Matter from space

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Summary. General Relativity offers the possibility to model attributes of matter, like mass, momentum, angular momentum, spin, chirality etc. from pure space, endowed only with a single field that represents its Riemannian geometry. I review this picture of 'Geometrodynamics' and comment on various developments after Einstein.

1 Introduction

Towards the end of his famous habilitation address, delivered on June 10th 1854 to the Philosophical Faculty of the University of Göttingen, Bernhard Riemann applied his mathematical ideas to physical space and developed the idea that it, even though of euclidean appearance at macroscopic scales, may well have a non-euclidean geometric structure in the sense of variable curvature if resolved below some yet unspecified microscopic scale. It is remarkable that in this connection he stressed that the measure for geometric ratios ("Massverhältnisse") would already be encoded in the very notion of space itself if the latter were considered to be a discrete entity, whereas in the continuous case the geometry must be regarded as being a contingent structure that depend on "acting forces".[1]

This suggestion was seized and radicalised by William Kingdon Clifford, who in his paper 'On the Space-Theory of Matter', read to the Cambridge Philosophical Society on February 21st 1870, took up the tough stance that *all* material properties and happenings may eventually be explained in terms of the curvature of space and its changes. In this seminal paper the 24-year old said ([4], reprinted on p. 71 of [44]):

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[1] "Es muß also entweder das dem Raume zugrude liegende Wirkliche eine diskrete Mannigfaltigkeit bilden, oder der Grund der Maßverhältnisse außerhalb, in darauf wirkenden bindenden Kräften gesucht werden." ([46], p. 20)

"I wish here to indicate a manner in which these speculations [Riemann's] may be applied to the investigation of physical phenomena. I hold in fact:

1. That small portions of space *are* in fact of a nature analogous to little hills on a surface which is on the average flat; namely, that the ordinary laws of geometry are not valid in them.
2. That this property of being curved or distorted is continually being passed from one portion of space to another after the manner of a wave.
3. That this variation of the curvature of space is what really happens in that phenomenon which we call the *motion of matter*, whether ponderable or etherial.
4. That in the physical world nothing else takes place but this variation, subject (possibly) to the law of continuity."

Fig. 1. Replica (by John Collier) of the portrait of William Kingdon Clifford at the London National Portrait Gallery.

In this contribution I wish to explain and comment on the status of this programme within General Relativity. This is not to suggest that present day physics offers even the slightest hope that this programme - understood in its radical sense - could succeed. But certain aspects of it certainly are realised, sometimes in a rather surprising fashion, and this is what I wish to talk about here.

That matter-free physical space should have physical properties at all seems to be quite against the view of Leibniz, Mach, and their modern followers, according to which space is a relational concept whose ontological status derives from that of the fundamental constituents of matter whose relations are considered. But at the same time it also seems to be a straightforward consequence of modern field theory, according to which fundamental fields are directly associated with space (or spacetime) rather than any space-filling material substance. Once the latter view is adopted, there seems to be no good reason to neglect the field that describes the geometry of space. This situation was frequently and eloquently described by Einstein, who empathetically wrote about the difficulties that one encounters in attempting to mentally emancipate the notion of a field from the idea of a substantial carrier whose physical states the field may describe. In doing this, the field describes the states of space itself, so that space becomes a dynamical agent, albeit one to which standard kinematical states of motion cannot be attributed, as Einstein stressed e.g. in his 1920 Leiden address "Äther und Relativitätstheorie" ([52] Vol. 7, Doc. 38, pp. 306-320). A famous and amusing cartoon is shown in Figure 2, whose caption quotes Einstein expressing a view close to that of Clifford's.

2 Geometrodynamics

The field equations of General Relativity with cosmological constant Λ read ($\kappa = 8\pi G/c^4$, where G is Newton's constant)

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = \kappa T_{\mu\nu} \,. \tag{1}$$

They form a system of ten quasilinear partial differential equations for the ten components $g_{\mu\nu}$ of the spacetime metric. These equations may be cast into the form of evolution equations. More precisely, the system (1) may be decomposed into a subsystem of four under-determined elliptic equations that merely constrain the initial data (the so-called 'constraints') and a complementary subsystem of six under-determined hyperbolic equations that drives the evolution. (The under-determination is in both cases a consequence of diffeomorphism invariance.) This split is made possible by foliating spacetime M into 3-dimensional spacelike leaves Σ_t via a one-parameter family of embeddings $\mathcal{E}_t : \Sigma \hookrightarrow M$ with images $\mathcal{E}_t(\Sigma) = \Sigma_t \subset M$; see Fig. 3. The object that undergoes evolution in this picture is the 3-dimensional Riemannian manifold (Σ, h) whose metric at time t is $h_t = \mathcal{E}_t^* g$, where g is the spacetime metric. In this evolutionary picture spacetime appears as space's history.

2.1 Hypersurface kinematics

Let us be more precise on what it means to say that spacetime is considered as the trajectory (history) of space. Let $\mathrm{Emb}(\Sigma, M)$ denote the space of smooth

"People slowly accustomed themselves to the idea that the physical states of space itself were the final physical reality...."
(A. Einstein, 1929)

Fig. 2. Cartoon of 1929 in *The New Yorker* by its first art editor Rea Irvin.

spacelike embeddings $\Sigma \to M$. We consider a curve $\mathbb{R} \ni t \to \mathcal{E}_t \in \mathrm{Emb}(\Sigma, M)$ corresponding to a one-parameter family of smooth embeddings with spacelike images. We assume the images $\mathcal{E}_t(\Sigma) =: \Sigma_t \subset M$ to be mutually disjoint and moreover $\hat{\mathcal{E}} : \mathbb{R} \times \Sigma \to M$, $(t, p) \mapsto \mathcal{E}_t(p)$, to be an embedding. (It is sometimes found convenient to relax this condition, but this is of no importance here). The Lorentz manifold $(\mathbb{R} \times \Sigma, \mathcal{E}^* g)$ may now be taken as ($\mathcal{E}$–dependent) representative of M (or at least some open part of it) on which the leaves of the above foliation simply correspond to the $t = \mathrm{const.}$ hypersurfaces. Let n denote a field of normalised timelike vectors normal to these leaves. n is unique up to orientation, so that the choice of n amounts to picking a 'future direction'.

The tangent vector $d\mathcal{E}_t/dt|_{t=0}$ at $\mathcal{E}_0 \in \mathrm{Emb}(\Sigma, M)$ corresponds to a vector field over $\mathcal{E}_0$ (i.e. section in $T(M)|_{\mathcal{E}_0(\Sigma)}$), given by
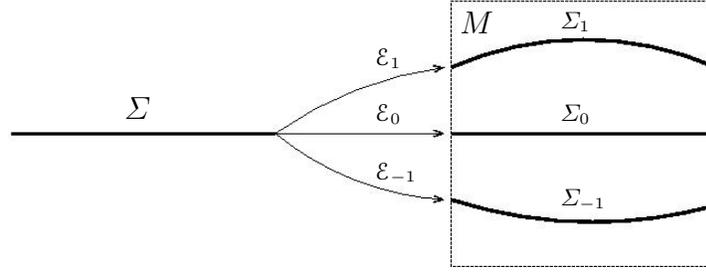
Fig. 3. Spacetime M is foliated by a one-parameter family of spacelike embeddings of the 3-manifold Σ. Here the image Σ_1 of Σ under $\mathcal{E}_{t=1}$ lies to the future (above) and $\Sigma_{-1} := \mathcal{E}_{t=-1}$ to the past (below) of $\Sigma_0 := \mathcal{E}_{t=0}(\Sigma)$.

$$\frac{d\mathcal{E}_t(p)}{dt}\bigg|_{t=0} =: \frac{\partial}{\partial t}\bigg|_{\mathcal{E}_0(p)} = \alpha n + \beta \tag{2}$$

with components (α, β) normal and tangential to $\Sigma_0 \subset M$. The functions α (one function), usually called the *lapse function*, and β (3 functions), usually called the *shift vector field*, combine the four-function worth of arbitrariness in moving the hypersurface Σ in spacetime; see Fig. 4.
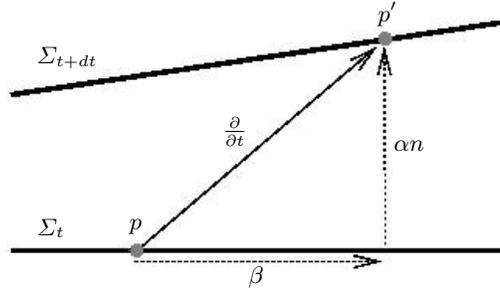

Fig. 4. For $q \in \Sigma$ the image points $p = \mathcal{E}_t(q)$ and $p' = \mathcal{E}_{t+dt}(q)$ are connected by the vector $\partial/\partial t|_p$ whose components tangential and normal to Σ_t are β (three functions) and αn (one function) respectively.

Conversely, each vector field V on M defines a vector field $X(V)$ on $\mathrm{Emb}(\Sigma, M)$, corresponding to the left action of $\mathrm{Diff}(M)$ on $\mathrm{Emb}(\Sigma, M)$ given by composition. In local coordinates y^μ on M and x^k on Σ it can be written as

$$X(V) = \int_\Sigma d^3x \, V^\mu(y(x)) \frac{\delta}{\delta y^\mu(x)} \, . \tag{3}$$

One easily verifies that $X : V \mapsto X(V)$ is a Lie homomorphism:

$$\big[X(V), X(W)\big] = X\big([V, W]\big) \, . \tag{4}$$

Alternatively, decomposing (3) into normal and tangential components with respect to the leaves of the embedding at which the tangent-vector field to Emb(Σ,M) is evaluated yields an embedding-dependent parametrisation of $X(V)$ in terms of (α, β),

$$X(\alpha, \beta) = \int_\Sigma d^3 x \Big(\alpha(x) n^\mu [y](x) + \beta^m(x) \partial_m y^\mu(x) \Big) \frac{\delta}{\delta y^\mu(x)} \,, \qquad (5)$$

where y in square brackets indicates the functional dependence of n on the embedding. The functional derivatives of n with respect to y can be computed (see the Appendix of [53]) from which the commutator of deformation generators follows:

$$\big[X(\alpha_1, \beta_1) \,, \, X(\alpha_2, \beta_2) \big] = - X(\alpha', \beta') \,, \qquad (6)$$

where

$$\alpha' = \beta_1(\alpha_2) - \beta_2(\alpha_1) \,, \qquad (7a)$$

$$\beta' = [\beta_1, \beta_2] + \sigma \alpha_1 \, \mathrm{grad}_h(\alpha_2) - \sigma \alpha_2 \mathrm{grad}_h(\alpha_1) \,. \qquad (7b)$$

Here we left open whether spacetime M is Lorentzian ($\sigma = 1$) or Euclidean ($\sigma = -1$), just in order to keep track how the signature of spacetime, $(-\sigma, +, +, +)$, enters. Note that the h-dependent gradient field for the scalar function α is given by $\mathrm{grad}_h(\alpha) = (h^{ab} \partial_b \alpha) \partial_a$. The geometric idea behind (7) is summarised in Figure 5.
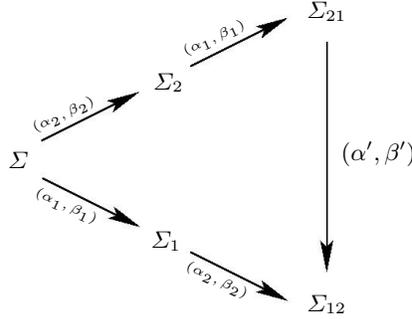

Fig. 5. An (infinitesimal) hypersurface deformation with parameters (α_1, β_1) that maps $\Sigma \mapsto \Sigma_1$, followed by one with parameters (α_2, β_2) that maps $\Sigma_1 \mapsto \Sigma_{12}$ differs by one with parameters (α', β') given by (7) from that in which the maps with the same parameters are composed in the opposite order.

