Complementarity of representations in quantum mechanics

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Abstract

We show that Bohr's principle of complementarity between position and momentum descriptions can be formulated rigorously as a claim about the existence of representations of the CCRs. In particular, in any representation where the position operator has eigenstates, there is no momentum operator, and vice versa. Equivalently, if there are nonzero projections corresponding to sharp position values, all spectral projections of the momentum operator map onto the zero element.

1 Introduction

Niels Bohr's principle of complementarity has both a positive and a negative tenet. According to the positive tenet, a particle can have *either* a sharp position *or* a sharp momentum. According to the negative tenet, a particle can never have *both* a sharp position *and* a sharp momentum. In this paper, I show that both tenets of Bohr's complementarity principle correspond to straightforward mathematical facts about the existence of representations of the canonical commutation relations (CCRs). In particular, there is a (nonregular) "position representation" with a complete set of position eigenstates, and there is a (nonregular) "momentum representation" with a complete set of momentum eigenstates. However, in any representation with propositions ascribing a precise position value, there is no momentum operator, and all propositions attributing a momentum value to the particle are *contradictory*.

As a foil to my account, I consider two criticisms — one corresponding to each tenet — of Bohr's complementarity principle. The criticism of the negative tenet is well-known: It is claimed that Bohr makes a suspect inference from lack of joint measurability to lack of joint reality. The results of this paper, however, show that one has no need to appeal to suspect philosophical doctrines in order to provide a solid foundation for the negative tenet of complementarity. The criticism of the positive tenet — although less well-known — is much more difficult to overcome. In particular, it is a simple mathematical fact that there are no states in the standard Schrödinger representation of the CCRs in which a particle has a precise position or momentum.

However, this second criticism provides the key to unlocking both sides of the complementarity principle: If we want to maintain that particles can have precise positions or momenta, we must abandon one of the assumptions that entails (via the Stone-von Neumann theorem) the uniqueness of the Schrödinger representation. Once we drop this assumption, we will see that there is a representation of the CCRs which has the resources to describe precise position values, and there is a representation of the CCRs which has the resources to describe precise momentum values. However, there is no single representation of the CCRs that has the resources to describe *both* precise position values *and* precise momentum values. (Along the way, I argue that the "problem" of inequivalent representations is not peculiar to quantum field theory, but arises already in elementary quantum mechanics.)

Preliminaries:

Let μ denote the Lebesgue measure on $\mathbb{R}$. We say that a Borel function f from $\mathbb{R}$ to $\mathbb{C}$ is *square-integrable* just in case $\int_{\mathbb{R}} |f|^2 d\mu < \infty$. If f, g are square-integrable, we write $f \sim g$ just in case $\int_{\mathbb{R}} |f-g|^2 d\mu = 0$, and we let $[f] = \{g : g \sim f\}$. Let $L_2(\mathbb{R})$ denote the Hilbert space of equivalence classes (mod $\sim$) of square-integrable functions from $\mathbb{R}$ into $\mathbb{C}$, with inner product given by $\langle [f],[g] \rangle = \int_{\mathbb{R}} \overline{f} g \, d\mu$. Let $\Sigma(\mathbb{R})$ denote the σ-algebra of Borel subsets of $\mathbb{R}$. We define an equivalence relation $\approx$ on $\Sigma(\mathbb{R})$ by setting $S_1 \approx S_2$ just in case $\mu(S_1 - S_2) + \mu(S_2 - S_1) = 0$.

Let ι be the identity function, $\iota(x) = x$, on $\mathbb{R}$. Let Q denote the self-adjoint operator defined on a dense domain of $L_2(\mathbb{R})$ by $Q[f] = [\iota \cdot f]$, and let P denote the self-adjoint operator defined on a dense domain of $L_2(\mathbb{R})$ by $P[f] = -i[f']$. (We set $\hbar = 1$ throughout.) Since Q and P satisfy the canonical commutation relation $[Q,P] = iI$, they can be taken as corresponding, respectively, to the position and momentum observables of a particle with one degree of freedom. For $S \in \Sigma(\mathbb{R})$, we define the projection operator $E^Q(S)$ on $L_2(\mathbb{R})$ by setting $E^Q(S)[f] := [\chi_S \cdot f]$, where χ_S is the characteristic function of S. In this case, E^Q gives the canonical spectral measure for the self-adjoint operator Q.

Finally, let $\mathcal{L}$ denote the lattice of projection operators on $L_2(\mathbb{R})$, equipped with the standard operations $\wedge, \vee, \perp$ corresponding respectively to subspace intersection, span, and orthocomplement.

