Coordinate formalism on Hilbert manifolds

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Abstract

Infinite-dimensional manifolds modelled on arbitrary Hilbert spaces of functions are considered. It is shown that changes in model rather than changes of charts within the same model make coordinate formalisms on finite and infinite-dimensional manifolds deeply similar. In this context the infinite-dimensional counterparts of simple notions such as basis, dual basis, orthogonal basis, etc. are shown to be closely related to the choice of a model. It is also shown that in this formalism a single tensor equation on an infinite-dimensional manifold produces a family of functional equations on different spaces of functions.

1 Introduction

Methods of differential geometry have proven to be very useful in the theory of gauge fields and especially in gravity. It is then natural to assume that they must be even more important in the string theory, which aims to develop into a "Theory of Everything".

Geometrization of string fields leads to infinite-dimensional manifolds. These usually are manifolds of curves on a finite dimensional manifold. When the strings are closed, they are the well known loop spaces.

Despite a very intense investigation of infinite-dimensional manifolds and many results, up until now we do not understand clearly how to work with them. In particular, we do not know what is an appropriate generalization of the integration procedure on infinite-dimensional manifolds. Probably related to this lack of knowledge is our trouble understanding quantum theory based on the Feynman integral.

In addition we are not sure whether the manifolds that we consider a physically acceptable. In particular, if we are serious about "moving" physics to an infinite-dimensional background, the concept of ordinary space-time must be derived from it. In this respect the manifold of curves on space-time is not an appropriate concept as it needs the space-time to be defined in advance.

In such a situation any analysis bringing a new light on the notion of an infinite-dimensional manifold is important.

In the paper a coordinate formalism on infinite-dimensional Hilbert manifolds is developed and a coordinate version of tensor algebra is constructed.

A Hilbert manifold is locally diffeomorphic to a separable Hilbert space. In a most general scenario images of charts on an infinite-dimensional Hilbert manifold belong to different Hilbert spaces. This fact is often neglected as all such models are isomorphic. That is, we can assume that there is a fixed Hilbert space model. The situation is similar in the case of a finite-dimensional manifold. In the latter case images of charts belong to n-dimensional vector spaces each isomorphic to R^n. There is, however, an important difference. In applications we work with specific functional representations of the elements of a Hilbert manifold. Functions behave very differently with respect to any kind of analyzis. Therefore, a particular choice of Hilbert model may become important. For instance, the role of different Sobolev spaces is well understood in analysis of partial differential equations. In quantum theory, the right choice of represenation often permits us to simplify the problem in hand as well as to gain an important insight. As no particular Hilbert space is large enough to contain all functions useful in applications, we are lead to consider several Hilbert models at once.

From this there follows a more technical difference beween the finite and infinite-dimensional settings. By fixing a chart on a finite-dimensional manifold, we describe points by columns of numbers. When fixing a chart on a Hilbert manifold instead, we describe points by elements of a Hilbert space H. These elements usually have a functional nature. They can be represented, in turn, by columns of numbers, their components in a basis on H.

The coordinate formalism developed in the paper seems to indicate a deep connection between finite and infinite-dimensional manifolds. It is shown that the choice of a Hilbert model even more than the choice of a chart within a given model is similar to the choice of coordinates on a finite-dimensional manifold.

Here is the plan of the paper. To begin with, we analyze the notion of a string in the string theory and describe a natural structure on the set of strings. This leads to the idea of a string manifold, which is a Hilbert manifold with a Riemannian metric and with charts taking values in arbitrary Hilbert spaces of functions. We remark here that despite their name, string manifolds at this stage have very little to do with the string theory.

In section 3 the case of a linear string manifold and its conjugate is carefully analyzed. In particular, the notions of a string basis, orthogonal string basis, dual string basis, and coordinate space are introduced.

In section 4 we study linear transformation of coordinates on a linear string manifold. An interesting example of coordinate transformations is carefully analyzed. In particular, a possible meaning of the square of the Dirac δ-function is explained.

Section 5 deals with the notion of a generalized eigenvalue problem. It is shown in particular that each eigenvalue problem on a string space generates a family of eigenvalue problems on different coordinate spaces.

Section 6 generalizes the results to the case of nonlinear functional coordinate transformations on a linear string manifold. We show in particular that an arbitrary coordinate space is a complete linear metric space.

Consideration of arbitrary local diffeomorphisms makes it possible to define a string manifold structure. This is done in section 7. There we also discuss tensor bundles and representation of tensors in local coordinates.

2 What is a string?

The set of physical strings in the string theory is considered as a factor-set $\Omega M = \Sigma/D$ of the set Σ of (smooth) functions on a given interval with values in a given space-time by the group D of diffeomorphisms of the interval. In other words, a string is a smooth curve in space-time. The string fields are then functionals on strings. The elements of Σ are called parametrised strings.

As a geometric object a string is an analogue of a point of a finite-dimentional manifold. A parametrized string $x^\alpha(s)$ is a functional analogue of the set of coordinates of a point in a given coordinate system. A parametrized string will be therefore called a coordinate function of the string, or, simply, a coordinate of the string.

Reparametrization is the following transformation of the string coordinate (index α is ommited): $\widetilde{x}(s) = x(\widetilde{s}(s)) = \int \delta(\widetilde{s}(s) - t)x(t)dt$. Here $\widetilde{s} = \widetilde{s}(s)$ is a new parameter and the integral is understood as the convolution $*$, i.e. $\int \delta(\widetilde{s}(s) - t)x(t)dt = \delta(\widetilde{s}(s)) * x(s)$. The finite-dimensional analogue of reparametrization is obtained by replacing the δ-function by the Kroeneker δ-symbol, the parameters s, $\widetilde{s}$ by the natural indices i, $j = j(i)$, and the convolution by the sum. This gives $\widetilde{x}^i = \sum_l \delta_l^{j(i)} x^l$. Therefore, reparametrization reduces to a permutation of coordinates of the point.

Clearly, permutation is a very special transformation of coordinates and a much larger group of general coordinate transformations preserves points of a finite-dimensional manifold.