2.2 Hamiltonian geometrodynamics

The idea of Hamiltonian Geometrodynamics is to realise these relations in terms of a Hamiltonian system on the phase space of physical fields. The

most simple case is that where the latter merely include the spatial metric h on Σ, so that the phase space is the cotangent bundle $T^*\mathrm{Riem}(\Sigma)$ over $\mathrm{Riem}(\Sigma)$. One then seeks a correspondence that associates to each pair (α, β) of lapse and shift a real-valued function on phase space:

$$(\alpha, \beta) \mapsto \big(H(\alpha, \beta) \,:\, T^*\mathrm{Riem}(\Sigma) \to \mathbb{R}\big)\,, \tag{8}$$

where

$$H(\alpha, \beta)[h, \pi] := \int_{\Sigma} d^3x \big(\alpha(x)\mathcal{H}[h, \pi](x) + h_{ab}(x)\beta^a(x)\mathcal{D}^b[h, \pi](x)\big)\,, \tag{9}$$

with integrands $\mathcal{H}[h, \pi](x)$ and $\mathcal{D}^b[h, \pi](x)$ yet to be determined. H should be regarded as distribution (here the test functions are α and β^a) with values in real-valued functions on $T^*\mathrm{Riem}(\Sigma)$. Now, the essential requirement is that the Poisson brackets between the $H(\alpha, \beta)$ are, up to a minus sign,[2] as in (6-7):

$$\big\{H(\alpha_1, \beta_1)\,,\, H(\alpha_2, \beta_2)\big\} = H(\alpha', \beta')\,. \tag{10}$$

For the integration of canonical initial data (h, π) with Hamiltonian $H(\alpha, \beta)$ we need to specify by hand a one-parameter (representing parameter time t) family of lapse functions $t \mapsto \alpha_t(x) = \alpha(t, x)$ and shift vector fields $t \mapsto \beta_t(x) = \beta(t, x)$. It is now clear that this freedom just corresponds to the freedom to foliate the spacetime to be constructed. The Hamiltonian equations of motion contain only the unknown functions $h_t(x) = h(t, x)$ and $\pi_t(x) = \pi(t, x)$ and should be regarded as evolution equations (in terms of parameter time t) for the one-parameter families of tensor fields $t \mapsto h_t$ and $t \mapsto \pi_t$. Once the integration is performed, the solution gives rise to solution of Einstein's equation: If $\beta_t^\flat$ is the one-form field corresponding to the vector field β_t via h_t, i.e. $\beta_t^\flat := h_t(\beta_t, \cdot)$, then the Lorentzian metric that satisfies Einstein's equation on the manifold $I \times \Sigma$, where I is the interval on the real line in which the parameter t takes its values, is given by

$$g = -\big(\alpha_t^2 - h_t(\beta_t, \beta_t)\big)dt \otimes dt + \beta_t^\flat \otimes dt + dt \otimes \beta_t^\flat + h_t\,. \tag{11}$$

However, this integration may not start from any arbitrary set of initial data (h, π). The data themselves need to satisfy a system of (under-determined elliptic) partial differential equations, the so-called 'constraints'. The reason for their existence as well as their analytic form will be explained in the next subsections.

2.3 Why constraints

From (10) alone follows a remarkable uniqueness result as regards the analytical structure of $H(\alpha, \beta)$ as functional of (h, π). Before stating it with all its

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[2]Due to the standard convention that the Hamiltonian action being defined as a *left* action, whereas the Lie bracket on a group is defined by the commutator of left-invariant vector fields which generate *right* translations.

hypotheses, we show why the constraints $\mathcal{H}[h, \pi] = 0$ and $\mathcal{D}^b[h, \pi] = 0$ must be imposed.

Consider the set of smooth real-valued functions on phase space, F : $T^*\mathrm{Riem}(\Sigma) \to \mathbb{R}$. They are acted upon by all $H(\alpha, \beta)$ via Poisson bracketing: $F \mapsto \{F, H(\alpha, \beta)\}$. This defines a map from (α, β) into the derivations of phase-space functions. We require this map to also respect the commutation relation (10), that is, we require

$$\{\{F, H(\alpha_1, \beta_1)\}, H(\alpha_2, \beta_2)\} - \{\{F, H(\alpha_2, \beta_2)\}, H(\alpha_1, \beta_1)\} = \{F, H\}(\alpha', \beta').$$
(12)

The crucial and somewhat subtle point to be observed here is the following: Up to now the parameters (α_1, β_1) and (α_2, β_2) were considered as given functions of $x \in \Sigma$, independent of the fields h and π, i.e. independent of the point of phase space. However, from (7b) we see that $\beta'(x)$ does depend on $h(x)$. This dependence should not give rise to extra terms $\propto \{F, \alpha'\}$ in the Poisson bracket, for, otherwise, the extra terms would prevent the map $(\alpha, \beta) \mapsto \{-, H(\alpha, \beta)\}$ from being a homomorphism from the algebraic structure of hypersurface deformations into the derivations of phase-space functions. This is necessary in order to interpret $\{-, H(\alpha, \beta)\}$ as a generator (on phase-space functions) of a *spacetime* evolution corresponding to a normal lapse α and tangential shift β. In other words, the evolution of observables from an initial hypersurface Σ_i to a final hypersurface Σ_f must be independent of the intermediate foliation ('integrability' or 'path independence' [53, 31, 32]). Therefore we placed the parameters (α', β') outside the Poisson bracket on the right-hand side of (12), to indicate that no differentiation with respect to h, π should act on them.

To see that this requirement implies the constraints, rewrite the left-hand side of (12) in the form

$$
\begin{aligned}
&\{\{F, H(\alpha_1, \beta_1)\}, H(\alpha_2, \beta_2)\} - \{\{F, H(\alpha_2, \beta_2)\}, H(\alpha_1, \beta_1)\} \\
&= \{F, \{H(\alpha_1, \beta_1), H(\alpha_2, \beta_2)\}\} \\
&= \{F, H(\alpha', \beta')\} \\
&= \{F, H\}(\alpha', \beta') + H(\{F, \alpha'\}, \{F, \beta'\}),
\end{aligned}
$$
(13)

where the first equality follows from the Jacobi identity, the second from (10), and the third from the Leibniz rule. Hence the requirement (12) is equivalent to

$$H(\{F, \alpha'\}, \{F, \beta'\}) = 0$$
(14)

for all phase-space functions F to be considered and all α', β' of the form (7). Since only β' depends on phase space, more precisely on h, this implies the vanishing of the phase-space functions $H(0, \{F, \beta'\})$ for all F and all β' of the form (7b). This can be shown to imply $H(0, \beta) = 0$, i.e. $\mathcal{D}[h, \pi] = 0$. Now, in turn, for this to be preserved under all evolutions we need $\{H(\alpha, \tilde{\beta}), H(0, \beta)\} = 0$, and hence in particular $\{H(\alpha, 0), H(0, \beta)\} = 0$ for all α, β, which implies

$H(\alpha, 0) = 0$, i.e. $\mathcal{H}[h, \pi] = 0$. So we see that the constraints indeed follow from the required integrability condition.

Sometimes the constraints $H(\alpha, \beta) = 0$ are split into the *Hamiltonian (or scalar) constraints*, $H(\alpha, 0) = 0$, and the *diffeomorphisms (or vector) constraints*, $H(0, \beta) = 0$. The relations (10) with (7) then show that the vector constraints form a Lie-subalgebra which, because of $\{H(0, \beta), H(\alpha, 0)\} = H\big(\beta(\alpha), 0\big) \neq H(0, \beta')$, is not an ideal. This means that the Hamiltonian vector fields for the scalar constraints are not tangent to the surface of vanishing vector constraints, except where it intersects the surface of vanishing scalar constraints. This implies that the scalar constraints do not act on the solution space for the vector constraints, so that one simply cannot first reduce the vector constraints and then, on the solutions of that, search for solutions to the scalar constraints.

2.4 Uniqueness of Einstein's geometrodynamics

It is sometimes stated that the relations (10) together with (7) determine the function $H(\alpha, \beta) : T^*\mathrm{Riem}(\Sigma) \to \mathbb{R}$, i.e. the integrands $\mathcal{H}[h, \pi]$ and $\mathcal{D}[h, \pi]$, uniquely up to two free parameters, which may be identified with the gravitational and the cosmological constants. This is a mathematical overstatement if read literally, since the result can only be shown if certain additional assumptions are made concerning the action of $H(\alpha, \beta)$ on the basic variables h and π. The uniqueness result then obtained is still remarkable.

The first such assumption concerns the intended ('semantic' or 'physical') meaning of $H(0, \beta)$, namely that the action of $H(0, \beta)\}$ on h or π is that of an infinitesimal spatial diffeomorphism of Σ. Hence it should be the spatial Lie derivative, L_β, applied to h or π. It then follows from the general Hamiltonian theory that $H(0, \beta)$ is given by the *momentum map* that maps the vector field β (viewed as element of the Lie algebra of the group of spatial diffeomorphisms) into the function on phase space given by the contraction of the momentum with the β-induced vector field $h \to L_\beta h$ on $\mathrm{Riem}(\Sigma)$:

$$H(0, \beta) = \int_\Sigma d^3x\, \pi^{ab}(L_\beta h)_{ab} = -2\int_\Sigma d^3x (\nabla_a \pi^{ab})h_{bc}\beta^c\,. \qquad (15)$$

Comparison with (9) yields

$$\mathcal{D}^b[h, \pi] = -2\nabla_a \pi^{ab}\,. \qquad (16)$$

The second assumption concerns the intended ('semantic' or 'physical') meaning of $H(\alpha, 0)$, namely that $\{-, H(\alpha, 0)\}$ acting on h or π is that of an infinitesimal 'timelike' diffeomorphism of M normal to the leaves $\mathcal{E}_t(\Sigma)$. If M were given, it is easy to prove that we would have $L_{\alpha n}h = 2\alpha\, K$, where n is the timelike field of normals to the leaves $\mathcal{E}_t(\Sigma)$ and K is their extrinsic curvature. Hence one requires

$$\{h, H(\alpha, 0)\} = 2\alpha \, K \,. \tag{17}$$

Note that both sides are symmetric covariant tensor fields over Σ. The important fact to be observed here is that α appears without differentiation. This means that $H(\alpha, 0)$ is an ultralocal functional of π, which is further assumed to be a polynomial. Note that at this moment we do not assume any definite relation between π and K. Rather, this relation is a consequence of (17) once the analytic form of $H(\alpha, 0)$ is determined.

The Hamiltonian evolution so obtained is precisely that of General Relativity (without matter) with two free parameters, which may be identified with the gravitational constant $\kappa = 8\pi G/c^4$ and the cosmological constant Λ. The proof of the theorem is given in [36], which improves on earlier versions [53, 32] in that the latter assume in addition that $\mathcal{H}[h, \pi]$ be an even function of π, corresponding to the requirement of time reversibility of the generated evolution. This was overcome in [36] by the clever move to write the condition set by $\{H(\alpha_1, 0), H(\alpha_2, 0)\} = H(0, \beta')$ (the right-hand side being already known) on $H(\alpha, 0)$ in terms of the corresponding Lagrangian functional L, which is then immediately seen to turn into a condition which is *linear* in L, so that terms with even powers in velocity decouple form those with odd powers. There is a slight topological subtlety remaining which is further discussed in [28]. The two points which are important for us here are:

1. *The dynamics of the gravitational field as given by Einstein's equations can be fully understood in term of the constraints.*
2. *Modulo some technical assumptions spelled out above, the constraints for pure gravity follow from the kinematical relation (10) with (7), once one specifies and gravitational phase space to be $T^*\mathrm{Riem}(\Sigma)$, i.e. the gravitational configuration space to be $\mathrm{Riem}(\Sigma)$.*

2.5 What the constraints look like

Rather than writing down the constraints in terms of h and π, we shall use the simple relation between π and K that follows from (17) for given $H(\alpha, 0)$, the reason being that K has the simple interpretation as extrinsic curvature (also called second fundamental form) of the images of Σ in M, which is rather intuitive. From the determination of $H(\alpha, 0)$ the h-dependent relation between π and K, in terms of components, turns out to be

$$\pi^{ab} = \sqrt{\det\{h_{nm}\}} \, h^{ac} h^{bd} \big(K_{cd} - h_{cd} h^{ij} K_{ij}\big) \,. \tag{18}$$

In terms of h and K the constraints then assume the form

$$\big(h^{ac} h^{bd} \, K_{ab} K_{cd} - (h^{ab} K_{ab})^2\big) - \big(R(h) - 2\Lambda\big) = -\, (2\kappa)\rho \,, \tag{19a}$$

$$\nabla_b \big(K^{ab} - h^{ab} \, h^{nm} K_{nm}\big) \;=\; (c\kappa) \, j^a \,, \tag{19b}$$

The right-hand sides of both equations (19) are zero in the matter free (vacuum) case which we consider here. But we think it is instructive to know what

it will be in the presence of matter: Here ρ and j represent the matter's energy and momentum densities on Σ respectively.[3]Moreover, $R(h)$ is the Ricci scalar for h, Λ is the cosmological constant, and $\kappa = 8\pi G/c^4$ as in (1).

