Conformal Transformations of Space-Time as Vector Bundle Automorphisms

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Abstract

Conformal group of Minkowski space-time M is considered as a group of bundle automorphisms of a vector bundle U over M. 4-component spin-vectors (4-spinors) are sections of a subbundle of the tangent bundle over U. Isotropic 4-vectors are images of 4-spinors under projection. This leads to a particularly clear interpretation of the spin properties of Nature.

1 Introduction

According to Einstein, space-time continuum is a pseudo-Riemannian manifold. Although it is a real manifold, complex numbers have proven to be very useful in gravity and even more so in physics on space-time. This naturally led to the idea of incorporating complex numbers into the structure of space-time manifold.

The most far reaching and successful advance in this direction is the twistor theory of Penrose [1, 2]. In this theory, the (conformally compactified and complexified) Minkowski space-time is identified with the Grassmanian $G_{2,4}(C)$ of complex 2-planes in C^4.

The drawback of twistors is that they appear to be powerful in a rather restrictive class of conformally flat and half-conformally flat metrics on space-time. In addition, the idea that the usual manifold of points has to be replaced with a manifold of complex 2-dimensional surfaces has not been readily accepted by physicists. As a result, twistors are generally considered a very useful tool for dealing with a significant number of linear and non-linear differential equations relevant to physics. The essence of the tool is the twistor transform [2], which permits interpretation of equations on space-time in terms of holomorphic data on the associated twistor space.

The success of twistor theory contributes to the feeling that complex structure is something immanent to space-time and should be considered as more than just a tool for applications.

In this paper, the basic geometry of spinors and twistors is revisited and a simple geometric interpretation of conformal symmetry is proposed. The conformal group appears as a group of fibre preserving transformations of a certain vector bundle $\pi : U \to \bar{M}$ over the space-time manifold $\bar{M}$. The total space U possesses a natural complex structure.

In this light the spin properties of nature also obtain a clear geometric interpretation. Usually spin-vectors are considered as objects of algebraic nature. In fact, they are elements of the space of spin representation of Lorentz group. In Penrose's approach, spin-vectors are interpreted as isotropic flags [3]. This gives a semi-geometric representation of spin-vectors in Minkovski space-time.

In the paper 4-component spin-vectors (4-spinors) appear as vectors on the total space U of the above mentioned vector bundle. In fact, they are restricted to $\bar{M}$ sections of the tangent bundle $T(U)$ over U. Isotropic 4-vectors are images of 4-spinors under projection π.

The entire approach has been inspired by the above mentioned strong belief that the space-time manifold must be extended to a complex manifold. It was the main method used by Einstein to interpret physics on space-time as a physics of space-time. The fact that complex numbers are used extensively in describing physics favors the idea of a complex space-time manifold.

On the other hand, any analytic (pseudo-) Riemannian space-time can be isometrically embedded into a complex 4-manifold with a nice metric structure, e.g., (indefinite-) Kähler, on it (see [4] and references there). So the idea of extension of both space-time and Riemannian structure is easily fulfilled. The problem is to find a correct extension. That means, in particular, to relate the extension to conformal symmetry, which is the main underlying theme of the twistor theory. It also means to find a natural geometric representation of 4-component spin-vectors within the extension formalism.

Yet another major factor that led to the proposed interpretation was an analysis of similarities and differences between conformal group action on the spheres S^2 and S^4, and on (conformally compactified) Minkowski space-time $\bar{M} \simeq S^1 \times S^3$. We start with the bundles $\pi : C^2_* \to CP^1 \simeq S^2$ and $\pi : C^4_* \to HP^1 \simeq S^4$ that provide a clear geometric framework for conformal group action on S^2 and S^4. Later these bundles are generalized to a universal bundle construction applicable to $\bar{M}$ as well.

2 Geometry of spin

In the simplest case, a space-time manifold is a four-dimensional space M of vectors $X = (x^0, ..., x^3)$. To measure distances and angles on M an inner product of vectors is introduced: $(X, Y) = \eta_{ik}x^i y^k$. The norm $(X, X) = \|X\|^2_M$ must be independent of the choice of coordinates in space-time. This leads to Minkowski tensor η_{ik} of signature $(+, -, -, -)$ and to the group of (restricted) Lorentz transformations $SO(1, 3)$ preserving $\|X\|^2_M$. Space M with the above metric is called Minkowski space.

