Coordinate formalism on Hilbert manifolds: string bases of eigenvectors

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Abstract

Coordinate formalism on Hilbert manifolds developed in [1], [2] is further analyzed. The main subject here is a comparison of the ordinary and the string bases of eigenvectors of a linear operator as introduced in [1]. It is shown that the string basis of eigenvectors is a natural generalization of its classical counterpart. It is also shown that the developed formalism forces us to consider any Hermitian operator with continuous spectrum as a restriction to a space of square integrable functions of a self-adjoint operator defined on a space of generalized functions. In the formalism functional coordinate transformations preserving the norm of strings are now linear isometries rather than the unitary transformations.

1 Introduction

In [1], [2] a coordinate formalism on abstract Hilbert manifolds has been developed. The formalism is based on consideration of infinite-dimensional Hilbert manifolds modeled on a given collection of Hilbert spaces rather than on a single Hilbert model. This possibility is usually neglected as all infinite-dimensional separable Hilbert models are isomorphic. However, unlike the finite dimensional case, different spaces of functions must be treated differently in analysis, as elements of these spaces have different analytic properties.

The developed coordinate formalism has shown a deep and unusual similarity with the standard formalism of finite dimensional differentiable manifolds. In particular, the *new* infinite-dimensional counterparts of the classical notions of basis, dual basis, Riemannian metric, orthogonal basis, etc., have been found. They all represent a natural and nontrivial extension of the corresponding classical notions and in general cannot be reduced to the last ones or to their standard infinite-dimensional analogues.

The introduced formalism seems to be especially useful when working with generalized functions. In particular, as shown in [1] the generalized eigenvalue problem for an arbitrary linear operator on a Hilbert space of functions can be considered as a (functional) coordinate transformation of the classical eigenvalue problem. This last aspect of the formalism will be analyzed in more details in the paper.

In section 2 we will review the main definitions of [1].

In section 3 the classical notion of a basis of eigenvectors and a string basis of eigenvectors of a linear operator will be compared. In particular, it will be shown in what sense the string basis of eigenvectors generalizes the ordinary one. We also discuss how the developed formalism leads to a necessary extension of an arbitrary Hermitian operator on a Hilbert space to a more general self-adjoint operator on the string space.

The findings are summarized in conclusion.

2 Review of linear algebra on the string space

Here the main definitions introduced in [1] are given. Please, refer to [1] for details.

Definition. The *string space* $\mathbf{S}$ is an abstract vector space that is also a differentiable manifold linearly diffeomorphic to an infinite-dimensional separable Hilbert space.

Definition. A *Hilbert space of functions* is either a Hilbert space H, elements of which are equivalence classes of maps between two given subsets of R^n or the Hilbert space H^* dual to H. Two elements $f, g \in H$ are called *equivalent* if the norm of $f-g$ in H is zero.

Definition. A linear isomorphism e_H from a Hilbert space H of functions onto $\mathbf{S}$ is called a *string basis* on $\mathbf{S}$.

Definition. Given e_H the function $\varphi \in H$ such that $\Phi = e_H \varphi$ will be called a *coordinate* (or an *H-coordinate*) of a string $\Phi \in \mathbf{S}$. The space H itself is called a *coordinate space.*

Definition. Let $\mathbf{S}^*$ be the dual string space. A linear isomorphism e_{H^*} of H^* onto $\mathbf{S}^*$ is called a *string basis on* $\mathbf{S}^*$.

Definition. The basis e_{H^*} is called *dual* to the basis e_H if for any string $\Phi = e_{Hk}\varphi^k$ and for any functional $F = e^k_{H^*} f_k$ the following is true: $F(\Phi) = f(\varphi)$.

Remark. In general case we have

$$F(\Phi) = e_{H^*} f(e_H \varphi) = e^*_H e_{H^*} f(\varphi), \tag{2.1}$$

where $e^*_H : \mathbf{S}^* \longrightarrow H^*$ is the adjoint of e_H. Therefore, e_{H^*} is the dual string basis if $e^*_H e_{H^*} : H^* \longrightarrow H^*$ is the identity operator.

