
This connection has mainly been studied in the setting of the algebraic approach to two dimensional chiral conformal field theory, i.e. chiral conformal nets. We will come back to this point in section 3.3.

3 Chiral Conformal Nets

In the following section we introduce two dimensional conformal field theory (CFT_2) in the geometric setting of a compact picture $S^1 \times S^1$ and its symmetry group, the so-called Moebius group. The definition of a chiral CFT_2 in the aforementioned frame of algebraic QFT, i.e. the chiral conformal net, is stated. As an example and working model the $U(1)$-chiral-current-algebra on the circle S^1 is presented. We end with the Bisognano-Wichmann property for chiral conformal nets, and the geometric identification of the associated modular structure for the $U(1)$-model in the vacuum sector via the KMS-property.

3.1 Chiral Conformal Field Theory in 2 dimensions

Conformal Field Theory in two dimensions (CFT_2) [19] provides a well-suited realm for algebraic quantum field theory [3], especially for problems concerning the geometric identification of the modular structure [5, 7].
Two dimensional Minkowskian CFT_2 may be represented on the product of two circles, $S^1 \times S^1$-spacetime (the "compact-picture"). The global symmetry group of the CFT_2 is the Moebius-group $PSU(1,1) \times PSU(1,1)$. In CFT_2 there are so-called chiral fields invariant under lightlike translations. The corresponding theory of chiral fields may be considered as localized on one circle only. We will therefore concentrate on one of the groups:

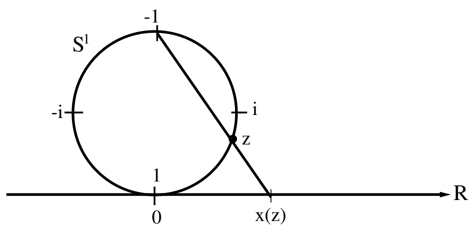
$$PSU(1,1) := SU(1,1)/\{\pm 1\},$$

$$SU(1,1) := \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \middle| \, \alpha, \beta \in \mathbb{C}, \ |\alpha|^2 - |\beta|^2 = 1 \right\},$$

which is the conformal extension of the Poincaré group in $1+1$ dimensions [12]. As an example the Lorentz-boosts reduce to the form:

$$\Lambda(2\pi t) = \begin{pmatrix} ch(2\pi t) & sh(2\pi t) \\ sh(2\pi t) & ch(2\pi t) \end{pmatrix}. \tag{4}$$

One may equivalently present the CFT_2 on a product of lines, $\mathbb{R} \times \mathbb{R}$-spacetime (the "non-compact-picture"). The coordinate transformation from the circle to the line is provided by the stereographic projection (Cayley-transformation) [9]:



The global symmetry group $PSU(1,1)$ maps under this projection isomorphically onto the real group $PSL(2,\mathbb{R})$ [9]:

$$x \longmapsto \hat{g}(x) := \frac{ax+b}{cx+d}, \quad ad-bc=1,\ x \in \mathbb{R}.$$

On the lightcone elements $L_\pm := \{x_\pm := x_0 \pm x_1 | \, (x_0, x_1) \in \mathbb{R}^{1+1}\}$ these boosts become scaling transformations, i.e. dilations (7) :

$$\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \Lambda(2\pi t) \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \tag{5}$$

$$\Longrightarrow \begin{pmatrix} y_0 + y_1 \\ y_1 - y_0 \end{pmatrix} = \begin{pmatrix} e^{2\pi t} & 0 \\ 0 & e^{-2\pi t} \end{pmatrix} \begin{pmatrix} x_0 + x_1 \\ x_0 - x_1 \end{pmatrix} = \begin{pmatrix} e^{2\pi t}(x_0 + x_1) \\ e^{-2\pi t}(x_0 - x_1) \end{pmatrix}. \tag{6}$$

Therefore, of special interest is the one-parameter group of dilations which have the following form:

$$\hat{Dil}(t)x = e^{-2\pi t}x,\ x,t \in \mathbb{R}, \quad \hat{Dil}(t) \in PSL(2,\mathbb{R}). \tag{7}$$