The first bracket on the left-hand side of (19a) contains an h-dependent bilinear form in K. It can be seen as the kinetic term in the Hamiltonian of the gravitational field. Usually the kinetic term is positive definite, but this time it is not! Hence we wish to understand this bilinear form in more detail. In particular: Under what conditions on K is it positive or negative definite? This can be answered in terms of the eigenvalues of K. To make this precise, let $\tilde{K}$ be the endomorphism field which is obtained from K by raising one index (which one does not matter due to symmetry) using h. We may now unambiguously speak of the eigenvalues of $\tilde{K}$, a triple for each space point. Each triple we collect in an eigenvalue vector $\boldsymbol{\lambda} \in \mathbb{R}^3$. In terms of $\tilde{K}$ the bilinear form reads $\text{Tr}(\tilde{K}^2) - (\text{Tr}(\tilde{K}))^2$, which equals $\|\boldsymbol{\lambda}\|^2 - (\boldsymbol{\lambda} \cdot \mathbf{d})^2$ in terms of $\boldsymbol{\lambda}$. Here the dot product and the norm are the usual ones in $\mathbb{R}^3$ and $\mathbf{d}$ is the 'diagonal vector' with unit entries $(1, 1, 1)$. Hence the bilinear form is positive definite iff[4] the modulus of the cosine of the angle between $\boldsymbol{\lambda}$ and $\mathbf{d}$ is less than $1/\sqrt{3}$ and negative definite iff it is greater than $1/\sqrt{3}$. In other words, the bilinear form is negative definite on those $\tilde{K}$ whose eigenvalue vector lies in the interior of a double cone whose vertex is the origin, whose symmetry axis is the 'diagonal' generated by $\mathbf{d}$, and whose opening angle (angle between symmetry axis and boundary) is $\arccos(1/\sqrt{3}) \approx 54.7°$. Note that this opening angle is just the one at which the boundary of each cone just contains all three positive or negative coordinate half-axes. It properly contains the maximal cones contained in the positive and negative octants, whose opening angle is $\arccos(\sqrt{2/3}) \approx 35.3°$.[5] Hence strictly positive or strictly negative eigenvalues of $\tilde{K}$ imply a negative definite value of the bilinear form, but the converse is not true.[6]

The preceding discussion shows that the bilinear form is not of a definite nature. In fact, it is a $(1 + 5)$ – dimensional Lorentzian metric on the six-

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[3]Recall that the symmetry of the energy-momentum tensor for the matter implies that the momentum density is c^{-2} times the energy current density (energy per unit surface area and unit time).

[4]In this article we use 'iff' as abbreviation for 'if and only if'.

[5]The maximal cone touches the three 2-planes $\lambda_i = 0$ at the bisecting lines $\lambda_j = \lambda_k$, where i, j, k is any of the three cyclic permutations of $1, 2, 3$. Hence the cosine of the opening angle is the scalar product between $(1, 1, 1)\sqrt{3}$ and, say, $(1, 1, 0)\sqrt{2}$, which is $\sqrt{2/3}$.

[6]Had we done the very same analysis in terms of π rather than K we would have found that in eigenvalue space (now of the endomorphism $\tilde{\pi}$) the opening angle of the cone inside which the bilinear form is negative definite and outside which it is positive definite is now *precisely* the maximal one $\arccos(\sqrt{2/3})$ (see previous footnote). Indeed, rewriting the bilinear form in terms of π using (18), it is positively proportional to $\text{Tr}(\tilde{\pi}^2) - \frac{1}{2}(\text{Tr}(\tilde{\pi}))^2$. It is the one-half in front of the second term that causes this interesting coincidence.

dimensional space of positive definite bilinear forms over a real 3-dimensional vector space (the tangent space to Σ), which is known as the DeWitt metric since DeWitt's seminal paper [6] on canonical quantum gravity. Parametrising it by h_{ab} or (τ, r_{ab}) it can be written as

$$G^{ab\,cd}dh_{ab} \otimes dh_{cd} = -(32/3)\,d\tau \otimes d\tau + \tau^2\,\mathrm{Tr}(r^{-1}dr \otimes r^{-1}dr)\,, \qquad (20a)$$

where

$$r_{ab} := [\det(h)]^{-1/3}h_{ab}, \quad \tau := [\det(h)]^{1/4}\,, \qquad (20b)$$

and

$$G^{ab\,cd} = \tfrac{1}{2}\sqrt{\det(h)}\big(h^{ac}h^{bd} + h^{ad}h^{bc} - 2h^{ab}h^{cd}\big)\,. \qquad (20c)$$

The form (20a) clearly reveals it geometric meaning as a warped-product metric of 'cosmological type' on the manifold $\mathbb{R} \times SL(3,\mathbb{R})/SO(3)$, where the five-dimensional homogeneous space $SL(3,\mathbb{R})/SO(3)$, parametrised by r_{ab}, carries its left invariant metric $\mathrm{Tr}(r^{-1}dr \otimes r^{-1}dr) = r^{ac}r^{bd}\,dr_{ab} \otimes dr_{cd}$.

This pointwise Lorentzian metric induces a metric on the infinite-dimensional manifold $\mathrm{Riem}(\Sigma)$, known as Wheeler-DeWitt metric, through

$$\mathcal{G}(k, \ell) = \int_\Sigma d^3x\, G^{qb\,cd}k_{ab}\ell_{cd} \qquad (21)$$

where the tensor fields k and ℓ are now considered as tangent vectors at $h \in \mathrm{Riem}(\Sigma)$. See [28] and references therein for a recent review on geometric aspects associated with this metric and its rôle in Geometrodynamics.

To end this brief sketch of Geometrodynamics let me just stress its (admittedly somewhat crude) analogy to relativistic point mechanics. The latter takes place in Minkowski space which is endowed with an absolute (i.e. non dynamical) geometry through the Minkowski metric. Here the configuration space is $\mathrm{Riem}(\Sigma)$, which is also endowed with an absolute geometry through the Wheeler-DeWitt metric, although it is not true that the Einstein equations correspond to geodesic motion in it. However, the deviation from geodesic motion derives from a force that corresponds to a vector field on $\mathrm{Riem}(\Sigma)$ given by $-2(R_{ab} - \tfrac{1}{4}h_{ab}R)$, where R_{ab} and R are the Ricci tensor and scalar for h [24] respectively.

2.6 Vacuum Data

Following Clifford's dictum, we shall in the following be interested in vacuum data, that is data (h, K) that satisfy (19) for vanishing right-hand sides. Upon evolution these give rise to solutions $g_{\mu\nu}$ to Einstein's equations (1) for $T_{\mu\nu} = 0$.

An important non-trivial observation is that the system (19) does not impose any topological obstruction on Σ. That means that for any topological 3-manifold Σ there are data (h, K) that satisfy (19) with vanishing right-hand side. This result can be understood as an immediate consequence of a famous

theorem proved in [33], that states that *any* smooth function $f : \Sigma \to \mathbb{R}$ which is negative somewhere can be the scalar curvature for some Riemannian metric. Given that strong result, we may indeed always solve (19) for $\rho = 0$ and $j = 0$ as follows: First we make the Ansatz $K = \alpha h$ for some constant α and some $h \in \mathrm{Riem}(\Sigma)$. This solves (19b), whatever α, h will be. Given the space-time interpretation of K as extrinsic curvature, this means that the initial Σ will be a totally umbillic hypersurface in the spacetime M that is going to evolve from the data. Next we solve (19a) by fixing α so that $\alpha^2 > \Lambda/3$ and then choosing h so that $R(h) = 2\Lambda - 6\alpha^2$, which is possible by the result just cited because the right-hand side is negative by construction.

Simple but nevertheless very useful examples of vacuum data are provided by time-symmetric conformally flat ones. Time symmetry means that the initial 'velocity' of h vanishes, and hence that $K = 0$, so that (19b) is already satisfied. A vanishing extrinsic curvature is equivalent to saying that the hypersurface is totally geodesic, meaning that a geodesic in spacetime that initially starts on and tangent to $\Sigma \subset M$ will remain within Σ. This is to be expected since motions with vanishing initial velocity should be time-reflection symmetric, which here would imply the existence of an isometry of M (the history of space) that exchanges both sides of Σ in M and leaves Σ pointwise fixed. However, a fixed-point set of an isometry is always totally geodesic, for, if it were not, a geodesic starting on and tangent to Σ but taking off Σ eventually would be mapped by the isometry to a different geodesic with the same initial conditions, which contradicts the uniqueness theorem for solutions of the geodesic equation.

As time symmetry implies $K = 0$, we have automatically solved (19b) for $j = 0$. That h be conformally flat means that we may write $h = \phi^4 \delta$, where δ is the flat euclidean metric on Σ and $\phi : \Sigma \to \mathbb{R}_+$ is a positive real-valued function. The remaining constraint (19a) for $\rho = 0$ then simply reduces to Laplace's equation for ϕ:

$$\Delta\phi = 0\,, \tag{22}$$

where Δ denotes the Laplace operator with respect to the flat metric δ.

Usually one seeks solutions so that (Σ, h) is a manifold with a finite number of asymptotically flat ends. One such end is then associated with 'spatial infinity', which really just means that the solution under consideration represents a quasi isolated lump of geometry with a sufficiently large (compared to its own dimension) almost flat transition region to the ambient universe. According to Arnowitt, Deser, and Misner (see their review [1]) we can associate a (active gravitational) mass to each such end, which is defined as a limit of a flux integrals over 2-spheres pushed into the asymptotic region of the end in question. If the mass is measured in geometric units (i.e. it has the physical dimension of a length and is converted to a mass in ordinary units by multiplication with c^2/G), it is given by

$$m := \lim_{R\to\infty} \left\{ \frac{1}{16\pi} \int_{S_R} (\partial_j h_{ij} - \partial_i h_{jj}) n^i \, d\Omega \right\}\,, \tag{23}$$

where $S_R \subset \Sigma$ is a 2-sphere of radius R, outward-pointing normal n, and surface measure $d\Omega$. For later use we note in passing that if the asymptotically flat spacetime is globally stationary (i.e. admits a timelike Killing field K), the overall mass can also be written in the following simple form, known as 'Komar integral' [34]:

$$m := \lim_{R \to \infty} \left\{ \frac{-1}{8\pi} \int_{S_R} \star dK^\flat \right\}, \qquad (24)$$

where d denotes the exterior differential on spacetime, $K^\flat := g(K, \cdot)$ the one-form corresponding to K under the spacetime metric g (lowering the index) and $\star$ is the Hodge-duality map. A similar expression exists for the overall angular momentum of a rotationally symmetric spacetime, as we shall see later.

The celebrated 'positive-mass theorem' states in the vacuum case that for any Riemannian metric h which satisfies the constraints (19) with $\rho = 0$ and $j = 0$ for some K has $m \geq 0$ for each asymptotically flat end.[7] Moreover, $m = 0$ iff (Σ, h) is a spatial slice through Minkowski space. This already implies that the mass must be strictly positive if Σ is topologically different from $\mathbb{R}^3$: Non-trivial topology implies non-zero positive mass! This is supported by the generalisation of the Penrose-Hawking singularity theorems due to Gannon [16], which basically states that the geometric hypothesis of the existence of closed trapped surfaces in Σ in the former may be replaced by the purely topological hypothesis of Σ not being simply connected. This is our first example in GR of how attributes of matter (here mass) arise from pure geometry/topology.

2.7 Solution strategies

A variety of methods exist to construct interesting solutions to (22). On of them is the 'method of images' known from electrostatics [40]. It is based on the conformal properties of the Laplace operator, which are as follows: Let $\Sigma = \mathbb{R}^3 - \{\mathbf{x}_0\}$ and δ its usual flat metric. Consider a sphere S_0 of radius r_0 centred at $\mathbf{x}_0$. The 'inversion at S_0', denoted by $I_{(\mathbf{x}_0, r_0)}$, is a diffeomorphism of Σ that interchanges the exterior and the interior of $S_0 \subset \Sigma$ and leaves S_a pointwise fixed. In spherical polar coordinates centred at $\mathbf{x}_0$ it takes the simple form

$$I_{(\mathbf{x}_0, r_0)}(r, \theta, \varphi) = (r_0^2/r, \theta, \varphi). \qquad (25a)$$

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[7]Note that the definition of the ADM mass (23) just depends on the Riemannian metric h and is independent of K. But for the theorem to hold it is essential to require that h is such that there exists a K so that (h, K) satisfy the constraints. It is easy to write down metrics h with negative mass: Take e.g. (29) with negative m for $r > r_* > m/2$, smoothly interpolated within $m/2 < r < r_*$ to, say, the flat metric in $r < m/2$. The positive mass theorem implies that for such a metric no K can be found so that (h, K) satisfy the constraints.