In order to describe electrons, it is necessary to generalize vectors to spin-vectors. For this, we identify M with the space $H(2)$ of 2×2 Hermitian matrices via the following isomorphism:

$$X = (x^0, ..., x^3) \to p = \begin{pmatrix} x^0 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 - x^3 \end{pmatrix}. \tag{2.1}$$

Clearly, $\det p = \|X\|^2_M$. If $\det p = 0$, i.e. X is a null vector, then

$$\begin{pmatrix} x^0 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 - x^3 \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \begin{pmatrix} \bar{\xi} & \bar{\eta} \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}^+ \tag{2.2}$$

for some vector $\begin{pmatrix} \xi \\ \eta \end{pmatrix} \in C^2$. Vectors $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$ are called spin-vectors or spinors. Because of the above relation, spin-vectors are also called square-roots of vectors.

If $A \in SL(2, C)$ acts on C^2 and $\begin{pmatrix} \mu \\ \nu \end{pmatrix} = A \begin{pmatrix} \xi \\ \eta \end{pmatrix}$, then we have the induced action on M : $p \to p' = ApA^+$. This action preserves $\|X\|^2_M$ as $\det p = \det ApA^+$. It is therefore an action of the Lorentz group. If, on the other hand, $L \in SO(1, 3)$, then under the above isomorphism $X \to p$, LX

corresponds to ApA^+, where $A \in SL(2,C)$ is determined up to a sign. Therefore $SL(2,C)$ is a 2-1 covering of the Lorentz group $SO(1,3)$.

It is known that the (restricted) Lorentz group is isomorphic to the group $C(2)$ of (proper) conformal transformations of the sphere S^2. That is, it preserves angles between vectors tangent to the sphere. The relation between $C(2)$ and $SL(2,C)$ has a nice interpretation in terms of the stereographic projection. On Figure 1 (x,y,z) are Cartesian coordinates of a point P on the unit sphere $S^2 \subset R^3$. Center O of S^2 is at the origin of the coordinate system. Point ξ on the complex plane C through O is the corresponding complex coordinate. We have $\xi = \frac{x+iy}{1-z}$ and $SL(2,C)$, acting via fractional linear transformations on the complex plane C, induces the conformal group action on S^2.

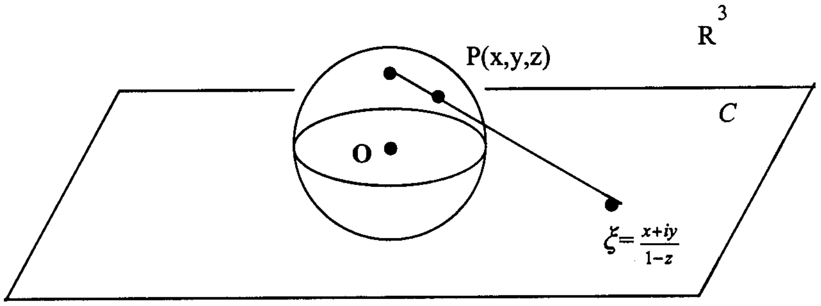

Figure 1

To clarify the geometry of this correspondence, assume that the (unit) sphere S^2 is embedded in the complex Euclidean space C^2 with coordinates (ξ, η). The north pole of S^2 is taken to be at the origin $\xi = \eta = 0$. The complex plane C is given by $\eta = 1$ (see Figure 2). Let $A \in SL(2,C)$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $\begin{bmatrix} \mu \\ \nu \end{bmatrix}$ denotes the complex line through $\begin{pmatrix} \mu \\ \nu \end{pmatrix}$ in C^2. Assume $\nu \neq 0$. Then $\begin{bmatrix} \mu \\ \nu \end{bmatrix} = \begin{bmatrix} \xi \\ 1 \end{bmatrix}$, where $\xi = \mu\nu^{-1}$ and

$$A \begin{bmatrix} \xi \\ 1 \end{bmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{bmatrix} \xi \\ 1 \end{bmatrix} = \begin{bmatrix} (a\xi + b)(c\xi + d)^{-1} \\ 1 \end{bmatrix}. \tag{2.3}$$

The case $\nu = 0, \mu \neq 0$ is treated similarly. The resulting fractional transformation $\xi \to (a\xi + b)(c\xi + d)^{-1}$ is a conformal transformation of the compactified complex plane $\bar{C}$ which is conformally equivalent to S^2.