On the real line the dilations are scaling transformations and are easy to handle. These mappings leave the points $\{0, \infty\} \in \mathbb{R}$ ($\{1, -1\} \in S^1$) invariant. Therefore the upper and lower semicircle, S^1_+ respectively S^1_- ($\mathbb{R}_+, \mathbb{R}_-$), are mapped by dilations onto themselves. One of the peculiar features of the conformal symmetry is that the dilation-group may be attached to an arbitrary, proper interval $\mathcal{I} \subset S^1$. This dilation has the limit points of the interval $\mathcal{I}$ as fixpoints. It can be construed as follows:

$$Dil_{\mathcal{I}}(t) := g_{\mathcal{I}}^{-1} Dil(t) g_{\mathcal{I}}, \quad g_{\mathcal{I}}, Dil_{\mathcal{I}} \in PSU(1,1),\ g_{\mathcal{I}}\mathcal{I} = S^1_+. \tag{8}$$

The interval $\mathcal{I}$ is mapped bijectively to the upper semicircle, dilated and mapped back. This is illustrated by the following diagram:

$$\begin{tikzcd} \mathcal{I} \arrow[r, "Dil_{\mathcal{I}}"] \arrow[d, "g_{\mathcal{I}}"'] & \mathcal{I} \\ S^1_+ \arrow[r, "Dil"'] & S^1_+ \arrow[u, "g_{\mathcal{I}}^{-1}"'] \end{tikzcd}$$

The spectrum generating algebra of reparametrizations of the circle is generated by the Virasoro algebra (with central charge c) $\mathfrak{L}_c$:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \quad n \in \mathbb{Z}. \tag{9}$$

The group $PSU(1,1)$ is globally well-defined on the circle and has the underlying generators L_{-1}, L_0, L_1, fulfilling a $sl(2,\mathbb{C})$-algebra:

$$[L_1, L_{-1}] = 2L_0, \quad [L_{\pm 1}, L_0] = \pm L_{\pm 1}. \tag{10}$$

But the Virasoro algebra $\mathfrak{L}_c$ contains an infinite number of $sl(2,\mathbb{C})$-algebras, generated by the modes $L_{-n}, L_0, L_n,\ n > 1$:

$$\begin{array}{rl} L_{-n} \mapsto \tilde{L}_{-n} := & \frac{1}{n}L_{-n} \\ L_0 \mapsto \tilde{L}_0 := & \frac{1}{n}L_0 + \frac{c}{24}\frac{(n^2-1)}{n} \\ L_{+n} \mapsto \tilde{L}_{+n} := & \frac{1}{n}L_{+n} \end{array} \quad \longmapsto \left\{ \begin{array}{l} [\tilde{L}_{+n}, \tilde{L}_{-n}] = 2\tilde{L}_0 \\ [\tilde{L}_{\pm n}, \tilde{L}_0] = \pm \tilde{L}_{\pm n} \end{array} \right\} sl(2,\mathbb{C}). \tag{11}$$

The corresponding finite transformations leaving the unit-circle S^1 invariant are of the form:

$$g_n(z) := \left(\frac{\alpha z^n + \beta}{\bar{\beta} z^n + \bar{\alpha}} \right)^{\frac{1}{n}}, \quad \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in PSU(1,1). \tag{12}$$

It is slightly more cumbersome to handle the transformations analogue to (12) in the non-compact picture [2]. For this reason we perform the calculations in the compact picture representation. It will, however, be instructive to discuss aspects of localization (see section 4.2) in the non-compact picture representation.
The definition of chiral CFT_2 within the algebraic approach is the following [14]. Spacetime regions reduce to proper intervals on the unitcircle. Let $\mathfrak{J}$ be the set of proper intervals on S^1. The net may be characterized by indexed von Neumann algebras $\mathfrak{J} \ni \mathcal{I} \to \mathcal{N}(\mathcal{I}) \subset \mathcal{B}(\mathcal{H}_0)$ on some Hilbert space $\mathcal{H}_0$ containing a unique ground state vector $|\Omega\rangle$. Isotony (now for intervals) and the spectrum condition are the same way as before. The theory contains a continuous unitary representation U_g of the Moebius group $g \in PSU(1,1)$ on $\mathcal{H}_0$. Every U_g invariant vector in $\mathcal{H}_0$ is a multiple of $|\Omega\rangle$. A comparison of the locality condition with the one given in section 2 shows that spacelike separation is expressed by disjoint intervals on the circle [12, 10].