There is a variant of this map that results from an additional antipodal reflection in the 2-spheres so that no fixed points exist:

$$I'_{(\mathbf{x}_0,r_0)}(r,\theta,\varphi) = (r_0^2/r, \pi - \theta, \varphi + \pi) \,. \tag{25b}$$

Associated to each of these self-maps of Σ are self-maps $J_{(\mathbf{x}_0,r_0)}$ and $J'_{(\mathbf{x}_0,r_0)}$ of the set of smooth real-valued functions on Σ, given by

$$J_{(\mathbf{x}_0,r_0)}(f) = (r_0/r)\,(f \circ I_{(\mathbf{x}_0,r_0)}) \tag{26}$$

and likewise with $I'_{(\mathbf{x}_0,r_0)}$ exchanged for $I_{(\mathbf{x}_0,r_0)}$ on the right-hand side in case of $J'_{(\mathbf{x}_0,r_0)}$. Now, the point is that these maps obey the following simple composition laws with the Laplace operator (considered as self-map of the set of smooth functions on Σ):

$$\Delta \circ J_{(\mathbf{x}_0,r_0)} = (r_0/r)^4\, J_{(\mathbf{x}_0,r_0)} \circ \Delta \tag{27}$$

and likewise with $J'_{(\mathbf{x}_0,r_0)}$ replacing $J_{(\mathbf{x}_0,r_0)}$. In particular, the last equation implies that $J_{(\mathbf{x}_0,r_0)}$ and $J'_{(\mathbf{x}_0,r_0)}$ map harmonic functions (i.e. functions ϕ satisfying $\Delta\phi = 0$) on Σ to harmonic functions on Σ. Note that Σ did not include the point $\mathbf{x}_0$ at which the sphere of inversion was centred. It is clear from (26) that the maps $J_{(\mathbf{x}_0,r_0)}$ will change the singular behaviour of the functions at $\mathbf{x}_0$. For example, the image of the constant function $f \equiv 1$ under either $J_{(\mathbf{x}_0,r_0)}$ or $J_{(\mathbf{x}_0,r_0)}$ is just the function $\mathbf{x} \mapsto r_0/\|\mathbf{x} - \mathbf{x}_0\|$, i.e. a pole of strength r_0 at $\mathbf{x}_0$. Iterating once more, a pole of strength a located at $\mathbf{a}$ is mapped via $J_{(\mathbf{x}_0,r_0)}$ resp. $J'_{(\mathbf{x}_0,r_0)}$ to a pole of strength $a/\|\mathbf{x}_0 - \mathbf{a}\|$ at $I_{(\mathbf{x}_0,r_0)}(\mathbf{a})$ resp. $I'_{(\mathbf{x}_0,r_0)}(\mathbf{a})$.

The general strategy is then as follows: Take a set S_i, $i = 1, \cdots N$, of N spheres with radii r_i and centres $\mathbf{x}_i$, so that each sphere S_i is disjoint from, and to the outside of, each other sphere S_j, $j \neq i$. Take the constant function $f \equiv 1$ and take the sum over the free group generated by all $J_{(\mathbf{x}_i,r_i)}$ (alternatively the $J'_{(\mathbf{x}_i,r_i)}$). This converges to an analytic function ϕ provided $(N-1)r_*/d < 1$, where $r_* = \max\{r_1, \cdots r_N\}$ and d is the infimum of euclidean distances from the centres $\mathbf{x}_i$ to points on the spheres S_j, $j \neq i$; see [40][19].[8] By construction the function ϕ is then invariant under each inversion map $J_{(\mathbf{x}_i,r_i)}$ (alternatively $J'_{(\mathbf{x}_i,r_i)}$). Consequently, the maps $I_{(\mathbf{x}_i,r_i)}$ (alternatively $I'_{(\mathbf{x}_i,r_i)}$) are isometries of the Metric $h := \phi^4\,\delta$, which is defined on the manifold Σ which one obtains by removing from $\mathbb{R}^3$ the centres $\mathbf{x}_i$ and all their image points under the free group generated by the inversions $I_{(\mathbf{x}_i,r_i)}$ or $I'_{(\mathbf{x}_i,r_i)}$. However, the topology of the manifold may be modified by suitable identifications using these isometries. For example, using $I'_{(\mathbf{x}_i,r_i)}$, we may excise the interiors of all spheres S_i and identify antipodal points on each remaining boundary component S_i. In this fashion we obtain a manifold with one end, which is the connected sum of N real-projective spaces minus a point (the point at spatial infinity).

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[8]Note that $(N-1)r_*/d < 1$ always holds in case of two spheres, $N = 2$, since the condition of being disjoint and to the exterior of each other implies $r_*/d < 1$.
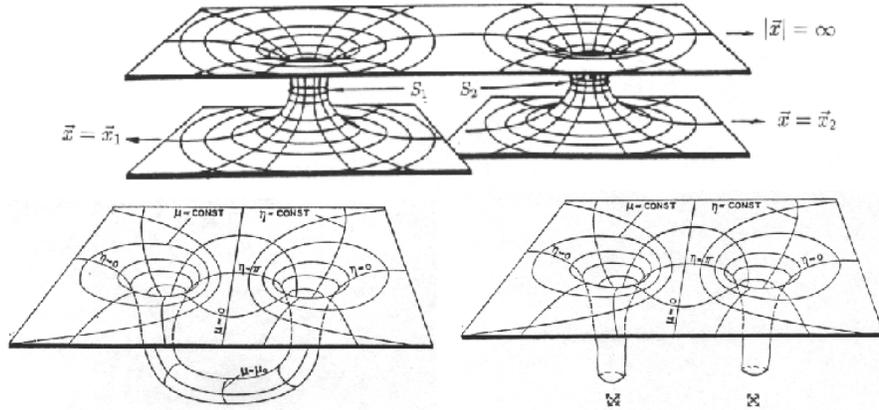
Fig. 6. Various topologies for data (Σ, h) representing two black holes momentarily at rest. The upper manifold has three asymptotically flat ends, one at spatial infinity and one each 'inside' the apparent horizons (= minimal surfaces) S_1, S_2. The lower two manifolds have only one end each. The lower left manifold (wormhole) is topologically $S^1 \times S^2 -$ {point} the lower right $\mathbb{R}\mathrm{P}^3 \# \mathbb{R}\mathrm{P}^3 -$ {point}, where $\#$ denotes connected sum. The crosswise arrows in the lower right picture indicate that the shown 2-sphere boundaries are closed off by antipodal identifications. The coordinates μ, η correspond to bispherical polar coordinates. No two of these three manifolds are locally isometric.

In general there are many topological options. Consider, for example, the simpler case of just two 2-spheres S_1 and S_2 of, say, equal radii, $r_1 = r_2$. We again excise their interiors and identify their boundaries. If we use the maps $I_{(\mathbf{x}_i, r_i)}$ for the data construction, we may identify S_1 with S_2 in an orientation *reversing* fashion (with respect to their induced orientations) so that the quotient space is orientable. This results in Misner's wormhole[39] whose data are often used in numerical studies of black-hole collisions. If instead we had used the maps $I'_{(\mathbf{x}_i, r_i)}$ we have two choices: either to identify antipodal points on each S_i separately, which results in the connected sum of two real-projective spaces, as explained above, or to identify S_1 with S_2, but now in an orientation preserving fashion (with respect to their induced orientations) so that the resulting manifold is a non-orientable version of Misner's wormhole discussed in [19]. The latter two manifolds are locally isometric but differ in their global topology, whereas they are not even locally isometric to the standard (orientable) Misner wormhole.

Let us turn to the simplest non-trivial example: a single black hole. it corresponds to the solution of (22) with a single pole at, say, $\mathbf{x}_0 = \mathbf{0}$ and asymptotic value $\phi \to 1$ for $r \to \infty$, where $r := \|\mathbf{x}\|$. Hence we have

$$\phi(\mathbf{x}) = 1 + \frac{m}{2r} \, . \tag{28}$$

It is easy to verify that the constant m just corresponds to the ADM mass defined via (23). The 3-dimensional Riemannian manifold (Σ, h) is now given by $\Sigma = \mathbb{R}^3 - \{\mathbf{0}\}$ and the metric, in polar coordinates centred at the origin,

$$h = \left(1 + \frac{m}{2r}\right)^4 \underbrace{\left(dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right)}_{= \delta} \qquad (29)$$

It allows for the two discrete isometries

$$I : \left(r, \theta, \varphi\right) \mapsto \left(m^2/4r\,,\,\theta\,,\,\varphi\right), \qquad (30)$$

$$J : \left(r, \theta, \varphi\right) \mapsto \left(m^2/4r\,,\,\pi - \theta\,,\,\varphi + \pi\right). \qquad (31)$$

The set of fixed points for I is the sphere $r = m/2$, whereas J acts freely (without fixed points). That the sphere $r = m/2$ is the fixed-point set of an isometry (I) implies that it is totally geodesic (has vanishing extrinsic curvature in Σ), as already discussed above. In particular it implies that $r = m/2$ is a minimal surface that joins two isometric halves. Hence (Σ, h) has two asymptotically flat ends, one for $r \mapsto \infty$ (spatial infinity) and one for $r \mapsto 0$, as shown on the left of Fig. 7. This is sometimes interpreted by saying that there is a singular pointlike mass source at $r = 0$, just like for the electric Coulomb field for a point charge. But this interpretation is deceptive. It is true that the Coulomb field is a vacuum solution to Maxwell's equations if the point at which the source sits is simply removed from space. But this removal of a point leaves a clear trace in that the resulting manifold is incomplete. This is different for the manifold $(\mathbb{R} - \{\mathbf{0}\}, h)$, with h given by (29), which is complete, due to the fact that the origin is infinitely far away in the metric h. Hence no point is missing and the solution can be regarded as a genuine vacuum solution.

2.8 The $\mathbb{R}\mathbf{P}^3$ geon

There is a different twist to this story. One might object against the fact that $\mathbb{R} - \{\mathbf{0}\}$ has *two* ends rather than just one (at spatial infinity). After all, what would the 'inner end' correspond to? A locally isometric manifold with just one end is obtained by taking the quotient of $\mathbb{R} - \{\mathbf{0}\}$ with respect to the freely acting group $\mathbb{Z}_2$ that is generated by the isometry J in (31). This identifies the region $r > m/2$ with the region $r < m/2$ and antipodal points on the minimal 2-sphere $r > m/2$. The resulting space is real projective 3-space, $\mathbb{R}\mathrm{P}^3$, minus a point, which clearly has just one end. Its full time evolution, i.e. the space-time emerging from it, can be obtained from the maximal evolution of the Schwarzschild data: h as in (29), $K = 0$, which is Kruskal spacetime (see [35] and/or Chapter 5.5. in [30]). A conformally rescaled version (Penrose Diagram) of Kruskal spacetime is depicted on the right of Fig. 7.

In Kruskal coordinates[9] (T, X, θ, φ), where T and X each range in $(-\infty, \infty)$ obeying $T^2 - X^2 < 1$, the Kruskal metric reads (as usual, we write $d\Omega^2$ for

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[9]Kruskal [35] uses (v, u) Hawking Ellis [30] (t', x') for what we call (T, X).
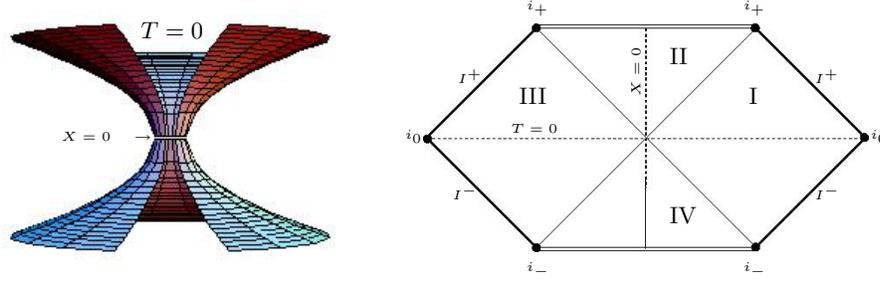
Fig. 7. To the right is the conformal (Penrose) diagram of Kruskal spacetime in which each point of this 2-dimensional representation corresponds to a 2-sphere (an orbit of the symmetry group of spatial rotations). The asymptotic regions are i_0 (spacelike infinity), $I^{\pm}$ (future/past lightlike infinity), and $i^{\pm}$ (future/past timelike infinity). The diamond and triangular shaped regions I and II correspond to the exterior ($r > 2m$) and interior ($0 < r < 2m$) Schwarzschild spacetime respectively, the interior being the black hole. The triangular region IV is the time reverse of II, a white hole. Region III is another asymptotically flat end isometric to the exterior Schwarzschild region I. The double horizontal lines on top an bottom represent the singularities ($r = 0$) of the black and white hole respectively. The left picture shows an embedding diagram of the hypersurface $T = 0$ (central horizontal line in the conformal diagram) that serves to visualise its geometry. Its minimal 2-sphere at the throat corresponds to the intersection of the hyperplanes $T = 0$ and $X = 0$ (bifurcate Killing Horizon).