Let now $\mathbf{S}$ be a Hilbert space with the inner product $(\cdot,\cdot)_S$.

Definition. A string basis e_H in $\mathbf{S}$ is called *orthonormal* if

$$(\Phi, \Psi)_S = f_\varphi(\psi), \tag{2.2}$$

where $f_\varphi = (\varphi, \cdot)$ is a regular functional and $\Phi = e_H\varphi$, $\Psi = e_H\psi$ as before. That is,

$$(\Phi, \Psi)_S = f_\varphi(\psi) = \int \varphi(x)\psi(x)d\mu(x), \tag{2.3}$$

where $\int$ here denotes an actual integral over a μ-measurable set $D \in R^n$.

Remark. A reason to call such a basis orthonormal is given in [1]. Not every coordinate Hilbert space H can produce an orthonormal string basis e_H. Equation (2.3) shows that orthonormality of a string basis imposes a symmetry between coordinates of the dual objects in the basis. In particular, if e_H is orthonormal, then H must be an L_2-space, i.e. a space $L_2(D,\mu)$ of square integrable functions on a μ-measurable set $D \in R^n$. Thus, Hilbert spaces l_2 and $L_2(R)$ are examples of coordinate spaces that admit an orthonormal string basis.

Definition. A linear coordinate transformation on $\mathbf{S}$ is an isomorphism $\omega : \widetilde{H} \longrightarrow H$ of Hilbert spaces which defines a new string basis $e_{\widetilde{H}} : \widetilde{H} \longrightarrow \mathbf{S}$ by $e_{\widetilde{H}} = e_H \circ \omega$.

Let φ be coordinate of a string Φ in the basis e_H and $\widetilde{\varphi}$ its coordinate in the basis $e_{\widetilde{H}}$. Then $\Phi = e_H\varphi = e_{\widetilde{H}}\widetilde{\varphi} = e_H\omega\widetilde{\varphi}$. That is, $\varphi = \omega\widetilde{\varphi}$ by the uniqueness of the decomposition.

Let now $\Phi, \Psi \in \mathbf{S}$ and let $\mathbf{A}$ be a linear operator on $\mathbf{S}$. Let $\Phi = e_H\varphi, \Psi = e_H\psi$ with $\varphi, \psi \in H$. The scalar product $(\Phi, \mathbf{A}\Psi)_S$ is independent of a basis and in a basis e_H reduces to

$$(\varphi, A\psi)_H = (\widehat{G}\varphi, A\psi), \tag{2.4}$$

where $\widehat{G} : H \longrightarrow H^*$ defines the metric on H.

If $\omega : \widetilde{H} \longrightarrow H$ is a linear coordinate transformation and $\varphi = \omega\widetilde{\varphi}, \psi = \omega\widetilde{\psi}$, then

$$(\widehat{G}\varphi, A\psi) = (\widehat{G}\omega\widetilde{\varphi}, A\omega\widetilde{\psi}) = (\omega^*\widehat{G}\omega\widetilde{\varphi}, \omega^{-1}A\omega\widetilde{\psi}) = (\widehat{G}_{\widetilde{H}}\widetilde{\varphi}, A_{\widetilde{H}}\widetilde{\psi}). \tag{2.5}$$

Therefore we have the following transformation laws:

$$\varphi = \omega\widetilde{\varphi} \tag{2.6}$$
$$\psi = \omega\widetilde{\psi} \tag{2.7}$$
$$\widehat{G}_{\widetilde{H}} = \omega^*\widehat{G}\omega \tag{2.8}$$
$$A_{\widetilde{H}} = \omega^{-1}A\omega. \tag{2.9}$$

More generally, we know that by definition $\mathbf{S}$ is a Hilbert manifold (modeled on itself). Let then (U_α, π_α) be an atlas on $\mathbf{S}$.