- Locality: $\mathcal{I}_{1,2} \in \mathfrak{J},\ \mathcal{I}_1 \cap \mathcal{I}_2 = \varnothing \Rightarrow [\mathcal{N}(\mathcal{I}_1), \mathcal{N}(\mathcal{I}_2)] = 0.$

3.2 The Weyl algebra of U(1)-currents on the circle $\mathbf{S^1}$

For the following discussion we would like to introduce the algebra of exponentials of the smeared $U(1)$-current-algebra on the circle as a model for a chiral conformal net [10]. The $U(1)$-currents $J(z)$ on the circle S^1 are the product of a Fermi field with its complex conjugated at one point $z \in S^1$. The constituting relation of the $U(1)$-algebra is the current-current commutation-relation (with the circle as base-space):

$$[J(z), J(w)] = -\partial_z \delta(z-w).$$

In order to bring the algebraic ansatz to the stage one has to smear the $U(1)$-currents with real smooth testfunctions on the circle:

$$J(f) := \int_{S^1} \frac{dz}{2\pi i} f(z)\, J(z).$$

The bounded operators $W(f) := e^{iJ(f)}$ fulfill the Weyl-relations:

$$W(f)^* = W(-f) \tag{13}$$

$$W(f)W(g) = e^{-\frac{1}{2}\sigma(f,g)} W(f+g), \tag{14}$$

$\sigma(f,g) := \int \frac{dz}{2\pi i} f(z)g'(z)$ being a symplectic 2-form on the testfunction space $\mathcal{S}(S^1)$. These operators give rise to a von Neumann algebra via the double commutant:

$$\mathcal{W}(\mathcal{I}) := \left\{ W(f) \middle| \, supp(f) \subseteq \mathcal{I} \subset S^1 \right\}'', \tag{15}$$

which is called (local) Weyl-algebra [10, 21]. The net of Weyl-algebras

$$\mathcal{I} \mapsto \mathcal{W}(\mathcal{I}), \quad \mathcal{I} \subset S^1$$

fulfills the postulates of algebraic quantum field theory, i.e it forms a chiral conformal net with respect to the vacuum state $|\Omega\rangle$. In particular the locality property transfers from the (unbounded) $U(1)$-currents $J(f)$ to the bounded Weyl-operators $W(f)$ [15] i.e. a Weyl element is localized in the region where its testfunction is non-zero.

3.3 Bisognano-Wichmann property for chiral nets

It was shown[11] that for arbitrary chiral conformal nets the representation of the one-parameter group of dilations (8) gives (a geometric realization of) the unique modular group $\{\Delta_{\mathcal{I}}^{it};\, t \in \mathbb{R}\}$ associated to the standard tuple $(\mathcal{N}(\mathcal{I}), |\Omega\rangle)$. It is a peculiarity of chiral CFT_2 that arbitrary intervals can be mapped onto the upper (or lower) semicircle and thereby allow to identify the modular group of algebras localized in these regions with the above constructed dilations (8).
In the case of the net of Weyl-algebras (15) one can verify the identification of the

[11] See e.g. [14].

modular group with the dilation subgroup explicitly with the help of the relation between the modular structure and the KMS-property. The vacuum expectation values of Weyl-operators obey the KMS-condition with respect to (the representation of) the one-parameter group of dilations (8) [17]:

$$\langle\Omega|W(f)\,U_{Dil_{\mathcal{I}}(t)}[W(g)]|\Omega\rangle \overset{\text{KMS}}{=} \langle\Omega|U_{Dil_{\mathcal{I}}(t+i)}[W(g)]\,W(f)|\Omega\rangle, \tag{16}$$

$W(f),\ W(g) \in \mathcal{W}(\mathcal{I}),\ \beta = 1$. In the case of the vacuum state this is a necessary and sufficient condition to identify uniquely the one-parameter group of dilations as the modular group [22] mentioned above.

4 Modular Origin of higher Virasoro-Modes

In this section we present briefly the Schroer and Wiesbrock ansatz [2] giving a modular interpretation for all $sl(2,\mathbb{C})$ algebras generated by the (modified) Virasoro modes $\tilde{L}_{n,0,-n}$ (11) within the $U(1)$-model. This result is based mainly on the construction of a new class of KMS-states invariant under the group generated by the modes $\tilde{L}_{n,0,-n}$ (12). The localization properties of these states are investigated and analyzed with respect to the general algebraic setting of conformal nets.

