Ephemeral point-events: is there a last remnant of physical objectivity?*

Massimo Pauri[1] and Michele Vallisneri[2]

[1] Dipartimento di Fisica, Università di Parma, 43100 Parma, Italy
[2] Theoretical Astrophysics 130-33, Caltech, Pasadena CA 91125, USA

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Abstract

For the past two decades, Einstein's Hole Argument (which deals with the apparent indeterminateness of general relativity due to the general covariance of the field equations) and its resolution in terms of *Leibniz equivalence* (the statement that Riemannian geometries related by active diffeomorphisms represent the same physical solution) have been the starting point for a lively philosophical debate on the objectivity of the point-events of space-time. It seems that Leibniz equivalence makes it impossible to consider the points of the space-time manifold as *physically individuated* without recourse to dynamical individuating fields. Various authors have posited that the metric field itself can be used in this way, but nobody so far has considered the problem of explicitly distilling the *metrical fingerprint* of point-events from the gauge-dependent components of the metric field. Working in the Hamiltonian formulation of general relativity, and building on the results of Lusanna and Pauri (2002), we show how Bergmann and Komar's *intrinsic pseudo-coordinates* (based on the value of curvature invariants) can be used to provide a physical individuation of point-events in terms of the true degrees of freedom (the *Dirac observables*) of the gravitational field, and we suggest how this conceptual individuation could in principle be implemented with a well-defined empirical procedure. We argue from these results that point-events retain a significant kind of physical objectivity.

*This Essay is dedicated with warm affection to Roberto Torretti on the occasion of his $70^{\rm th}$ Birthday. Most of the technical developments that underlay this work were introduced by Lusanna and Pauri (2002). Some of this material was also discussed at the international workshop *General covariance and the quantum: where do we stand?*, held at the University of Parma on June 21–23, 2001. We are deeply indebted to Luca Lusanna for a long series of enlightening discussions about the canonical reduction of general relativity and about the Bergmann–Komar intrinsic coordinates.

1 Introduction: Einstein, the Hole Argument, and the physical individuation of point-events

General relativity owes much of its mathematical beauty to its formulation in terms of the theory of pseudo–Riemannian manifolds. This beauty, however, carries a curse: at the mathematical level, even a *naked* manifold has well-defined points distinguishable in terms of coordinates, but in physics it is a widely held assumption that points can be distinguished only by the values of physical fields or as the positions of physical objects, including measuring devices. Any attempt to take the bare points seriously leads to well-known puzzles and quandaries.

Possibly the first puzzle of this kind (the Hole Argument, or *Lochbetrachtung*) crossed Albert Einstein's path repeatedly between 1913 and 1915. These were years of alternating joy and distress for Einstein, as he set out to create a theory of gravitation based on the guiding principle of general covariance, failed to do so, used the Hole Argument to convince himself that general covariance was physically inconsistent, formulated the short-lived Einstein–Grossmann (*Entwurf*) theory, and finally returned to his original conviction, having come, through the Hole Argument, to his explanation of the physical meaning of general covariance. Roberto Torretti wrote a beautiful account of Einstein's woes and triumphs in his masterly treatise *Relativity and Geometry* (Torretti, 1987), and more about this story can be found in John Norton's contribution to this very volume, as well as in many other papers by Norton (1984; 1987; 1988; 1989; 1992; 1993; 2001b; see also Howard and Norton, 1993) and by John Stachel (1980; 1986a; 1986b; 1993; 1999).

Einstein's "triumph" [to use Norton's wording (2001a)] over the Hole Argument and "over the space-time coordinate systems" came only after he adopted a very idealized model of physical measurement where all possible observations reduce to the intersections of the worldlines of observers, measuring instruments, and measured physical objects (*point-coincidence argument*). In Einstein's own words (1916):

> That the requirement of general covariance, *which takes away from space and time the last remnant of physical objectivity*, is a natural one, will be seen from the following reflexion. All our space-time verifications invariably amount to a determination of space-time coincidences. If, for example, events consisted merely in the motion of material points, then ultimately nothing would be observable but the meetings of two or more of these points. Moreover, the results of our measurings are nothing but verifications of such meetings of the material points of our measuring instruments with other material points, coincidences between the hands of a clock and points on the clock dial, and observed point-events happening at the same place at the same time. The introduction of a system of reference serves no other purpose than to facilitate the description of the totality of such coincidences.

The Hole Argument received new life with John Stachel's seminal paper (1980), which raised a rich philosophical debate[1] that is still alive today. Soon it became widely recognized that the Hole Argument was intimately tied with our conceptions of space and time, at least as they are represented by the mathematical models of general of relativity.

Of course, it is to philosophical preferences that we must defer the judgment on the ontological status of the notions that are introduced in physical theories to describe Nature; and this is especially true for the conditions that decide in favor of a literal or nonliteral interpretation of theoretical structures. So we shall not be concerned here with the metaphysical issue of the reality or nature of space-time, let alone of the *Raum* of our experience. We agree with Michael Friedman when he argues that the Hole Argument leaves an unsolved problem about the characterization of intrinsic space-time structure, rather than an ontological question about the existence of space-time ["avoiding quantification over 'bare' points ... appears to be a non-trivial mathematical problem" (Friedman, 1984)].

In this paper we offer our contribution to the clarification of this non-trivial problem. More precisely, we investigate the relation between the physical meaning of spatio-temporal localization and the unavoidable use of *arbitrary* coordinate systems in the practice of general relativity. Thus, we explore the limits on the objectivity of space-time that are imposed by the mathematical representation of spatio-temporal structure, in conjunction with the requirements of the empirical foundation of general relativity.

1.1 The Hole Argument

In its modern version, the Hole Argument runs as follows. Consider a general-relativistic spacetime, as specified by a four-dimensional mathematical manifold $\mathcal{M}_4$ and by a metrical tensor field g, which represents at the same time the chrono-geometrical structure of space-time and the potential for the gravitational field. The metric g is a solution of the generally-covariant Einstein equations. If any nongravitational physical fields are present, they are represented by tensor fields that are also dynamical fields, and that appear as sources in the Einstein equations.

Now assume that $\mathcal{M}_4$ contains a *hole* $\mathcal{H}$: that is, an open region where all the nongravitational fields are null. On $\mathcal{M}_4$ we can prescribe an *active*[2] diffeomorphism D_A (Norton, 1987; Stachel, 1993; Wald, 1984) that remaps the

[1] See Bartels (1994); Belot (1995, 1996); Brighouse (1994); Butterfield (1984, 1987, 1988, 1989); Chuang (1996a,b); Disalle (1994); Earman and Norton (1987); Fine (1984); Hofer (1996, 1998); Hofer and Cartwright (1993); Leeds (1995); Maudlin (1988, 1990); Norton (1984, 1987, 1988, 1989, 1992, 1993, 2001a,b); Rynasiewicz (1992, 1994, 1996a); Saunders (2001); Stein (1977); Teller (1991); Wilson (1993). In this paper we shall make no attempt to analyze or survey this discussion, not least because we believe that some debaters occasionally overstep the philosophical latitude allowed by the very structure of general relativity. Instead, we shall recall only the major points that can be seen as a premise to our discussion.

[2] As originally formulated by Einstein (1914), the Hole Argument does not rely on the effects of active diffeomorphisms in the modern geometrical sense, but rather on the following procedure. After taking a coordinate transformation $\hat{\xi}^\mu = f^\mu(\xi^\nu)$, we obtain the transformed metric $\hat{g}_{\sigma\rho}(\hat{\xi}^\mu)$, and then we consider the object $\hat{g}_{\sigma\rho}(\xi^\mu)$ defined by transferring the func-

points inside $\mathcal{H}$, but blends smoothly into the identity map outside $\mathcal{H}$ and on the boundary. Because the Einstein equations are generally covariant, if g is one of their solutions, so is the *drag-along* field $g' = D_A g$. By construction, for any point $x \in \mathcal{H}$ we have (geometrically) $g'(D_A x) = g(x)$, but of course $g'(x) \neq g(x)$ (also geometrically).

What is the correct interpretation of the new field g'? Clearly, the transformation entails an *active redistribution of the metric over the points of the manifold*, so the crucial question is whether, to what extent, and how the points of the manifold are primarily *individuated.*[3] In the mathematical literature about topological spaces, it is always implicitly assumed that the entities of the set can be distinguished and considered separately (provided the Hausdorff conditions are satisfied), otherwise one could not even talk about point mappings or homeomorphisms. It is well known, however, that the points of a homogeneous space cannot have any intrinsic *individuality.* As Hermann Weyl (1946) put it:

> There is no distinguishing objective property by which one could tell apart one point from all others in a homogeneous space: at this level, fixation of a point is possible only by a *demonstrative act* as indicated by terms like "this" and "there."

Quite aside from the phenomenological stance implicit in Weyl's words,[4] there is only one way to individuate points at the mathematical level that we are considering: namely by coordinatization, which transfers the individuality of n-tuples of real numbers to the elements of the topological set. Therefore, all the relevant transformations (including *active* diffeomorphisms) operated on the manifold $\mathcal{M}_4$, even if viewed in purely geometrical terms, *must* be constructible in terms of coordinate transformations (see for instance note 2). So we have necessarily crossed from the domain of *geometry* to *algebra*, and we can justify our use of the symbol x to denote a point of the manifold, as mathematically individuated by the chosen coordinates.