With the above embedding of S^2 into C^2 the entire picture has almost a mechanical interpretation. On Figure 2 $SL(2,C)$ acts on C^2 and therefore on the complex lines in C^2, i.e. on CP^1. This action gets passed through the "poles" (complex lines of C^2) to the sphere S^2. Location of the sphere is kept fixed, so S^2 can only transform along its surface. The resulting transformation of S^2 is conformal. In particular, for an action of the $SU(2)$ subgroup of $SL(2,C)$ the resulting transformation is an $SO(3)$ rotation yielding the isomorphism $SO(3) \simeq SU(2)/Z_2$.

It is worth noticing that this mechanical interpretation can be put in precise terms using the language of fibre bundles. Consider the CP^1-defining fibre bundle $\pi : C^2_* \to CP^1 \simeq S^2$, where $*$ in C_* and C^2_* means "take away zero". We would like to identify the space CP^1 of complex lines in C^2 with the parametrizing unit sphere S^2 embedded in C^2 in the way described above (i.e. with the north pole of S^2 at the origin of C^2). The group $SL(2,C)$ of fibre preserving transformations of the bundle then induces conformal transformations of the base S^2. The problem with this interpretation is that S^2 can not be embedded in the total space C^2_* in the required way. For simplicity we then remove the fibre l through $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in C^2_* and the corresponding point $L = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ of CP^1. This gives

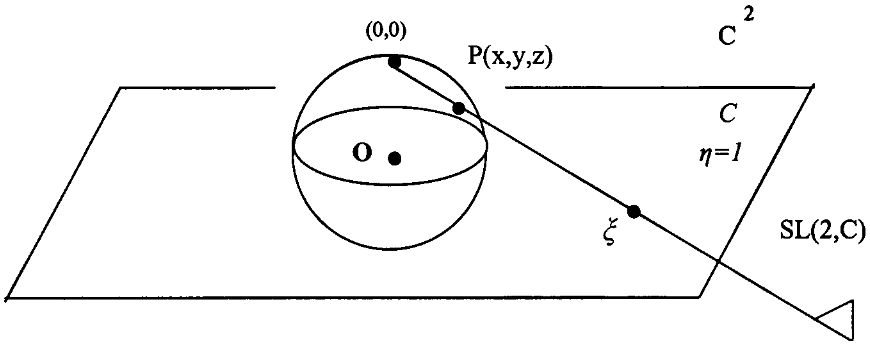

Figure 2

the subbundle $\pi : C'^2_* \to C$, where $C'^2_* = C^2_* - l$, and $C = CP^1 - L$. The base C is identified with the affine complex plane $\eta = 1$ (see Figure 2). Then the $SL(2,C)$ action on C'^2_* induces the action of conformal group $C(2)$ on $C = R^2$. The commutative diagram below summarizes the situation:

$$\begin{array}{ccc} C'^2_* & \xrightarrow{F} & C'^2_* \\ \pi \downarrow & & \downarrow \pi \\ C & \xrightarrow[f]{} & C \end{array}$$

Here $F \in SL(2,C)$ and $f \in C(2)$ is the induced action on C.

The sphere S^2 on Figure 2 is the conformal compactification of C. This means simply that conformal transformations on C (i.e. fractional linear transformations obtained above) can be extended to act on the compactification $\bar{C} \simeq S^2$.

We can similarly consider the HP^1-defining bundle $\pi : H^2_* \to HP^1 \simeq S^4$ with quaternion lines as fibres. Thus, let (p,q) be coordinates in H^2_*. By removing the quaternion line through $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in H^2_* and the corresponding point L below it in HP^1, we get the subbundle $\pi : H'^2_* \to H$. Here $HP^1 - L = H$ is identified with the affine quaternion space $q = 1$ in H^2_*. The group $SL(2,H)$ of fibre preserving transformations of the bundle induces the action of conformal group $C(4)$ on $H = R^4$. This gives the following diagram:

$$\begin{array}{ccc} H'^2_* & \xrightarrow{F} & H'^2_* \\ \pi \downarrow & & \downarrow \pi \\ H & \xrightarrow[f]{} & H \end{array}$$

On the diagram $F \in SL(2,H)$ and $f \in C(4)$ is the induced action.