As strings are analogues of points of a finite-dimensional manifold, it is natural to consider them as objects invariant under a functional analogue of the group of general coordinate transformations. This motivates the idea to consider strings as points of an infinite-dimensional differentiable manifold S. That is, S is locally diffeomorphic to an infinite-dimensional Banach space $\mathbf{S}$.

To be able to measure distances we will endove S with a Riemannian structure. For this S will be assumed to be a Hilbert manifold, i.e. $\mathbf{S}$ will be a Hilbert space. As discussed in introduction, the Hilbert model will not be fixed. So the model space $\mathbf{S}$ will be an abstract Hilbert space that can be identified with an arbitrary Hilbert space of functions within a given family. This will permit, in particular, to describe strings by singular functions (distributions) rather than by only the smooth ones. The manifold S will be called a string manifold. Admissible coordinate transformations on a string manifold are then diffeomorphisms of open sets in Hilbert spaces of functions.

3 String space and its conjugate

To begin implementing this program, consider the case of a linear string manifold $\mathbf{S}$, which we shall call a string space.

Definition. A *string space* $\mathbf{S}$ is an abstract vector space that is also a differentiable manifold linearly diffeomorphic to a separable Hilbert space. In this case we also say that $\mathbf{S}$ is a *Hilbertable* space.

The fact that $\mathbf{S}$ is an abstract vector space means that it consists of the elements of an unspecified nature. A choice of string coordinate map (see section 7) on $\mathbf{S}$ will be a way of realizing $\mathbf{S}$ as a particular Hilbert space of functions.

As any two infinite-dimensional separable Hilbert spaces are isomorphic, it follows that any two string spaces are linearly diffeomorphic. Throughout the paper any such linear diffeomorphism will be called an isomorphism of string spaces. We will also say that a Hilbert and a string space are isomorphic in a sense that they are linearly diffeomorphic.

Let us now define a Hilbert space of functions.

Definition. A *Hilbert space of functions* is either a Hilbert space H, elements of which are equivalence classes of maps between two given finite dimensional manifolds or the Hilbert space H^* dual to H. Two elements $f, g \in H$ are called *equivalent* if the norm of $f - g$ in H is zero.

Consider a linear map $e_H : H \longrightarrow \mathbf{S}$ from a Hilbert space of functions H into the string space $\mathbf{S}$. The action of e_H on $\varphi \in H$ will be written in one of the following ways:

$$e_H(\varphi) = (e_H, \varphi) = \int e_H(k)\varphi(k)dk = e_{Hk}\varphi^k. \tag{3.1}$$

The integral sign is used as a notation for the action of e_H on an element of H and in general does not refer to an actual integration. We also use an obvious and convenient generalization of the Einstein's summation convention over the repeated indices k one of which is above and one below. Once again, only in special cases does this notation refer to an actual summation or integration over k.

Definition. A linear isomorphism e_H from H onto $\mathbf{S}$ will be called a *string basis* or an H-basis (or simply a basis) on $\mathbf{S}$.

It is clear that any string Φ is the image of a unique element $a \in H$, i.e.

$$\Phi = \int e_H(k)a(k)dk. \tag{3.2}$$

Also, if

$$\int e_H(k)a(k)dk = 0, \tag{3.3}$$

then $a = 0$. This justifies the definition of a basis. It is worth noticing that the basis e_H defines the coordinate space H. In fact, it acquires the meaning only as a map on H. On the other hand, within H the map e_H can be choosen in different ways.

Let $\pi_H : \mathbf{S} \longrightarrow H$ be a global linear coordinate chart on $\mathbf{S}$ (which exists as $\mathbf{S}$ is isomorphic to a Hilbert space). Then an obvious example of an H-basis is the linear isomorphism $e_H = \delta_H$, such that $\delta_H \circ \pi_H \Phi = \Phi$. That is, $\delta_H = \pi_H^{-1}$. It is natural to call this basis a coordinate basis on $\mathbf{S}$.

Let $\mathbf{S}^*$ be the dual string space. That is, $\mathbf{S}^*$ is the space of all linear continuous functionals on strings. $\mathbf{S}^*$ can be considered as a linear string manifold with the chart $(\mathbf{S}^*, \pi_{H^*})$. Here H^* is dual of H and π_{H^*} is a linear isomorpism $\mathbf{S}^*$ onto H^*.

Definition. A linear isomorphism of H^* onto $\mathbf{S}^*$ will be called a *string basis on* $\mathbf{S}^*$ (or an H^* basis on $\mathbf{S}^*$).

We will denote such a basis by e_{H^*}. Decomposition of an element $F \in \mathbf{S}^*$ with respect to the basis will be written in one of the following ways:

$$F = e_{H^*}(f) = (e_{H^*}, f) = \int e_{H^*}(k)f(k)dk = e^k_{H^*} f_k. \tag{3.4}$$

Definition. The basis e_{H^*} will be called *dual* to the basis e_H if for any string $\Phi = e_{Hk}\varphi^k$ and for any functional $F = e^k_{H^*} f_k$ the following is true:

$$F(\Phi) = f(\varphi). \tag{3.5}$$

In general we have

$$F(\Phi) = e_{H^*} f(e_H \varphi) = e^*_H e_{H^*} f(\varphi), \tag{3.6}$$

where $e^*_H : \mathbf{S}^* \longrightarrow H^*$ is the adjoint of e_H. Therefore, e_{H^*} is the dual string basis if $e^*_H e_{H^*} : H^* \longrightarrow H^*$ is the identity operator. In this case we will also write

$$e_{Hl}{}^* e^k_{H^*} = \delta^k_l. \tag{3.7}$$

In special cases δ^k_l is the usual Kroeneker symbol or the δ- function.

Notice that in general the action of F on Φ can be written in the following way:

$$F(\Phi) = e^k_{H^*} f_k e_{Hl} \varphi^l = G(f, \varphi) = g^k_l f_k \varphi^l, \tag{3.8}$$

where G is a non-degenerate bilinear functional on $H^* \times H$.

The dual basis always exists. In fact, the bilinear functional G generates a linear isomorphism $\widehat{G} : H^* \longrightarrow H^*$ by $(\widehat{G}f, \cdot) = G(f, \cdot)$. Therefore $\widehat{G}f = \widetilde{f}$ can be considered as a new H^*-coordinate of the dual string F. This is the coordinate in the dual basis $\widetilde{e}_{H^*}$.