Let us go back to the effect of this *primary* individuation of manifold points. If we now think of the points of $\mathcal{H}$ as already *independently individuated* spatio-temporal *physical events* even before the metric is defined, then g and g' must be regarded as *physically distinct* solutions of the Einstein equations (after all, $g'(x) \neq g(x)$ at the *same* point x). This is a devastating conclusion for the causality, or better, *determinateness*[5] of the theory, because it implies that,

tional dependence of $\hat{g}_{\sigma\rho}(\hat{\xi}^{\mu})$ to the *old* coordinates x^{μ}. This is akin to obtaining an active diffeomorphism as the *dual* of a passive transformation.

[3]Consistently with our program, we shall not get involved in the deep philosophical issue of the *individuation* of entities in general. Throughout this essay, our notion of individuation will be deliberately restricted to the meaning that it can have at the mathematical level and, above all, within the conceptual context of a physical theory.

[4]One could contemplate stripping the argument from its phenomenological flavor by asserting that, after all, the demonstrative act also establishes an empirical coincidence. This view is taken, for instance, by Moritz Schlick (1917), who writes: "In order to fix a point in space, one must somehow directly or indirectly, *point to it* ... that is, one establishes a spatio-temporal coincidence of two otherwise [already] *separate* elements."

[5]We prefer to avoid the term *determinism*, because we believe that its metaphysical fla-

even after we completely specify a physical solution for the gravitational and nongravitational fields outside the hole (for example, on a Cauchy surface for the initial value problem), we are still unable to predict uniquely the physical solution within the hole. Clearly, if general relativity has to make any sense as a physical theory, there must be a way out of this foundational quandary, *independently of any philosophical consideration*.

In the modern understanding, the most widely embraced escape from the strictures of the Hole Argument (which is essentially an update to current mathematical terms of the naive solution adopted by Einstein), is to *deny that diffeomorphically related mathematical solutions represent physically distinct solutions*. With this assumption, *an entire equivalence class of diffeomorphically related mathematical solutions represents only one physical solution*.[6] This statement has come to be called [after Earman and Norton (1987)] *Leibniz equivalence*.

It should be clear from the beginning that this is an allusion to a *new Leibniz* adapted to the modern context of general relativity. Apart from the structural analogy, modern Leibnizian arguments proceed without any reference to the metaphysical premises of Leibniz's historical arguments.[7] The same should be said of the Newtonian arguments that underlie the modern version of *substantivalism* (see more below). Rynasiewicz (1996b) has properly remarked that, as it often portrayed in twentieth-century philosophical literature, even the opposition between *substantivalism* and *relationism* amounts to a historical misrepresentation of the classical Newton–Leibniz controversy [see also Dorato (2000)]. This is not irrelevant to the present considerations, for we find it rather arbitrary to transcribe Newtonian absolutism (or at least part of it) into the so-called *manifold substantivalism*, no less than to assert that general relativity is a *relational* theory in an allegedly Leibnizian sense. As emphasized by Rynasiewicz, the crucial point is that the historical debate presupposed a clear-cut distinction between *matter* and *space*, or between *content* and *container*; but by now these distinctions have been blurred by the emergence of the so-called *electromagnetic view of nature* in the late nineteenth century [for a detailed model-theoretical discussion of this point see also Friedman (1983)].

Still, although some might argue [as do Earman and Norton (1987)] that the metric tensor, *qua* physical field, cannot be regarded as the *container* of other physical fields, we argue that the metric field has ontological priority over all

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vor tends to overstate the issue at hand. This is especially true if *determinism* is taken in opposition to *indeterminism*, which is not mere absence of *determinism*.

[6] Of course, taken at face value this statement could be misinterpreted as the naive (and physically vacuous) assertion that metric tensors that have different descriptions in different coordinate systems are geometrically the same tensor (invariance with respect to *passive* diffeomorphisms $\mathrm{Diff}_P\, \mathcal{M}_4$). To formulate the Hole Argument, however, we have used *active* diffeomorphisms: although, as said before, these are generated by the drag-along of coordinate systems, they have the effect that the metric tensors g and $D_A g$ become *geometrically different* at each point $x \in \mathcal{H}$.

[7] More aptly, Friedman calls this Leibniz, stripped of his metaphysical assumptions, the *Leibniz of the positivists* (Friedman, 1983, p. 219; see also Friedman, 2001). A penetrating analysis of the *old* Leibniz versus the *new* one can be found in Earman (1979).

other fields. This preeminence has various reasons (Pauri, 1996), but the most important is that the metric field tells all other fields how to move causally. We also agree with Friedman (1983) that, in agreement with the general-relativistic practice of not counting the gravitational energy induced by the metric as a component of the total energy, we should regard the manifold, endowed with its metric, as space-time; and leave the task of representing matter to the stress-energy tensor. Because of this priority, beside the fact that the Hole *is* pure gravitational field, we maintain, unlike other authors [see for example Rovelli (1991, 1997, 1999)], that the issue of the individuation of points of the manifold as *physical* point-events[8] should be discussed primarily in the context of the vacuum gravitational field, without any recourse to nongravitational entities, except perhaps at the operational level. In this paper we shall indeed adopt this choice.

Stachel (1980; 1986a; 1986b; 1993; 1999) has given a very enlightening analysis of the meaning of general covariance and of its relations with the Hole Argument, expounding the conceptual consequences of Einstein's acceptance of modern Leibniz equivalence through the point-coincidence argument. Stachel stresses that asserting that g and $D_A g$ represent *one and the same gravitational field* is to imply that *the mathematical individuation of the points of the differentiable manifold by their coordinates has no physical content until a metric tensor is specified.* In particular, coordinates lose any *physical* significance whatsoever (Norton, 2001a). Furthermore, as Stachel emphasizes, if g and $D_A g$ must represent the same gravitational field, they cannot be physically distinguishable in any way. So when we act on g with D_A to create the drag-along field $D_A g$, no element of physical significance can be left behind: in particular, nothing that could identify a point x of the manifold as the *same* point of space-time for both g and $D_A g$. Instead, when x is mapped onto $x' = D_A x$, it brings over its identity, as specified by $g'(x') = g(x)$.

This conclusion leads Stachel to the conviction that space-time points *must* be *physically* individuated *before* space-time itself acquires a physical bearing, and that the metric plays in fact the role of *individuating field.* What is more, in practice even the topology of the underlying manifold cannot be introduced independently of the specific form of the metric tensor, a circumstance that makes it even more implausible to interpret the mere topological manifold as *substantival space-time* (*manifold substantivalism*).

Finally, it is essential to note, once again with Stachel, that simply because a theory has generally covariant equations, it does not follow that the points of the underlying manifold must lack any kind of physical individuation. Indeed, what really matters is that there can be no *nondynamical individuating field* that is specified *independently* of the dynamical fields, and in particular independently of the metric. If this was the case, a *relative* drag-along of the metric with respect to the (supposedly) individuating field would be physically significant

[8] There is an unfortunate ambiguity in the usage of the term *space-time points* in the literature: sometimes it refers to element of the mathematical structure that is the first layer of the space-time model, sometimes to the points interpreted as *physical* events: we will adopt the term *point-event* in the latter sense and simply *point* in the former.

and would generate an inescapable Hole problem. Thus, the absence of any nondynamical individuating field, as well as of any dynamical individuating field independent of the metric, is the crucial feature of the purely gravitational solutions of general relativity.

After a brief detour into the main themes of the philosophical debate on the Hole, we shall come back to Leibniz equivalence and argue that it bears little relation to the determinateness of general relativity, and that instead it amounts to the recognition that the mathematical representation of space-time contains superfluous structure, which must be isolated.

1.2 The philosophical debate on the Hole

The modern *substantivalist* position[9] is a statement of spatiotemporal realism: its adherents claim that the individual *points* of the manifold, for a given solution of the Einstein equations, represent *directly* the physical points of space-time, as they would occur in the actual or in some possible world.

Of course, as we have already emphasized, if we do assume that the points possess an individual existence of their own, then the rearrangement of the metric field against their background, as envisaged in the Hole Argument, would produce a true change in the physical state of space-time. For this reason, according to Earman and Norton (1987), substantivalism can be accused of turning general relativity into an *indeterministic* theory: if diffeomorphically related metric fields represent different physical states, then any prescription of initial data (outside the hole) would fail to determine a corresponding solution of the Einstein equations (inside the hole), because many are equally possible. Earman and Norton's intention is to confront the substantivalists with a dire dilemma: accept *indeterminism*, or abandon *substantivalism*.

There have been various attempts in the substantivalist camp to counter this threat of *indeterminism*. For example, Butterfield (1984; 1987; 1988; 1989) portrays diffeomorphic models as different possible worlds and invokes counterpart theory to argue that at most one can represent an actual space-time. Maudlin (1988; 1990) claims that a space-time can be properly represented by at most one of two diffeomorphically related solutions of Einstein's equations, because each space-time point carries metrical properties *essentially*, so these properties are *names* in the Kripkean sense of *rigid designators*: within a class of diffeomorphic models, only one specimen can represent a possible world, because a world in which a point bears metrical properties other than the ones it actually bears would be an impossible world.

Bartels (1994) objects to Maudlin that "with respect to the concrete spots of the metrical field in our world one can reasonably say that their metrical properties could not be otherwise than they actually are ... But to say the same with respect to manifold points in a model is highly problematic, because diffeomorphisms obviously generate permissible models in which the same manifold

[9]See Bartels (1994); Butterfield (1984, 1987, 1988, 1989); Disalle (1994); Fine (1984); Hofer (1996, 1998); Hofer and Cartwright (1993); Maudlin (1988, 1990); Stein (1977); Teller (1991).

points bear different metrical properties." Bartels then proposes to take a whole equivalence class of diffeomorphic image points of a point p as the representation of one and the same possible space-time point, because all the diffeomorphic image points of a certain point p in a model bear the *same individuating metrical fingerprint*. Yet, independently of any philosophical preference, this suggestion is technically not viable, for lacking any specific definition of such equivalence classes, it could even happen that an equivalence class, which is supposed to represent a *real point*, actually covers all points of the manifold. It seems therefore that the *essentialist* recourse to metrical fingerprints as an escape from the Hole Argument is doomed to fail, unless it is possible to give a consistent mathematical definition of *metrical fingerprint*. Even then, we still believe that it is necessary to accept *Leibniz equivalence*, at least as a starting point. At the end of our analysis, it should be apparent that the specific structure of the individuating *metrical fingerprint* leaves no room to sidestep the Hole Argument with any essentialist interpretation of point-events.