4.1 New KMS-states by Schroer and Wiesbrock

For the $U(1)$-model one can construct a state $|\Omega_n\rangle$, invariant under transformations of the form (for $n \in \mathbb{N}$):

$$g_n(z) := \left(\frac{\alpha z^n + \beta}{\bar{\beta} z^n + \bar{\alpha}}\right)^{\frac{1}{n}}, \quad \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in PSU(1,1) \tag{17}$$

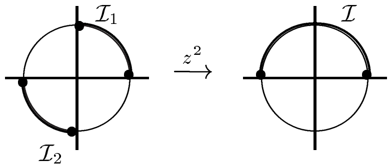
by a simple reparametrization of the unit-circle $S^1 \subset \mathbb{C}$ in terms of the conformal mapping $S^1 \ni z \longmapsto z^n,\ n \in \mathbb{N}$. The case $n = 1$ reproduces the situation discussed in section 3.
The correlation functions are defined in the following way:

$$\langle\Omega_n|\prod_{i=1}^{m} J(z_i)|\Omega_n\rangle := \prod_{i=1}^{m} n z_i^{n-1}\langle\Omega|\prod_{i=1}^{m} J(z_i^n)|\Omega\rangle. \tag{18}$$

Using the Moebius-invariance of the vacuum $|\Omega\rangle$ one can show the invariance of $|\Omega_n\rangle$ under transformations of the form (17)[12]. For the sake of simplicity we choose $n = 2$ for the rest of this paper.
It is easy to see that the transformation z^2 maps both intervals $\mathcal{I}_i,\ i = 1, 2$ to the upper semicircle.

[12]This is true for general conformal fields [11].

$$\mathcal{I}_i \xrightarrow{z^2} \mathcal{I}, \; i = 1, 2 \; ,$$



i.e. either interval $\mathcal{I}_i, \; i = 1, 2$ is part of the image of the square root of the upper semicircle, $(S^1_+)^{\frac{1}{2}} = \mathcal{I}_1 \cup \mathcal{I}_2$.
Hence, the Weyl-algebras $\mathcal{W}(\mathcal{I}_1)$ and $\mathcal{W}(\mathcal{I}_2)$ are both mapped to the Weyl-algebra localized on the upper semicircle:

$$\langle \Omega_2 | W(f) \, W(g) | \Omega_2 \rangle := \langle \Omega | W(f_{\frac{1}{2}}) \, W(g_{\frac{1}{2}}) | \Omega \rangle, \quad W(f), \, W(g) \in \mathcal{W}(\mathcal{I}_i), \; i = 1, 2. \tag{19}$$

This reads for the localized currents:

$$\langle \Omega_2 | J(f) \; J(g) | \Omega_2 \rangle = \langle \Omega | J(f_{\frac{1}{2}}) \; J(g_{\frac{1}{2}}) | \Omega \rangle \tag{20}$$

with $f_{\frac{1}{2}}(z) := f((z)^{\frac{1}{2}}), \; supp(f_{\frac{1}{2}}) \subset S^1_+$. Testfunctions localized in either $\mathcal{I}_1$ or $\mathcal{I}_2$ are transformed into testfunctions with support in S^1_+ (this, of course, is true for general intervals $\mathcal{I}$)[13].
The modified dilations in (17) act in the following way:

$$\begin{aligned} Dil_{2,\mathcal{I}^{\frac{1}{2}}}(t) \, (z) \quad &:= \quad \left(Dil_{\mathcal{I}}(t) \, (z)^2 \right)^{\frac{1}{2}} \\ &= \quad \left(g_{\mathcal{I}}^{-1} \, Dil(t) \, g_{\mathcal{I}} \, (z)^2 \right)^{\frac{1}{2}} . \end{aligned} \tag{21}$$

This again can be illustrated by the following diagram:

$$\begin{tikzcd} \mathcal{I}^{\frac{1}{2}} \arrow[rrr, "Dil_{2,\mathcal{I}^{\frac{1}{2}}}"] \arrow[dr, "(\bullet)^2"'] & & & \mathcal{I}^{\frac{1}{2}} \\ & \mathcal{I} \arrow[r, "Dil_{\mathcal{I}}"] \arrow[d, "g_{\mathcal{I}}"'] & \mathcal{I} \arrow[ur, "(\bullet)^{\frac{1}{2}}"'] & \\ & S^1_+ \arrow[r, "Dil"'] & S^1_+ \arrow[u, "g_{\mathcal{I}}^{-1}"'] & \end{tikzcd} \tag{22}$$