$d\theta^2 + \sin^2\theta\, d\varphi^2$):

$$g = \frac{32m^2}{\rho} \exp(-\rho/2m)\left(-dT^2 + dX^2\right) + r^2 d\Omega^2\,, \qquad (32)$$

where ρ is a function of T and X, implicitly defined by

$$\left((\rho/2m) - 1\right)\exp(\rho/2m) = X^2 - T^2\,. \qquad (33)$$

Here ρ corresponds to the usual radial coordinate, in terms of which the Schwarzschild metric reads

$$g = -\left(1 - \frac{2m}{\rho}\right)dt^2 + \frac{dr^2}{1 - \frac{2m}{\rho}} + \rho^2\, d\Omega^2 \qquad (34)$$

where $\rho > 2m$. It covers region I of the Kruskal spacetime. Setting

$$\rho = r\left(1 + \frac{m}{2r}\right)^2 \qquad (35)$$

so that the range $m/2 < r < \infty$ covers the range $2m < \rho < \infty$ twice, we obtain the 'isotropic form'

$$g = -\left(\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}}\right)^2 dt^2 + \left(1 + \frac{m}{2r}\right)^4\left(dr^2 + r^2\, d\Omega^2\right) \qquad (36)$$

which covers regions I and III of the Kruskal manifold.

The Kruskal metric (32) is spherically symmetric and allows for the additional Killing field[10]

$$K = \tfrac{1}{4m}\left(X\partial_T + T\partial_X\right),\tag{37}$$

which is timelike for $|X| > |T|$ and spacelike for $|X| < |T|$.

The maximal time development of the $\mathbb{R}\mathrm{P}^3$ initial data set is now obtained by making the following identification on the Kruskal manifold:

$$J : (T, X, \theta, \varphi) \mapsto (T, -X, \pi - \theta, \varphi + \pi).\tag{38}$$

It generates a freely acting group $\mathbb{Z}_2$ of smooth isometries which preserve space- as well as time-orientation. Hence the quotient is a smooth space- and time-orientable manifold, the $\mathbb{R}\mathrm{P}^3 geon$.[11] Its conformal diagram is just given by cutting away the $X < 0$ part (everything to the left of the vertical $X = 0$ line) in Fig. 7 and taking into account that each point on the remaining edge, $X = 0$, now corresponds to a 2-sphere with antipodal identification, i.e. a $\mathbb{R}\mathrm{P}^2$ (which is not orientable). The spacelike hypersurface $T = 0$ has now the topology of the once punctured $\mathbb{R}\mathrm{P}^3$. In the left picture of Fig. 7 this corresponds to cutting away the lower half and eliminating the inner boundary 2-sphere $X = 0$ by identifying antipodal points. The latter then becomes a minimal one-sided non-orientable surface in the orientable space-section of topology $\mathbb{R}\mathrm{P}^3 - \{\text{point}\}$. The $\mathbb{R}\mathrm{P}^3$ geon isometrically contains the exterior Schwarzschild spacetime (region I) with timelike Killing field K. But K ceases to exits globally on the geon spacetime since it reverses direction under (38).

3 X without X

3.1 Mass without mass

At the end of Section 2.6 we already explained in what sense (active gravitational) mass emerges from pure topology and the constraints implied by Einstein's equation. Physically this just means that localised configurations of

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[10]That K is Killing is immediate, since (33) shows that ρ depends only on the combination $X^2 - T^2$ which is clearly annihilated by K.

[11]The $\mathbb{R}\mathrm{P}^3$ geon is different from the two mutually different 'elliptic interpretations' of the Kruskal spacetime discussed in the literature by Rindler, Gibbons, and others. In [47] the identification map considered is $J' : (T, X, \theta, \varphi) \mapsto (-T, -X, \theta, \varphi)$, which gives rise to singularities on the set of fixed-points (a 2-sphere) $T = X = 0$. Gibbons [17] takes $J'' : (T, X, \theta, \varphi) \mapsto (-T, -X, \pi - \theta, \varphi + \pi)$, which is fixed-point free, preserves the Killing field (37) (which our map J does not), but does not preserve time-orientation. J'' was already considered in 1957 by Misner & Wheeler (Section 4.2 in [41]), albeit in isotropic Schwarzschild coordinates already mentioned above, which only cover the exterior regions I and III of the Kruskal manifold.

overall non-vanishing mass/energy may be formed from the gravitational field alone. With some care one may say that such solutions represent bounded states of gravitons ('graviton balls'). However, they cannot be stable since Gravitational solitons do not exist (in four spacetime dimensions)!

If Σ is topologically non-trivial, Gannon's theorem [16] (already discussed above) implies in full generality that the spacetime is singular (geodesically incomplete). The non-existence of vacuum, stationary, asymptotically flat spacetimes with non-vanishing mass, where the spacetime topology is $\mathbb{R} \times \Sigma$ and where Σ has only one end (spatial infinity), follows immediately from the expression (24) for the overall mass. Indeed, converting the surface integral (24) into a space integral via Stokes' theorem and using that $d \star dK^\flat$ is proportional to the spacetime's Ricci tensor shows[12] that the expression vanishes identically due to the source-free Einstein equation. This generalises an older result due to Einstein & Pauli [10] and is known as 'Lichnerowicz theorem', since Lichnerowicz first generalised the Einstein & Pauli result from static to stationary spacetimes, albeit using a far more involved argument than that given here (see [37], livre premier, chapitre VIII).

Most interestingly, this non-existence result ceases to be true in higher dimensions, as is exemplified by the existence of so-called Kaluza-Klein monopoles [50, 29], which are non-trivial, regular, static, and 'asymptotically flat' solutions to the source-free Einstein equations in a five-dimensional spacetime. The crucial point to be observed here is that the Kaluza-Klein spacetime is 'asymptotically flat' in the sense that it is asymptotically flat in the ordinary sense for three spatial directions, but not in the added fourth spatial direction, which is topologically a circle. Had asymptotic flatness in n dimensional spacetime been required for all $n-1$ spatial directions, no such solution could exist [5].

In this connection it is interesting to note that in their paper [10], Einstein & Pauli actually claim to show the non-existence of soliton-like solutions in all higher dimensional Kaluza-Klein theories even though they require asymptotic flatness in three spatial directions. But closer inspection reveals that their proof, albeit correct, invokes an additional and physically unjustified topological hypothesis that is violated by Kaluza-Klein monopoles. This is explained in more detail in [27]. Hence we may take Kaluza-Klein monopoles as a good example for the generation of mass and also magnetic charge in the framework of pure (higher dimensional!) General Relativity without any sources.

3.2 Momenta without momenta

Source free solutions with linear and angular momenta are also not difficult to obtain. Let us here just note a simple way of how to arrive at flux-integral expressions for these quantities. Let again (h, π) be a data set which is asymptotically flat on Σ with one end. Let ξ be a vector field on Σ that tends to the

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[12]One uses the Killing identity $\nabla_a \nabla_b K_c = K_d R^d{}_{a\,bc}$ to convert the second derivatives of $K^\flat$ into terms involving no derivatives and the Riemann tensor.

generator of an asymptotic isometry at infinity, that is, either a translation or a rotation. The corresponding linear or angular momentum is then just given by the usual *momentum map* corresponding to ξ:

$$\xi \mapsto \int_{\Sigma} d^3x\, \pi^{ab}\, L_{\xi} h_{ab} =: p_{\xi}\,, \tag{39}$$

where the right hand side is considered as a function on phase space $T^*\mathrm{Riem}(\Sigma)$. Using the momentum constraint, $\nabla_a \pi^{ab} = 0$, an integration by parts in (39) converts it into a flux integral at spatial infinity which, re-expressing π in terms of K, reads

$$p_{\xi} := \lim_{R \to \infty} \left\{ \frac{1}{8\pi} \int_{S_R} \left(K_{ij} - h_{ij} h^{ab} K_{ab}\right) \xi^i n^j\, d\Omega \right\}. \tag{40}$$

This is the well known expression for the ADM (Arnowitt, Deser, Misner) linear and angular momentum in geometric units [13].

Obviously there cannot exist a non-trivial asymptotically flat initial data set with an exact translational symmetry (because that translation could shift any local lump of curvature arbitrarily far into the asymptotically flat region, so that the curvature must be zero). But there may be such data sets with exact rotational symmetry. In that case, if ξ is the rotational Killing field and $\xi^{\flat} := g(\xi, \cdot)$ its associated one form in spacetime, a much simpler expression for angular momentum is given by the Komar integral [34]:

$$p_{\xi} := \lim_{R \to \infty} \left\{ \frac{1}{16\pi} \int_{S_R} \star\, d\xi^{\flat} \right\}, \tag{41}$$

where, as before, d is the exterior differential in spacetime and $\star$ the Hodge dual with respect to the spacetime metric g. Again $d \star d\xi^{\flat}$ is proportional to the spacetime's Ricci tensor and hence zero, since spacetime is assumed to satisfy the source free Einstein equation. Hence Stokes' theorem implies that if Σ has only one end (spatial infinity) and the solution is regular in the interior, p_{ξ} must be zero. Therefore there cannot exist regular data set which give rise to rotationally symmetric solutions with non-vanishing angular momenta. A minimal relaxation is given by data sets which are *locally* symmetric, that is, in which a rotational Killing field exists up to sign. This slight topological generalisation indeed suffices to render the non-existence argument just given insufficient. Such data sets with net angular momentum have been constructed in [14].

3.3 Charge without charge

One case of 'charge without charge' is clearly given by the Kaluza-Klein monopoles mentioned above. Here we wish to stick to four spacetime di-

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[13]That is, linear momentum has the unit of length (like mass) and angular momentum of length-squared. The are converted into ordinary units through multiplication with c/G.

mensions and ask whether electric or magnetic charge can emerge from the Einstein-Maxwell equations without sources for the Maxwell field (in distinction to above, the energy-momentum tensor for the Maxwell field now acts as a source for the gravitational field).

If F is the 2-form on spacetime that represents the electromagnetic field, then the electric and magnetic charges q_e and q_m inside a 2-sphere S are respectively given by

$$q_e \;=\; \frac{1}{4\pi} \int_S \star dF \,, \tag{42a}$$

$$q_m \;=\; \frac{1}{4\pi} \int_S dF \,. \tag{42b}$$

Since $dF = 0$ (homogeneous Maxwell equation) and $d \star F = 0$ (inhomogeneous Maxwell equation with vanishing sources) these integrals depend only on the homology class on S. This seems to imply that if spacetime has a regular interior, i.e. is of topology $\mathbb{R} \times \Sigma$ and Σ has only one end (spatial infinity), there will be no global net charge. The only possibility seems to be that there are local charges, like, e.g., if Σ has a wormhole topology $S^1 \times S^2 - \{\text{point}\}$, as shown by the lower-left drawing in Fig. 6, where the flux lines thread through the wormhole. The homology class of 2-spheres that contain both wormhole mouths has zero charge, whereas the two individual wormhole mouths have equal and opposite charges associated to them.

However, this is not the only possibility! Our argument above relied on Stokes' theorem, which for ordinary forms presumes that the underlying manifold is orientable. On a non-orientable manifold it only holds true for forms of density weight one, i.e. sections of the tensor bundle of forms twisted with the (now non trivial) orientation bundle; see, e.g., § 7 of [3]. This means that the argument for the non-existence of global charges can be extended to the non-orientable case for $\star F$ (which is a two form of density weight one) but not to F (which is a two form of density weight zero). Hence net electric charges cannot, but magnetic[14] can exist [49][14]. A simple illustration of how orientability comes into this is given by Fig. 8.