Let us now have a look at Roberto Torretti's reaction to some of these positions, and to the Hole Argument in general. In his recent book *The Philosophy of Physics* (Torretti, 1999), Torretti argues that "the [Hole] argument forgets the fact, so clearly set forth by Newton, that points in a structured manifold have no individuality apart from their structural relations." He then quotes Newton's *De Gravitatione* (Hall and Hall, 1962):

> Perhaps now it is maybe expected that I should define extension as substance or accident or else nothing at all. But by no means, for it has its own manner of existence which fits neither substance nor accidents [...] Moreover the immobility of space will be best exemplified by duration. For just as the parts of duration derive their individuality from their order, so that (for example) if yesterday could change places with today and become the latter of the two, it would lose its individuality and would no longer be yesterday, but today; so the parts of space derive their character from their positions, so that if any two could change their positions, they would change their character at the same time and each would be converted numerically into the other *qua* individuals. The parts of duration and space are only understood to be the same as they really are because of their mutual and positions (*propter solum ordinem et positiones inter se*); nor do they have any other principle of individuation besides this order and position which consequently cannot be altered.

Earlier (Torretti, 1987), Torretti had downplayed the issue of the physical individuation of space-time points, noticing that

> [...] the idea that space-time points are what they are only by virtue of the metric structure to which they belong agrees well with the thesis, common to Leibniz *and* Newton, that "it is only by their mutual order and position that the parts of time and space are understood to be the very same which in truth they are," for "they do

> not possess any principle of individuation apart from this order and these positions."

Torretti goes on to point out that making this assumption entails very important consequences: for instance, "it is obviously meaningless to speak in General Relativity of a space-time point at which the metric is not defined," it becomes impossible to hold that "the metric of a relativistic space-time is not a matter of fact, but of mere convention" (geometric conventionalism), and serious problems arise for the [Kripke's] "fashionable semantic theory that conceives of proper names as 'rigid designators', denoting the same individual in many alternative diversely structured 'possible worlds.' Proper names cannot function in this was if the very individuals which are their referents owe their identity to the structure in which they are enmeshed."

In conclusion, Torretti proposes a more equitable "way of dealing with Einstein's [Hole Argument], which does not assume that space-time points can only be physically distinguished by means of their metric properties and relations." To reject the Hole Argument, he argues, it is enough to note that two physical objects can be distinguished either *empirically* (basically, because our direct experience suggest they differ) or *rationally* ("if they are equated to or represented by structurally unequal conceptual systems"). The two physical situations envisaged in the Hole Argument are both observationally indistinguishable (in short, because of the point-coincidence argument) and conceptually indistinguishable (because structurally isomorphic): they are

> [...] as far as our assumptions go, perfectly indiscernible, and therefore must be regarded as identical. In the view I have just put forward, *the onus of individuating the points of space-time does not rest with the metric, which is a structural feature of the world. The role of structure is not to individuate, but to specify; and of course it cannot perform this role beyond what its own specific identity will permit, that is, "up to isomorphism."* It is only on nonconceptual grounds that two isomorphic structures can be held to represent two really different things.

In essence, in 1983 Torretti was satisfied with a structuralist view *à la Newton*, conjoined with the modern understanding of *Leibniz equivalence*.

However, as Friedman has remarked (1984, p. 663), if we stick to simple Leibniz equivalence, "how do we describe this physical situation *intrinsically*?" What is the meaning of point-events as the local elements of space-time? We believe that the task of describing the physical situation *intrinsically* is worth pursuing. To this end, we can take advantage of the fact the points of general-relativistic space-times, quite unlike the points of the homogeneous Newtonian space, are endowed with a remarkably rich *non–point-like* texture[10] provided by the *metric field*. This texture can be exploited for the purpose of the physical individuation of points, for it is now the *dynamical* metric field that characterizes

[10]More important, as we shall see, the *physical* individuation of points as *point-events* is necessarily *nonlocal* in terms of the manifold points.

their "mutual order and positions." Furthermore, as we shall see, the need to connect the formal structure of the theory to the empirical requirements of measurements leads necessarily to a refinement of Leibniz equivalence.

Following this line of thought, we shall argue that there is a specific technical sense in which a procedure of point individuation follows directly from the Hamiltonian formulation of general relativity as a gauge theory. In particular, we will show that the individuation of points originates directly from the *effective degrees of freedom of the gravitational field*, which come to play the role of *basic metrical fingerprints*.

1.3 What is the metrical fingerprint of point-events?

Now, how is it that the metric field can realize concretely its would-be role of physical individuator? After all, we know very well that only a subset of the ten components of the metric is physically significant. It seems then plausible that only this part of the metric might serve as individuating field, while the remaining components would carry physically spurious information.

We move from the analysis given by Bergmann and Komar,[11] who suggest that (in the absence of matter fields) the value of four invariant scalar fields built from contractions of the Weyl tensor can be used as *intrinsic pseudo-coordinates* that are invariant under diffeomorphic transformations. Stachel (1993) reprises this suggestion, but he does not pursue it further.[12]

Our considerations are based on the technical premises laid down by Lusanna and Pauri (2002) with the purpose of extending and clarifying the Bergmann–Komar–Stachel program within the Hamiltonian formulation of general relativity as a gauge theory. Three circumstances make the recourse to the Hamiltonian formalism especially propitious.

1. It is evident that the Hole Argument is inextricably entangled with the initial-value problem of general relativity, but, strangely enough, the Hole Argument has never been explicitly discussed in that context in a systematic way. Possibly the reason is that most authors have implicitly adopted the Lagrangian approach (the *manifold way*), where the initial-value problem is intractable because of the nonhyperbolic nature of Einstein's equations.[13]

[11] See Bergmann (1960, 1962, 1971, 1977); Bergmann and Komar (1960, 1972); Komar (1955, 1958).

[12] To our knowledge, Bergmann and Komar did not follow up on their suggestion, either. The last organic presentation of the issue seems to be Bergmann's *Handbuch* article (Bergmann, 1962, p. 252–255).

[13] Actually, David Hilbert was the first person to discuss the Cauchy problem for the Einstein equations and to realize its connection to the Hole phenomenology (Hilbert, 1917). He discussed the issue in the context of a general-relativistic generalization of Mie's special-relativistic nonlinear electrodynamics, and pointed out the necessity of fixing a special geometrically adapted (*Gaussian* in his terms, or geodesic normal as known today) coordinate system to assure the causality of the theory. In this connection see Howard and Norton (1993). However, as noted by Stachel (1992), Hilbert's analysis was incomplete and neglected important related problems.

2. Only in the Hamiltonian approach can we isolate the gauge variables, which carry the descriptive arbitrariness of the theory, from the (Dirac) observables, which are the right candidates to become the dynamical individuating fields.

3. Finally, in the context of the Hamiltonian formalism, we can resort to Bergmann and Komar's theory of general coordinate-group symmetries (Bergmann and Komar, 1972) to clarify the significance of active diffeomorphisms as *on-shell dynamical symmetries* of the Einstein equations. This step is crucial: to understand fully the role played by *active* diffeomorphisms in the Hole Argument, it is necessary to interpret them as the *manifold-way* counterparts of the Hamiltonian gauge transformations, which are *passive*[14] by definition.

2 Mathematical development: general relativity as a gauge theory and the physical individuation of point-events

This section provides the technical foundations for our analysis of the physical individuation of point-events in general relativity. We start off with a brief, qualitative outline of general relativity as a constrained Hamiltonian theory (especially for the benefit of the philosophers of science who have not had the chance of studying it in detail): Sec. 2.1 introduces constrained Hamiltonian theories in general, while Sec. 2.2 specializes to the case of gravity. Sec. 2.3 discusses the relation between the gauge transformations of the Hamiltonian formalism and the dynamical symmetries of the Einstein equations. Finally, in Sec. 2.4 we present the theory of the Komar–Bergmann intrinsic coordinates, and we explore their link with gauge freedom in general relativity and their significance for the physical individuation of space-time points.

2.1 The constrained Hamiltonian formalism

As most other fundamental theories in modern physics, general relativity falls under the chapter of *gauge theories*. To use the very general definition given by Henneaux and Teitelboim (1992):

> These are theories in which the physical system being dealt with is described by more variables than there are physically independent degrees of freedom. The physically meaningful degrees of freedom

[14] This passive view of active diffeomorphisms is not equivalent to their *dual* representation by the corresponding passive coordinate transformations [as shown, for instance, by Wald (1984); see also footnote 2]. By rights, the active diffeomorphisms should be considered as passive transformations on the function space of metric fields, rather than on the space-time manifold.

> then reemerge as being those invariant under a transformation connecting the variables (gauge transformation). Thus, one introduces extra variables to make the description more transparent, and brings in at the same time a gauge symmetry to extract the physically relevant content.

