Coordinate formalism on abstract Hilbert space

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Abstract

Coordinate formalism on Hilbert manifolds developed in [1] is reviewed. The results of [1] are applied to the simpliest case of a Hilbert manifold: the abstract Hilbert space. In particular, functional transformations preserving properties of various linear operators on Hilbert spaces are found. Any generalized solution of an arbitrary linear differential equation with constant coefficients is shown to be related to a regular solution by a (functional) coordinate transformation. The results also suggest a way of using generalized functions to solve nonlinear problems on Hilbert spaces.

1 Introduction

In [1] a coordinate formalism on abstract infinite-dimensional Hilbert manifolds has been introduced. By letting images of charts on a Hilbert manifold belong to arbitrary Hilbert spaces of functions we were able to find the new infinite-dimensional counterparts of notions of a basis, dual basis, orthogonal basis etc. We found that the choice of a functional Hilbert model for a Hilbert manifold (rather than the choice of a chart within a given model) is deeply similar to the choice of coordinates on a finite dimensional manifold.

In this paper we continue investigating the obtained formalism. The main attention here is on the simpliest case of a Hilbert manifold: the abstract Hilbert space.

In section 2 the main definitions of [1] are briefly reviewed.

In section 3 we consider the coordinate transformations preserving locality of a given operator. As the particular solutions of the locality equations we obtain the Fourier transform.

In section 4 the coordinate transformations preserving the derivative operator are investigated. It is shown in particular that the generalized and the regular solutions of linear differential equations with constant coefficients are related by a change of functional coordinates.

In section 5 we analyze the coordinate transformations preserving the operator of multiplication by a function. It is shown that except for the trivial transformations locality of the operator of multiplication by a function can not be preserved.

In section 6 more general coordinate transformations are considered. In particular, we show how the results of section 4 can be generalized to the case of linear differential equations with the non-constant coefficients. We also start investigating here the case of nonlinear differential equations.

The results are briefly summarized in the conclusion.

2 Linear algebra on the string space

Definition. The *string space* $\mathbf{S}$ is an abstract vector space that is also a differentiable manifold linearly diffeomorphic to an infinite-dimensional separable Hilbert space.

Definition. A *Hilbert space of functions* is either a Hilbert space H, elements of which are equivalence classes of maps between two given subsets of R^n or the Hilbert space H^* dual to H. Two elements $f, g \in H$ are called *equivalent* if the norm of $f - g$ in H is zero.

Definition. A linear isomorphism e_H from a Hilbert space H of functions onto $\mathbf{S}$ will be called a *string basis* on $\mathbf{S}$.

Notation. The action of e_H on $\varphi \in H$ will be written in one of the following ways:

$$e_H(\varphi) = (e_H, \varphi) = \int e_H(k)\varphi(k)dk = e_{Hk}\varphi^k. \tag{2.1}$$

The integral sign is used as a notation for the action of e_H on an element of H and in general does not refer to an actual integration.

Definition. Given e_H the function $\varphi \in H$ such that $\Phi = e_H\varphi$ will be called a *coordinate* (or an *H-coordinate*) of a string $\Phi \in \mathbf{S}$. The space H itself will be called a *coordinate space*.

Definition. Let $\mathbf{S}^*$ be the dual string space. A linear isomorphism e_{H^*} of H^* onto $\mathbf{S}^*$ will be called a *string basis on* $\mathbf{S}^*$.

Notation. Decomposition of an element $F \in \mathbf{S}^*$ with respect to the basis will be written in one of the following ways:

$$F = e_{H^*}(f) = (e_{H^*}, f) = \int e_{H^*}(k)f(k)dk = e^k_{H^*}f_k. \tag{2.2}$$

Definition. The basis e_{H^*} will be called *dual* to the basis e_H if for any string $\Phi = e_{Hk}\varphi^k$ and for any functional $F = e^k_{H^*}f_k$ the following is true: $F(\Phi) = f(\varphi)$.

Remark. In general case we have

$$F(\Phi) = e_{H^*}f(e_H\varphi) = e^*_H e_{H^*}f(\varphi), \tag{2.3}$$

where $e_H^* : \mathbf{S}^* \longrightarrow H^*$ is the adjoint of e_H. Therefore, e_{H^*} is the dual string basis if $e_H^* e_{H^*} : H^* \longrightarrow H^*$ is the identity operator.