We have:

$$F = (\widetilde{e}_{H^*}, \widetilde{f}) = (\widetilde{e}_{H^*}, \widehat{G}f) = (\widehat{G}^*\widetilde{e}_{H^*}, f) = (e_{H^*}, f). \tag{3.9}$$

Therefore we obtain

$$\widetilde{e}_{H^*} = \left(\widehat{G}^*\right)^{-1} e_{H^*}. \tag{3.10}$$

By definition the string space $\mathbf{S}$ is linearly isomorphic to a separable Hilbert space H of functions. The latter will be called an H-coordinate space. Any linear isomorphism $\pi_H : \mathbf{S} \longrightarrow H$ induces the Hilbert structure on $\mathbf{S}$ itself. In fact, linear structures on $\mathbf{S}$ and H are the same. Also, let $\Phi, \Psi \in \mathbf{S}$, $\varphi, \psi \in H$ and $\varphi = \pi_H \Phi$, $\psi = \pi_H \Psi$. Then define the inner product $(,)_S$ on $\mathbf{S}$ by $(\Phi, \Psi)_S = (\varphi, \psi)_H$, where $(,)_H$ is the inner product on H. It is clear that with this inner product $\mathbf{S}$ is a Hilbert space and π_H becomes an isomorphism of Hilbert spaces.

Let us assume that H is a real Hilbert space. We have:

$$(\Phi, \Psi)_S = \mathbf{G}(\Phi, \Psi) = G(\varphi, \psi) = g_{kl}\varphi^k\psi^l. \tag{3.11}$$

Here $G : H \times H \longrightarrow R$ is a bilinear form defining the inner product on H and $\mathbf{G} : \mathbf{S} \times \mathbf{S} \longrightarrow R$ is the induced bilinear form. The expression on the right is a convenient form of writing the action of G on $H \times H$. Notice also that G is the representation of $\mathbf{G}$ in the basis e_H.

The bilinear form $\mathbf{G} : \mathbf{S} \times \mathbf{S} \longrightarrow R$ generates a linear isomorphism $\widehat{\mathbf{G}} : \mathbf{S} \longrightarrow \mathbf{S}^*$ by $\mathbf{G}(\Phi, \Psi) = (\widehat{\mathbf{G}}\Phi, \Psi)$.

Definition. A string basis e_H in $\mathbf{S}$ will be called *orthonormal* if $\widehat{\mathbf{G}}e_{Hk} = e^k_{H^*}$, where $e^k_{H^*}$ is the dual of e_{Hk}.

In other words it means that coordinates f_k of the functional $F(\cdot) = (\Phi, \cdot)_S \in \mathbf{S}^*$ in the dual basis satisfy the equation

$$f_k = \varphi^k. \tag{3.12}$$

To be more precise, in any basis e_H we have

$$(\Phi, \Psi)_S = \widehat{\mathbf{G}}(e_H\varphi, e_H\psi) = e^*_H\widehat{\mathbf{G}}e_H\varphi(\psi), \tag{3.13}$$

where $e^*_H\widehat{\mathbf{G}}e_H$ maps H onto H^*. If e_H is orthonormal, then $e^*_H\widehat{\mathbf{G}}e_H\varphi = \varphi$. In this case we will also write

$$(\Phi, \Psi)_S = \varphi^k\psi^k = \delta_{kl}\varphi^k\psi^l. \tag{3.14}$$

In a special case δ_{kl} can be the Kroeneker symbol or Dirac's δ- function $\delta(k-l)$.

It is important to realize that not every coordinate Hilbert space H can produce an orthonormal string basis e_H. Assume, for example, that H contains the δ-function as a coordinate φ of a string $\Phi \in \mathbf{S}$ (example of such H is given below). Then $\delta_{kl}\varphi^k\varphi^l$ is not defined, that is, δ-function is not a coordinate function of a string in orthonormal basis.

This does not contradict the well known existence of an orthonormal basis in any separable Hilbert space. In fact, the meaning of a string basis is quite different from the meaning of a classical basis on a Hilbert space. Namely, a string basis on $\mathbf{S}$ permits us to represent an invariant with respect to functional transformations object (string) in terms of a function, which is an element of a Hilbert space. A basis on a Hilbert space in turn permits us to represent this function in terms of numbers, components of the function in the basis.

The orthonormality of a string basis imposes a symmetry between coordinates of the dual objects in the basis (see 3.11). In particular, if e_H is orthonormal, then H must be an L_2-space (i.e. a space $L_2(D, d\mu)$ of square integrable functions on a measurable set D with respect to a measure $d\mu$). Thus, Hilbert spaces l_2 and $L_2(R)$ are examples of coordinate spaces that admit an orthonormal string basis.

Theorem. The choice of a coordinate Hilbert space determines the corresponding string basis up to a unitary transformation.

Proof. Let e_H and $\widetilde{e}_H$ be two string bases on $\mathbf{S}$ with the same coordinate space H. Then $(\Phi, \Psi)_S = G(\varphi, \psi) = \widetilde{G}(\widetilde{\varphi}, \widetilde{\psi}) = \widetilde{G}(U\varphi, U\psi)$. As $\widetilde{G} = G$ we have $G(\varphi, \psi) = G(U\varphi, U\psi)$, that is, U is a unitary transformation. Therefore $e_H = \widetilde{e}_H U$, i.e. the basis e_H is determined up to a unitary transformation on H.

Notice that formula (3.14) suggests that coordinate Hilbert spaces that possess an orhonormal basis are analogues of the rectangular Cartesian coordinate systems in Euclidean space. More general formula (3.11) shows that other coordinate Hilbert spaces are analogues of oblique Cartesian coordinate systems in Euclidean space.