The mathematical development of gauge theories starts when we realize that the Lagrangian action principle, $\delta \int \mathcal{L}(q, \dot{q})\, dt = 0$, yields Euler–Lagrange equations that are not hyperbolic, because they cannot be solved for all the accelerations. Technically, the same condition that makes it so (the singularity of the Hessian matrix[15] $\partial^2 \mathcal{L}/\partial \dot{q}^k \partial \dot{q}^h$) means also that, when we move from the Lagrangian to the Hamiltonian formulation, the momenta are not all functionally independent, but satisfy some conditions known as (primary) *constraints*. *Secondary* constraints arise when we require that the primary constraints be preserved through evolution.[16] There is no strong distinction between primary and secondary constraints in the role that they come to play in the unfolding of constrained dynamics.

The existence of constraints implies that not all the points of phase space represent physically meaningful states: rather, we are restricted to the *constraint surface* where all the constraints are satisfied. The dimensionality of the constraint surface is given by the number of the original canonical variables, minus the number of functionally independent constraints.

Generally, the constraints are given as functions of the canonical variables which vanish on the constraint surface; technically, these functions are said to be *weakly* zero[17] (≈ 0). Note that weakly vanishing functions may have nonvanishing derivatives in directions normal to the constraint surface, so their Poisson brackets with some of the canonical variables may well be nonzero. If instead all the derivatives vanish, the functions are said to be *strongly* zero, and they can be freely inserted in any Poisson bracket without changing the result.

When used as generators of canonical transformations, some constraints, known as *first class*,[18] will map points on the constraint surface to points on the same surface; these transformations are known as *gauge transformations*. *Second class* constraints, on the contrary, will generate transformations that map allowed physical states (points on the constraint surface) onto disallowed states (points off the constraint surface). Since second-class constraints do not show up in the Hamiltonian formulation of general relativity, we will disregard them in the rest of this exposition.

[15] Throughout this section we shall outline the constrained Hamiltonian theory in the simpler case of a finite number of degrees of freedom. For field theories (such as general relativity) there are, as always, additional subtleties.

[16] Tertiary constraints follow from the conservation of secondary constraints, and so on. In physically interesting theories this chain ends before we run out of all the original degrees of freedom.

[17] Conversely, any *weakly* vanishing function is a linear combination of the *weakly* vanishing functions that define the constraint surface.

[18] A function of the canonical variables is defined to be *first class* if its Poisson brackets with all the constraints are strongly or weakly zero. It is defined to be *second class* if its Poisson bracket with at least one constraint is not zero.

To obtain the correct dynamics for the constrained system, we need to modify the Hamiltonian variational principle to enforce the constraints; we do this by adding the constraint functions to the Hamiltonian, after multiplying them by arbitrary functions of time (the *Lagrange–Dirac multipliers*). Because the first-class constraints generate gauge transformations on the constraint surface, different choices of the Lagrange–Dirac multipliers will generate evolutions of the canonical variables that differ by gauge transformations. If, with Dirac, we make the reasonable demand that the evolution of all *physical variables* should be unique,[19] then the points of the constraint surface that sit on the same *gauge orbit* (that is, that are linked by gauge transformations) *must describe the same physical state.* Conversely, only the functions in phase space that are invariant with respect to gauge transformations can describe physical quantities.[20]

To eliminate this ambiguity and create a one-to-one mapping between points in phase space and physical states, we can impose further constraints, known as *gauge conditions.* The gauge conditions can be defined by arbitrary functions of the variables of the constraint surface, except that they must define a *reduced phase space* that intersects each gauge orbit exactly once. In other words, given a point on the constraint surface, there must be a gauge transformation that takes it into the reduced phase space; conversely, if we apply a gauge transformation to a point in the reduced phase space, we take it out of the gauge. Abstractly, reduced phase space is the quotient of the constraint surface by the group of gauge transformations and it represents the space of variation of true degrees of freedom of the theory.

The number of independent gauge conditions must be equal to the number of independent first-class constraints. Because of their role, the gauge conditions cannot commute (have vanishing Poisson bracket) with the original first-class constraints; so the set of the first-class constraints, with the addition of the gauge conditions, becomes a set of *second-class* constraints. After this *canonical reduction* is performed, the theory is completely determined: each physical state corresponds to one and only one set of canonical variables that satisfies the constraints and the gauge conditions. Then we are also able to determine the Lagrange–Dirac multipliers, so no arbitrary functions of time appear anymore in the Hamiltonian.

At this stage, we can invoke the *Shanmugadhasan transformation* (Shanmugadhasan, 1973) to put the gauge conditions into an especially meaningful functional form. The Shanmugadhasan transformation has the effect of reshuffling all the first-class constraints into a set of Abelian canonical momenta. The surface where these momenta are zero is just the original constraint surface, and the conjugate canonical variables are the gauge functions, whose gauge fixing determines the reduced phase space. The so-called *Dirac observables* are just a

[19] Otherwise we would have to envisage *real* physical variables that are indeterminate and therefore not *observable*, and ultimately not measurable.

[20] Of course, in many cases (such as electromagnetism) we know the observable quantities from the beginning, because we have gauge-independent dynamical equations for the fields (e. g., the Maxwell equations). Then the distinction between observables and gauge variables that follows from the first-class constraints must reproduce this situation.

Darboux basis for the reduced phase space.[21] Note that the entire procedure of canonical reduction is performed *off shell*, that is, without reference to the actual solution of the Hamilton equations.

Thus, after reducing twice the dimension of the initial phase space by the number of independent constraints (once to go to the constraint surface, once again when the gauge conditions are enforced to obtain the reduced phase space), we are at the end of our long trip. Under the action of the Hamiltonian, the Dirac observables *evolve deterministically* within the reduced phase space, and the indeterminateness of the nonhyperbolic Euler–Lagrange equations has been converted into the physically harmless arbitrariness of the gauge fixing.

2.2 General relativity as a constrained Hamiltonian theory

The standard progression of general-relativity textbooks takes us through a dense barrage of differential geometry until we have gathered enough foundations to lay down the vacuum Einstein equations,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0; \tag{1}$$

on this mountaintop we can draw a breath of relief, and contemplate the beauty of general relativity. These equations can be derived as Euler–Lagrange equations from the Lagrangian variation of the Einstein–Hilbert action

$$S = \int d^4x \sqrt{-g} R, \tag{2}$$

where the independent components of the metric field $g_{\mu\nu}$ serve as configuration variables. However, the Eqs. (1) cannot be solved as they are written, because they are not hyperbolic: only two equations out of ten are evolution equations for the “accelerations” of the metric. The reason is that the action is invariant under general coordinate transformations (the *passive diffeomorphisms* $\mathrm{Diff}_P\, \mathcal{M}_4$), so the Hessian matrix has vanishing determinant (Sundermayer, 1982). From the Lagrangian point of view, to solve the Eqs. (1) we need to remove the diffeomorphism invariance by fixing the coordinate system completely.[22]

[21] While the Poisson brackets of the *Dirac* observables with the original constraints vanished only *weakly*, the reduced phase space is equipped with a new Poisson–Dirac algebra given by the so-called *Dirac brackets* (denoted by $\{\cdot,\cdot\}^*$), and the Dirac brackets of the observables with the *Abelianized* constraints and their conjugate variables vanish *strongly*. This is precisely the purpose of the Shanmugadhasan transformation, which creates a true *projection* from the original constraint surface to the reduced phase space.

[22] In the Lagrangian formalism (*manifold way*), the counting of degrees of freedom goes as follows: the ten Einstein equations can be rearranged as four *Lagrangian constraints* (restrictions on the initial Cauchy data), four *Bianchi identities* (which vanish identically), and two dynamical second-order equations. Therefore, of the ten independent components of the metric tensor, two are deterministic dynamical degrees of freedom, four are bound by the Lagrangian constraints, and the remaining four are completely indeterminate until the coordinates are fixed.

Let us now turn to the Hamiltonian formalism, where the gauge symmetry of the system is fully manifest. Although several variations are possible, we will outline the standard ADM formalism [named after Arnowitt, Deser and Misner (1962)]. Before we attempt to solve the Cauchy problem for the Einstein equations, we need to perform a $3+1$ *split* of spacetime: that is, we need to assume that the spacetime $(\mathcal{M}_4, {}^4g)$ is globally hyperbolic, and that it can be *foliated* by a family of spacelike Cauchy surfaces Σ_τ, indexed by the *parameter time* τ. This means essentially that we view the global spacetime as representing the (parameter) time development of a three-dimensional Riemannian metric ${}^3g$ on a fixed tridimensional manifold Σ_τ. The three-metric is a classical field which depends on the three spatial coordinates[23] σ^a on Σ_τ, and evolves with the parameter time τ.

To complete the $3+1$ split, we need to specify the packing of the surfaces Σ_τ in proper (physical) time, and the *physical correspondence* between the points on each surface (loosely, we need to keep track of which point is which as we progress through time). These choices are achieved by specifying the *lapse function* N and the shift vector N^a. Only now the four-metric can be reconstructed from the τ dependence of the three-metric, the lapse, and the shift.