The sphere S^4 itself is the conformal compactification of the Euclidean space R^4. Notice also that $HP^1 \simeq S^4$ can be considered as a totally real submanifold of the Grassmanian $G_{2,4}(C)$.

In case of a Lorentzian metric we can also embed the conformally compactified Minkowski space $\bar{M} \simeq S^1 \times S^3$ into the Grassmanian $G_{2,4}(C)$. For this we use the isomorphism $M \to H(2)$ as in (2.1). Namely, we identify each point p with the corresponding complex plane $\begin{bmatrix} p \\ I \end{bmatrix}$ through the columns of the 4×2 matrix $\begin{pmatrix} p \\ I \end{pmatrix}$. Let $SU(2,2)$ be the group of 4×4 matrices acting on C^4 and preserving the form $\Phi(Z) = 2Im(z_1\bar{z}_3 + z_2\bar{z}_4)$. The action of $SU(2,2)$ on C^4 induces the action of conformal group $C(1,3)$ on M. If $g \in SU(2,2)$, then

$$g\begin{bmatrix} p \\ I \end{bmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{bmatrix} p \\ I \end{bmatrix} = \begin{bmatrix} (Ap+B)(Cp+D)^{-1} \\ I \end{bmatrix}. \tag{2.4}$$

In particular, when $C = B = 0$, and $A \in SL(2, C)$, $p \to ApA^{+}$, i.e. we recover the group of Lorentz transformations.

The action of $C(1,3)$ can be extended to the compactified Minkowski space $\bar{M} \simeq S^1 \times S^3$, and the corresponding embedding $\bar{M} \hookrightarrow G_{2,4}(C)$ is totally real [2].

Is there any fibre bundle structure in this case similar to the ones we have in two and four dimensional Euclidean cases? The bundle structure $C^4_* \to \bar{M}$ would not be possible here as the 2-C subspaces of C^4 parametrized by $\bar{M}$ intersect nontrivially.

To get a bundle structure consider a disjoint union of the C^2 subspaces of C^4 representing elements of $G_{2,4}(C)$ and the resulting universal bundle $C^2 \to \widetilde{U} \to G_{2,4}(C)$. The embedding $\varphi : \bar{M} \to G_{2,4}(C)$ produces the pullback bundle $L \to U \to \bar{M}$. Here a typical fibre L is a "real" C^2 plane (i.e. a plane $\begin{bmatrix} p \\ I \end{bmatrix}$ or $\begin{bmatrix} I \\ p \end{bmatrix}$ for some $p \in H(2)$). The total space U is simply a "portion" of the universal bundle $\pi : \widetilde{U} \to G_{2,4}(C)$ above the submanifold $\varphi(\bar{M})$ of $G_{2,4}(C)$.

By the above, the action of $SU(2,2)$ on C^4_* and therefore on U induces the conformal group action on M and on the conformal compactification $\bar{M} \simeq S^1 \times S^3$ of M. Therefore, given an ($SU(2,2)$-generated) automorphism F of U and the induced map $f \in C(1,3)$ we have the following commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{F} & U \\ \pi \downarrow & & \downarrow \pi \\ \bar{M} & \xrightarrow[f]{} & \bar{M} \end{array}$$

Notice that unlike the earlier diagrams, the base manifold here is compact (i.e. it is conformally compactified). We could similarly "compactify" the previous diagrams by embedding the bundles $\pi : C^2_* \to S^2$ and $\pi : H^4_* \to S^4$ into the universal bundles on CP^1 and HP^1 respectively.

3 Bundle algebra

Based on the previous section we accept the following hypothesis: The (compactified) Minkowski space-time manifold is the base of a fibre bundle $L \to U \to \bar{M}$. Here $\bar{M}$ is embedded in the Grassmanian $G_{2,4}(C)$ as a totally real 4-dimensional manifold (see Section 2). The bundle itself is the subbundle of the universal bundle over $G_{2,N}(C)$. It is the pullback bundle induced by the embedding $\varphi : \bar{M} \to G_{2,4}(C)$. The typical fibre L is a real complex two-dimensional plane, i.e. a plane $\begin{bmatrix} p \\ I \end{bmatrix}$ or $\begin{bmatrix} I \\ p \end{bmatrix}$, where $p \in H(2)$. For simplicity, the non compact version M of Minkowski space will be often used. It is then identified with the manifold of real planes $\begin{bmatrix} p \\ I \end{bmatrix}$, where $p \in H(2)$. The compactification step will be always assumed.