4 Transformation of coordinates on S

Consider a change of coordinates on $\mathbf{S}$, i.e. an isomorphism $\omega : \widetilde{H} \longrightarrow H$ of Hilbert spaces which defines a new string basis $e_{\widetilde{H}} : \widetilde{H} \longrightarrow \mathbf{S}$ by $e_{\widetilde{H}} = e_H \circ \omega$. Let φ be coordinate of a string Φ in the basis e_H and $\widetilde{\varphi}$ its coordinate in the basis $e_{\widetilde{H}}$. Then $\Phi = e_H\varphi = e_{\widetilde{H}}\widetilde{\varphi} = e_H\omega\widetilde{\varphi}$. That is, $\varphi = \omega\widetilde{\varphi}$ by the uniqueness of the decomposition. This provides a transformation law of string coordinates under a change of coordinates.

For the metric we have:

$$(\Phi, \Psi)_S = (\widehat{G}\varphi, \psi) = (\widehat{\widetilde{G}}\omega\widetilde{\varphi}, \omega\widetilde{\psi}), \tag{4.1}$$

where $\widehat{G} : H \longrightarrow H^*$ and $\widehat{\widetilde{G}} : \widetilde{H} \longrightarrow \widetilde{H}^*$ are operators defining inner products on H and $\widetilde{H}$. Then

$$\widehat{G} = \omega^*\widehat{\widetilde{G}}\omega, \tag{4.2}$$

where ω^* is the adjoint of ω. This is the transformation law of the metric under a change of coordinates.

By fixing $\widetilde{H}$ to be, say $L_2(R)$ we see that coordinate representation of the metric in any coordinate space can be obtained from the L_2-metric by means of isomorphism ω.

Notice also, that $\widehat{G}^{-1}$ defines a metric on H^*. In fact, if $f, g \in H^*$, $f = \widehat{G}\varphi$, $g = \widehat{G}\psi$, we can define

$$(f, g)_{H^*} = (\widehat{G}^{-1}f, g) = (\psi, \varphi)_H, \tag{4.3}$$

which gives a metric on H^*.

Example. Let us see the string coordinate transformations in action. Let W be the Schwartz space of infinitely differentiable rapidly decreasing functions on R. That is, functions $\varphi \in W$ satisfy inequalities of the form $\left|x^k\varphi^{(n)}(x)\right| \leq C_{kn}$ for some constants C_{kn} and any $k, n = 0, 1, 2, ...$ Let W^* be the dual space of continuous linear functionals on W. Let us find a Hilbert space which contains W^* as a topological subspace. For this consider a linear transformation ρ from W^* into W given by $(\rho f)(x) = \int f(y)e^{-(x-y)^2-x^2}dy$ for any $f \in W^*$.

Theorem. $\rho(W^*) \subset W$ and ρ is injective.

Proof. It is known (see [1]) that every functional $f \in W^*$ acts as follows:

$$(f, \varphi) = \int F(x)\varphi^{(m)}(x)dx, \tag{4.4}$$

where F is a continuous function on R of power growth and $\varphi^{(m)}(x)$ is the derivative of order m of the function $\varphi(x) \in W$. Therefore,

$$\int f(y)e^{-(x-y)^2-x^2}dy = \int F(y)\frac{d^m}{dy^m}e^{-(x-y)^2-x^2}dy = \int F(x-z)(-1)^{m+1}\frac{d^m}{dz^m}e^{-z^2-x^2}dz \qquad (4.5)$$

where $z = x - y$. As F is continuous of power growth, integration gives an element of W.

To check injectivity of ρ assume $\rho f = 0$. Then

$$\int F(y)\frac{d^m}{dy^m}e^{-(x-y)^2}dy = 0. \qquad (4.6)$$

Differentiating an arbitrary number k of times under the integral sign and changing to $z = x - y$ we have:

$$\int F(x-z)\frac{d^{m+k}}{dz^{m+k}}e^{-z^2}dz = 0. \qquad (4.7)$$

Let us use the fact that

$$\frac{d^{m+k}}{dz^{m+k}}e^{-\frac{z^2}{2}} = H_{m+k}(z)(-1)^{m+k}e^{-\frac{z^2}{2}}, \qquad (4.8)$$

where $H_{m+k}(z)$ are Hermite polynomials. It follows that

$$\int F(x-z)e^{-\frac{z^2}{2}}\varphi_{m+k}(z)dz = 0, \qquad (4.9)$$

where $\varphi_{m+k}(z) = H_{m+k}(z)e^{-\frac{z^2}{2}}$ is a complete orthonormal system of functions in $L_2(R)$. Therefore, if $i > m$ all Fourier coefficients $c_i(x)$ of the function $F(x-z)e^{-\frac{z^2}{2}} \in L_2(R)$ are equal to zero. That is, almost everywhere, and by continuity of F everywhere, we have

$$F(x-z)e^{-\frac{z^2}{2}} = \sum_{i=0}^{m-1} c_i(x)H_i(z)e^{-\frac{z^2}{2}}. \qquad (4.10)$$

Therefore F is a polynomial function of z of degree $m-1$. As $x - z = y$ it follows that F is a polynomial function of y as well. Thus,

$$(f,\varphi) = \int P_{m-1}(y)\varphi^{(m)}(y)dy, \qquad (4.11)$$

where P_{m-1} is a polynomial function of degree $m-1$. Integration by parts m times gives then $f = 0$ proving injectivity of ρ.

Assume the strong topology on W^*. We could, however, choose the weak topology as well as the weak and strong topologies on W^* are equivalent [1]. Let us define a topology on the linear space $\widetilde{W}$=$\rho(W^*)$ by declaring open sets on $\widetilde{W}$ to be images of open sets on W^* under the action of ρ. As ρ is a bijection, this indeed defines a topology on $\widetilde{W}$. In this topology ρ and ρ^{-1} are continuous.

Theorem. The embedding $\widetilde{W} \subset W$ is continuous.

Proof. Topology on W may be defined by the countable system of norms

$$\|\varphi\|_p = \sup_{x\in R;k,q\leq p}\left|x^k\varphi^{(q)}(x)\right|, \qquad (4.12)$$

where k, q, p are nonnegative integers and $\varphi^{(q)}$ is the derivative of φ of order q. The strong topology on W^* is defined by taking as neighborhoods of zero the sets of functionals $f \in W^*$ for which

$$\sup_{\varphi \in B} |(f, \varphi)| < \epsilon. \tag{4.13}$$

Here $B \subset W$ is any bounded set (that is, a set bounded with respect to each norm $\|\cdot\|_p$) and $\epsilon > 0$ is any number.