$\mathcal{I}^{\frac{1}{2}}$ stands for either interval $\mathcal{I}_i, \; i = 1, 2$ and $\mathcal{I}$ is an arbitrary proper interval on S^1. One can check that the Weyl-algebras $\mathcal{W}(\mathcal{I}_1)$ and $\mathcal{W}(\mathcal{I}_2)$ are both invariant

[13] One should mention here that for defining the Weyl-algebras with respect to the state $|\Omega_2\rangle$, one has to use a modified symplectic 2-form here with regard to the modified 2-point functions of the currents in (20).

under $Dil_{2,\mathcal{I}^{\frac{1}{2}}}(t)$.
The KMS-property for the state $|\Omega_2\rangle$

$$\langle\Omega_2|W(f)\,U_{Dil_{2,\mathcal{I}^{\frac{1}{2}}}(t)}[W(g)]|\Omega_2\rangle \overset{\text{KMS}}{=} \langle\Omega_2|U_{Dil_{2,\mathcal{I}^{\frac{1}{2}}}(t+i)}[W(g)]\,W(f)|\Omega_2\rangle \qquad (23)$$

transfers directly from the KMS-property of the vacuum using equation (19):

$$\langle\Omega|W(f_{\frac{1}{2}})\,U_{Dil_{\mathcal{I}}(t)}[W(g_{\frac{1}{2}})]|\Omega\rangle \overset{\text{KMS}}{=} \langle\Omega|U_{Dil_{\mathcal{I}}(t+i)}[W(g_{\frac{1}{2}})]\,W(f_{\frac{1}{2}})|\Omega\rangle. \qquad (24)$$

The KMS-property implies the faithfulness of the state defined by $|\Omega_2\rangle$ on either algebra $\mathcal{W}(\mathcal{I}_i),\ i=1,2$ [22]:

$$\omega_2(W(f)) := \langle\Omega_2|W(f)|\Omega_2\rangle$$

and therefore identifies the modified dilation group $Dil_{2,\mathcal{I}^{\frac{1}{2}}}(t)$ with the modular group for both standard tuples $(\mathcal{W}(\mathcal{I}_i),|\Omega_2\rangle),\ i=1,2$.

4.2 Localization properties

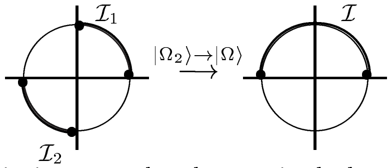
The locality condition for chiral conformal nets which we presented in section 3.1 is:

$$\mathcal{I}_1\cap\mathcal{I}_2=\varnothing \Rightarrow [\mathcal{N}(\mathcal{I}_1),\mathcal{N}(\mathcal{I}_2)]=0, \qquad (25)$$

because the criteria of spacelike separation is expressed by disjoint intervals on the circle. This seemingly natural condition does not hold in the setting described above. Consider the special case that the currents are localized in the first and third quarter circle, i.e. $supp(f)\subset\mathcal{I}_1$ respectively $supp(g)\subset\mathcal{I}_2$. Hence, they lie in disjoint regions of S^1 and therefore one should expect the left hand side of equation (26) to vanish. But by an explicit calculation one finds that the right hand side of this expression can still be nonzero:

$$\langle\Omega_2|[J(f),J(g)]|\Omega_2\rangle \overset{(18)}{=} \langle\Omega|[J(f_{\frac{1}{2}}),J(g_{\frac{1}{2}})]|\Omega\rangle. \qquad (26)$$

This is due to the fact that the supports of the transformed testfunctions lying in the upper semicircle may have a nonzero overlap.