Notation. The action of F on Φ in any bases e_H on $\mathbf{S}$ and e_{H^*} on $\mathbf{S}^*$ will be written in one of the following ways:

$$F(\Phi) = e_{H^*}^k f_k e_{Hl} \varphi^l = G(f, \varphi) = g_l^k f_k \varphi^l, \tag{2.4}$$

where G is a non-degenerate bilinear functional on $H^* \times H$.

Notation. Assume that H is a real Hilbert space. We have:

$$(\Phi, \Psi)_S = \mathbf{G}(\Phi, \Psi) = G(\varphi, \psi) = g_{kl} \varphi^k \psi^l. \tag{2.5}$$

Here $G : H \times H \longrightarrow R$ is a bilinear form defining the inner product on H and $\mathbf{G} : \mathbf{S} \times \mathbf{S} \longrightarrow R$ is the induced bilinear form. The expression on the right is a convenient form of writing the action of G on $H \times H$.

Definition. A string basis e_H in $\mathbf{S}$ will be called *orthonormal* if

$$(\Phi, \Psi)_S = f_\varphi(\psi), \tag{2.6}$$

where $f_\varphi = (\varphi, \cdot)$ is a regular functional and $\Phi = e_H \varphi$, $\Psi = e_H \psi$ as before. That is,

$$(\Phi, \Psi) = f_\varphi(\psi) = \int \varphi(x)\psi(x)dx, \tag{2.7}$$

where $\int$ here denotes an actual integral over a measurable set $D \in R^n$.

Remark. Not every coordinate Hilbert space H can produce an orthonormal string basis e_H. Equation (2.7) shows that orthonormality of a string basis imposes a symmetry between coordinates of the dual objects in the basis. In particular, if e_H is orthonormal, then H must be an L_2-space, i.e. a space $L_2(D)$ of square integrable functions on a measurable set $D \in R^n$. Thus, Hilbert spaces l_2 and $L_2(R)$ are examples of coordinate spaces that admit an orthonormal string basis.

Definition. A linear coordinate transformation on $\mathbf{S}$ is an isomorphism $\omega : \widetilde{H} \longrightarrow H$ of Hilbert spaces which defines a new string basis $e_{\widetilde{H}} : \widetilde{H} \longrightarrow \mathbf{S}$ by $e_{\widetilde{H}} = e_H \circ \omega$.

Let φ be coordinate of a string Φ in the basis e_H and $\widetilde{\varphi}$ its coordinate in the basis $e_{\widetilde{H}}$. Then $\Phi = e_H\varphi = e_{\widetilde{H}}\widetilde{\varphi} = e_H\omega\widetilde{\varphi}$. That is, $\varphi = \omega\widetilde{\varphi}$ by the uniqueness of the decomposition.

Let now $\Phi, \Psi \in \mathbf{S}$ and let $\mathbf{A}$ be a linear operator on $\mathbf{S}$. Let $\Phi = e_H\varphi, \Psi = e_H\psi$ with $\varphi, \psi \in H$. The scalar product $(\Phi, \mathbf{A}\Psi)_S$ is independent of a basis and in a basis e_H reduces to

$$(\varphi, A\psi)_H = (\widehat{G}\varphi, A\psi), \tag{2.8}$$

where $\widehat{G} : H \longrightarrow H^*$ defines the metric on H.

If $\omega : \widetilde{H} \longrightarrow H$ is a linear coordinate transformation and $\varphi = \omega\widetilde{\varphi}, \psi = \omega\widetilde{\psi}$, then

$$(\widehat{G}\varphi, A\psi) = (\widehat{G}\omega\widetilde{\varphi}, A\widetilde{\psi}) = (\omega^*\widehat{G}\omega\widetilde{\varphi}, \omega^{-1}A\omega\widetilde{\psi}) = (\widehat{G}_{\widetilde{H}}\widetilde{\varphi}, A_{\widetilde{H}}\widetilde{\psi}). \tag{2.9}$$

Therefore we have the following transformation laws:

$$\varphi = \omega\widetilde{\varphi} \tag{2.10}$$

$$\psi = \omega\widetilde{\psi} \tag{2.11}$$

$$\widehat{G}_{\widetilde{H}} = \omega^*\widehat{G}\omega \tag{2.12}$$

$$A_{\widetilde{H}} = \omega^{-1}A\omega. \tag{2.13}$$

More generally, we know that by definition $\mathbf{S}$ is a Hilbert manifold (modelled on itself). Let then (U_α, π_α) be an atlas on $\mathbf{S}$.