To prove that the embedding $\widetilde{W} \subset W$ is continuous, we need to show that for any neighborhood O of zero in W a neighborhood $\widetilde{O}$ of zero in $\widetilde{W}$ can be found such that $\widetilde{O} \subset O$. Any neighborhood of zero $\widetilde{O}$ is given by taking the image under ρ of a neighborhood O^* of zero in W^*. By (4.13), any neighborhood O^* is given as the set of functionals $f \in W^*$ such that $\sup_{\varphi \in B} |(f, \varphi)| < \epsilon$ for some bounded $B \subset W$ and some $\epsilon > 0$.

Let then O be given by $\sup_{x \in R; k,q \leq p} \left| x^k \varphi^{(q)}(x) \right| < K$ for some p and K. Consider O^* defined by $\sup_{\varphi \in B} |(f, \varphi)| < \epsilon$. Let us show that for some choice of B and ϵ the neighborhood $\widetilde{O} = \rho(O^*)$ is a subset of O.

For this, let

$$\psi(x) = \int f(y) e^{-(x-y)^2 - x^2} dy \in \widetilde{O}, \tag{4.14}$$

where $f \in O^*$, i.e. $\sup_{\varphi \in B} |(f, \varphi)| < \epsilon$. Then

$$\|\psi\|_p = \sup_{x \in R; k,q \leq p} \left| x^k \psi^{(q)}(x) \right| = \sup_{x \in R; k,q \leq p} \left| \int f(y) x^k \left(e^{-(x-y)^2 - x^2} \right)^{(q)} dy \right|. \tag{4.15}$$

Consider $\varphi_x(y) = x^k \left(e^{-(x-y)^2 - x^2} \right)^{(q)}$. For every x the function $\varphi_x(y)$ is in W. The set A of functions φ_x parametrised by x is bounded in W with respect to each norm $\|\cdot\|_l$ that define W. That is, A is bounded in W. Let then take $B = A$ and $\epsilon = K$. Then $\|\psi\|_p < K$ and so $\psi \in O$. Therefore $\widetilde{O} \subset O$, which proves the theorem.

Theorem. The embedding $W \subset W^*$ is continuous.

Proof. We assume here that every $\psi \in W$ is identified with the regular functional $f_\psi \in W^*$ defined by $(f_\psi, \varphi) = \int \psi(x) \varphi(x) dx$.

If O is a neighborhood of zero in W^*, then $\sup_{\varphi \in B} |(f, \varphi)| < \epsilon$ for some bounded B and some ϵ. Consider the neighborhood $\widetilde{O} \subset W$ of functions ψ given by $\sup_x |\psi(x)| < K$. Then

$$\sup_{\varphi \in B} |(f_\psi, \varphi)| < K \sup_{\varphi \in B} \int |\varphi(x)| \, dx = K \cdot L_B \tag{4.16}$$

for some constant L_B. Here we used the fact that any set bounded in W is bounded in $L_1(R)$. Therefore taking $K = \frac{\epsilon}{L_B}$ we obtain the desired inclusion $\widetilde{O} \subset O$.

Let us now narrow down the domain of the operator ρ to the subspace $L_2^*(R)$ of W^*.

Theorem. The embedding $L_2^*(R) \subset W^*$ is continuous.

Proof. Any neighborhood $O \subset W^*$ is given by $\sup_{\varphi \in B} |(f, \varphi)| < \epsilon$ for some bounded set $B \subset W \subset L_2(R)$ and some $\epsilon > 0$. Also, any set bounded in W is bounded in $L_2(R)$. This is so because the embedding $W \subset L_2(R)$ is continuous. Let then A be a ball in $L_2(R)$ that contains B. Consider the neighborhood $\widetilde{O}$ in $L_2^*(R)$ given by $\sup_{\varphi \in A} |(f, \varphi)| < \epsilon$. Then as $B \subset A$ we also have $\sup_{\varphi \in B} |(f, \varphi)| < \epsilon$. We therefore obtain the desired inclusion $\widetilde{O} \subset O$.

In particular, the restriction $\rho|_{L_2^*}$ of ρ on $L_2^*(R)$ is a continuous operator $\rho : L_2^*(R) \longrightarrow W$ which we will denote again by ρ.

Let $H = \rho L_2^*(R)$. Then ρ induces the Hilbert structure on H.

Theorem. With this Hilbert structure the embedding $H \subset W$ is continuous.

Proof. It is clear that $H \subset W$ as a set. As W is a first-countable space, it is enough to check that if $\varphi_n \in H$ and $\varphi_n \longrightarrow 0$ in H, then $\varphi_n \longrightarrow 0$ in W. Now, $\varphi_n \longrightarrow 0$ in H means $\varphi_n = \rho(f_n)$, where $f_n \in L_2^*(R)$ and $f_n \longrightarrow 0$ in $L_2^*(R)$. But then as $L_2^*(R) \subset W^*$ is a continuous embedding, $f_n \longrightarrow 0$ in W^*. Also, we have seen that $\rho(W^*) = \widetilde{W}$ is a topological subspace of W. Therefore $\varphi_n = \rho(f_n) \longrightarrow 0$ in W.

Theorem. The embedding $W^* \subset H^*$ is continuous.

Proof. We assume that topology on H^* is strong. It is clear that $W^* \subset H^*$ as the set because any functional continuous on W is continuous on $H \subset W$. Any neighborhood O of zero in H^* is a set of functionals $f \in H^*$ such that $\|f\|_{H^*} = \sup_{\psi \in B} |(f, \psi)| < \epsilon$, where B is a unit ball in H. As the embedding $H \subset W$ is continuous, B is bounded in W as well. Consider the neighborhood $\widetilde{O}$ in W^* given by $\sup_{\psi \in B} |(f, \psi)| < \epsilon$. Clearly then $\widetilde{O} \subset O$.

As a result, we have a chain of topological embeddings:

$$H \subset W \subset L_2(R) \subset L_2^*(R) \subset W^* \subset H^*. \tag{4.17}$$

Here we identify $L_2(R)$ and $L_2^*(R)$ by identifying in the usual way each $L_2(R)$-function ψ with the corresponding regular functional $(\psi, \cdot)_H$.