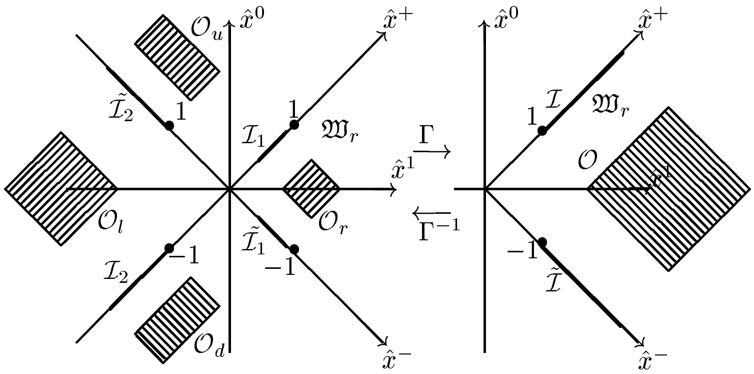


Note, however, that it is in general only required that the testfunctions have a non-zero overlap. Their domain still need not be identical. This shows that the local commutativity condition as defined above fails for the $|\Omega_2\rangle$ state since disjoint intervals may get connected in a non-local fashion by this state. But the above example implicitly contains a way how the usual locality condition has to be modified. In terms of localized algebras the following modified locality condition for the case of $n>1$ may be defined by:

- Locality:
  Let $\mathcal{A}(\mathcal{I})$, $\mathcal{A}(\tilde{\mathcal{I}})$ be algebras localized in the interval $\mathcal{I} \subset S^1$ and $\tilde{\mathcal{I}} \subset S^1$, respectively. Locality with respect to the state $|\Omega_n\rangle$, $n > 1 \in \mathbb{N}$ is expressed by:

$$1)\, \mathcal{I} \cap \tilde{\mathcal{I}} = \varnothing \text{ and}$$
$$2)\, \forall_{\substack{z \in \mathcal{I},\\ w \in \tilde{\mathcal{I}}}} : \arg(z) - \arg(w) \neq 0 \bmod(\tfrac{2\pi}{n}) \Rightarrow \left[\mathcal{A}(\mathcal{I}), \mathcal{A}(\tilde{\mathcal{I}})\right] = 0. \qquad (27)$$

So spacelike separation in a theory with the modified Moebius group (17) as its symmetry is expressed by disjoint intervals modulo $2\pi/n$. In the non-compact representation this modification is also taking care of the dependencies between disjoint intervals introduced by the new KMS-states but its geometrical interpretation is distorted by the Cayley-transformation and becomes less intuitive. Nevertheless it manages to circumvent the locality problem described above for both geometrical settings, since the chosen intervals $\mathcal{I}_{1,2}$ (26) in the presented example are exactly those that are interdependent with respect to the new state $|\Omega_2\rangle$, and therefore do not agree with the modification 2) in (27).
The following example serves to illustrate the mutual dependence of spacelike separated disconnected spacetime regions in the more intuitive 2d non-compact picture $\mathbb{R} \times \mathbb{R}$. We again set $n = 2$. The regions $\mathcal{O}_x$, $x = u, d, l, r$ (see below) of localization of the field algebras $\mathcal{A}(\mathcal{O}_x)$ generate an interesting "tilling" of the minkowskian plane shown below.



The intervals $\mathcal{I}_1$, $\mathcal{I}_2$ and $\tilde{\mathcal{I}}_1$, $\tilde{\mathcal{I}}_2$ are the localization regions on $\mathbb{R} \times \mathbb{R}$, i.e. the lightcones. The mappings $\Gamma(x)$, $\Gamma^{-1}(x)$, $x \in \mathbb{R}$ correspond on each sector to the complex mappings z^2, $z^{\frac{1}{2}}$ respectively. Region $\mathcal{O}$ is the image of the regions $\mathcal{O}_x$, $x = u, d, l, r$ under the map $\Gamma \times \Gamma$. It can now be seen that algebras localized in the regions $\mathcal{O}_r$ and $\mathcal{O}_l$, which lie in each others causal complement (in the sense of minkowskian geometry), get interdependent when mapped onto the region $\mathcal{O}$. Exactly this happens in equation (18) defining the geometrical state $|\Omega_n\rangle$ (when extended to both sectors). This non-local feature of the new (KMS-) states made a modification of the locality condition necessary.