2 A problem with position-momentum complementarity

The complementarity between position and momentum is sometimes mistakenly equated with the *uncertainty* relation:

$$(\Delta_\psi Q) \cdot (\Delta_\psi P) \geq 1/2\,, \tag{1}$$

where $\Delta_\psi B$ is the dispersion of B in ψ. But Eqn. 1 says nothing about when Q and P can *possess* values; at best, it only tells us that there is a reciprocal relation between our *knowledge* of the value of Q and our *knowledge* of the value of P. To infer from this that Q and P cannot simultaneously possess values would be to lapse into positivism.

A more promising analysis of complementarity is suggested by Bub and Clifton's [2] classification of "no collapse" interpretations of quantum mechanics. According to this analysis, we can think of Bohr's complementarity interpretation as a no collapse interpretation in which the measured observable R and a state ψ determines a unique maximal sublattice $\mathcal{L}(\psi, R)$ of the lattice $\mathcal{L}$ of all subspaces of the relevant Hilbert space. $\mathcal{L}(\psi, R)$ should be thought of as containing the propositions that have a definite truth value when R is measured in the state ψ. In particular, $\mathcal{L}(\psi, R)$ always contains all propositions ascribing a value to R (i.e. the spectral projections of R), and ψ can be decomposed into a mixture of 2-valued homomorphisms (i.e. "truth valuations") on $\mathcal{L}(\psi, R)$. Thus, we can think of ψ as representing our *ignorance* of the possessed value of R. We would then say that observables R and R' are complementary just in case propositions attributing a value to R' are never contained in $\mathcal{L}(\psi, R)$ and vice versa.

But there is a serious difficulty in using this analysis to explicate Bohr's notion of position-momentum complementarity. In order to see this, note that for any $S_1, S_2 \in \Sigma(\mathbb{R})$, if $S_1 \approx S_2$ then $(\chi_{S_1} \cdot f) \sim (\chi_{S_2} \cdot f)$, and thus $E^Q(S_1)[f] = E^Q(S_2)[f]$. Since this is true for any $[f] \in L^2(\mathbb{R})$, it follows that $E^Q(S_1) = E^Q(S_2)$ when $S_1 \approx S_2$. In particular $E^Q(\{\lambda\}) = \mathbf{0}$, for any $\lambda \in \mathbb{R}$, since $\{\lambda\} \approx \emptyset$. This enables us to formulate a very simple "proof" that particles cannot have sharp positions.

Mathematical Fact: $E^Q(\{\lambda\}) = \mathbf{0}$; i.e. $E^Q(\{\lambda\})$ is the contradictory proposition.

Interpretive Assumption: $E^Q(\{\lambda\})$ represents the proposition "The particle is located at the point λ."

Conclusion: It is always false that the particle is located at λ.

What is more, it follows from the interpretive assumption that "being located in S_1" is literally *the same property as* "being located in S_2," whenever $S_1 \approx S_2$. Thus, any attempt to attribute a position to the particle would force us to revise the classical notion of location in space.

Halvorson [5] argues that we can solve these difficulties by reinterpreting elements of $\mathcal{L}$ as "experimental propositions" rather than as "property ascriptions," and by introducing non-countably additive (i.e. non-vector) states on $\mathcal{L}$. In particular, suppose that we adopt the alternative interpretive assumption:

Interpretive Assumption-2: $E^Q(S)$ means "A measurement of the position of the particle is certain to yield a value in S."

Then, $E^Q(\{\lambda\}) = \mathbf{0}$ does *not* entail that a particle cannot be located at λ, but only that no position measurement can be certain to show that it is located at λ. Moreover, there is a non-countably additive state h on $\mathcal{L}$ such that $h(E^Q(S)) = 1$ for *all* open neighborhoods S of λ. Thus, we could think of h as representing a state in which the particle is located at λ.

However, this solution is not fully satisfactory. In particular, there is still no proposition in the "object language" $\mathcal{L}$ which can express the claim that the particle is located at λ. [A pure state h on $\mathcal{L}$ is countably additive if and only if there is a unique minimal element $E \in \mathcal{L}$ such that $h(E) = 1$. Therefore, if we drop countable additivity, we allow for there to be more states than can be described in the language $\mathcal{L}$ of the theory.] Thus, the standard language $\mathcal{L}$ of quantum mechanics is incapable of describing particles as having precise positions or momenta. And, if we believe (as Bohr does) that particles *can* have precise positions or momenta, we are forced to conclude that quantum mechanics is *descriptively incomplete* (see [11]).