The meaning of this chain will be discussed in the conclusion. For now let us use the fact that H^* includes all functionals from W^*. Consider the Hilbert space H^* as a coordinate space for the string space $\mathbf{S}$. Let $\Phi \in \mathbf{S}$ be a string with coordinate function equal to the δ- function. Since H^* is Hilbert, the square of the δ- function must be finite. Indeed, using (4.3) with $\rho = \omega^{-1}$ we have:

$$(\Phi, \Phi)_S = (\delta, \delta)_{H^*} = (\rho\rho^*\delta, \delta) = \int e^{-(x-y)^2-x^2} e^{-(y-z)^2-z^2} \delta(x)\delta(z)dydxdz = \int e^{-4y^2}dy = \frac{\sqrt{\pi}}{2}. \tag{4.18}$$

The fact the norm of the δ- function in H^* is finite is of course related to the fact that the metric $g_{xz} = \int e^{-(x-y)^2-x^2} e^{-(y-z)^2-z^2} dy$ is a smooth function of x and z and is capable of "compensating" singularities of the product of two δ- functions. In the case of L_2 spaces the metric g_{xz} is equal to $\delta(x - z)$. Therefore δ-function can not be a coordinate of a string and the norm of it is not defined.

Assume instead, that a coordinate space consists only of the smooth functions. That is, it is obtained by some smoothing of the elements of L_2 (as the space H in the example). Then the metric in the new coordinates is obtained by a singularization of the $\delta(x - z)$ metric on L_2.

5 Generalized eigenvalue problem

Let A be a linear operator on a linear topological space V. If V is finite-dimensional and A is, say, Hermitian, then a basis in V exists, such that each vector of it is an eigenfunction of A. If V is infinite-dimensional, this statement is no longer true. Yet quite often there exists a basis of "generalized eigenfunctions" of A in the sense of the definition below (see [2]):

Definition. A linear functional F on V, such that

$$F(A\Phi) = \lambda F(\Phi) \tag{5.1}$$

for every $\Phi \in V$, is called a *generalized eigenfunction of A corresponding to the eigenvalue λ.*

Assume now that V is the string space $\mathbf{S}$ and e_H is an H- basis on $\mathbf{S}$. Assume F is a generalized eigenfunction of a linear operator $\mathbf{A}$ on $\mathbf{S}$. Then

$$F(\mathbf{A}e_H\varphi) = \lambda F(e_H\varphi), \tag{5.2}$$

where $e_H\varphi = \Phi$. Therefore

$$e_H^* F(e_H^{-1}\mathbf{A}e_H\varphi) = \lambda e_H^* F(\varphi). \tag{5.3}$$

Here $e_H^{-1}\mathbf{A}e_H$ is the representation of $\mathbf{A}$ in the basis e_H.

By defining $e_H^* F = f$ we have

$$f(e_H^{-1}\mathbf{A}e_H\varphi) = \lambda f(\varphi) \tag{5.4}$$

Notice that the last equation desribes not just one eigenvalue problem, but a family of such problems, one for each string basis e_H. As we change e_H, the operator $A = e_H^{-1}\mathbf{A}e_H$ in general changes as well, as do the eigenfuctions f.

Example. Let W be the Schwartz space. Although it is not possible to introduce a Hilbert metric on W, this will not play any role in what follows. If necessary, one could replace W with the Hilbert space $H \subset W$ from the previous example. Consider the operator of differentiation $A = i\frac{d}{dx}$ on W. The generalized eigenvalue problem for A is

$$f(i\frac{d}{dx}\varphi) = \lambda f(\varphi), \tag{5.5}$$

where $f \in W^*$. The functionals

$$f(x) = e^{-i\lambda x} \tag{5.6}$$

are the eigenvectors of A. Let us now consider a coordinate change $\rho : W \longrightarrow W$ given by the Fourier transform. That is,

$$\psi(k) = (\rho\varphi)(k) = \int \varphi(x)e^{ikx}dx. \tag{5.7}$$

The Fourier transform is a toplinear automorphism of W. The inverse transform is given by

$$(\omega\psi)(x) = \frac{1}{2\pi}\int \psi(k)e^{-ikx}dk. \tag{5.8}$$

According to (5.4) the eigenvalue problem in new coordinates is

$$\omega^* f(\rho A\omega\psi) = \lambda\omega^* f(\psi). \tag{5.9}$$

We have

$$A\omega\psi = i\frac{d}{dx}\frac{1}{2\pi}\int \psi(k)e^{-ikx}dk = \frac{1}{2\pi}\int k\psi(k)e^{-ikx}dk. \tag{5.10}$$

Therefore,

$$(\rho A\omega\psi)(k) = k\psi(k). \tag{5.11}$$

So, the eigenvalue problem in new coordinates is as follows:

$$g(k\psi) = \lambda g(\psi). \tag{5.12}$$

Thus, we have the eigenvalue problem for the operator of multiplication by the variable. The eigenvectors here are given by

$$g(k) = \delta(k-\lambda). \tag{5.13}$$

Notice that $g = \omega^* f$ is as it should be. Indeed,

$$(\omega^* f)(k) = \frac{1}{2\pi} \int f(x) e^{ikx} dx = \frac{1}{2\pi} \int e^{-i\lambda x} e^{ikx} dx = \delta(k - \lambda). \tag{5.14}$$

As a result, the eigenvalue problems (5.5), and (5.12) are two coordinate expressions of a single eigenvalue problem

$$F(\mathbf{A}\Phi) = \lambda F(\Phi) \tag{5.15}$$

for an operator $\mathbf{A}$ on $\mathbf{S}$.

Notice that in such an invariant approach we are not free to define A on a specific Hilbert space H. In fact, the choice of e_H dictates not just H, but the specific coordinate representation A of $\mathbf{A}$. On the other hand, if the space H and an operator A on it are given, then by changing the basis we can find the unique representation of $\mathbf{A}$ in any basis.