Definition. A collection of quadruples $(U_\alpha, \pi_\alpha, \omega_\alpha, H_\alpha)$, where each H_α is a Hilbert space of functions and ω_α is an isomorphism of $\mathbf{S}$ onto H_α will be called a *functional atlas* on $\mathbf{S}$. A collection of all compatible functional atlaces on $\mathbf{S}$ will be called a *coordinate structure* on $\mathbf{S}$.

Definition. Let (U_α, π_α) be a chart on $\mathbf{S}$. If $p \in U_\alpha$, then $\omega_\alpha \circ \pi_\alpha(p)$ is called the *coordinate* of p. The isomorphisms $\omega_\beta \circ \pi_\beta \circ (\omega_\alpha \circ \pi_\alpha)^{-1} : \omega_\alpha \circ \pi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \omega_\beta \circ \pi_\beta(U_\alpha \cap U_\beta)$ are called *coordinate transformations* on $\mathbf{S}$.

As $\mathbf{S}$ is a differentiable manifold one can also introduce the tangent bundle structure $\tau : T\mathbf{S} \longrightarrow \mathbf{S}$ and the bundle $\tau^r_s : T^r_s\mathbf{S} \longrightarrow \mathbf{S}$ of tensors of rank (r, s).

3 Coordinate transformations preserving locality of operators

In section 5 of [1] it was shown that a single eigenvalue problem for an operator on the string space leads to a family of eigenvalue problems in particular string bases. We now raise a more general question of describing changes of functional equations under transformations of string coordinates. In particular, we will be interested in differential and algebraic equations.

When transforming a functional equation it is often necessary to preserve some of the properties of the equation. In particular, it is often useful to preserve locality of the operators in the equation. To define locality of operators let us start with the following

Definition. A generalized function is *concentrated at a point* if it is equal to zero on every test function that is equal to zero on a neighborhood of the point.

The structure of such functionals is given by the following theorem (see [2]):

Theorem. If the fundamental space contains all infinitely differentiable functions of bounded support at least in some neighborhood of a given point x_0, then every generalized function concentrated at x_0 has the form

$$f = \sum_{|q| \leq r} a_q D^q \delta(x - x_0). \tag{3.1}$$

Here $x = (x_1, ..., x_n)$ is a point in R^n, r is a nonnegative integer, $q = (q_1, ..., q_n)$ is a set of nonnegative integers, $|q| = q_1 + ... + q_n$, and $D^q = \frac{\partial^{|q|}}{\partial {x_1}^{q_1} ... \partial {x_n}^{q_n}}$.

Definition. Let H be a coordinate space of functions on R^n. We shall say that a continuous operator $A : H \longrightarrow H$ is local if

$$(Af)(x) = \int \sum_{|q| \leq r} a_q(x) D^q \delta(y - x) f(y) dy. \tag{3.2}$$

Here $\delta(y - x)$ denotes the δ-function of the diagonal (x, x) in R^{2n}. Assume first that f is an infinitely differentiable function of bounded support and $a_q(x) D^q f(x)$ is integrable. Then formula (3.2) is understood by requiring the validity of "integration by parts" which reduces (3.2) to

$$(Af)(x) = \int \sum_{|q|\leq r} (-1)^{|q|} a_q(x) D^q f(x) dx, \tag{3.3}$$

with integration over the entire space R^n. More generally, let $f \in H$ be any generalized function on R^n. Then (3.2) is understood by requiring that

$$(Af, \varphi) = (f, B\varphi), \tag{3.4}$$

where φ is any smooth function of bounded support on R^n and

$$(B\varphi)(x) = \int \sum_{|q|\leq r} (-1)^{|q|} a_q(x) D^q \varphi(x) dx. \tag{3.5}$$

It is easy to see that locality of an operator is not an invariant property, that is, it depends on a particular choice of coordinates. We now want to describe such coordinate transformations that preserve locality of linear operators.

Suppose then that $A : H \longrightarrow H$ is a local linear operator, $\omega : \widetilde{H} \longrightarrow H$ is a transformation of coordinates, and $A_{\widetilde{H}} = \omega^{-1} A \omega : \widetilde{H} \longrightarrow \widetilde{H}$ is the transformed operator.