3 The Solution: Inequivalent Representations

Why should we take $L_2(\mathbb{R})$ as the state space for a particle with one degree of freedom? And, why is the lattice $\mathcal{L}$ of subspaces for $L_2(\mathbb{R})$ supposed to give us all possible properties of a particle with one degree of freedom? One answer to these questions is that $L_2(\mathbb{R})$ and $\mathcal{L}$ supply the elements needed for an empirically adequate model of such a particle. However, this answer is not sufficient, because it leaves open the possibility that there are inequivalent formalisms which could also model the phenomena. In order to supply a fully convincing answer, we would need a uniqueness theorem which shows that *any* empirically adequate (or physically reasonable) model is equivalent to the standard model.

Now, in the case of elementary quantum mechanics we do have a uniqueness theorem. In order to state this precisely, we first define one parameter groups of unitary operators on $L_2(\mathbb{R})$ by setting $U_a = \exp\{iaQ\}$ and $V_b = \exp\{ibP\}$ for all $a, b \in \mathbb{R}$. Then, as is well-known, these groups satisfy the *Weyl form* of the CCRs.

$$U_a V_b = e^{-iab} V_b U_a, \qquad (a, b \in \mathbb{R}). \tag{2}$$

More generally, we say that any pair $(\{U_a\}, \{V_b\})$ of one-parameter groups of unitary operators acting a Hilbert space $\mathcal{H}$ give a representation of the Weyl form of the CCRs just in case they satisfy Eqn. 2. Furthermore, we say that the representation is *irreducible* just in case no nontrivial subspaces of $\mathcal{H}$ are left invariant by all operators $\{U_a, V_b : a, b \in \mathbb{R}\}$. We say that two representations $(\{U_a\}, \{V_b\})$ and $(\{\tilde{U}_a\}, \{\tilde{V}_b\})$ on Hilbert spaces $\mathcal{H}$ and $\tilde{\mathcal{H}}$ are *unitarily equivalent* just in case there is a unitary operator $W : \mathcal{H} \mapsto \tilde{\mathcal{H}}$ such that

$$W U_a W^* = \tilde{U}_a, \qquad W V_b W^* = \tilde{V}_b, \tag{3}$$

for all $a, b \in \mathbb{R}$. Finally, a representation of the Weyl form of the CCRs is said to be *regular* just in case $a \mapsto U_a$ and $b \mapsto V_b$ are continuous. (Where $a \mapsto U_a$ is continuous just in case $a \mapsto \langle \varphi, U_a \varphi \rangle$ is continuous for any $\varphi \in \mathcal{H}$.) In this case, it follows that there is a unique irreducible representation of the CCRs.

Stone-von Neumann Uniqueness Theorem ([12]). *Every irreducible regular representation of the CCRs is unitarily equivalent to the Schrödinger representation on $L_2(\mathbb{R})$.*

The Stone-von Neumann uniqueness theorem seems to force our interpretive hand: There is a unique representation of the CCRs, and the "language" $\mathcal{L}$ of that representation (i.e. the lattice of projection operators) either has no sentences that ascribe precise positions or momenta to particles, or (if one thinks of projection operators as ascribing properties) those sentences are necessarily false. Thus, if quantum mechanics is descriptively complete, particles do not have precise positions or momenta.

However, this analysis is simply wrong. The Stone-von Neumann theorem cannot be used to prove that particles do not have precise positions or momenta, because *the regularity assumption begs the question against position-momentum complementarity.* In particular, Stone's theorem [6, Thm. 5.6.36] entails that a representation of the CCRs is regular if and only if the self-adjoint generators Q of $\{U_a : a \in \mathbb{R}\}$ and P of $\{V_b : b \in \mathbb{R}\}$ exist on $\mathcal{H}$. However, if complementarity is correct, then Q and P cannot both have sharp values. Why, then, do we need to assume that both operators exist in one representation space? What is more, we saw in the previous section that if *both* operators do exist, then *neither* can possess a sharp value. Does this not give us a reason to rethink the regularity assumption?

Let us formulate the previous argument more explicitly. First, let DC denote the claim that the language of quantum mechanics is descriptively complete:

Descriptive Completeness (DC): A particle can have a property E only if there is a corresponding projection operator $\hat{E}$ in some representation of the CCRs.