Let us find first of all the largest subgroup G of the group of linear transformations $GL(4, C)$ of C^4 which induce fibre preserving transformations of the bundle $\pi : U \to \bar{M}$. We have

$$g\begin{bmatrix} p \\ I \end{bmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{bmatrix} p \\ I \end{bmatrix} = \begin{bmatrix} (Ap+B)(Cp+D)^{-1} \\ I \end{bmatrix}. \tag{3.1}$$

Transformation g will be fibre preserving iff

$$[(Ap+B)(Cp+D)^{-1}]^{+} = (Ap+B)(Cp+D)^{-1}. \tag{3.2}$$

In other words,

$$(pA^{+}+B^{+})(Cp+D) = (pC^{+}+D^{+})(Ap+B), \tag{3.3}$$

or,

$$pA^{+}Cp - pC^{+}Ap + pA^{+}D + B^{+}Cp - D^{+}Ap - pC^{+}B + B^{+}D - D^{+}B = 0. \tag{3.4}$$

This yields the following three equations:

$$B^{+}D = D^{+}B, \tag{3.5}$$

$$A^{+}C = C^{+}A, \tag{3.6}$$

and

$$p(A^{+}D - C^{+}B) = (D^{+}A - B^{+}C)p. \tag{3.7}$$

The last equation must be true for every p. In particular, $A^{+}D - C^{+}B = D^{+}A - B^{+}C$, i.e. $A^{+}D - C^{+}B$ is hermitian. Moreover, since the space $Mat(2\times 2, C)$ of complex 2×2 matrices has an Hermitian basis, (3.7) yields

$$A^{+}D - C^{+}B = D^{+}A - B^{+}C = rI, \tag{3.8}$$

where r must be a real number not equal to zero. It is easy to check that the set G of such matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ forms a group. If we impose an extra condition $\det g = 1$ in addition to (3.5)-(3.7), we obtain the group $SU(2,2)$ discussed earlier. As always, it is possible to satisfy $\det g = 1$ without changing $(Ap+B)(Cp+D)^{-1}$ by multiplying A, B, C, and D by an appropriate real number. Therefore, the linear group of fibre preserving transformations of the bundle $\pi : U \to \bar{M}$ can be identified with the group $SU(2,2)$.

The group $SU(2,2)$ is usually defined as the group of transformations preserving Hermitian form Φ of signature $(+,+,-,-)$ in C^4. In appropriate coordinates this form is given by $\Phi = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}$, so that $\Phi(Z) = 2Im(z_1\bar{z}_3 + z_2\bar{z}_4)$ for any vector $Z = (z_1, z_2, z_3, z_4) \in C^4$. This form defines a quadric Q_7 (of dimension 7) given by $\Phi(Z) = 0$. In canonical coordinates equation of the quadric reduces to $|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 = 0$. We claim that Q_7 is the set of points in C^4 swept by the real planes in C^4, i.e. by the fibres of $\pi : U \to \bar{M}$.

In fact, if $p = \begin{pmatrix} x & z \\ \bar{z} & y \end{pmatrix}$, where $x, y \in R$ and $z \in C$, then solving the system of equations

$$\alpha \begin{pmatrix} x \\ \bar{z} \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} z \\ y \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in C^4, \tag{3.9}$$

we have $\alpha = c$, $\beta = d$, and

$$\left\{ \begin{array}{c} cx + dz = a \\ c\bar{z} + dy = b \end{array} \right\}. \tag{3.10}$$

Determinant of the system is zero and solvability requires $Im(a\bar{c} + b\bar{d}) = 0$. Replacing (a, b, c, d) with (z_1, z_2, z_3, z_4), we have $Im(z_1\bar{z}_3 + z_2\bar{z}_4) = 0$ i.e. $\Phi(Z) = 0$. This condition is also sufficient, if we include the planes $\begin{bmatrix} I \\ p \end{bmatrix}$. As a result, points of the quadric Q_7 and only these points are reachable by the real planes $\begin{bmatrix} p \\ I \end{bmatrix}$ and $\begin{bmatrix} I \\ p \end{bmatrix}$ in C^4.