The $(3+1)$-split Einstein equations are obtained from the Lagrangian variation of the *ADM action*,

$$S_{ADM} = \int d\tau\, N \int_{\Sigma_\tau} d\sigma^a \sqrt{-g}\, [R + K_{ab}K^{ab} - K^2] + \text{surface terms}^{24}, \tag{3}$$

where R is the scalar curvature of the three-metric g_{ab}, where the *extrinsic curvature* K_{ab} is essentially the τ derivative of g_{ab}, and where $K = K_a{}^a$. The ten configuration variables are N, N^a, and the six independent components of g_{ab}. The Legendre transformation yields the momenta

$$\pi^{ab} = -\sqrt{-g}[K^{ab} - Kg^{ab}] \quad \text{(conjugated to } g_{ab}\text{)}, \tag{4}$$

$$\pi_0 = 0,\ \pi_a = 0 \quad \text{(conjugated to } N, N^a\text{)}. \tag{5}$$

Phase space is indexed by the 20 variables (N, π_0), (N^a, π_a), (g_{ab}, π^{ab}), but the conditions (5) on the momenta conjugated to lapse and shift must be understood as the *primary constraints* of the theory, and therefore should be written as $\pi^A \approx 0$. By requesting that the primary constraints be preserved through dynamical evolution, we obtain the secondary constraints,

$$\mathcal{H}_0 \equiv \frac{1}{\sqrt{-g}}\left[\pi^{ab}\pi_{ab} - \frac{1}{2}(\pi_a{}^a)^2\right] - \sqrt{-g}\, R \approx 0 \quad \text{(superhamiltonian constraint)}, \tag{6}$$

$$\mathcal{H}_a \equiv -2\pi_a{}^b{}_{|b} \approx 0 \quad \text{(supermomentum constraints)}, \tag{7}$$

[23]From now on, it will be our convention to drop all the "3" indices which denote tensors on the spatial manifold; furthermore, we will use lowercase latin indices to enumerate the spatial coordinates, and uppercase latin indices to enumerate parameter time plus the spatial coordinates.

[24]Within the rest of this paper, we shall always neglect these terms.

where the bar denotes covariant differentiation on Σ_τ. Altogether, the primary and secondary constraints restrict the allowable physical states to a 12-dimensional constraint surface Γ_{12} in phase space. The π_A and the $\mathcal{H}_A$ are all first-class constraints, and generate *gauge* transformations on the constraint surface: the effect of the π_A is to change the lapse and shift, while $\mathcal{H}_0$ and the $\mathcal{H}_a$ respectively induce normal deformations of the surfaces Σ_τ, and generate transitions from a three-coordinate system to another. There are no second-class constraints.

The *Dirac Hamiltonian* (which rules the constrained dynamics) can be written purely in terms of the constraints:[25]

$$H_D = \int d\sigma^a [N^A \mathcal{H}_A + \lambda^A \pi_A], \tag{8}$$

where the λ^A are Lagrange–Dirac multipliers. At this stage we have already restored the hyperbolicity of the (Hamilton) equations of motion, but at the price of introducing the four arbitrary functions of time[26] λ^A:

$$\dot{N}^A \approx \lambda^A, \quad \dot{g}_{ab} \approx f_{ab}[g, \pi|\lambda], \tag{9}$$

$$\dot{\pi}^A \approx 0, \quad \dot{\pi}_{ab} \approx h_{ab}[g, \pi|\lambda]. \tag{10}$$

To remove this arbitrariness, we must fix the gauge as follows. The first step is the gauge fixing to the secondary constraints: we choose four functions χ_A of the g and π (but not of N^A!) that satisfy the orbit conditions,[27] $\det|\{\chi_A, \mathcal{H}_B\}| \neq 0$, and we impose $\chi_A \approx 0$ on the constraint surface. It turns out that the requirement of time constancy for the gauge fixings χ_A fixes the gauge with respect to the primary constraints. Finally, the requirement of time constancy for these latter gauge fixings determines the multipliers λ^A. So the choice of the four constraints χ_A is sufficient to remove all the gauge arbitrariness.

Under the Shanmugadhasan transformation proposed by Lusanna (2000; 2001), the superhamiltonian constraint corresponds[28] to a new canonical pair, where the gauge parameter is the conformal factor of g (which is proportional to $\det g$) and the new constraint is the conformal-factor momentum π_ϕ, which determines the normal deformations of Σ_τ. The corresponding gauge fixing, $\chi_0 \approx 0$, has the effect of selecting the shape of Σ_τ. The supermomentum

[25] Even before adding the constraints, the canonical Hamiltonian can be written as $H_C = \int d\sigma^a N^A \mathcal{H}_A$, so we could formally absorb the Lagrange–Dirac multipliers relative to the $\mathcal{H}_A$ into the definition of the N^A. Still, lapse and shift are not arbitrary functions, but dynamical variables! The fact that the Hamiltonian vanishes on the constraint surface is a general feature of generally covariant theories. See for instance Henneaux and Teitelboim (1992).

[26] The λ^A are also arbitrary functions of the spatial coordinates σ^a, although in a slightly different sense: loosely speaking, there are four arbitrary multipliers *at each spatial location*, so the spatial coordinates, together with "^{A}," play the role of generalized degree-of-freedom indexes.

[27] These conditions implement the Lorentz signature of the reconstructed four-metric, by inheriting the signature already implicit in the superhamiltonian and supermomentum.

[28] In practice, this transformation requires the solution of the superhamiltonian constraint, but so far this result has proved elusive.

constraints correspond to three canonical pairs where the new constraints are the three longitudinal components of π^{ab}, while the gauge parameters are just the three-coordinates on Σ_τ. The corresponding gauge fixings, $\chi_a \approx 0$, have the effect of selecting the coordinate system on Σ_τ. After the gauge parameters have been fixed, the second-order time-constancy requirement (mentioned above) has the effect of providing partial differential equations for the lapse and shift, in a manner compatible with the shape of Σ_τ and with the choice of the three-coordinates.

At the end of the canonical reduction procedure, the 12 degrees of freedom of the constraint surface are reduced to four, the *Dirac observables* q^r, p_s $(r, s = 1, 2)$ that index the reduced phase space Ψ_4, and that represent the two *true dynamical degrees of freedom of the gravitational field.* Each gauge fixing creates a realization of Ψ_4, with a canonical structure implemented by the Dirac brackets associated to that gauge. The Dirac observables satisfy the final Hamilton equations,

$$\dot{q}^r = \{q^r, H_{\mathrm{ADM}}\}^*, \quad \dot{p}_s = \{p_s, H_{\mathrm{ADM}}\}^*, \tag{11}$$

where H_{ADM} is intended as the restriction of the ADM Hamiltonian to Ψ_4 and where the $\{\cdot, \cdot\}^*$ are the Dirac Brackets. In general, $q^r(\tau, \sigma^a)$ and $p_s(\tau, \sigma^a)$ are highly nonlocal[29]; they are neither tensors nor invariants under space-time diffeomorphisms, because their functional form depends on the gauge fixing. As we shall see, *on shell* (when the dynamical variables are restricted to the values that they can have as solutions of the Hamilton–Dirac equations) the gauge fixing is equivalent to the choice of a set of four-dimensional coordinates.

According to Lusanna and Pauri (2002), the Shanmugadhasan transformation proposed by Lusanna (2000; 2001) allows the (loose) interpretation of the Dirac observables as representing the *tidal effects* of the gravitational field. Obviously, in general relativity there are no gravitational *forces* in the common sense. Yet, we can introduce the general-relativistic analogs of inertial forces with respect to the worldlines of nongeodesic observers (Abramowicz, 1993; Abramowicz, Nurowski and Wex, 1993). The physical meaning of the eight gauge transformations is just to modify the *inertial* (reference-frame–induced) effects; however, the *presentation* of both the tidal effects and the inertial forces depends on the gauge fixings, just as the functional form of the Dirac observables does.

2.3 Gauge groups and dynamical symmetries in the general theory of relativity

Not all the transformations generated by the first-class constraints (the *off-shell Hamiltonian gauge group* $\mathcal{G}_8$) are true, *harmless* gauge transformations in the sense introduced by Dirac, because some of them will join points of the constraint surface that represent different four-geometries,[30] and therefore different

[29]Because in general relativity the Shanmugadhasan transformation is highly nonlocal.

[30]The quotient of the constraint surface with respect to the off-shell Hamiltonian gauge transformations is the so-called *reduced off-shell conformal superspace* $\Gamma_4 = \Gamma_{12}/\mathcal{G}_8$. Each

physical states. This property follows from the fact that, in the Dirac Hamiltonian, among the eight multipliers only four are arbitrary Lagrange–Dirac multipliers (the other four are the dynamical variables lapse and shift), and that the correct gauge-fixing procedure starts by giving only the four gauge fixings for the secondary constraints. Going *on shell* (that is, restricting our consideration to the solutions of the Hamilton–Dirac equations) we introduce a functional dependence among the group descriptors of $\mathcal{G}_8$, creating a four-dimensional subgroup $\mathcal{G}_4^{\mathrm{dyn}}$ (the *on-shell Hamiltonian gauge group*) whose transformations are also *dynamical symmetries* of the Hamilton–Dirac equations (dynamical symmetries are defined as the transformations that map solutions of the equations of motion onto other solutions; as such, they are an on-shell concept).

In the context of the *Lagrangian* formalism, the (passive) dynamical symmetries of the Einstein equations were studied by Bergmann and Komar (1972), who showed that the largest group of such transformations is not $\mathrm{Diff}_P\,\mathcal{M}_4$ $[\xi'^{\mu} = f^{\mu}(\xi^{\nu})]$ but rather the group Q of transformations of the form $\xi'^{\mu} = f^{\mu}(\xi^{\nu}, g_{\alpha\beta})$. These transformations map points on points, but associate with a given point x an image point x' that depends also on the metric field g. Hence the elements of Q should be considered as mappings from the functional space of metric fields onto itself.

Bergmann and Komar showed that the passive diffeomorphisms, $\mathrm{Diff}_P\,\mathcal{M}_4$, are a *nonnormal subgroup* of Q. We have just met another nonnormal subgroup of Q: it is the on-shell Hamiltonian gauge group $\mathcal{G}_4^{\mathrm{dyn}}$, or rather its *Legendre pullback* to configuration space, which Bergmann and Komar call Q_{can}. The subgroups $\mathrm{Diff}_P\,\mathcal{M}_4$ and Q_{can} have a nonempty intersection, which consists of all the passive coordinate transformations that respect the $3+1$ splitting of the ADM construction.