Definition. A collection of quadruples $(U_\alpha, \pi_\alpha, \omega_\alpha, H_\alpha)$, where each H_α is a Hilbert space of functions and ω_α is an isomorphism of $\mathbf{S}$ onto H_α is called a *functional atlas* on $\mathbf{S}$. A collection of all compatible functional atlaces on $\mathbf{S}$ is called a *coordinate structure* on $\mathbf{S}$.

Definition. Let (U_α, π_α) be a chart on $\mathbf{S}$. If $p \in U_\alpha$, then $\omega_\alpha \circ \pi_\alpha(p)$ is called the *coordinate* of p. The isomorphisms $\omega_\beta \circ \pi_\beta \circ (\omega_\alpha \circ \pi_\alpha)^{-1} : \omega_\alpha \circ \pi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \omega_\beta \circ \pi_\beta(U_\alpha \cap U_\beta)$ are called *coordinate transformations* on $\mathbf{S}$.

As $\mathbf{S}$ is a differentiable manifold one can also introduce the tangent bundle structure $\tau : T\mathbf{S} \longrightarrow \mathbf{S}$ and the bundle $\tau^r_s : T^r_s\mathbf{S} \longrightarrow \mathbf{S}$ of tensors of rank (r, s).

3 String basis of eigenvectors of a linear operator on S

Let $\mathbf{A} : \mathbf{S} \longrightarrow \mathbf{S}$ be a linear operator on the string space. In [1] the following definition was given:

Definition. A string basis e_H is a *basis of eigenvectors* of a linear operator $\mathbf{A} : \mathbf{S} \longrightarrow \mathbf{S}$ with eigenvalues (eigenvalue function) $\lambda = \lambda(k)$ if

$$\mathbf{A}e_H(\varphi) = e_H(\lambda\varphi) \tag{3.1}$$

for any $\varphi \in H$.

As any string basis, a string basis of eigenvectors of $\mathbf{A}$ is a linear map from H onto $\mathbf{S}$ and λ is a function of k.

By rewriting (3.1) as

$$e_H{}^{-1}\mathbf{A}e_H(\varphi) = \lambda\varphi \tag{3.2}$$

we see that the problem of finding a basis of eigenvectors of $\mathbf{A}$ is equivalent to the problem of finding such a string basis e_H in which the action of $\mathbf{A}$ reduces to multiplication by a function λ. In particular case of an l_2-basis this yields the classical problem of finding a basis of eigenvectors of a linear operator.

Let us compare the notions of a string basis of eigenvectors and of the ordinary basis of eigenvectors of the operator $\mathbf{A}$ in more detail. The operators that correspond to physical observables in quantum physics are Hermitian operators acting on a space L_2. With the goal to apply our analysis to physics, assume then that A is an Hermitian operator on $L_2(R)$. Then

$$(f, Ag)_{L_2} = (Af, g)_{L_2} \tag{3.3}$$

for any $f, g \in L_2(R)$. Therefore,

$$(\Phi, \mathbf{A}\Psi)_S = (\mathbf{A}\Phi, \Psi)_S, \tag{3.4}$$

where $\Phi = e_{L_2}f, \Psi = e_{L_2}g, \mathbf{A} = e_{L_2}Ae_{L_2}{}^{-1}$, and e_{L_2} is an L_2-string basis on $\mathbf{S}$.

It follows that the same will be true in any string basis e_H, that is,

$$(\varphi, A_H\psi)_H = (A_H\varphi, \psi)_H, \tag{3.5}$$

where $\Phi = e_H\varphi, \Psi = e_H\psi, \mathbf{A} = e_H A_H e_H{}^{-1}$, and e_H is any string basis on $\mathbf{S}$.