As stated above, in the non-orientable case, Stokes theorem (here in three dimensions) continues to apply to two-forms of density weight one (e.g. the Hodge duals of one forms) and does not apply to two-forms of density weight zero, like the magnetic two form or, equivalently, its Hodge dual, which is a pseudo-vector field $\mathbf{B}$ of zero divergence. We apply Stokes' theorem to suitable orientable submanifolds as explained in the caption to Fig. 8. We obtain, denoting the flux of $\mathbf{B}$ through a surface S with orientation O by $\Phi(\mathbf{B}, S, O)$,

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[14]The distinction between electric and magnetic is conventional in Einstein-Maxwell theory without sources for the Maxwell field, since the energy-momentum tensor T for the latter is invariant under duality rotations which rotate between F and $\star F$ according to $\omega \mapsto \exp(i\varphi)\,\omega$, where $\omega := F + i \star F$. Since $T_{\mu\nu} \propto \omega_{\mu\lambda}\bar{\omega}_\nu{}^\lambda$, where an overbar denotes complex conjugation, the invariance of T is immediate.
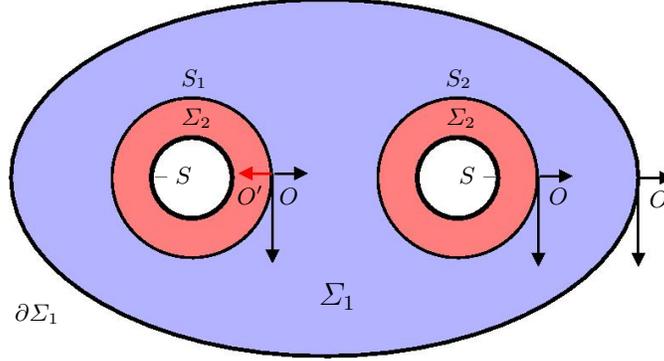
Fig. 8. Consider the three-dimensional region that one obtains by rotating this figure about the central horizontal axis of symmetry. The two inner boundary spheres S are to be identified in a way so that their induced orientations O match, e.g. by simple translation. (In this two-dimensional picture an orientation is represented by an ordered two-leg, where the ordering is according to the different lengths of the legs.) This results in a *non-orientable* manifold with single outer boundary component $\partial\Sigma_1$, corresponding to the non orientable wormhole. In the text we apply Stokes' theorem twice to two orientable submanifolds: First, to the heavier shaded region bounded by the outer 2-sphere $\partial\Sigma_1$ with orientation O and the inner two 2-spheres S_1 and S_2 with like orientations O as indicated. Second, to the lightly shaded cylindrical region labelled by Σ_2 that is bounded by the two 2-spheres S_1 and S_2 with opposite orientations O' and O respectively.

$$\Phi(\mathbf{B}, \partial\Sigma_1, O) + \Phi(\mathbf{B}, S_1, O) + \Phi(\mathbf{B}, S_2, O) = 0 \tag{43}$$

in the first case, and

$$\Phi(\mathbf{B}, S_1, O') + \Phi(\mathbf{B}, S_2, O) = 0 \tag{44}$$

in the second. Using the obvious fact that the flux integral changes sign if the orientation is reversed, i.e., $\Phi(\mathbf{B}, S_1, O') = -\Phi(\mathbf{B}, S_1, O)$, we get

$$\Phi(\mathbf{B}, \partial\Sigma_1, O) = -2\,\Phi(\mathbf{B}, S_1, O)\,. \tag{45}$$

So in order to get a non-zero global charge, we just need to find a divergenceless pseudo-vector field on Σ_1 with non vanishing flux through S, which can be arranged.

Note that the trick played here in using non orientable Σ would not work for the Komar integrals (24) (41), since the Hodge map $\star$ turns the ordinary two form $dK^\flat$ (or $d\xi^\flat$) of density weight zero into the two form $\star dK^\flat$ (or $\star d\xi^\flat$) of density weight one, so that Stokes' theorem continues to hold in these cases for for non-orientable Σ by the result cited above.

3.4 Spin without spin

In my opinion the by far most surprising case of '$X without X$' if that where X stands for *spin*, i.e., half-integral angular momentum. It was certainly not anticipated by Misner, Thorne, and Wheeler, who in their otherwise most comprehensive book [42] were quite lost in trying to answer their own question of how "to find a natural place for spin $1/2$ in Einstein's standard geometro-dynamics (Box 44.3 in [42]). A surprising answer was offered 8 years later, in 1980, by John Friedman and Rafael Sorkin [15].

It is often said that the need to go from the group $SO(3)$ of spatial rotations to its double (= universal) cover, $SU(2)$, is quantum-mechanical in origin and cannot be understood on a classical basis. In some sense the mathematical facts underlying the idea of 'spin $1/2$ from gravity' disprove this statement. They imply that if the 3-manifold Σ has a certain topological characteristic, the asymptotic symmetry group for isolated systems (modelled by spatially asymptotically flat data) is not the Poincaré group (inhomogeneous Lorentz group) in the sense of [2], but rather its double (= universal) covers – for purely topological reasons! Let us try to explain all this in more detail.

Recall that in Quantum Mechanics the possibility for this enlargement (central extension) of a classical symmetry group has its origin in the assumption that the phase of the complex wave function is a redundant piece of description (i.e. unobservable), at least for states describing isolated systems, so that symmetry groups should merely act on the space of rays rather than on Hilbert space by proper representations. Hence it is sufficient for the symmetry group to be implemented by so-called ray representations, which in case of the rotation group are in bijective correspondence to proper representations of its double (= universal) cover group. Accordingly, in Quantum Mechanics, there exist physically relevant systems whose state spaces support proper representations of $SU(2)$ but not of $SO(3)$: These are just the systems whose angular momentum is an odd multiple of $\hbar/2$. We will say that such systems admit *spinorial states*.

Spinorial states are not necessarily tight to the usage of spinors. They also have a place in ordinary Schrödinger quantisation, i.e. for systems whose quantum state space is represented by the Hilbert space of square integrable functions over the classical configuration space Q, which here and in what follows is understood to be the reduced configuration space in case constraints existed initially. Then, spinorial states exist if the following conditions hold:

S1 Q is not simply connected.
S2 The (say left) action $SO(3) \times Q \to Q$, $(g,q) \mapsto g \cdot q$, of the ordinary rotation group $SO(3)$ on the classical configuration space Q is such that if $\gamma : [0, 2\pi] \mapsto SO(3)$ is any full 360-degree rotation about some axis, then the loop $\Gamma := \gamma \cdot q$ in Q, based at $q \in Q$, is not contractible, i.e. defines a non-trivial element (of order two since γ traversed twice is contractible in $SO(3)$) in $\pi_1(Q, q)$, the fundamental group of Q based at q. It is not hard to see that this property (of being non-contractible) is independent of the

basepoint $q \in Q$ within the same path component of Q, though it may vary if one goes from one path component to another (as in the Skyrme model mentioned below). See [20], in particular the proof of Lemma 1.

The reason for the existence of spinorial states in such situations lies in possible generalisations of Schrödinger quantisation if the domain for the wave function is a space, Q, whose fundamental group is non trivial. The idea of generalisation is to define the Schrödinger function on the universal cover space $\bar{Q}$ (i.e. the Hilbert space is the space of square-integrable functions on $\bar{Q}$) but to restrict the observables to those that commute with the unitary action of the deck transformations. The latter then form a discrete gauge group isomorphic to the fundamental group of Q. The Hilbert space decomposes into superselection sectors which are labelled by the equivalence classes of unitary irreducible representations. The sector labelled by the trivial class is isomorphic to that of ordinary Schrödinger quantisation on Q, whereas the other sectors are acquired through the generalisation discussed here.

This is related to, but not identical with, another generalisation that is usually mentioned in the context of geometric quantisation. There one generalises Schrödinger quantisation by considering quantum states as square-integrable *sections* in a complex line bundle over Q (rather than just complex-valued functions on Q). This leads to additional sectors labelled by the equivalence classes of complex line bundles, which are classified by $H^2(Q, \mathbb{Z})$, the second cohomology group of Q with integer coefficients (see, e.g., [56]).

In generalised Schrödinger quantisation spinorial states will correspond to particular such new sectors. To make this more precise in the geometric-quantisation picture, we recall that $H^2(Q, \mathbb{Z})$, being a finitely generated abelian group, has the structure

$$H^2(Q, \mathbb{Z}) \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{\text{free part}} \oplus \underbrace{\mathbb{Z}_{p_1} \oplus \cdots \mathbb{Z}_{p_n}}_{\text{torsion part}} . \qquad (46)$$

The number of factors $\mathbb{Z}$ in the free part is called the second Betti number and the number n of cyclic groups the second torsion number. For this to be well defined we have to agree that each of the integers p_i should be a power of a prime.[15] Spinorial states are then given by sections in all those line bundles which represent *a particular* $\mathbb{Z}_2$ factor in the torsion part of its decomposition according to (46) non trivially.

We said 'a particular $\mathbb{Z}_2$ factor'. Which one? The answer is: That one, which is generated by the 360-degree rotation according to criterion S2 above. To understand this, we remark that the torsion part of $H^2(Q, \mathbb{Z})$ can be understood in terms of the fundamental group. More precisely, the torsion part of $H^2(Q, \mathbb{Z})$ is isomorphic to the torsion part of the abelianisation of the

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[15]A classic theorem on finite abelian groups states that if p, q are integers, $\mathbb{Z}_{pq}$ is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_q$ iff p and q are coprime.

fundamental group.[16] Given that isomorphism, we can now identify the $\mathbb{Z}_2$ factor in $H^2(Q,\mathbb{Z})$ with the $\mathbb{Z}_2$ subgroup of the fundamental group that is generated by 360-degree rotations, as explained by S2.

A simple illustrative example of this is given by the rigid rotor, that is a system whose configuration space Q is the group manifold $SO(3)$, which as manifold is isomorphic to $\mathbb{R}\mathrm{P}^3$. The action of physical rotations is then given by left translation. Here we have $H^2(Q,\mathbb{Z}) \cong \mathbb{Z}_2$, i.e. it is pure torsion and, in fact, isomorphic to the fundamental group. Quantisation then leads to two sectors: Those containing states of integral spin, which are represented by ordinary square integrable functions on Q, and those containing half-integral spin, represented by square integrable sections in the unique non-trivial line bundle over $Q \cong \mathbb{R}\mathrm{P}^3$.

More sophisticated field theoretic examples for this mechanism are given by so-called non-linear sigma models, in which the physical states are given by maps from physical space into some non-linear space of field values, like, e.g., a sphere. A particular such model is the Skyrme model [48] in which the target space is the three-sphere S^3. Configurations of finite energy must map spatial infinity (physical space is $\mathbb{R}^3$) into a single point of S^3 so that Q decomposes into a countably infinity of path components according to the winding number of that map. In the Skyrme model, which serves to give an effective description of baryons, this winding number corresponds to the baryon number. The fundamental group of each path component is isomorphic to the fourth homotopy group of the target space S^3, which is again just $\mathbb{Z}_2$. One can now prove that the loops traced through by 360-degree rotations are contractible in the components of even winding numbers and non-contractible in the components of odd winding numbers [20]. Hence spinorial states exist for odd baryon numbers, as one should expect on physical grounds.

These examples differ from those in General Relativity insofar as in the latter spinorial states usually exist only in non-abelian sectors, i.e. sectors that correspond to higher-dimensional unitary irreducible representations of the fundamental group [22]. An example will be mentioned below. For that reason we made the distinction between the first and the second (geometric quantisation) method of generalising Schrödinger quantisation, since non-abelian sectors are obtained in the first, but not in the second method, which is only sensitive to the abelianisation of the fundamental group. That the restriction to abelian sectors is unnecessary and unwarranted is further discussed in [23]

The geometric-topological situation underlying the existence of spinorial states in General Relativity is this [15]: Consider a 3-manifold Σ with one regular end, so as to describe an asymptotically flat isolated system without internal infinities. Here 'regular' means that that the one-point compactifi-

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[16]This follows in two steps: First one recalls $H^2(Q,\mathbb{Z})$ is isomorphic to the direct sum of the free part of $H_2(Q,\mathbb{Z})$ and the torsion part of $H_1(Q,\mathbb{Z})$ (universal coefficient theorem). Second one uses that $H_1(Q,\mathbb{Z})$ is isomorphic to the abelianisation of the fundamental group (Hurewicz' theorem).

cation $\bar{\Sigma}$ of Σ is again a manifold. This means that Σ contains a compact subset the complement of which is a cylinder $\mathbb{R} \times S^2$. A physical rotation of the system so represented is then given by a diffeomorphism whose support is entirely on that cylinder and rotates the S^2 at one end relative to the S^2 at the other end by full 360 degrees; see Fig. 9.
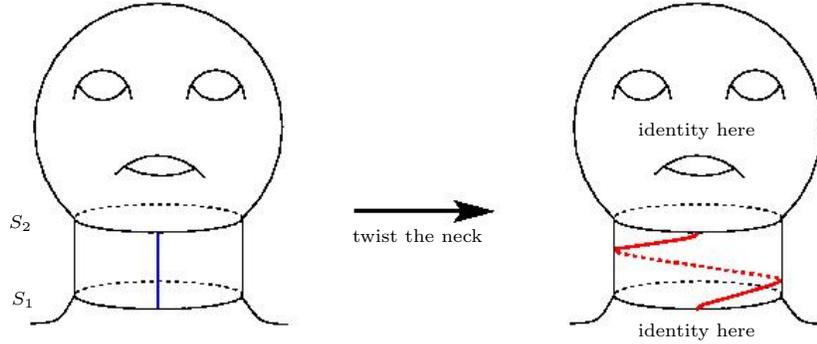