The operator $A_{\widetilde{H}}$ will be local if

$$\sum_{|q|\leq r} \omega^{-1}(x,y) a_q(y) D^q \delta(z-y) \omega(z,u) f(u) dy dz du = \sum_{|q|\leq s} b_q(x) D^q \delta(y-x) f(y) dy, \tag{3.6}$$

where notations are as in (3.1) and the integral symbol is omitted. That is,

$$\sum_{|q|\leq r} a_q(x) D^q \delta(z-x) \omega(z,y) dz = \sum_{|q|\leq s} \omega(x,z) b_q(z) D^q \delta(y-z) dz. \tag{3.7}$$

In the simpliest case when H and $\widetilde{H}$ are spaces of functions of one variable and the kernels of A and $A_{\widetilde{H}}$ contain only one term of the form $a_q(x) D^q \delta(y-x)$ each, the equation (3.7) reduces to

$$a(x) \frac{\partial^n}{\partial z^n} \delta(z-x) \omega(z,y) dz = \omega(x,z) b(z) \frac{\partial^m}{\partial y^m} \delta(y-z) dz. \tag{3.8}$$

If particular, when $n = 1$ and $m = 0$ (3.8) yields

$$a(x) \frac{\partial}{\partial z} \delta(z-x) \omega(z,y) dz = \omega(x,z) b(z) \delta(y-z) dz. \tag{3.9}$$

Assuming that ω is a smooth solution, "integration by parts" gives

$$-a(x)\frac{\partial\omega(x,y)}{\partial x}=\omega(x,y)b(y). \tag{3.10}$$

Solving (3.10) we obtain

$$\omega(x,y)=F(y)e^{-c(x)b(y)}, \tag{3.11}$$

where $c(x)=\int\frac{dy}{a(y)}$ and $F(y)$ is an arbitrary smooth function. To be a coordinate transformation ω must be an isomorphism as well. In particular, Fourier transform is a solution of (3.10) with

$$\omega(x,y)=e^{ixy}. \tag{3.12}$$

Coordinate transformations satisfying (3.10) preserve locality of the first order differential operators on H by transforming them into operators of multiplication.

In the case of a more general equation (3.8), we have:

$$a(x)\frac{\partial^n\omega(x,y)}{\partial x^n}=(-1)^n\frac{\partial^m(\omega(x,y)b(y))}{\partial y^m}. \tag{3.13}$$

Solutions of (3.13) for different values of n and m produce coordinate transformations preserving locality of various differential operators.

4 Coordinate transformations preserving derivatives

Among solutions of (3.13) those preserving the order q of derivatives are of particular interest. To describe such transformations it is enough to obtain solutions of (3.13) with $n=m=1$. Let us assume here that the coefficients a and b in (3.13) are constants. Then up to a constant coefficient which we assume to be equal to one we obtain the following equation:

$$\frac{\partial\omega(x,y)}{\partial x}+\frac{\partial\omega(x,y)}{\partial y}=0. \tag{4.1}$$

The smooth solutions of (4.1) are given by

$$\omega(x,y)=f(x-y), \tag{4.2}$$

where f is an arbitrary infinitely differentiable function on R. In particular, the function

$$\omega(x,y)=e^{-(x-y)^2} \tag{4.3}$$

satisfies (4.1). Also, in section 4 of [1] it was verified that the corresponding transformation is injective. When Hilbert structure on $\widetilde{H}=\omega^{-1}(H)$ is induced by ω, this transformation becomes an isomorphism of Hilbert spaces. Therefore, it provides an example of a coordinate transformation that preserves derivatives.

Let now H be a Hilbert space of functions on R^n. We are looking for a nontrivial transformation preserving all partial derivative operators on H. Applying equation (3.7) to this case we obtain:

$$\frac{\partial\omega(x,y)}{\partial x_i}+\frac{\partial\omega(x,y)}{\partial y_i}=0. \tag{4.4}$$

Here i changes from 1 to n. The function

$$\omega(x,y)=e^{-(x-y)^2} \tag{4.5}$$

with $x=(x_1,...,x_n)$ and $y=(y_1,...,y_n)$ satisfies (4.4) and the corresponding transformation is injective inducing a Hilbert structure on the space $\widetilde{H}=\omega^{-1}(H)$.