Hermicity of $\mathbf{A}$ leads to severe restrictions on possible spaces H on which a basis of eigenvectors of $\mathbf{A}$ is defined. In fact, if e_H is a string basis of eigenvectors of an Hermitian operator $\mathbf{A}$ on $\mathbf{S}$, then

$$(\Phi, \mathbf{A}\Psi)_S = (\varphi, \lambda\psi)_H = \int g(x,y)\varphi(x)\lambda(y)\psi(y)dxdy, \tag{3.6}$$

were the integral symbol is used for the action of bilinear metric functional G with kernel g (see section 3 of [1] for details). Hermicity gives then

$$\int g(x,y)(\lambda(x) - \lambda(y))\varphi(x)\psi(y)dxdy = 0 \tag{3.7}$$

for any $\varphi, \psi \in H$. It follows that the bilinear functional given by $g(x,y)(\lambda(x) - \lambda(y))$ is equal to zero. This is satisfied wherever λ is a constant. Assuming it is not the case, for example assuming λ is defined on R and is not constant on any interval $(a,b) \in R$, we conclude that $g(x,y)$ must be equal to $a(x)\delta(y-x)$, where δ denotes Dirac δ-function. If λ is a sequence instead, then assuming that no two values in the sequence are equal we obtain $g(x,y) = a_x\delta_{y,x}$, where $\delta_{y,x}$ is the δ-symbol.

As a result, if $\mathbf{A}$ is Hermitian and λ is defined on a subset $D \in R$ and is not constant on any open interval in D, or (whenever D is discrete) if no two values of λ are equal, the coordinate space H must be an L_2-space. This means that the string basis of eigenvectors in this case is orthogonal (as defined in section 1). In particular case of $H = l_2$ this simply means orthogonality of the eigenvectors corresponding to different eigenvalues of an Hermitian operator on l_2.

In an orthogonal string basis of eigenvectors of $\mathbf{A}$ with $H = L_2(R)$ we have

$$(\Phi, \mathbf{A}\Psi)_S = (f, \lambda g)_{L_2(R)} = \int f(x)\lambda(x)g(x)dx, \tag{3.8}$$

where $f, g \in L_2(R)$. In particular, it is impossible to choose $f(x)$ to be a (generalized) eigenvector of the operator A of multiplication by the function $\lambda(x)$. Indeed, the eigenvectors of such an operator would be δ-functions, which do not belong to $L_2(R)$.

Assume instead that $H = l_2$. Let e_{l_2} be a string basis of eigenvectors of $\mathbf{A}$. Let $\varphi = (\varphi_1, \varphi_2, ...) \in l_2$. The action of e_{l_2} on φ reduces to the matrix multiplication of the matrix $e_{l_2} = (e_1, e_2, ...)$ of linearly independent vectors by the column of components of φ. That is, $e_{l_2}\varphi = e_1\varphi_1 + e_2\varphi_2 + ...$ Therefore,

$$\mathbf{A}e_{l_2}\varphi = e_{l_2}(\lambda\varphi) = e_{l_2}(\lambda_1\varphi_1, \lambda_2\varphi_2, ...) = e_1\lambda_1\varphi_1 + e_2\lambda_2\varphi_2 + ... \tag{3.9}$$

Let now $e_{l_2} = (e_1, e_2, ...)$ be the ordinary basis of eigenvectors of $\mathbf{A}$ instead, and $\varphi = e_1\varphi_1 + e_2\varphi_2 + ...$ Then

$$\mathbf{A}(e_1\varphi_1 + e_2\varphi_2 + ...) = \lambda_1 e_1\varphi_1 + \lambda_2 e_2\varphi_2 + ... = e_1\lambda_1\varphi_1 + e_2\lambda_2\varphi_2 + ... \tag{3.10}$$

Comparison of (3.9) and (3.10) shows that when $H = l_2$, the string basis of eigenvectors of $\mathbf{A}$ can be identified with the basis of eigenvectors of $\mathbf{A}$ in the ordinary sense. In this case φ in the expression

$$(\varphi, \lambda\psi)_{l_2} \tag{3.11}$$

can be chosen to be an eigenvector of the operator of multiplication by the function λ.