4.3 Multilocalized Algebras

The aforementioned modified locality condition is an ad hoc ansatz to control the non-locality of the new states $|\Omega_n\rangle$. Another and more elegant way is to introduce multilocalized algebras, i.e.

$$\mathcal{W}(\mathcal{I}^{\frac{1}{2}}) := (\mathcal{W}(\mathcal{I}_1) \cup \mathcal{W}(\mathcal{I}_2))''\,.$$

These von Neumann algebras are localized in both intervals $\mathcal{I}_1$ *and* $\mathcal{I}_2$[14]. They therefore contain the geometrical information expressed by the modification of the locality condition, since the intervals $\mathcal{I}_{1,2}$ invariant under the modified dilations (21) are also the ones interdependent with respect to the new KMS-state $|\Omega_2\rangle$. Thus, the introduction of multilocalized algebras seems to be the natural choice with respect to the non-locality of $|\Omega_2\rangle$, because the locality condition (27) reduces to the usual one if one starts out with multilocalized algebras in the first place: Let $\mathcal{A}(\mathcal{I}_1^{\frac{1}{2}})$, $\mathcal{A}(\mathcal{I}_2^{\frac{1}{2}})$ be field-algebras, (multi)localized in the region $\mathcal{I}_1^{\frac{1}{2}} \subset S^1$ and $\mathcal{I}_2^{\frac{1}{2}} \subset S^1$, respectively. Locality with respect to the state $|\Omega_2\rangle$ is expressed by:

$$\mathcal{I}_1^{\frac{1}{2}} \cap \mathcal{I}_2^{\frac{1}{2}} = \emptyset \Rightarrow \left[\mathcal{A}(\mathcal{I}_1^{\frac{1}{2}}), \mathcal{A}(\mathcal{I}_2^{\frac{1}{2}})\right] = 0. \tag{28}$$

But, taking a closer look one finds that the new states suffer a serious problem on these multilocalized algebras. They are non-faithful states[15], i.e. there are non-zero elements of the algebra that are mapped to zero. One of these elements can be easily reconstructed:

$$\langle\Omega_2|1 - \mathcal{W}(f)\mathcal{W}(g)|\Omega_2\rangle = \langle\Omega|1 - \mathcal{W}(f_{\frac{1}{2}})\mathcal{W}(g_{\frac{1}{2}})|\Omega\rangle.$$

In this equation the right hand side is zero for $f = -g$, $supp(f) \subset \mathcal{I}_1$, $supp(g) \subset \mathcal{I}_2$ and $supp(f) = -supp(g)$! The very last condition implies that the supports of the transformed testfunctions $f_{\frac{1}{2}}$ and $g_{\frac{1}{2}}$ are *identical* on the upper semicircle. For the locality problem it is only required that the supports of transformed testfunctions have a non-vanishing overlap on the upper semicircle. As a consequence the problem of faithfulness and locality are closely related but appear not to be identical. The loss of faithfulness has serious consequences for the identification of the modified dilations (21) as the modular group of the theory, since these dilations no longer act as automorphisms on the multilocalized algebras. Therefore there is no immediate geometrical interpretation of the modular group. In this situation the first thing to try is to solve the faithfulness problem and then further study the locality properties.
One ansatz followed by Schroer *et al.* [18] is to reduce to an algebra localized in only one interval, either $\mathcal{I}_1$ respectively $\mathcal{I}_2$. On both the modified dilations act as the modular group. But which one to choose is not clear from any physical reasons. Of course, the non-local properties of the state stay the same but one

[14]As before, $\mathcal{I}_1$ and $\mathcal{I}_2$ are the image of the interval $\mathcal{I}$ under the map $z^{\frac{1}{2}}$.
[15]This was not noticed till [18].

forgets about the relation to algebras localized in different regions.
Another approach is to project out the problematic parts of the multilocalized algebras. As in general there always exists a projection E in a von Neumann algebra $\mathcal{N}$ rendering a non-faithful state faithful on the reduced von Neumann algebra $E\mathcal{N}E$. The hope is that as the locality and faithfulness problem are related but not identical one can get rid of the former and still find some interesting aspects about the non-local structure of the states. But the mathematical details of this ansatz are still work in progress and it is believed that this finally leads to the case discused by Schroer mentioned above.
It seems that restoring faithfulness always ensures the usual locality properties. Therefore we are eventually lead to the conjecture that in the described setting faithfulness ensures the proper locality structure. This is also natural from the point of view that it is widely believed that physical senseful states ought to be faithful.

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