Theorem. The generalized solutions of any linear differential equation with constant coefficients (either ordinary or partial) are coordinate transformations of the corresponding regular solutions. That is, let L be a polynomial function of n variables. Let $u,v\in\widetilde{H}$ be functionals on the space K of functions of n variables which are infinitely differentiable and have bounded supports. Assume that u is a generalized solution of

$$L\left(\frac{\partial}{\partial x_1},...,\frac{\partial}{\partial x_n}\right)u=v. \tag{4.6}$$

Then there exists a regular solution φ of

$$L\left(\frac{\partial}{\partial x_1},...,\frac{\partial}{\partial x_n}\right)\varphi=\psi, \tag{4.7}$$

where $\varphi = \omega u$ and $\psi = \omega v$ and ω is as in (4.5).

Proof. Consider first the simpliest case of the ordinary differential equation

$$\frac{d}{dx}u(x) = v(x). \tag{4.8}$$

Assume u is a generalized solution of (4.8). Define $\varphi = \omega u$ and $\psi = \omega v$, where ω is as in (4.3). Notice that φ, ψ are infinitely differentiable. In fact, any functional on the space K of infinitely differentiable functions of bounded support acts as follows (see [2]):

$$(f, \varphi) = \int F(x)\varphi^{(m)}(x)dx, \tag{4.9}$$

where F is a continuous function on R. Applying ω to f shows that the result is a smooth function.

As $\omega^{-1}\frac{d}{dx}\omega = \frac{d}{dx}$, we have

$$\omega^{-1}\frac{d}{dx}\omega u = v. \tag{4.10}$$

That is,

$$\frac{d}{dx}\varphi(x) = \psi(x) \tag{4.11}$$

proving the theorem in this case. The higher order derivatives can be treated similarly as

$$\omega^{-1}\frac{d^n}{dx^n}\omega = \omega^{-1}\frac{d}{dx}\omega\omega^{-1}\frac{d}{dx}\omega...\omega^{-1}\frac{d}{dx}\omega. \tag{4.12}$$

That is, transformation ω preserves derivatives of any order. Generalization to the case of several variables is straightforward.

5 Coordinate transformations preserving products of functions

It is now natural to investigate changes of equations containing products of functions under transformations of string coordinates. Consider the simpliest algebraic equation

$$a(x)f(x) = h(x), \tag{5.1}$$

where f is an unknown (generalized) function of a single variable and $h \in H$. To investigate transformation properties of this equation we need to interpret it as a tensor equation on the string space $\mathbf{S}$. The right hand side is a function. Therefore this must be a "vector equation" (i.e. both sides must be $(1,0)$-tensors on the string space). If f is to be a function as well, a must be a $(1,1)$-tensor. That is, the "correct" equation is:

$$a(x)\delta(x-y)f(y)dy = h(x). \tag{5.2}$$

To preserve the product-like form of the equation we need such a coordinate transformation $\omega : \widetilde{H} \longrightarrow H$ that

$$\omega^{-1}(u,x)a(x)\delta(x-y)\omega(y,z)dxdy = b(u)\delta(u-z). \tag{5.3}$$

In this case the equation (5.1) in new coordinates is

$$b(x)\varphi(x) = \psi(x), \tag{5.4}$$

where $h = \omega\psi$, $f = \omega\varphi$, and $\varphi, \psi \in \widetilde{H}$.

Equation (5.3) is a particular case of equation (3.13) with $n = m = 0$. It yields

$$a(x)\omega(x,y) = \omega(x,y)b(y). \tag{5.5}$$

If $a(x) = b(y) = C$, this equation is satisfied for any ω. Otherwise ω must be a local transformation. The first case is trivial. In the second case we have

$$\omega(x,y) = \sum_{|q|\leq r} a_q(x)D^q\delta(y-x). \tag{5.6}$$

Assume first that only one term in the sum (5.6) is present, i.e.

$$\omega(x,y) = a_n(x)D^n\delta(y-x). \tag{5.7}$$

For now let us leave the question of invertibility of ω aside. Applying equation (5.5) to a function φ after "integration by parts" we have

$$a(x)a_n(x)D^n\varphi(x) = a_n(x)D^n(b(x)\varphi(x)), \tag{5.8}$$

or,

$$a(x)a_n(x)D^n\varphi(x) = a_n(x)\sum_{i+k=n}\frac{k!}{i!(k-i)!}D^i b(x)D^k\varphi(x)). \tag{5.9}$$