Let SP denote the claim that particles *can* have sharp positions, and let SM denote the claim that particles *can* have sharp momenta. Finally, let R denote the claim that any physically reasonable representation of the CCRs must be regular. Then the Stone-von Neumann theorem shows that:

$$\mathrm{DC} \wedge \mathrm{R} \Longrightarrow \neg\mathrm{SP} \wedge \neg\mathrm{SM}$$

However, this is equivalent to:

$$\mathrm{DC} \wedge (\mathrm{SP} \vee \mathrm{SM}) \Longrightarrow \neg\mathrm{R}$$

Thus, it is logically coherent to maintain the descriptive completeness of quantum mechanics along with the claim that particles can have precise positions or momenta; to do so, we must reject the assumption that any physically reasonable representation of the CCRs needs to be regular. [It should be noted that the regularity assumption can be replaced with the assumption that the Hilbert space $\mathcal{H}$ is separable (see [10]). However, the warrant for separability is even shakier than the warrant for regularity.] In order to show that this logical possibility is real, we now construct nonregular representations of the CCRs in which there are *contingent* statements attributing precise position (or momentum) values to a particle.

3.1 The Position Representation.

Let $l_2(\mathbb{R})$ denote the (nonseparable) Hilbert space of square-summable functions from $\mathbb{R}$ into $\mathbb{C}$. That is, an element f of $l_2(\mathbb{R})$ is supported on a countable subset S_f of $\mathbb{R}$ and $\|f\| := \sum_{x\in S_f} |f(x)|^2 < \infty$. The inner product on $l_2(\mathbb{R})$ is given by

$$\langle f, g\rangle = \sum_{x\in S_f\cap S_g} \overline{f(x)}g(x)\,. \tag{4}$$

For each $\lambda \in \mathbb{R}$, let φ_λ denote the characteristic function of $\{\lambda\}$. Thus, the set $\{\varphi_\lambda : \lambda \in \mathbb{R}\}$ is an orthonormal basis for $l_2(\mathbb{R})$. For each $a \in \mathbb{R}$, define U_a on the set $\{\varphi_\lambda : \lambda \in \mathbb{R}\}$ by

$$U_a\varphi_\lambda := e^{ia\lambda}\varphi_\lambda\,. \tag{5}$$

Since U_a maps $\{\varphi_\lambda : \lambda \in \mathbb{R}\}$ onto an orthonormal basis for $l_2(\mathbb{R})$, U_a extends uniquely to a unitary operator on $\mathcal{H}$. Similarly, define V_b on $\{\varphi_\lambda : \lambda \in \mathbb{R}\}$ by

$$V_b\varphi_\lambda := \varphi_{\lambda-b}\,. \tag{6}$$

Then V_b extends uniquely to a unitary operator on $l_2(\mathbb{R})$. Now, a straightforward calculation shows that,

$$U_aV_b\varphi_\lambda \;=\; e^{-iab}V_bU_a\varphi_\lambda\,, \tag{7}$$

for any $a, b \in \mathbb{R}$. Thus, the operators $\{U_a : a \in \mathbb{R}\}$ and $\{V_b : b \in \mathbb{R}\}$ give a representation of the Weyl form of CCRs on $l_2(\mathbb{R})$.

Furthermore, $\lim_{a\to 0}\langle\varphi_\lambda, U_a\varphi_\lambda\rangle = \lim_{a\to 0} e^{ia\lambda} = 1$, for any $\lambda \in \mathbb{R}$. Thus, $a \mapsto U_a$ is continuous, and Stone's theorem entails that there is a self-adjoint operator Q on $l_2(\mathbb{R})$ such that $U_a = \exp\{iaQ\}$ for all $a \in \mathbb{R}$. In particular,

$$Q\varphi_\lambda = -i\lim_{a\to 0} a^{-1}(U_a - I)\varphi_\lambda = -i\lim_{a\to 0} a^{-1}(e^{ia\lambda} - 1)\varphi_\lambda = \lambda\varphi_\lambda, \tag{8}$$

for each $\lambda \in \mathbb{R}$. On the other hand, we have:

$$\langle\varphi_\lambda, V_b\varphi_\lambda\rangle \;=\; \begin{cases} 0 & \text{when } b \neq 0, \\ 1 & \text{when } b = 0. \end{cases}$$

Thus, $b \mapsto V_b$ is not continuous, and there is no self-adjoint operator P such that $V_b = \exp\{ibP\}$ for all $b \in \mathbb{R}$. In other words, the momentum operator *does not exist* in this representation.

3.2 The Momentum Representation.

By means of a completely analogous construction, we can obtain a representation of the CCRs in which, for each $\lambda \in \mathbb{R}$, there is a vector φ_λ such that $P\varphi_\lambda = \lambda\varphi_\lambda$ (see [1, 3]). In this case, however, it is not possible to define a position operator.