Looking in the other direction (from configuration space to phase space), Q_{can} represents the part of Q that is *projectable* into phase-space transformations. It follows that the subgroup Q_{can} is defined by a particular choice of the four functionally independent descriptors that are the manifold counterparts of the four independent descriptors of $\mathcal{G}_4^{\mathrm{dyn}}$.

All these groups are just different representations of the descriptive arbitrariness of general relativity, so we expect that they should all generate the same partition of the space $\mathrm{Riem}\,\mathcal{M}_4$ of solutions of the Einstein–ADM equations into equivalence classes. Indeed, Bergmann and Komar showed that

$$\mathrm{Geom}\,\mathcal{M}_4 = \frac{\mathrm{Riem}\,\mathcal{M}_4}{\mathrm{Diff}_P\,\mathcal{M}_4} = \frac{\mathrm{Riem}\,\mathcal{M}_4}{Q_{\mathrm{can}}} = \frac{\mathrm{Riem}\,\mathcal{M}_4}{Q}, \tag{12}$$

which is mathematically possible because both $\mathrm{Diff}_P\,\mathcal{M}_4$ and Q_{can} are nonnormal subgroups of Q.

Only one detail is missing: what is the status of the active diffeomorphisms $\mathrm{Diff}_A\,\mathcal{M}_4$ within this representation? Intuitively, it seems that active and pas-

point of Γ_4 (a *Hamiltonian off-shell* or *kinematical* gravitational field) is an equivalence class known as *off-shell conformal three-geometry* for the space-like hypersurfaces Σ_τ. It is not a four-geometry, because it contains all the off-shell three-geometries connected by Hamiltonian gauge transformations.

sive diffeomorphisms make up all the operations that can be defined on the space-time manifold; however, nobody so far has studied in detail the mathematical structure of the group Q. It is however easy to show (Lusanna and Pauri, 2002) that at least the infinitesimal active diffeomorphisms belong to Q, because they can be interpreted as passive transformations with the following procedure.

Consider an infinitesimal (passive) transformation of the type $\xi'^{\mu} = \xi^{\mu} + X^{\mu}(\xi, g)$. This will induce the usual formal local variation of the metric tensor,

$$\bar{\delta} g_{\mu\nu} = -\big(X_{\mu;\nu}(\xi, g) + X_{\nu;\mu}(\xi, g)\big). \tag{13}$$

Therefore, if $\bar{\delta} g_{\mu\nu}$ is the variation of the metric tensor associated with the infinitesimal *active* diffeomorphism, the solution $X^{\mu}(\xi, g)$ of these Killing-type equations identifies a corresponding passive Bergmann–Komar dynamical symmetry of Q. This should imply that all the active diffeomorphisms connected with the identity in $\mathrm{Diff}_A\, \mathcal{M}_4$ can be reinterpreted as elements of a nonnormal subgroup of the generalized passive transformations of Q. Clearly this subgroup is disjoint from the subgroup $\mathrm{Diff}_P\, \mathcal{M}_4$: note that this is possible because diffeomorphism groups do not possess a canonical identity. Given this, we could naturally guess that Q_{can} is a mix of passive and active diffeomorphisms, because the active and passive diffeomorphisms, being nonnormal subgroups of Q, should, as it were, fill Q densely in a suitable topology.

Finally, we complete Eq. (12): because obviously we have

$$\mathrm{Geom}\, \mathcal{M}_4 = \frac{\mathrm{Riem}\, \mathcal{M}_4}{\mathrm{Diff}_P\, \mathcal{M}_4} = \frac{\mathrm{Riem}\, \mathcal{M}_4}{\mathrm{Diff}_A\, \mathcal{M}_4}, \tag{14}$$

we obtain the final definition of the equivalence classes of *on-shell* or *dynamical* gravitational fields,

$$\mathrm{Geom}\, \mathcal{M}_4 = \frac{\mathrm{Riem}\, \mathcal{M}_4}{\mathrm{Diff}_P\, \mathcal{M}_4} = \frac{\mathrm{Riem}\, \mathcal{M}_4}{\mathrm{Diff}_A\, \mathcal{M}_4} = \frac{\mathrm{Riem}\, \mathcal{M}_4}{Q_{\mathrm{can}}} = \frac{\mathrm{Riem}\, \mathcal{M}_4}{Q}. \tag{15}$$

In other words, any of the groups $\mathrm{Diff}_P\, \mathcal{M}_4$, $\mathrm{Diff}_A\, \mathcal{M}_4$, Q_{can}, and Q can be used to implement Leibniz equivalence *on shell*.

2.4 The Bergmann–Komar invariants: metrical structure and the physical individuation of points in the (un)real world

Let us now take a quick detour back to four-dimensional (so to speak) general relativity. We note with Bergmann and Komar[31] that for a vacuum solution of the Einstein equations, in the hypothesis that spacetime admits *no symmetries*, there are exactly four functionally independent scalars that can be written using

[31] See Bergmann (1960, 1962, 1971, 1977); Bergmann and Komar (1960).

the lowest possible derivatives of the metric.[32] These are the four Weyl scalars (the eigenvalues of the Weyl tensor), here written in Petrov compressed notation,

$$w_1 = \mathrm{Tr}\,(gWgW), \tag{16}$$
$$w_2 = \mathrm{Tr}\,(gW\epsilon W), \tag{17}$$
$$w_3 = \mathrm{Tr}\,(gWgWgW), \tag{18}$$
$$w_4 = \mathrm{Tr}\,(gWgW\epsilon W), \tag{19}$$

where g is the *four*-metric, W is the Weyl tensor, and ϵ is the Levi–Civita totally antisymmetric tensor.

Bergmann and Komar then propose that we build a set of *intrinsic coordinates* for the point-events of space-time as four functions of the w_T,

$$\hat{I}^{[A]} = \hat{I}^{[A]}\big[w_T[g(x), \partial g(x)]\big]. \tag{20}$$

Indeed, under the hypothesis of no space-time symmetries,[33] the $\hat{I}^{[A]}$ can be used to label the point-events of spacetime, at least locally.[34] What is more, the value of the intrinsic coordinates at a point–event can be extracted (in principle) by an actual experiment designed to measure the w_T (see Sec. 3). Because they are functionals of scalars, the $\hat{I}^{[A]}$ are invariant under passive diffeomorphisms (therefore they do not define a coordinate chart in the usual sense), and by construction they are also constant under the drag-along of tensor fields induced by active diffeomorphisms.[35]

The metric can be rewritten with respect to the intrinsic coordinates:

$$\hat{g}^{[AB]} = \frac{\delta \hat{I}^{[A]}}{\delta x^\mu}\frac{\delta \hat{I}^{[B]}}{\delta x^\nu} g^{\mu\nu}. \tag{21}$$

[32]The fact that there are just *four* independent invariants is crucial for the purpose of point individuation, and it should not be regarded as a coincidence. After all, recall that in general space-times with matter there are 14 invariants of this kind! (Géhénieau and Debever, 1956)

[33]Our attempt to use intrinsic coordinates to provide a physical individuation of point-events would *prima facie* fail in the presence of symmetries, when the $\hat{I}^{[A]}$ become degenerate. This objection was originally raised by Norton (1988) as a critique to manifold-plus-further-structure (MPFS) substantivalism [according to which the points of the manifold, conjoined with additional local structure such as the metric field, can be considered physically real; see for instance Maudlin (1988)]. Several responses are possible. First, although to this day all the *known* exact solutions of the Einstein equations admit one or more symmetries, these mathematical models are very idealized and simplified; in a realistic situation (for instance, even with two masses) space-time is filled with the excitations of the gravitational degrees of freedom, and admits no symmetries at all. Second, the parameters of the symmetry transformations can be used as supplementary individuating fields, since, as noticed by Stachel (1993), they also depend on metric field, through its isometries. Third, and most important, in our analysis of the physical individuation of points we are arguing a question of principle, and therefore we must consider *generic* solutions of the Einstein equations rather than the null-measure set of solutions with symmetries.

[34]Problems might arise if we try to extend the labels to the entire space-time: for instance, the coordinates might turn out to be multivalued.

[35]Already at this stage, we see that this is just the right method to realize the equivalence class of points to which Bartels was alluding (Bartels, 1994).

The $\hat{g}^{[AB]}$ represent the ten *invariant scalar components* of the metric; of course they are not all independent, but they should satisfy six functional restrictions that follow from the Einstein equations. However, Eq. (21) is deceiving, because the $\hat{g}^{[AB]}$ are functionals of the metric and of *its partial derivatives* (through the $\hat{I}^{[A]}$). It should be noted that, in a sense, the freedom to express the metric using any set of coordinates is still present in the choice of the four functions $\hat{I}^{[A]}$ of the Weyl scalars. What is more, given any coordinatization of a spacetime without symmetries, it is possible to reproduce the tensorial components of the metric using a suitable set of $\hat{I}^{[A]}$.