Choose φ so that the derivatives $\varphi^{(k)}$ form a (classical) basis on $\widetilde{H}$. Then by equating the coefficients of $\varphi^{(n)}$ we have $a(x) = b(x)$. If $n > 0$, we also have $b'(x) = 0$. That is, if $n > 0$, then $a(x) = b(x) = C$ and

$$\omega(x,y) = a_n(x)D^n\delta(y-x). \tag{5.10}$$

If $n = 0$ instead, then $a(x) = b(x)$ can be any. In this case, however,

$$\omega(x,y) = a_0(x)\delta(y-x), \tag{5.11}$$

i.e. the transformation is simply multiplication by a function.

In more general case when ω is as in (5.6) a similar analysis gives the same result: whether $a(x) = b(x) = C$, or $\omega(x,y) = a_0(x)\delta(x-y)$.

We therefore have the following

Theorem. Unless $a(x)$ is a constant or $\omega(x,y) = a_0(x)\delta(x-y)$, it is impossible to preserve the product form of (5.2) under coordinate transformations.

In particular, the product of nonconstant functions of one and the same variable is not an invariant operation.

One could refer to the operator $a(x)\delta(x-y)$ in (5.2) as the operator of multiplication by $a(x)$. Clearly, it is a local operator. The theorem then says that locality of this operator can be preserved only in trivial cases when $a(x) = C$ or ω itself is an operator of multiplication by a function.

On the other hand, consider the equation

$$a(x)f(y) = h(x,y), \tag{5.12}$$

where a and f are functions of a single variable and h is a function of two variables. This equation can be viewed as a tensor equation on the string space. The left hand side represents then a tensor product of two "vectors". The right hand side is a $(2,0)$-tensor. Therefore, any coordinate transformation preserves this form of the equation. In particular, generalized solutions of this equation can be transformed into regular solutions by transformation of coordinates.

6 More general coordinate transformations

In section 4 we have studied coordinate transformations preserving linear differential operators with constant coefficients. Here we will investigate the case of linear differential operators with non-constant coefficients. We will also begin analyzing coordinate transformations of nonlinear differential equations. We start with the following

Example. Consider the equation (3.13) with $n = m = 1$ assuming $a(x)$ and $b(y)$ are functions. In this case the equation reads

$$a(x)\frac{\partial \omega(x,y)}{\partial x} + \frac{\partial(\omega(x,y)b(y))}{\partial y} = 0. \tag{6.1}$$

Let us look for a solution in the form

$$\omega(x,y) = e^{f(x)g(y)}. \tag{6.2}$$

Then (6.1) yields

$$a(x)f'(x)g(y) + b(y)f(x)g'(y) + b'(y) = 0. \tag{6.3}$$

If $b(y) = 1$, (6.3) is a separable equation and we have

$$\frac{a(x)f'(x)}{f(x)} = -\frac{g'(y)}{g(y)} = C, \tag{6.4}$$

where C is a constant. Solving this we have,

$$\omega(x,y) = e^{Ce^{\int \frac{C_1}{a(x)}dx}e^{C_2 y}}. \tag{6.5}$$

Taking for example $C = C_1 = C_2 = 1$ and $a(x) = x$, we have

$$\omega(x,y) = e^{xe^y}. \tag{6.6}$$

The corresponding transformation can be shown to be invertible on an appropriate space of functions. As we see it transforms the operator $x\delta'(y-x)$ into $\delta'(y-x)$. That is,

$$\omega : x\psi'(x) \longrightarrow \psi'(x) \tag{6.7}$$

for *any* function ψ on the space of definition of ω.

Example. As another example consider (3.13) with $n=2, m=0$. We have:

$$a(x)\frac{\partial^2 \omega(x,y)}{\partial x^2} = \omega(x,y)b(y) \tag{6.8}$$

Looking for a solution in the form $\omega(x,y) = e^{f(x,y)}$ we obtain:

$$f_{xx} + f_x^2 = \frac{b(y)}{a(x)}, \tag{6.9}$$

where f_x, f_{xx} denote partial derivatives of $f(x,y)$ with respect to x. Using $g(x,y) = f_x(x,y)$ we obtain a first order differential equation

$$g_x + g^2 = \frac{b(y)}{a(x)}. \tag{6.10}$$

Existence and regularity of solutions of (6.10) is known from the general theory. In particular, when $b(y) = y^2$ and $a(x) = 1$ we are back to the Fourier-like transform as in (3.11).