Thus, we have shown that it *is* possible to give a standard Hilbert space description in which a particle has a precise position in the continuum; and it *is* possible to give a standard Hilbert space description in which a particle has a precise, numerical momentum value. However, the representations we constructed have a curious feature: In the position representation we cannot define a momentum operator, and in the momentum representation we cannot define a position operator. We now show that this "complementarity" between position and momentum holds in any representation of the CCRs.

Theorem 1. *In any representation of the Weyl form of the CCRs, if Q exists and has an eigenvector then P does not exist. If P exists and has an eigenvector then Q does not exist.*

Proof. We show that if there is a common eigenvector φ for $\{V_b : b \in \mathbb{R}\}$ then $a \mapsto U_a$ is not continuous. (The other half of the theorem follows by symmetry.) Indeed, if φ is a common eigenvector for $\{V_b : b \in \mathbb{R}\}$ then,

$$e^{iab}\langle \varphi, U_a\varphi\rangle = \langle \varphi, V_{-b}U_aV_b\varphi\rangle = \langle \varphi, U_a\varphi\rangle\,, \tag{9}$$

for all $a, b \in \mathbb{R}$. But this is possible only if $\langle \varphi, U_a\varphi\rangle = 0$ when $a \neq 0$. Since $\langle \varphi, U_a\varphi\rangle = 1$ when $a = 0$, it follows that $a \mapsto U_a$ is not continuous. $\square$

Of course, this theorem also shows that the position and momentum representations are inequivalent. And it shows (using the contrapositive of each part) that the momentum and position representations are both inequivalent to the Schrödinger representation. (This latter fact is already clear, though, since the Hilbert spaces differ in dimension.)

4 Nonexistent Quantities

I have claimed that if there are eigenstates for the position observable, then the momentum observable "does not exist." What was actually shown is that if there are eigenstates for the position observable, then $b \mapsto V_b$ is not continuous; and so we cannot reconstruct the momentum operator by taking the derivative $-i(\partial V_t/\partial t)|_{t=0}$. However, this failure of continuity does not itself carry a natural physical interpretation.

In this section, I give another perspective on the complementarity between position and momentum. In particular, we shall see that, in *every* representation of the CCRs, there are projection-valued measures E^Q, E^P representing the position and momentum observables. However, the properties (in particular, the sets of measure zero) of these measures vary between representations. Thus, the complementarity between position and momentum can be formulated as a claim about the relation between E^Q, E^P in representations of the CCRs. In particular, our main result shows that if $E^Q(\{\lambda\}) \neq \mathbf{0}$ for some $\lambda \in \mathbb{R}$ then $E^P(\mathbb{R}) = \mathbf{0}$. That is, if it is possible for a particle to have a point location, then it is impossible for that particle to have any momentum value in $\mathbb{R}$.

4.1 Preliminaries.

Let $BC(\mathbb{R})$ denote the C^*-algebra of continuous functions from $\mathbb{R}$ into $\mathbb{C}$ that are bounded in the norm

$$\|f\| = \sup_{x \in \mathbb{R}} |f(x)|. \tag{10}$$

Elements of $BC(\mathbb{R})$ are called *bounded continuous functions.* Recall that the space of pure states of $BC(\mathbb{R})$ (equipped with the weak* topology) is homeomorphic to the Stone-Čech compactification of $\mathbb{R}$. In particular, for each pure state ω of $BC(\mathbb{R})$ there is a unique ultrafilter $\mathcal{U}$ on $\mathbb{R}$ such that $\omega(f) = \lim_{\mathcal{U}} f$.

Let X be a Hausdorff topological space, and let π be a mapping of $\mathbb{R}$ into X. We say that (π, X) is a *compactification* of $\mathbb{R}$ just in case X is compact, and π is a continuous embedding of $\mathbb{R}$ onto a dense subset of X. [We do not require π to be a homeomorphism of $\mathbb{R}$ onto $\pi(\mathbb{R})$.] There is a one-to-one correspondence between C^*-subalgebras of $BC(\mathbb{R})$ and compactifications of $\mathbb{R}$ (see [4, p. 16]). In particular, since each $\lambda \in \mathbb{R}$ gives rise to a principal ultrafilter $\mathcal{U}_\lambda$ on $\mathbb{R}$, there is a natural injection of $\mathbb{R}$ into the space $\sigma(\mathcal{A})$ of pure states of $\mathcal{A}$. Furthermore, $\sigma(\mathcal{A})$ is compact, and $\mathcal{A}$ is naturally isomorphic to the continuous functions $C(\sigma(\mathcal{A}))$ on $\sigma(\mathcal{A})$. Thus, a subalgebra of $BC(\mathbb{R})$ is naturally isomorphic to the continuous functions on a compact space $X = \mathbb{R} \cup (X \backslash \mathbb{R})$, where the elements in $X \backslash \mathbb{R}$ can be thought of as "points at infinity."