This seems to be very similar to the usual change of matrix representation of a linear operator acting on a finite-dimensional vector space. There is, however, an important difference. As in the finite-dimensional case a linear operator acting on a Hilbert space H can be represented by an (infinite) matrix. For this we simply choose a basis on H and find the matrix representation of A in this basis. What we are doing here is different. Whenever H is the space of functions, we consider these functions as coordinates of invariant elements of the string space $\mathbf{S}$. This passage to $\mathbf{S}$ permits us to consider all possible functional Hilbert spaces at once as images of different coordinate charts on $\mathbf{S}$. This is very useful for two reasons. First of all, it relates objects of different nature by asserting that they are coordinate representations of one and the same invariant object on the string space. Second, a particular choice of a coordinate chart can simplify significantly the problem in hand. We also see that the particular properties of functions in H are no longer important as we change these properties by changing coordinates.

6 Nonlinear coordinate transformations on S

Up until now we considered coordinate transformations that were isomorphisms of coordinate Hilbert spaces. To have a differentiable manifold structure on $\mathbf{S}$ we need to consider local nonlinear diffeomorphisms of Hilbert spaces as well.

Theorem. Let ρ be a bijection of a Hilbert space H onto a Hilbert space $V = \rho(H)$. Then ρ induces on V a new metric structure such that V with this structure is a complete linear metric space.

Proof. Given $f = \rho(\varphi)$ and $g = \rho(\psi)$ we define the distance from f to g by

$$d_V(f, g) = \|\varphi - \psi\|_H . \tag{6.1}$$

We clearly have

$$\begin{aligned} d_V(f,f) &= 0,\ d_V(f,g) > 0\ for f \neq g && (6.2)\\ d_V(f,g) &= d_V(g,f) && (6.3)\\ d_V(f,h) &\leq d_V(f,g) + d_V(g,h). && (6.4) \end{aligned}$$

That is, d_V is indeed a metric. When the topology on V is defined by d_V, the operator ρ becomes a diffeomorphism from H onto V.

It is also clear that V is complete in this topology. In fact, $d_V(f_n, f_m) \longrightarrow 0$ means that $\|\varphi_n - \varphi_m\|_H \longrightarrow 0$, where $f_n = \rho(\varphi_n)$. By completeness of H, $\varphi_n \longrightarrow \varphi$ in H. Therefore $f_n \longrightarrow f = \rho(\varphi)$ in V.

In general we shall consider local diffeomorphisms of Hilbert spaces. That is, $\rho : N \longrightarrow \widetilde{H}$ will be defined on a neighborhood N of a point $\varphi \in H$. In this case using (6.1) and taking $h = f - g$ and $\omega = \rho^{-1}$ we have:

$$d_V^2(f,g) = \|\omega f - \omega g\|_H^2 = (\widehat{G}_H(\omega f - \omega g), \omega f - \omega g) = (\widehat{G}_H(Lh + \alpha h), Lh + \alpha h) = (L^* G_H L h, h) + \beta h. \tag{6.5}$$

Here $\widehat{G}_H : H \longrightarrow H^*$ defines the metric on H, the linear isomorpism $L : \widetilde{H} \longrightarrow H$ is the derivative of ω, and $\alpha, \beta \longrightarrow 0$ as $\|h\|_{\widetilde{H}} \longrightarrow 0$.Thus, given the string bases e_H and $e_{\widetilde{H}}$,

$$\widehat{G}_{\widetilde{H}} = L^* \widehat{G}_H L \tag{6.6}$$

defines a Hilbert metric on the tangent space $T_g \widetilde{H}$ to $\widetilde{H}$ at the point g.

7 String manifolds

We define a string manifold S as a manifold modelled on $\mathbf{S}$ and furnished with a coordinate structure (see below) and a Riemannian metric. Before introducing a notion of coordiante structure let us recall the definition of a differential manifold that works in infinte-dimensional case. See [3] for details.

Definition. Let S be a set. Let U_α (with α changing in some indexing set) be a collection of subsets of S that covers S. For each α let π_α be a bijection of U_α onto an open subset $\pi_\alpha(U_\alpha)$ of $\mathbf{S}$. Assume that for any $\alpha, \beta, \pi_\alpha(U_\alpha \cap U_\beta)$ is open in $\mathbf{S}$. Assume also that for each pair of indices α, β the map $\pi_\beta \pi_\alpha^{-1} : \pi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \pi_\beta(U_\alpha \cap U_\beta)$ is a C^∞-isomorphism. Then the collection of pairs (U_α, π_α) is called a C^∞ *atlas* on S.

Notice that by taking U_α to be open, we give S a topology. In this topology π_α are homeomorphisms.

Definition. Each pair (U_α, π_α) is called a *chart* of the atlas.

Definition. Given an open set $U \subset S$ and a homeomorphism $\pi : U \longrightarrow \mathbf{U}$ onto an open set of $\mathbf{S}$ we say that (U, π) is *compatible with the atlas* (U_α, π_α), if each map $\pi_\alpha \pi^{-1}$, whenever defined, is a diffeomorphism.

Definition. Two atlaces are *compatible* if each chart of one is compatible with the other atlas.

Compatibility is an equivalence relation between the atlaces.

Definition. An equivalence class of C^∞-atlaces with respect to compatibility relation is said to define a *structure of an abstract C^∞- Hilbert manifold.*

So locally S looks like $\mathbf{S}$. Let us now introduce a coordinate formalism on S.

Definition. Let (U_α, π_α) be an atlas on S. Consider a collection of quadruples $(U_\alpha, \pi_\alpha, \omega_\alpha, H_\alpha)$, where each H_α is a Hilbert space of functions and ω_α is an isomorphism of $\mathbf{S}$ onto H_α. Such a collection will be called a *functional atlas* on S. A collection of all compatible functional atlaces on S will be called a *coordinate structure* on S.

That is, S with a coordinate structure can be thought of as a Hilbert manifold S with differentiable structure defined by the atlaces $(U_\alpha, \omega_\alpha \circ \pi_\alpha)$. We prefer, however, to distinguish between an abstract Hilbert manifold and a Hilbert manifold with a coordinate structure.