Real planes $\begin{bmatrix} p \\ I \end{bmatrix}$ (as well as the planes $\begin{bmatrix} I \\ p \end{bmatrix}$) intersect nontrivially in C^4, so that we have no uniqueness property for the system above. That is why the bundle $\pi : U \to \bar{M}$ can not be embedded in the space C^4 where the planes live. Let us impose an extra condition $xy - |z|^2 = 0$ on p. Then the corresponding system has a unique solution for any point $Z = (z_1, z_2, z_3, z_4) \in Q_7$ for which $z_1\bar{z}_3 + z_2\bar{z}_4 \neq 0$. In fact, solving the system (3.10) in this case we have:

$$x = \frac{|z_1|^2}{z_1\bar{z}_3 + z_2\bar{z}_4}, \; y = \frac{|z_2|^2}{z_1\bar{z}_3 + z_2\bar{z}_4}, \; z = \frac{z_1\bar{z}_2}{z_1\bar{z}_3 + z_2\bar{z}_4}. \tag{3.11}$$

Notice that the solvability condition $Im(z_1\bar{z}_3 + z_2\bar{z}_4) = 0$ ensures that $z_1\bar{z}_3 + z_2\bar{z}_4$ is real.

In the solution above we assumed that $z_1\bar{z}_3 + z_2\bar{z}_4 \neq 0$. Let S be the subset of Q_7, where this condition is violated. Then any point $Z \in Q_7 \backslash S$ has a unique real plane $\begin{bmatrix} p \\ I \end{bmatrix}$ passing through Z and such that $\det p = 0$.

Condition $\det p = xy - |z|^2 = 0$ defines a light cone Q_3 at the origin of M. The above result shows that there is a well defined projection from $Q_7 \backslash S$ into Q_3 along real planes in C^4.

Real planes defined by $\det p = 0$ represent a special family of real planes. By considering light cones at every point of M and imposing the corresponding conditions on real planes we can obtain at each point of M a light cone Q'_3 and the corresponding quadric Q'_7 above it with a well defined projection from $Q'_7 \backslash S$ into Q'_3.

In fact, the group $SU(2,2)$ acts transitively on real planes, i.e. on $\bar{M}$. So if we start with the real planes satisfying $\det p = 0$ and apply $SU(2,2)$ transformations, we are going to get all possible real planes. Moreover, translations form a subgroup in $SU(2,2)$ which acts transitively on M. By applying a translation $\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$ to the plane $\begin{bmatrix} p \\ I \end{bmatrix}$ we obtain a new plane $\begin{bmatrix} p+B \\ I \end{bmatrix}$. Translation of the cone $\det p = 0$ gives a new cone Q'_3 defined by $\det(p - B) = 0$. An easy check shows that the planes above it can intersect only along the set S and that they sweep out the entire quadric Q_7.

More generally, consider $G_{2,4}(C)$ as a quadric Q in CP^5 obtained via Plücker embedding. Let $p \in \bar{M} \subset Q$ and let $\overline{T}_p Q$ be the tangent space to Q at p. Then $\overline{T}_p Q \cap Q \cap M$ can be identified with the light cone $Q_3(p)$ at p (see [2]). Cosequently, $T_p U \cap U$ can be identified with the quadric $Q_7(p)$ of the real C^2 planes above $Q_3(p)$, i.e. with $\pi^{-1}(Q_3(p))$. Notice also that this identification defines a complex structure on U. Consider now the light cone $Q_3(0) \in T_0\bar{M}$ at the origin of $\bar{M}$. Here $T_0\bar{M}$ is the tangent space to $\bar{M}$ at $p = 0$. Let $Q_7(0) = \pi^{-1}(Q_3(0)) \in T_0 U$. As $SU(2,2)$ acts on C^4 it generates the action on the fibres of U. On the quadric $Q_7(0)$ this action consists of translations $Q_7(0) \to Q_7(p)$ and of $SU(2,2)$-transformations preserving $Q_7(p)$. The induced action on $\bar{M}$ is conformal. On the light cone $Q_3(0)$ it consists of translations $Q_3(0) \to Q_3(p)$ and of the point p preserving conformal transformations. On a tangent space $T_p U$ at a given point $p \in \bar{M}$ we have an action of the group G_p of $SU(2,2)$-transformations preserving $Q_7(p)$ and the induced action of the point p preserving conformal transformations on $T_p\bar{M}$.