We see from the previous examples that in solving equation (3.13) we need to take into account the specifics of a problem in hand. A type of coordinate transformation especially useful to treat the problem is determined by a type of problem itself.

A very important question is whether we can apply the developed coordinate formalism to nonlinear differential equations. It is known that the theory of generalized functions has been mainly successful with the linear problems. The difficulty of corse lies in defining the product of generalized functions. To see what kind of solution can be offered in the new context consider the following

Example. Consider a differential equation containing the square of derivative of an unknown function, i.e. containing the term

$$\varphi'(x) \cdot \varphi'(x), \tag{6.11}$$

where $\varphi \in H$. To use the coordinate formalism we need to interpret this term as a tensor. We have:

$$\varphi'(x) \cdot \varphi'(x) = \delta(x-y)\delta'(u-x)\delta'(v-y)\varphi(u)\varphi(v)dydudv, \tag{6.12}$$

where as before we omit the integral symbol. Therefore, this term is the convolution of the $(1,2)$-tensor

$$c^x_{uv} = \delta(x-y)\delta'(u-x)\delta'(v-y)dy \tag{6.13}$$

with the pair of strings $\varphi^u = \varphi(u)$. With this interpretation we can easily obtain the transformation properties of c^x_{uv}. Denote

$$c^x_{uv}\varphi^u\varphi^v = \psi^x, \tag{6.14}$$

where the meaning of notations is described in section 2. Assume now that $\omega : \widetilde{H} \longrightarrow H$ is a coordinate transformation and $\omega\widetilde{\varphi} = \varphi$. Denote $\omega(x,y) = \omega^x_y$. As ψ^x is a "vector", we have

$$c^x_{uv}\omega^u_{u'}\widetilde{\varphi^{u'}}\omega^v_{v'}\widetilde{\varphi^{v'}} = \omega^x_{x'}\widetilde{\psi^{x'}}. \tag{6.15}$$

That is,

$$c^{x'}_{u'v'} = {\omega^{-1}}^{x'}_{x}\,{\omega^*}^v_{v'}c^x_{uv}\omega^u_{u'}, \tag{6.16}$$

where ω^* is the adjoint of ω. After "integration by parts" we obtain

$$c^x_{u'v'} = {\omega^*}^v_{v'}c^x_{uv}\omega^u_{u'} = \frac{\partial\omega(v',x)}{\partial x}\frac{\partial\omega(x,u')}{\partial x}. \tag{6.17}$$

By specifying the desired form of $c^{x'}_{u'v'}$ we obtain a nonlinear partial differential equation for the transformation ω. Existence of interesting solutions of this and similar equations is under investigation.

7 Conclusion

The main idea of the coordinate formalism introduced in [1] is to relate different spaces of functions by considering them as coordinate representations of an invariant "string" space. This last one is simply the abstract infinite-dimensional separable Hilbert space. This approach turns out to be very similar in spirit to the nineteenth century introduction of vectors. However, it can not be reduced to consideration of elements of an infinite-dimensional Hilbert space as vectors. In fact, given a Hilbert space of functions elements of such space are vectors. The objects that we call strings are more general. They are defined for all Hilbert spaces of functions (i.e. coordinate spaces) at once and do not depend on a choice of such space.

A particular choice of a coordinate space can be useful for a problem in hand. Therefore, special transformations of coordinates become important. In particular, transformations from the spaces of regular functions to the spaces of generalized functions provide a new insight on the theory of generalized functions.

Here we saw how preservation of different properties of linear operators led to different types of coordinate transformations. For example, preserving locality of operators in the simpliest case leads to the Fourier-like transformations as in (3.11).

By requiring preservation of the derivative operator we were able to relate the generalized and the regular solutions to linear partial differential equations with constant coefficients.

The results of the last section suggest that the nonlinear problems can be approached in the same fashion. For this it must be possible to interpret a given nonlinear equation as a tensor equation on the string space. Then different "nonlinearities" are interpreted as convolutions of tensors on the space. A more complete analysis in this direction is, however, a subject for a different paper.

References

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