We now define a specific C^*-subalgebra of $BC(\mathbb{R})$. For each $a \in \mathbb{R}$, let u_a denote the function given by $u_a(x) = e^{iax}$, $(x \in \mathbb{R})$. Let $AP(\mathbb{R})$ denote the C^*-subalgebra of $BC(\mathbb{R})$ generated by $\{u_a : a \in \mathbb{R}\}$. That is, $AP(\mathbb{R})$ consists of uniform limits of trigonometric polynomials of the form

$$f(x) = \sum_{j=1}^{n} c_j e^{i a_j x}, \tag{11}$$

where $c_j \in \mathbb{C}$ and $a_j \in \mathbb{R}$. Elements of $AP(\mathbb{R})$ are called *almost periodic functions.* The compactification $b\mathbb{R}$ of $\mathbb{R}$ corresponding to $AP(\mathbb{R})$ is called the *Bohr compactification* of $\mathbb{R}$ (after the mathematician Harald Bohr). Thus, $AP(\mathbb{R})$ is isomorphic to $C(b\mathbb{R})$.

There is a binary operation $\hat{+}$ on $b\mathbb{R}$ such that $(b\mathbb{R}, \hat{+}, 0)$ is a compact topological group in which $(\mathbb{R}, +, 0)$ is embedded as a dense subgroup [8, p. 30]. Let μ denote the unique (normalized) translation-invariant measure (i.e. *Haar measure*) on the Borel σ-algebra $\Sigma(b\mathbb{R})$ of $b\mathbb{R}$. Now define the *invariant mean* ω_μ on $C(b\mathbb{R})$ by setting

$$\omega_\mu(f) := \int_{b\mathbb{R}} f \, d\mu \,, \qquad (f \in C(b\mathbb{R})). \tag{12}$$

The invariant mean can also be defined explicitly by

$$\omega_\mu(f) = \lim_{N\to\infty} \frac{1}{2N} \int_{-N}^{N} f(x) \, dx \,, \qquad (f \in AP(\mathbb{R})). \tag{13}$$

Clearly, $\omega_\mu(u_a) = 0$ for all $a \neq 0$; and since $\{u_a : a \in \mathbb{R}\}$ is linearly dense in $C(b\mathbb{R})$, it follows that ω_μ is the unique state of $C(b\mathbb{R})$ with this property.

4.2 The Weyl algebra

There is a unique minimal C^*-algebra $\mathcal{A}[\mathbb{R}^2]$ containing two one-parameter groups $\{U_a : a \in \mathbb{R}\}$ and $\{V_b : b \in \mathbb{R}\}$ of unitary operators obeying the Weyl form of the CCRs [7]. The abelian C^*-subalgebra of $\mathcal{A}[\mathbb{R}^2]$ generated by $\{U_a : a \in \mathbb{R}\}$ is naturally isomorphic to $C(b\mathbb{R})$ ($\approx AP(\mathbb{R})$), and the same is true of the C^*-subalgebra generated by $\{V_b : b \in \mathbb{R}\}$.

Recall that a representation $(\pi, \mathcal{H})$ of a C^*-algebra $\mathcal{A}$ consists of a $*$-homo-morphism π of $\mathcal{A}$ into the C^*-algebra of bounded linear operators on $\mathcal{H}$. Thus, the irreducible representations of the Weyl form of the CCRs are in one-to-one correspondence with the irreducible representations of $\mathcal{A}[\mathbb{R}^2]$. Recall also that two representations $(\pi, \mathcal{H})$ and $(\tilde{\pi}, \tilde{\mathcal{H}})$ of $\mathcal{A}$ are *unitarily equivalent* just in case there is a unitary operator $W : \mathcal{H} \mapsto \tilde{\mathcal{H}}$ such that $W\pi(A)W^* = \tilde{\pi}(A)$ for all $A \in \mathcal{A}$.

Proposition 1. *For any irreducible representation $(\pi, \mathcal{H})$ of $\mathcal{A}[\mathbb{R}^2]$ there is a corresponding pair (E^Q, E^P) of projection-valued measures on $b\mathbb{R}$ such that*

$$\pi(U_a) = \int_{b\mathbb{R}} u_a(\lambda) \, dE^Q_\lambda, \qquad \pi(V_b) = \int_{b\mathbb{R}} u_b(\lambda) \, dE^P_\lambda, \tag{14}$$

for all $a, b \in \mathbb{R}$.