Decomposing the w_T with the $3+1$ splitting outlined in Sec. 2.2, we realize [again with Bergmann and Komar (1960)] that the four Weyl scalars w_T *do not depend on lapse and shift*. This circumstance is crucial, because it means that *we can use suitable functions of the w_T as gauge fixings to the secondary constraints*[36] (Lusanna and Pauri, 2002). To do so, we first write the Bergmann–Komar intrinsic coordinates as functionals of the ADM variables,

$$\hat{I}^{[A]}[w_T(g,\partial g)] \equiv \hat{Z}^{[A]}[w_T(g,\pi)]; \tag{22}$$

we then select a *completely arbitrary* coordinate system $\sigma^A \equiv [\tau, \sigma^a]$ adapted to the Σ_τ surfaces; finally, we apply the gauge fixing Γ defined by

$$\chi^A \equiv \sigma^A - \hat{Z}^{[A]}\big[w_T[(g(\sigma^B), \pi(\sigma^D)]\big] \approx 0; \tag{23}$$

of course the functions $\hat{Z}^{[A]}$ must be chosen to satisfy the orbit conditions $\{\hat{Z}^{[A]}, \mathcal{H}_B\} \neq 0$, which ensure the independence of the χ^A and carry information about the Lorentz signature. The effect is that the evolution of the *Dirac observables*, whose dependence on space (and on parameter time) is indexed by the chosen coordinates σ^A, reproduces the σ^A as the Bergmann–Komar intrinsic coordinates:

$$\sigma^A = \hat{Z}^{[A]}[w_T(q^r(\sigma^B), p_s(\sigma^C)|\Gamma)], \tag{24}$$

where the notation $w_T(q,p|\Gamma)$ represents the functional form that the Weyl scalars w_T and the *Dirac observables* q^r, p_s assume in the chosen gauge. Eq. (24) is just an identity with respect to the σ^A. The price that we have paid for this achievement is of course that we have broken general covariance!

At first this result may sound surprising: diffeomorphism-invariant quantities, such as the intrinsic coordinates, are known as *Bergmann observables*, and are often identified with the only locally measurable variables of the pure gravitational field (because being diffeomorphism invariants they can be obtained using the coordinate system corresponding to any experimental arrangement). From the Hamiltonian viewpoint, however, they are gauge-dependent[37] quantities that (in a sense) can be arranged to assume any functional dependence on Σ_τ.

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[36] Please refer back to Sec. 2.2, just after Eq. (10).

[37] *Canonical reduction* (which creates the distinction between gauge-dependent quantities and Dirac observables) is made *off shell*, that is, *before* solving the equations of motion. It is not known so far whether suitable diffeomorphism-invariant intrinsic coordinates can also become *Dirac observables on shell*, that is, on the space of solutions to the equations of motion. See however Sec. 4.

The crucial point to remember here is that the gauge transformations of $\mathcal{G}_8$ can actually link different four-geometries; correspondingly, a complete gauge fixing can modify the value of diffeomorphism-invariant quantities.[38] So we can take any four-geometry, find its Cauchy data on Σ_τ, and then move along its $\mathcal{G}_8$ gauge orbit to create any arbitrary structure for the Weyl scalars; but the final point on the constraint surface will represent a *different* four-geometry. On the other hand, the on-shell Hamiltonian gauge group $\mathcal{G}_4^{\mathrm{dyn}}$ contains only transformations that are counterparts of active or passive *projectable* diffeomorphisms (the ones that are compatible with the $3+1$ split).

After canonical reduction and only for the solutions of the equations of motion, Eq. (24) becomes a *strong* relation, and it amounts to a *definition of the four coordinates* σ^A, providing a *physical individuation of any point–event, in the gauge-fixed coordinate system, in terms of the true dynamical gravitational degrees of freedom.*

The virtue of this elaborate setup is not that it selects a set of physically preferred coordinates, because by modifying the functions $I^{[A]}$ we have the possibility of implementing any coordinate transformation. So diffeomorphism invariance reappears under a different semblance: we find exactly the same functional freedom whether we choose a set of coordinates on $\mathcal{M}_4$, the functions $Z^{[A]}$, or the gauge fixing. Thus, it turns out that, *on shell*, at the Hamiltonian level as well as the Lagrangian level, *gauge fixing is clearly synonymous with the selection of manifold coordinates.* Instead, we are now able to claim that *any* coordinatization of the manifold can be seen as embodying the physical individuation of points, because it can be implemented[39] as the Komar–Bergmann intrinsic coordinates after we choose the correct $Z^{[A]}$ and we select the correct gauge. The byproduct of the gauge fixing is the identification of the form of the physical degrees of freedom as nonlocal functionals of the metric and curvature.

Summarizing, each of the point-events of spacetime is endowed with its own physical individuation (the right metrical fingerprint!) as the value, as it were, of the four canonical coordinates (just four!), or *Dirac observables* which describe the dynamical degrees of freedom of the gravitational field. However, these degrees of freedom are unresolveably entangled with the structure of the metric manifold in a way that is strongly gauge dependent.

As a final consideration, let us point out that Eq. (24) is a numerical identity that has an *inbuilt noncommutative structure*, deriving from the Dirac–Poisson structure on its right-hand side. The meaning of this structure is not clear at the classical level, but we believe that it could be relevant to the quantization of general relativity.

[38] Each three-metric in the conformal gauge orbit has a different three-Riemann tensor, and different three-curvature scalars. Since four-tensors and four-curvature scalars depend on lapse, shift, their gradients, and on the conformal-factor momentum, most of these objects are in general gauge variables from the Hamiltonian point of view.

[39] Again, at least locally.

3 The individuation of points in the real world

The philosophical analysis of the general-relativistic notion of space-time is developed most often (and this paper is no exception) on the *geometrodynamical* formulation of general relativity, which pictures matter following the straightest lines, so to speak, in a curved space-time arena deformed by gravitation. There are many reasons for this preference: the geometric theory is indeed very beautiful, and it appears to complete and extend more fully the critique of space-time structure begun with special relativity. Within this paradigm, the prototype solution is a strongly curved vacuum space-time with no symmetries. For such a space-time, coordinate system are freely interchangeable, and of course they are almost completely irrelevant to the physical individuation of points. For such a space-time, the philosophical arguments about the Hole Argument and about general covariance carry their full weight.

However, our universe is not a strongly curved space-time, and it is not a vacuum solution: rather, it resembles most closely the flat space-time of special relativity, and it contains much matter, organized in structures at many scales. Although we know, in theory, that all coordinate frames are equally acceptable, in this real physical world we manage to keep the time, keep our orientation, navigate the solar system, and make sense of the universe with a handful of very special coordinate systems. These systems are precisely the ones that recognize that gravity is weak (so it can be treated as a correction to flat space-time) and that matter with structure is available to provide useful points of reference (in a relational sense).

Indeed, Soffel (1989) defines the purpose of *astrometry* (the theory of constructing reference frames) as "the materialization of a global, nonrotating, quasi-inertial reference frame, in the form of a fundamental catalogue of stellar positions and proper motions." On a smaller scale, the preferred reference frames are those that provide a simple, understandable form for the dynamical equations that rule the motions of celestial bodies. In the case of the solar system, a suitable reference frame is the barycentric post–Newtonian frame, where the metric deviates from the Minkowski metric by simple corrections, and where the equations of motion are slightly modified Newtonian equations (Soffel, 1989).

Are these coordinate systems *methodologically preferred* because of their convenience? If so, can they confer identity to the point-events of space-time? Both questions deserve some investigation; however, we should note that they do not refer directly to the philosophical analysis of general relativity in the generic case, but rather in the case of a specific solution (our universe). So we should be cautious when we discuss the connection between the physical individuation of points (as we have outlined it) and the *theory of measurement* in general relativity, with its many real-world applications (such as time transport, geographic positioning and solar-system navigation). The *practice* (but not the theory) of general-relativistic measurements is necessarily a consequence of the particular solution of the Einstein equations that we happen to inhabit.[40]

Still, we wish to draw a scenario of how the physical individuation of points could be implemented (in principle) as an experimental setup and protocol for positioning and orientation. This construction, which could also be discussed more abstractly as a system of axioms[41] for the empirical foundation of general relativity, *closes the coordinative circuit* that joins the mathematical formulation of general relativity (and in particular of the Hamiltonian initial-value problem) to the practice of general-relativistic measurement, and to the physical individuation of space-time points. Three steps are necessary.

1. We define a *radar-gauge system of coordinates* in a finite four-dimensional volume, by means of a network of artificial satellites similar to the Global Positioning System (Ashby and Spilker, 1995). The GPS is a constellation of 24 satellites on quasicircular 20-km-high orbits around the Earth; each GPS satellite carries an atomic clock accurate to the nanosecond, and continuously broadcasts its own position and time,[42] as computed within an accurate model of its motion in the gravitational field of the Earth. By comparing the signals received from four satellites at a given instant of time (*pseudo-ranging*), the *GPS receivers* on the surface of the Earth are able to determine their radar distance from the satellites, and therefore to compute their own latitude, longitude, and altitude with a precision of a few tens of meters, and to track the international standard time with a maximum error of a few nanoseconds.

   The GPS receivers are able to determine their actual position (that is, the set of their four post–Newtonian, geocentric coordinates, with the time coordinate rescaled to the international standard time), because the entire GPS system is predicated on the advance knowledge of the gravitational field of the Earth and of the trajectories of the satellites, which in turn allows the *coordinate* synchronization of the satellite clocks to post–

[40] On the contrary, the physical individuation of points events by the analysis of the local metric fingerprint would be very relevant to orientation and navigation in a hypothetical world that is devoid of matter, and where gravity is very strong and unpredictable.

[41] We owe the classical paper on the axiomatics of general relativity to Ehlers, Pirani and Schild (1972), who start out by defining basic objects such as light rays, freely falling test particles, standard clocks, and so on. In their scheme, light-ranging measurements are then used to reveal the *conformal* structure of spacetime, while the free fall of test bodies is used to map out the *projective* structure. Under an axiom of compatibility [well corroborated by experiment; see Perlick (1994)] these two classes of observations determine completely the structure of spacetime.