Let (U_α, π_α) be a chart on S. For each $p \in U_\alpha$, $\pi_\alpha(p) \in \mathbf{S}$. Usually one introduces the i-th coordinate function p^i on a Hilbert manifold by choosing a basis $\{e_i\}$ on $\mathbf{S}$ and taking the i-th component of $\pi_\alpha(p)$ in the basis. Thus, we obtain coordinate functions by identifying $\mathbf{S}$ with the

Hilbert space l_2 of sequences. The coordinate structure in the definition above is more general in that we are able to identify **S** with any Hilbert space of functions and not only l_2. In this case, if $p \in U_\alpha$, then $\omega_\alpha \circ \pi_\alpha(p)$ will be called the coordinate map or simply the coordinate of p. The isomorphisms $\omega_\beta \circ \pi_\beta \circ (\omega_\alpha \circ \pi_\alpha)^{-1} : \omega_\alpha \circ \pi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \omega_\beta \circ \pi_\beta(U_\alpha \cap U_\beta)$ are called coordinate transformations on S.

As S is a differentiable manifold we can introduce the tangent bundle structure $\tau : TS \longrightarrow S$ and the bundle $\tau_s^r : T_s^r S \longrightarrow S$ of tensors of rank (r, s). Consider in particular the bundle $\tau_2 : ST_2S \longrightarrow S$ of symmetric $(0, 2)$ tensors.

Definition. A Riemannian metric on S is a section $g : S \longrightarrow ST_2S$ of $\tau_2 : ST_2S \longrightarrow S$, such that g_p is positive definite for every $p \in S$, i.e. $g_p(\Phi, \Phi) > 0$ for every $p \in S$ and any $\Phi \in T_pS$. Here T_pS is the tangent space to S at p.

Definition. A Hilbert manifold with coordinate structure and Riemannian metric is called a string manifold.

Coordinate structure on a string manifold permits one to obtain a functional description of any tensor. Namely, let $\mathbf{G}_p(F_1, ..., F_r, \Phi_1, ..., \Phi_s)$ be an (r, s)-tensor on S .

Definition. The coordinate map $\omega_\alpha \circ \pi_\alpha : U_\alpha \longrightarrow H_\alpha$ for each $p \in U_\alpha$ yelds the linear map of tangent spaces $d\rho_\alpha : T_{\omega_\alpha \circ \pi_\alpha(p)} H_\alpha \longrightarrow T_pS$, where $\rho_\alpha = \pi_\alpha^{-1} \circ \omega_\alpha^{-1}$. This map is called a local coordinate string basis on S.

Let $e_{H_\alpha} = e_{H_\alpha}(p)$ be such a basis and $e_{H_\alpha^*} = e_{H_\alpha^*}(p)$ be the corresponding dual basis. Notice that for each p the map e_{H_α} is a string basis as defined in section 2. Therefore, the local dual basis is defined for each p as before and is a function of p.

We now have $F_i = e_{H_\alpha^*} f_i$, and $\Phi_j = e_{H_\alpha} \varphi_j$ for any $F_i \in T_p^*S$, $\Phi_j \in T_pS$ and some $f_i \in H_\alpha^*, \varphi_j \in H_\alpha$. Therefore $\mathbf{G}_p(F_1, ..., F_r, \Phi_1, ..., \Phi_s) = G_p(f_1, ..., f_r, \varphi_1, ..., \varphi_s)$ defining component functions of the (r, s)-tensor $\mathbf{G}_p$ in the local coordinate basis e_{H_α}.

8 Concluding remarks

The notion of an infinite-dimensional manifold is a direct generalization of its finite dimensional counterpart. Yet many techniques available to us in a finite dimensional setting can not be easily generalized to the case of infinite dimensions.

Perhaps this is so because we are trying to generalize in a wrong way. To explain, consider for example a passage from the space of n-columns R^n to the Hilbert space of sequences l_2. A significant difference between these two spaces is the notion of convergence needed to identify a sequence of numbers as an element of l_2. This notion is exactly the new entity related to infinite dimensionality of l_2. In setting up a differentiable structure we do not pay that much attention to the kind of convergence available on the model space l_2. Even when, say, several Hilbert spaces of sequences are simultaneously considered as models of a Hilbert manifold, differentiable structure is not affected by the difference between them. It is concerned only with the differentiability of maps.

It is advocated here that a much more productive approach to infinite-dimensional manifolds is to use the difference between Hilbert models even if the manifold structure itself is not altered by changing a model.

A particular choice of a model is significant in applications where properties of functional objects depend on the type of space of functions used. By considering several models at once we can reduce the seemingly unrelated problems on different spaces of functions to equivalence classes of problems on the string space. The generalized eigenvalue problem considered in section 5 provides an example.

We also obtain the possibility to reformulate a problem given on one space in terms of another space. A good example of the usefulness of such a reformulation is provided by the theory of generalized functions (distributions). In this theory operations that could not be defined by themselves on spaces containing singular distributions are defined first on fundamental spaces of "good" functions. Then they are "transplanted" to the much larger dual spaces.

In the approach advocated here this passage from a space to its dual is a coordinate transformation and the operations themselves could be defined in invariant manner on the string space.

Notice also, that a coordinate transformation on $\mathbf{S}$ can alter analytic properties of elements of the coordinate space by changing, for example, singular distributions to infinite differentiable functions and vice versa. This was explored in example of section 4.

Choosing appropriate coordinates on $\mathbf{S}$ for a problem in hand is as useful (if not more) as choosing canonical coordinates for a finite dimensional problem. For example, Fourier and Laplace transforms reduce differentiation of functions to multiplication by a variable providing an algebraic approach to solving differential equations.

We also see that in this approach the finite and infinite-dimensional manifolds become related in a new way. The notions of a string basis, dual string basis, orthogonal string basis, local coordinate basis are the clear analogues of their finite dimensional counterparts. Simultaneously they provide us with the power of changing the coordinate spaces and the corresponding functional description of invariant objects (tensors).

The main spaces of functions used in the paper are Hilbert spaces. The role of countably normed spaces like the Schwartz space W in this setting is not completely understood. The symmetric way in which H and W appear in the chain (4.17) makes us think that such spaces should be considered as possible coordinate spaces as well. This is, however, the subject for a different paper.

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