Proof. Define a pair α^Q, α^P of representations of $C(b\mathbb{R})$ on $\mathcal{H}$ by setting $\alpha^Q(u_a) = \pi(U_a)$ and $\alpha^P(u_a) = \pi(V_a)$ for all $a \in \mathbb{R}$. The result then follows immediately from Theorem 5.2.6 of [6]. □

The heuristic integrals in Eqn. 14 can be interpreted rigorously as quantified statements about families of complex Borel measures on $b\mathbb{R}$. In particular, for any vector $\varphi \in \mathcal{H}$ we define a measure μ_φ on $b\mathbb{R}$ by setting

$$\mu_\varphi(S) := \langle \varphi, E^Q(S)\varphi \rangle, \qquad (S \in \Sigma(b\mathbb{R})). \tag{15}$$

Then, we define the left part of Eqn. 14 as the statement:

$$\langle \varphi, \pi(U_a)\varphi \rangle = \int_{b\mathbb{R}} u_a(\lambda)\, d\mu_\varphi(\lambda), \qquad \forall \varphi \in \mathcal{H}. \tag{16}$$

The right part of Eqn. 14 makes a similar statement about complex Borel measures defined in terms of E^P and vectors in $\mathcal{H}$.

We also claim (without proof) that $a \mapsto \pi(U_a)$ is continuous iff. $E^Q(\mathbb{R}) = \mathbf{I}$. In this case, $\{E^Q_\lambda : \lambda \in \mathbb{R}\}$ gives the spectral resolution of Q:

$$Q = \int_{\mathbb{R}} \lambda \, dE^Q_\lambda. \tag{17}$$

Similarly, $b \mapsto \pi(V_b)$ is continuous if and only if $E^P(\mathbb{R}) = \mathbf{I}$, in which case $\{E^P_\lambda : \lambda \in \mathbb{R}\}$ gives the spectral resolution of P.

Irreducible representations of $\mathcal{A}[\mathbb{R}^2]$ are in one-to-one correspondence with pure states of $\mathcal{A}[\mathbb{R}^2]$. In particular, for any pure state ω of $\mathcal{A}[\mathbb{R}^2]$, the GNS construction provides an irreducible representation $(\pi_\omega, \mathcal{H}_\omega)$ of $\mathcal{A}[\mathbb{R}^2]$ such that ω is represented by a cyclic vector $\Omega \in \mathcal{H}_\omega$ [6, Thm. 4.5.2; Thm. 10.2.3]. In fact, we can directly produce both the position and momentum representations by means of the GNS construction: For each $\lambda \in \mathbb{R}$, there is a unique pure state ω_λ of $\mathcal{A}[\mathbb{R}^2]$ such that $\omega_\lambda(U_a) = e^{ia\lambda}$ for all $a \in \mathbb{R}$; moreover, it follows that $\omega_\lambda(V_b) = 0$ when $b \neq 0$ (see [1]). Thus, $\omega_\lambda(A) = \langle \varphi_\lambda, A\varphi_\lambda \rangle$ whenever $A = U_a$ or $A = V_a$, for some $a \in \mathbb{R}$. Since $\{V_b\varphi_\lambda : b \in \mathbb{R}\}$ is also linearly dense in $l_2(\mathbb{R})$, the position representation is unitarily equivalent to the GNS representation induced by ω_λ [6, Prop. 4.5.3]. A completely analogous construction can be used to obtain a GNS representation that is unitarily equivalent to the momentum representation.

Theorem 2. *Let $(\pi, \mathcal{H})$ be an irreducible representation of $\mathcal{A}[\mathbb{R}^2]$, and let (E^Q, E^P) be the corresponding pair of projection-valued measures. If $E^Q(\{\lambda\}) \neq \mathbf{0}$ for some $\lambda \in \mathbb{R}$ then $E^P(\mathbb{R}) = \mathbf{0}$. If $E^P(\{\lambda\}) \neq \mathbf{0}$ for some $\lambda \in \mathbb{R}$ then $E^Q(\mathbb{R}) = \mathbf{0}$.*

Proof. The proof splits into two parts: (1.) If $E^Q(\{\lambda\}) \neq \mathbf{0}$ then $(\pi, \mathcal{H})$ is unitarily equivalent to the position representation. (2.) If $(\pi, \mathcal{H})$ is equivalent to the position representation then $E^P(\mathbb{R}) = \mathbf{0}$.