Fig. 9. A full 360-degree rotation of the part of the manifold above the 2-sphere S_2 relative to the part below the 2-sphere S_1 is given by a diffeomorphism with support on the cylinder region bounded by S_1 and S_2 that rotates one bounding sphere relative to the other by 360 degrees ('twisting the neck' by 360 degrees).
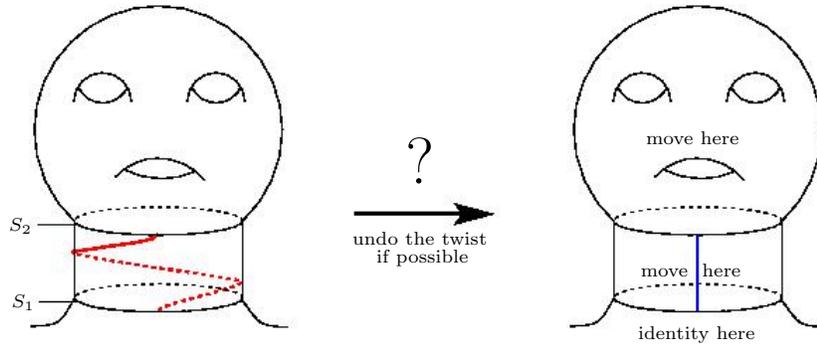

Fig. 10. Keeping all points below S_1 fixed but now allowing the points above S_2 to move, the neck-twist may or may not be continuously undone through a continuous sequence of diffeomorphisms whose support is entirely above the sphere S_1. If it cannot be undone in this fashion, the manifold above S_2, or rather its one-point compactification, is called 'spinorial'. Do not be misled to think that you can just undo it by rigidly rotating the upper part in the embedding space shown here, since this will generally not define a diffeomorphism of the manifold itself.

The questions is now this: Is this diffeomorphism in the identity component of those diffeomorphisms that fix the 2-sphere at 'spatial infinity'? See Fig. 10 for further illustration. The answer to this question just depends on the topology on Σ and is now known for all 3 manifolds.[17] Roughly speaking, the generic case is that spinorial spates are allowed. More precisely, those 3-manifolds $\bar{\Sigma}$ (from now on we represent the manifolds by their one-point compactifications in order to talk about closed spaces) which do not allow for spinorial states are connected sums of lens spaces and handles ($S^1 \times S^2$). This is a very nice (though rather non-trivial) result insofar, as the non-spinoriality of these spaces as well as their connected sums is easy to visualise. Hence one may say that there are no other non-spinorial manifolds than the 'obvious' ones.

Take, for example, the simplest lens space[18] $L(2,1)$, which is just real projective 3-space $\mathbb{R}P^3$. It can be imagined as a solid ball B in $\mathbb{R}^3$ with antipodal points on the 2-sphere boundary identified. Think of an inner point, say the centre, of B as the point at infinity, surround it by a small spherical shell whose inner boundary is the 2-sphere S_1 and outer boundary the 2-sphere S_2 (above we called it a 'cylinder' since its topology is $\mathbb{R} \times S^2$). Now perform a full 360-degree rotation of S_1 against S_2 with support inside the shell. Can this diffeomorphism been undone through a continuous sequence of diffeomorphims that fix all points inside the inner sphere S_1? Clearly it can: Just rigidly rotate the outside to undo it. The crucial point is that this rigid rotation is compatible with the boundary identification and hence does indeed define a diffeomorphism of $\mathbb{R}P^3$. Essentially the same argument applies to all other 'obvious cases'.

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[17]To decide this entails some subtle issues, like whether to diffeomorphisms that are homotopic (continuously connected through a one-parameter family of *continuous* maps) are also isotopic (continuously connected through a one-parameter family of *homeomorphisms*) and then also diffeotopic (continuously connected through a one-parameter family of *diffeomorphims*). The crucial question is whether homotopy implies isotopy, which is not at all obvious since on a homotopy the interpolating maps connecting two diffeomorphism are just required to be continuous, that is, they need not be continuously invertible as for an isotopy. For example, the inversion $I(\mathbf{x}) = -\mathbf{x}$ in $\mathbb{R}^n$ is clearly not isotopic to the identity, but homotopic to it via $\phi_t(\mathbf{x}) = (1 - 2t)\mathbf{x}$ for $t \in [0,1]$. Then $\phi_0 = \mathrm{id}$, $\phi_1 = I$, and only at $t = 1/2$ does the map ϕ_t cease to be invertible.

[18] The definition of lens spaces $L(p,q)$ in 3 dimensions is $L(p,q) = S^3/\sim$, where (p,q) is a pair of positive coprime integers with $p > 1$, $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$, and $(z_1, z_2) \sim (z_1', z_2') \Leftrightarrow z_1' = \exp(2\pi i/p)z_1$, and $z_2' = \exp(2\pi i\, q/p)z_2$. One way to picture the space is to take a solid ball in $\mathbb{R}^3$ and identify each points on the upper hemisphere with a points on the lower hemisphere after a rotation by $2\pi q/p$ about the vertical symmetry axis. In this way each set of p equidistant points on the equator is identified to a single point. The fundamental group of $L(p,q)$ is $\mathbb{Z}_p$, i.e. independent of q, and the higher homotopy groups are those of its universal cover, S^3. This does, however, not imply that $L(p,q)$ is homotopy equivalent, or even homeomorphic, to $L(p,q)$. The precise relation will be stated below.

In contrast, it is much more difficult to prove that such an undoing is impossible, i.e. the spinoriality of a given manifold. Needless to say, the fact that you cannot easily visualise a possible undoing of a 360 degree twist does not mean it does not exist. A simple and instructive example is given by the spherical space form S^3/D_8^*, where D_8^* is the 8-element non-abelian subgroup of $SU(2)$ that doubly covers (via the double cover $SU(2) \to SO(3)$) the 4-element abelian subgroup of $SO(3)$ that is given by the identity and the three 180-degree rotations about the mutually perpendicular x, y, and z axes. Identifying S^3 with $SU(2)$ the quotient S^3/D_8^* is defined by letting D_8^* act through, say, right translations. Since $SU(2)$ is also the group of unit quaternions, D_8^* can be identified with its subgroup $\{\pm 1, \pm i, \pm j, \pm k\}$, where i, j, k denote the usual unit quaternions (they square to -1 and $ij = k$ and also cyclic permutations thereof). A way to visualise S^3/D_8^* is given in Fig. 11. Note that if the 2-dimensional boundary of the cube is smoothly deformed to
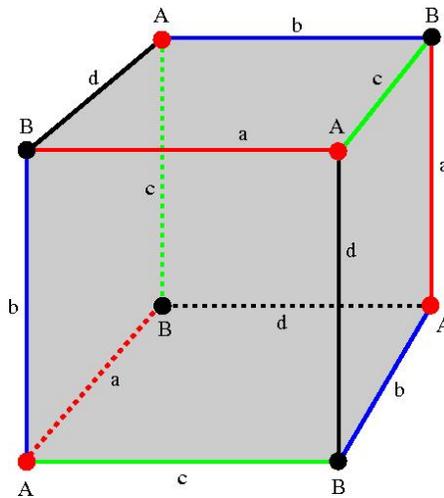

Fig. 11. The manifold S^3/D_8^* is obtained from a solid cube by identifying opposite faces after a relative 90-degree rotation about the axis connecting their midpoints. In the picture shown here the identifying motion between opposite faces is a right screw, giving rise to the identifications of edges and vertices as labelled in the picture.

a round 2-sphere a rigid rotation in the embedding $\mathbb{R}^3$ would still not be compatible with the boundary identifications. In fact, it is known that S^3/D_8^* is spinorial (see [21] for more information and references). Here we just remark that D_8^* has five equivalence classes of unitary irreducible representations: Four one-dimensional and a single two-dimensional one. The spinorial sector is that corresponding to the latter, that is, it is a non-abelian sector.

Another remarkable property of S^3/D_8^* is that it is *chiral*, that is, it does not admit for orientation reversing self-diffeomorphisms; see [21] for some in-

formation which 3-manifolds are chiral and [43] for a recent systematic investigation of chirality in all dimensions. This means that if we had chosen the map that identifies opposite faces of the cube shown in Fig. 11 to be a left rather than right screw, we would have obtained a manifold that is not orientation-preserving diffeomorphic to the the one originally obtained, though they are clearly orientation-reversing diffeomorphic as they are related by a simple reflection at the origin of the embedding $\mathbb{R}^3$. Being chiral seems to be more the rule than the exception for 3-manifolds [21].

4 Further developments

In the last subsection we have learnt that the fundamental group of the configuration space of the gravitational field will give rise to sectors with potentially interesting physical interpretations. Hence it seems natural to generally ask: What is the fundamental group of the configuration space associated to a manifold Σ? The last question can be given an elegant abstract answer, though not one that will always allow an easy characterisation (determination of the isomorphicity class) of the group. The abstract answer is in terms of a presentation of a certain mapping-class group and comes about as follows: Consider the 3-manifold Σ which we assume to have one regular end. Hence its one-point compactification, $\bar{\Sigma}$, is a manifold. Next consider the group of diffeomorphisms $\mathrm{Diff}_\mathrm{F}(\Sigma)$ that fix a prescribed point $p \in \bar{\Sigma}$ as well as all vectors in the tangent space at this point. It is useful to think of p as the 'point at infinity', i.e. the point that we added for compactification, for then it is intuitively clear that $\mathrm{Diff}_\mathrm{F}(\bar{\Sigma})$ corresponds to those diffeomorphism of Σ that tend to the identity as one moves to infinity within the single end. In order to have that picture in mind, we will from now on write ∞ for the added point p. The configuration space of the gravitational field on Σ can then be identified with the space of Riemannian metrics on $\bar{\Sigma}$, $\mathrm{Riem}(\bar{\Sigma})$, modulo the identifications induced by $\mathrm{Diff}_\mathrm{F}(\bar{\Sigma})$, i.e.

$$Q(\Sigma) = \mathrm{Riem}(\bar{\Sigma})/\mathrm{Diff}_\mathrm{F}(\bar{\Sigma}) \,. \tag{47}$$

Now, it is true that $\mathrm{Diff}_\mathrm{F}(\bar{\Sigma})$ acts freely on $\mathrm{Riem}(\bar{\Sigma})$ (there are no non-trivial isometries on a Riemannian manifold that fix a point and the frame at that point) and that this action admits a slice (see [9]). Hence $\mathrm{Riem}(\bar{\Sigma})$ is a principle fibre bundle with group $\mathrm{Diff}_\mathrm{F}(\bar{\Sigma})$ and base $Q(\Sigma)$ ([12, 13]). But $\mathrm{Riem}(\bar{\Sigma})$ is contractible (being an open positive convex cone in the vector space of smooth sections of symmetric tensor fields of rank two over $\bar{\Sigma}$). Hence the long exact-sequence of homotopy groups for the fibration $\mathrm{Diff}_\mathrm{F}(\bar{\Sigma}) \to \mathrm{Riem}(\bar{\Sigma}) \to Q(\Sigma)$ implies the isomorphicity of the n th homotopy group of the fibre $\mathrm{Diff}_\mathrm{F}(\bar{\Sigma})$ with the $n+1$ st homotopy group of the base $Q(\Sigma)$. In particular, the first homotopy group (i.e. the fundamental group) of $Q(\Sigma)$ is isomorphic to the zeroth homotopy group of the group $\mathrm{Diff}_\mathrm{F}(\bar{\Sigma})$. However, the latter is just

the quotient $\mathrm{Diff}_\mathrm{F}(\bar{\Sigma})/\mathrm{Diff}_\mathrm{F}^0(\bar{\Sigma})$, where $\mathrm{Diff}_\mathrm{F}^0(\bar{\Sigma}) \subset \mathrm{Diff}_\mathrm{F}(\bar{\Sigma})$ is the normal subgroup formed by the connected component of $\mathrm{Diff}_\mathrm{F}(\bar{\Sigma})$ that contains the identity. In this way we finally arrive at the result that the fundamental group of $Q(\Sigma)$ is isomorphic to a *mapping-class* group:

$$\pi_1\big(Q(\Sigma)\big) \cong \mathrm{Diff}_\mathrm{F}(\bar{\Sigma})/\mathrm{Diff}_\mathrm{F}^0(\bar{\Sigma}). \tag{48}$$