4 What is spin?

The bundle structure $\pi : U \to \bar{M}$ together with the group of automorphisms on it leads to a particularly clear interpretation of spin. A 4-spinor Z at a point $p \in \bar{M}$, for example, is a vector of $T_p U \simeq C^4$. It is an element of the space of fundamental representation of the group G_p of $SU(2,2)$-transformations preserving $Q_7(p)$. Historically, such objects go back to Dirac's 4-component spinors [5]. If Z is isotropic, projection $\pi(Z)$ of Z is defined and belongs to $T_p(\bar{M})$ (identified with M). Transformation of Z under the action of G_p on $T_p(U)$ is seen on $T_p(\bar{M})$ as a point p preserving conformal transformation. In particular, for the subgroup L of G_p given by the matrices $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$, with $A \in SL(2,C)$, $X = \pi(Z)$ transforms as a vector under the action of the group of Lorentz.

Transformation properties of 4-spinors under rotations now have a simple geometric interpretation. In a nutshell they are due to the fact that a real plane C^2 and the flipped upside down plane have the same image under the bundle projection. In particular, given a point $p \in M$ consider a rotation of a vector V of $T_p(M)$ through the angle of 2π. To be specific, consider the matrix $A = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \in SU(2)$ and the corresponding matrix $g = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \in G_p$. Under the action of g any vector Z in the real plane defined by V will transform as

$$Z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \longrightarrow \begin{pmatrix} e^{i\varphi} z_1 \\ e^{-i\varphi} z_2 \\ e^{i\varphi} z_3 \\ e^{-i\varphi} z_4 \end{pmatrix}. \tag{4.1}$$

According to (3.11) the components of vector $V = \pi(Z)$ will transform as

$$x' = x,\ y' = y,\ z' = e^{2i\varphi} z. \tag{4.2}$$

As φ changes from 0 to π, gZ changes from Z to $-Z$. Instead, $\tilde{g}V = \pi(gZ)$ changes from V to $V' = V$ describing a rotation through 2π in the z-plane. This is so because the set of vectors $\{-Z\}$ generates the same real plane as the set Z.

The matrix form of the relation between vectors and 4-spinors is clarified as well: a single point p of $\bar{M}$ is the image under π of the complex 2-dimensional plane – the fibre $\pi^{-1}(p)$. Planes are described by pairs of vectors, i.e. by matrices. So the nature of projection π leads to the correspondence *points of M $\longleftrightarrow$ hermitian 2x2 matrices.*

5 Concluding remarks

The main hypothesis advocated here is that the space-time manifold is the base of the fibre bundle $\pi : U \to \bar{M}$ described above.

Unlike the twistor theory, points of space-time in this approach are not secondary objects. Complex structure appears on the total space U of points rather than on Grassmanian $G_{2,4}(C)$. In this respect it is more similar to the bundle $CP^3 \to HP^1$ used in obtaining the instanton solutions of Yang-Mills equations [6]. The space-time manifold itself does not need to be conformally flat or half-coformally flat (see below).

When accepted, this hypothesis leads to a very natural interpretation of conformal symmetry. Conformal group action is induced on $\bar{M}$ from the action of a group of fibre-preserving transformations on U. The latter group is generated by the action of $SU(2,2)$ on C^4.

With this interpretation 4-component spin-vectors turn out to be vectors tangent to U. Whenever they can be projected on $T_p\bar{M}$ by π, they yield vectors which are transformed by a conformal transformation.

There are several remaining questions:

More general fibre preserving transformations of U are possible than those generated by the $SU(2,2)$ action on C^4. It is important to investigate the induced action of these transformations on $\bar{M}$.

The bundle $\pi : U \to \bar{M}$ can be defined even if M has no conformal symmetry at all. Consequences of this, in particular, a possibility to include the full gravity in this context must be studied.

By the above U has a complex structure. It is useful to know when this complex structure is integrable giving U as a complex 4-dimensional manifold.

What are the consequences of this interpretation of conformal symmetry in physics?

Some of these questions are addressed in a forthcoming paper [7].

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