(1.) Let $(\pi, \mathcal{H})$ be an irreducible representation of $\mathcal{A}[\mathbb{R}^2]$ such that $E^Q(\{\lambda\}) \neq \mathbf{0}$. Let φ be a unit vector in the range of $E^Q(\{\lambda\})$, and define a Borel measure ν on $b\mathbb{R}$ by setting

$$\nu(S) := \langle \varphi, E^Q(S)\varphi \rangle, \qquad (S \in \Sigma(b\mathbb{R})). \tag{18}$$

Then ν is concentrated on $\{\lambda\}$, and Prop. 1 entails that

$$\langle \varphi, \pi(U_a)\varphi \rangle = \int_{b\mathbb{R}} u_a \, d\nu = u_a(\lambda) = e^{ia\lambda}. \tag{19}$$

Since $(\pi, \mathcal{H})$ is irreducible, φ is cyclic for π. It follows then from the uniqueness of the GNS representation [6, Prop. 4.5.4], that $(\pi, \mathcal{H})$ is unitarily equivalent to the GNS representation of $\mathcal{A}[\mathbb{R}^2]$ induced by the state ω_λ, and thus to the position representation.

(2.) Suppose that $(\pi, \mathcal{H})$ is unitarily equivalent to the position representation. In particular, we can suppose that $\mathcal{H}$ $(= l_2(\mathbb{R}))$ is spanned by an orthonormal family of vectors $\{\varphi_\lambda : \lambda \in \mathbb{R}\}$ where $\pi(V_b)\varphi_\lambda = \varphi_{\lambda - b}$ for all $b, \lambda \in \mathbb{R}$. Thus, to show that $E^P(\mathbb{R}) = \mathbf{0}$, it will suffice to show that $E^P(\mathbb{R})\varphi_\lambda = 0$ for all $\lambda \in \mathbb{R}$. Fix λ and let $\varphi = \varphi_\lambda$. Define a Borel probability measure ν on $b\mathbb{R}$ by setting

$$\nu(S) := \|E^P(S)\varphi\|^2 = \langle \varphi, E^P(S)\varphi \rangle, \qquad (S \in \Sigma(b\mathbb{R})). \tag{20}$$

Define a state ω_ν of $C(b\mathbb{R})$ by

$$\omega_\nu(f) := \int_{b\mathbb{R}} f \, d\nu, \qquad (f \in C(b\mathbb{R})). \tag{21}$$

Then Prop. 1 entails that

$$\omega_\nu(u_a) = \int_{b\mathbb{R}} u_a \, d\nu = \langle \varphi, \pi(V_a)\varphi \rangle = 0, \tag{22}$$

for all $a \in \mathbb{R}$. Since $\{u_a : a \in \mathbb{R}\}$ generates $C(b\mathbb{R})$, it follows that ω_ν is the invariant mean. By the Riesz representation theorem [9, Thm. 2.14], there is a unique probability measure on $b\mathbb{R}$ corresponding to each state of $C(b\mathbb{R})$. Thus, ν is the Haar measure on $b\mathbb{R}$, and it follows from translation-invariance (and countable-additivity) that $\nu(\mathbb{R}) = 0$. $\square$

5 Conclusion

It is well-known that the existence of inequivalent representations raises significant issues for the interpretation of quantum *field* theory. However, it would be wrong to think (on the basis of the Stone-von Neumann uniqueness theorem) that the issue of inequivalent representations has no significance for elementary quantum mechanics. Indeed, it is only by employing nonregular representations of the CCRs that we can make sense of Bohr's views about position-momentum complementarity.

Still, one might wonder whether nonregular representations of the CCRs have any *empirical* relevance. In particular, don't we have purely empirical grounds for preferring the Schrödinger representation to these representations? But this question betrays a misunderstanding of the nature of representations. The abstract algebra of observables $\mathcal{A}[\mathbb{R}^2]$ carries the full empirical content of the quantum theory of a single particle (with one degree of freedom). (We have not mentioned dynamics, but this can also be defined in a representation-indendent manner.) In particular, $\mathcal{A}[\mathbb{R}^2]$ has enough observables to describe our measurement procedures, and enough states to describe each laboratory preparation. Thus, a representation does not add anything further to this already given empirical content, and one representation cannot be preferable to another on empirical grounds alone.

However, by saying that any two representations of $\mathcal{A}[\mathbb{R}^2]$ are empirically equivalent, we are not committing ourselves to the further claim that there are no *physically significant* differences between representations. In particular, each representation of $\mathcal{A}[\mathbb{R}^2]$ comes with a certain set of "sharp propositions" (i.e. projection operators) which can be used to describe "how things are" independent of our experimental interventions. Moreover, different representations give very different stories about how things are. For example, while the Schrödinger representation says that two locations are the same if they differ only by a set of Lebesgue measure zero, the position representation permits us to maintain the classical picture of the spatial continuum in which particles can have precise positions.

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