This is a very interesting result in its own right. It contains the mathematical challenge to characterise $\mathrm{Diff}_\mathrm{F}(\bar{\Sigma})/\mathrm{Diff}_\mathrm{F}^0(\bar{\Sigma})$. A way to attack this problem is to use the fact that each element in $\mathrm{Diff}_\mathrm{F}(\bar{\Sigma})$ defines a self-map $\pi_1(\bar{\Sigma}, \infty)$ just by mapping loops based at ∞ to their image loops, which are again based at ∞ since elements of $\mathrm{Diff}_\mathrm{F}(\bar{\Sigma})$ keep that point fixed. Since in this fashion homotopic loops are mapped to homotopic loops, this defines indeed a map on $\pi_1(\bar{\Sigma}, \infty)$ which is, in fact, an automorphism. Moreover, elements in the identity component $\mathrm{Diff}_\mathrm{F}^0(\bar{\Sigma})$ give rise to the trivial automorphisms. This is obvious, since the images of a loop under continuously related diffeomorphisms will in particular result in homotopic loops. Hence we have in fact a homomorphism from $\mathrm{Diff}_\mathrm{F}(\bar{\Sigma})/\mathrm{Diff}_\mathrm{F}^0(\bar{\Sigma})$ into the automorphism group of $\pi_1(\bar{\Sigma}, \infty)$:

$$h : \mathrm{Diff}_\mathrm{F}(\bar{\Sigma})/\mathrm{Diff}_\mathrm{F}^0(\bar{\Sigma}) \to \mathrm{Aut}\big(\pi_1(\bar{\Sigma}, \infty)\big). \tag{49}$$

The strategy is now this: Assume we know a presentation of $\mathrm{Aut}\big(\pi_1(\bar{\Sigma}, \infty)\big)$, that is, a characterisation of this group in terms of generators and relations. Then we aim to make useful statements about the kernel and image of the map in (49) so as to be able to derive a presentation for $\mathrm{Diff}_\mathrm{F}(\bar{\Sigma})/\mathrm{Diff}_\mathrm{F}^0(\bar{\Sigma})$. A simple but non-trivial example will be given below. We recall that $\bar{\Sigma}$ is a unique connected sum of prime manifolds and that $\pi_1(\bar{\Sigma})$ is the free product of the fundamental groups of the primes. Since (finite) presentations for the automorphism group of a free product can be derived if (finite) presentations for the automorphism groups of the factors are known [18], we in principle only need to know the latter.

Another mathematically interesting aspect connected with (48) is the fact that $\mathrm{Diff}_\mathrm{F}(\bar{\Sigma})/\mathrm{Diff}_\mathrm{F}^0(\bar{\Sigma})$ is a topological invariant of $\bar{\Sigma}$ which is *not* a homotopy invariant [38]. Hence (48) implies that $\pi_1\big(Q(\Sigma)\big)$, too, is a topological invariant of $\bar{\Sigma}$ which is not homotopy invariant, i.e. it might tell apart 3-manifolds which are homotopically equivalent but not homeomorphic. There are indeed examples for this to happen. Here is one: Recall that lens spaces (see footnote 18) $L(p, q)$ and $L(p, q')$ are homotopy equivalent iff $qq' = \pm n^2 \,(\mathrm{mod}\,p)$ for some integer n (theorem 10 in [54]) and homeomorphic[19] iff (all four possibilities) $q' = \pm q^{\pm 1} \,(\mathrm{mod}\,p)$ (here all four possibilities of combinations of $\pm$ signs are considered). On the other hand, the mapping-class group $\mathrm{Diff}_\mathrm{F}(\bar{\Sigma})/\mathrm{Diff}_\mathrm{F}^0(\bar{\Sigma})$ for $L(p, q)$ is $\mathbb{Z} \times \mathbb{Z}$ if $q^2 = 1 \,(\mathrm{mod}\,p)$ with $q \neq \pm 1 \,(\mathrm{mod}\,p)$ and $\mathbb{Z}$ in the remaining cases for $p > 2$ (see Table IV on p. 591 of [55]). Take now, as an example,

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[19]As regards the notion of chirality, an interesting refinement of this statement is that $L(p, q)$ and $L(p, q')$ are *orientation-preserving* homeomorphic iff $q' = q^{\pm 1} \,(\mathrm{mod}\,p)$ [45].

$p = 15$, $q = 1$, and $q' = 4$. Then the foregoing implies that $L(15,1)$ and $L(15,4)$ are homotopic but not homeomorphic and have different mapping-class groups.

Finally we give an example for a presentation and its pseudo-physical interpretation for $\mathrm{Diff}_\mathrm{F}(\bar{\Sigma})/\mathrm{Diff}_\mathrm{F}^0(\bar{\Sigma})$. Consider the connected sum (denoted by $\#$) of two real projective spaces $\mathbb{R}\mathrm{P}^3$. This manifold may be visualised as explained in Fig. 12.
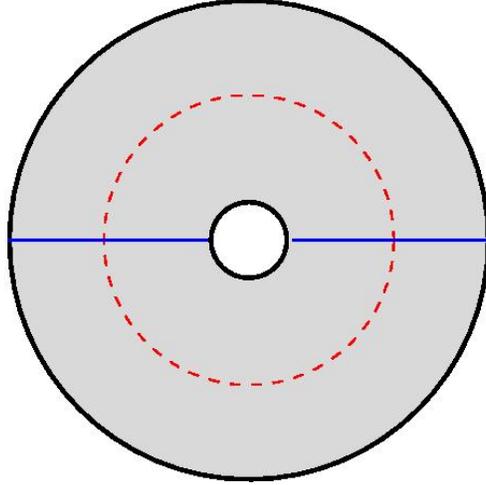

Fig. 12. The connected sum $\mathbb{R}\mathrm{P}^3 \# \mathbb{R}\mathrm{P}^3$ between two real projective spaces may be visualised as a spherical shell (here the grey-shaded region) where antipodal points on each of the two 2-sphere boundaries, S_1 and S_2, are identified. The 2-dimensional figure here should be rotated about the horizontal symmetry axis. The two horizontal line segments shown form a circle in view of the antipodal identifications. It shows that $\mathbb{R}\mathrm{P}^3 \# \mathbb{R}\mathrm{P}^3$ is a circle bundle over $\mathbb{R}\mathrm{P}^2$. The dotted circle, which upon rotation of the figure becomes a 2-sphere can be thought of as the 2-sphere along which the connected sum between the two individual $\mathbb{R}\mathrm{P}^3$ manifolds is taken.

The fundamental group of $\mathbb{R}\mathrm{P}^3 \# \mathbb{R}\mathrm{P}^3$ is the twofold free product $\mathbb{Z}_2 * \mathbb{Z}_2$ of the fundamental group $\mathbb{Z}_2$ of the factors $\mathbb{R}\mathrm{P}^3$. With respect to the generators a, b shown in Fig. 13 or, alternatively, with respect to the generators a, c, where $c := ab$ corresponds to the loop shown by the two horizontal segments in Fig. 12, two alternative presentation of the fundamental group are given by

$$\pi_1(\mathbb{R}\mathrm{P}^3 \# \mathbb{R}\mathrm{P}^3) = \underbrace{\langle a, b \mid a^2 = b^2 = 1 \rangle}_{\cong \mathbb{Z}_2 \times \mathbb{Z}_2} = \underbrace{\langle a, c \mid a^2 = 1, \; aca^{-1} = c^{-1} \rangle}_{\cong \mathbb{Z}_2 \ltimes \mathbb{Z}} \quad (50)$$

Turning to (49) we first remark that the automorphism group of $\mathbb{Z}_2 * \mathbb{Z}_2$ is itself isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$,

$$\mathrm{Aut}(\mathbb{Z}_2 * \mathbb{Z}_2) \cong \mathbb{Z}_2 * \mathbb{Z}_2 = \langle E, S \mid E^2 = S^2 = 1 \rangle, \quad (51)$$
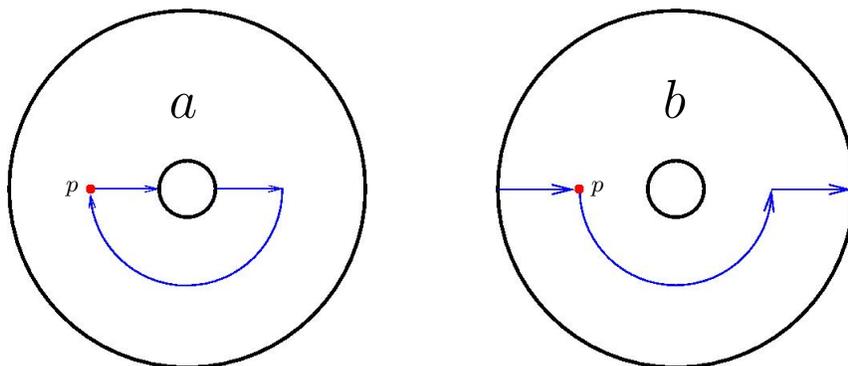
Fig. 13. Shown are two closed loops based at some point p whose homotopy classes generate the fundamental group $\mathbb{Z}_2 * \mathbb{Z}_2 \cong \mathbb{Z}_2 \ltimes \mathbb{Z}$ of the manifold $\mathbb{RP}^3 \# \mathbb{RP}^3$.

where the two generators E, S can be identified by stating their action on the generators a, b of the fundamental group:

$$E : (a, b) \mapsto (b, a), \quad S : (a, b) \mapsto (a, aba^{-1}). \tag{52}$$

It may now be shown that the map h in (49) is an isomorphism so that the fundamental group of the configuration space $Q(\Sigma)$ is the free product $\mathbb{Z}_2 * \mathbb{Z}_2$. Injectivity of h is not so obvious (but true) whereas surjectivity can be shown by visualising diffeomorphisms that actually realise the generators E and S of (52). For example, E can be realised by an inversion on the sphere along which the connected sum is taken (see Fig 12) (which is orientation reversing) followed by a simple reflection along a symmetry plane (so as to restore orientation preservation). Its 'physical' meaning is that of an exchange of the two diffeomorphic factors (primes) in the connected-sum decomposition. The map for S is a little harder to visualise since it mixes points between the two factors; see [26] for pictures. It can roughly be described as sliding one factor through the other and back to its original position. Here we wish to focus on the following: Given the generalisations of Schrödinger quantisations outlined above, we are naturally interested in the equivalence classes of unitary irreducible representations of $\mathbb{Z}_2 * \mathbb{Z}_2$. They can be obtained by elementary means and are represented by the four obvious one-dimensional representations where $E \mapsto \pm 1$ and $S \mapsto \pm 1$, and a continuum of 2-dimensional ones where

$$E \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad S \mapsto \begin{pmatrix} \cos\tau & \sin\tau \\ \sin\tau & -\cos\tau \end{pmatrix}, \quad \tau \in (0, \pi). \tag{53}$$

No higher dimensional ones occur. The one-dimensional representations already show that both statistics sectors exist. Moreover, the two-dimensional representations show that the diffeomorphims representing S mix the statistics sectors by an angle τ that depends on the representation class. All this

may be read as indication against a classical 'spin-statistics correlation' that one might have expected from experience with other non-linear field theories, e.g. following [11][51]. Such a connection can therefore only exist in certain sectors and the question can (and has) be asked how these sectors are selected [7, 8]. See [21, 25] for other examples with explicit presentations of $\mathrm{Diff}_{\mathrm{F}}(\bar{\Sigma})/\mathrm{Diff}_{\mathrm{F}}^{0}(\bar{\Sigma})$ where $\bar{\Sigma}$ is either the n fold connected sum of real projective spaces $\mathbb{R}\mathrm{P}^{3}$ or handles $S^{1} \times S^{2}$ and also some general statements.

From what has been said so far it clearly emerges that the enormous topological variety and complexity of 3-manifolds leave their structural traces in General Relativity, which can be used to model some of the properties in pure gravity that are usually associated with ordinary matter. This is indeed made practical use of, e.g. in modelling scattering and merging processes of black holes with data corresponding to wormhole topologies. But one should also say that the physical relevance of much of what I said later is not at all established. The aim of my presentation was to alert to the existence of these structures, leaving their physical relevance open for the time being. Somehow all this may remind one Tait's beautiful idea to model the discrete structural properties of material atoms on the properties of knots in physical space, which he thought of as knotted vortex lines in the all-embracing hypothetical ether medium. But whereas there was never formulated a fundamental dynamical theory of the ether[20] there is a well formulated and well tested theory of geometrodynamics: General Relativity. In that sense we are in a much better position than Tait was in the mid 1880s.

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[20]Maxwell's equations were thought of as a kind of effective theory that describes things on a coarse-grained scale, so that e.g. the vortex knots could be approximated by point particles.

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