We note here that both the Ehlers–Piran–Schild axiomatics (based on idealized primitive physical objects and operations) and our discussion of coordinate systems and metric field measurements in terms of technological instruments (GPS satellites) imply that the *coordination* of the mathematical theory of general relativity to the physical quantities defined operationally cannot be excised from the wider context of a comprehensive theory of physical reality, where the idealized primitive objects and operations of Ehlers, Pirani and Schild are, in essence, implemented by our technological instruments.

[42] More precisely, the clocks on the satellites are biased to yield the international standard time; that is, the proper time elapsed on the *geoid*, the surface of constant effective gravitational potential that sits very close to the surface of the Earth (at sea level).

Newtonian time. If, as in our case, the geometry of space-time and the motion of the satellites are not known in advance, it would be still possible for the receivers to obtain four, as it were, *conventional* coordinates by operating a full-ranging protocol (involving bidirectional communication) to four super-GPS satellites that broadcast the time of their standard, unsynchronized clocks. The problem of patching the coordinates obtained from different four-tuples of satellites is analog to deriving the coordinate transformations between overlapping patches within an atlas of a differential manifold, and it should be tractable by maintaining full-ranging communication between the satellites themselves.

Summarizing, our super-GPS constellation provides a radar-gauge system of coordinates (without any direct metrical significance) for all the point-events within a finite region of spacetime[43]:

$$\sigma_R^A \equiv (\tau_R, \sigma_R^A); \quad \tau_R = 0 \text{ defines } \Sigma_{\tau_R}. \tag{25}$$

2. By means of repeated measurements of the motion of four test particles[44] (see Ciufolini and Wheeler 1995, pp. 34–36; see also Rovelli 2001) and gyroscopes (to measure N^A!), with technologies similar to the Gravity Probe B space mission (GPB), suitable spacecraft could then *measure the components of the four-metric with respect to the radar-gauge coordinates*,

$${}^4g_{R(A,B)}(\tau_R, \sigma_R^a), \tag{26}$$

and by measuring the spatial and temporal variation of ${}^4g$, we could then compute (in principle) the components of the Weyl tensor, and *the Weyl invariant scalars.*[45]

3. By steps 1 and 2, we have obtained a slicing of spacetime into surfaces Σ_{τ_R}, and a set of coordinates σ^a on the surfaces, both defined *operationally*; furthermore, we have determined the components of the metric and the local value of the Weyl scalars with respect to the σ^A. We can then solve (in principle) for the functions $\hat{Z}^{[A]}$ that reproduce the radar-gauge coordinates as *radar-gauge intrinsic coordinates*,

$$\sigma_R^A = \hat{Z}^{[A]}\big[w_T[g(\sigma_R^B), \pi(\sigma_R^C)]\big]. \tag{27}$$

Finally, we can impose the gauge fixing that enforces this particular system of intrinsic coordinates,

$$\chi^A \equiv \sigma^A - \hat{Z}^{[A]}\big[w_T[g(\sigma^B), \pi(\sigma^C)]\big] \approx 0; \tag{28}$$

at the end of the canonical reduction procedure, we obtain the structure of the Dirac observables q^r, p_s as nonlocal functionals of g and π, and we

[43] Within the Ehlers–Pirani–Schild axiomatics, this corresponds to determining the *conformal structure* of spacetime.

[44] For vacuum gravitational fields. Six test particles are needed in general space-times.

[45] Within the Ehlers–Pirani–Schild axiomatics, this corresponds to determining the *projective structure* of spacetime.

reconstruct the intrinsic coordinates as functions of the Dirac observables in each point-event of space-time:

$$\sigma_R^A = \hat{Z}^{[A]}\big[w_T[q^r(\sigma_R^B), p_s(\sigma_R^C)]\big]. \tag{29}$$

Thus, the radar-gauge coordinates are legitimized as intrinsic coordinates that, because of their well-defined dependence on the Dirac observables, can endow the point-events of space-time with physical individuality. Of course, the particular form of this dependence, and the particular *presentation* of the *true* degrees of freedom of the gravitational field is gauge dependent.

This procedure closes the *coordinative circuit* of general relativity, linking individuation to experimentation.

4 Conclusion: finding the last remnant of physical objectivity

From the point of view of the constrained Hamiltonian formalism, general relativity is a gauge theory like any other; however, it is radically different from the physical point of view. In addition to creating the distinction between what is observable[46] and what is not, the gauge freedom of general relativity is unavoidably entangled with the definition–constitution of the very *stage*, space-time, where the *play* of physics is enacted. In other words, the gauge mechanism has the double role of making the dynamics unique (as in all gauge theories), and of fixing the spatio-temporal reference background *at the mathematical level*.

In gauge theories such as electromagnetism, we can rely from the beginning on empirically validated, gauge-invariant dynamical equations for the *local* fields. This is not the case for general relativity: in order to get dynamical equations for the basic field in a *local* form, we must pay the price of general covariance, which weakens the objectivity that the spatiotemporal description could have had *a priori*. Recalling the definition of gauge theory given by Henneaux and Teitelboim (see the beginning of Sec. 2.1), we could say that the introduction of extra variables does make the mathematical description of general relativity more transparent, but it also makes its physical interpretation more obscure and intriguing, at least at first sight.

By now, it should be clear that the Hole Argument has nothing to do with the alleged *indeterminism* of general relativity as a dynamical theory. In our discussion of the initial-value problem within the Hamiltonian framework we have shown that, *on shell*, a complete gauge-fixing (which could in theory concern the whole space-time) is equivalent to the choice of an atlas of coordinate charts on the space-time manifold, and in particular *within the Hole*. At the same time, we have seen that the active diffeomorphisms of the manifold can be interpreted as passive Hamiltonian gauge transformations. Because the gauge

[46]In the Dirac or Bergmann sense.

must be fixed *before* the initial-value problem can be solved to obtain a solution (outside and inside the Hole), it makes little sense to apply active diffeomorphisms to an already generated solution to obtain an allegedly "different" space-time. Conversely, it should be possible to generate these "different" solutions by appropriate choices of the initial gauge fixing.

In addition, we have established that within the Hamiltonian framework we can use a gauge-fixing procedure based on the Bergmann–Komar intrinsic coordinates to turn the *primary* mathematical individuation of manifold *points* into a *physical* individuation of *point-events* that is directly associated with the value of the gravitational degrees of freedom (Dirac observables). The price to pay is the breaking of general covariance. General covariance thus represents a horizon of *a priori* possibilities for the physical constitution of the space-time, possibilities that must be actualized within any given solution of the dynamical equations. What here we called *physical constitution* embodies at the same time the *chrono-geometrical*, the *gravitational*, and the *causal* properties of the space-time stage.

We have shown that this conceptual physical individuation can be implemented (at least in principle) with a well-defined empirical procedure that closes the *coordinative circuit*. We believe that these results cast some light over the *intrinsic structure* of the general relativistic space-time that had disappeared within Leibniz equivalence and that was the object of Michael Friedman's non-trivial question.

In 1972, Bergmann and Komar wrote (Bergmann and Komar, 1972):

> [...] in general relativity the identity of a world point is not preserved under the theory's widest invariance group. This assertion forms the basis for the conjecture that some physical theory of the future may teach us how to dispense with world points as the ultimate constituents of space-time altogether.

Indeed, would it be possible to build a fundamental theory that is grounded in the reduced phase space parametrized by the Dirac observables? This would be an abstract and highly nonlocal theory of gravitation that would admit an infinity of gauge-related, spatio-temporally local realizations. From the mathematical point of view, however, this theory would be just an especially perspicuous instantiation of the relation between canonical structure and locality that pervades contemporary theoretical physics nearly everywhere.

On the other hand, beyond the mathematical transparency and the latitude of choices guaranteed by general covariance, we need to remember that *local* spatio-temporal realizations of the abstract theory would still be needed for implementation of measurements in practice; conversely, any real-world experimental setting entails the choice of a definite *local realization*, with a corresponding gauge fixing that breaks general covariance.

Can this basic freedom in the choice of the *local realizations* be equated with a "taking away from space and time the last remnant of physical objectivity", as Einstein suggested? We believe that if we strip the physical situation from Einstein's "spatial obsession" about *realism as locality (and separability)*, a significant kind of spatio-temporal objectivity survives. It is true that the *functional dependence* of the Dirac observables upon the spatio-temporal coordinates depends on the particular choice of the latter (or equivalently, of the gauge); yet, there is no *a priori physical* individuation of the points independently of the metric field, so we cannot say that the physical-individuation procedures corresponding to different gauges individuate physical point-events that are *really* different. Given the conventional nature of the primary mathematical individuation of manifold points through n-tuples of real numbers, we could say instead that the *real point-events* are constituted by the nonlocal values of gravitational degrees of freedom, while the underlying point structure of the mathematical manifold may be changed at will.

In this connection, let us observe that we could venture even farther, and claim [with Lusanna and Pauri (2002)] that the physical content of the general-relativistic gravitational field in the absence of matter *is* just the very identification of the points of space-time by means of the four independent phase-space degrees of freedom of the gravitational field. We have not endorsed this stronger position in this paper.

Furthermore, looking at the Bergmann–Komar *intrinsic components* of the metric [see Eq. (21)], and calculating the Dirac brackets of the Weyl scalars, it might be possible to define four diffeomorphically invariant and canonically conjugated variables that are also Dirac observables *on shell*. This achievement would unify the general-covariant and the Dirac–Bergmann–Komar notion of observable, and would provide explicit evidence for the objectivity of point-event individuation. Finally, the procedure of individuation that we have outlined transfers, as it were, the noncommutative Poisson–Dirac structure of the Dirac observables onto the individuated point-events; the physical implications of this circumstance might deserve some attention in view of the quantization of general relativity.

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