Generalizing the results to arbitrary Hilbert spaces H of functions, we have the following:

a). If e_H is a string eigenbasis of an operator and the eigenvalue function λ is not constant on any open region of the domain of its definition, or, if no value is taken by λ more than once, then H must be an L_2 space.

b). The notion of a string basis of eigenvectors of a linear operator on $\mathbf{S}$ is a direct generalization of the notion of a basis of eigenvectors of a linear operator on H to which the former reduces when $H = l_2$.

c). If $H = L_2(R)$, φ in the expression $(\varphi, \lambda\psi)_{L_2(R)} = (\Phi, \mathbf{A}\Psi)_S$ can not be chosen to be an eigenvector of the operator $A = \lambda\cdot$ of multiplication by λ. If $H = l_2$ instead, such a choice of φ is possible.

In the $H = L_2(R)$ case the eigenvectors of $A = \lambda\cdot$ are generalized eigenvectors, i.e. they are functionals F on a space K of test functions, such that

$$F(A\varphi) = \lambda_0 F(\varphi) \tag{3.12}$$

for any $\varphi \in K$ (see [3],[1]).

It would be therefore natural to be able to consider such a coordinate space H for which the corresponding dual space H^* contains the eigenfunctions of A_H, where A_H is the operator of multiplication by λ. By the above this in general requires consideration of non-Hermitian operators.

Example. We know that $A_H = x$ (operator of multiplication by the variable) has no eigenvectors on $L_2(R)$. Consider then a different Hilbert space H with a metric G. We have:

$$(G\varphi, A_H\psi) = \int g(k,m)\varphi(k)m\psi(m)dkdm. \tag{3.13}$$

By the above,

$$\int g(k,m)\varphi(k)m\psi(m)dkdm \neq \int g(k,m)k\varphi(k)\psi(m)dkdm \tag{3.14}$$

unless H is an L_2 space.

On the other hand, if $g(k,m)$ defines a regular functional, then

$$\int g(k,m)\varphi(k)m\psi(m)dkdm = \int mg(k,m)\varphi(k)\psi(m)dkdm. \tag{3.15}$$

So by taking

$$F = \int g(k,m)\varphi(k)dk, \tag{3.16}$$

we have:

$$F(m\psi) = (mF)(\psi). \tag{3.17}$$

That is, $A_H = m\cdot$, although not Hermitian, is self-adjoint.

If now $F = \int g(k,m)\varphi(k)dk$ is not regular, we can define the action of A_H on it by the standard in the theory of generalized functions procedure, i.e. by taking

$$(mF)(\varphi) = F(m\varphi). \tag{3.18}$$

The resulting operator of multiplication by m is defined on H^* and is self-adjoint.

Notice now that the L_2 spaces are special in that any such space can be identidied with its dual. For this we idenify each functional $(f,\cdot)_{L_2}$ with the corresponding function f. As a result, Hermitian operators on L_2 can be identified with the corresponding self-adjoint operators. Namely, let A^+ be the

Hermitian conjugate operator and let A^* be the adjoint operator of a given operator A on a Hilbert space H. Let $\widehat{G}: H \longrightarrow H^*$ define a metric on H. Then the relationship between the operators is as follows (see section 5 of [1] for example):

$$A^+ = \widehat{G}^{-1} A^* \widehat{G}. \tag{3.19}$$

On L_2-spaces $\widehat{G}$ can be considered as the identical transformation and it becomes possible to identify A^+ and A^*. On the other hand, in coordinate spaces different than L_2, A^+ and A^* are not the same.

We can therefore accomodate both the Hermicity of operators on L_2 and the inclusion of generalized eigenvectors of operators on H into the space H itself. For this we apply a coordinate transform to extend the action of an Hermitian operator on L_2 to the action of the corresponding self-adjoint operator on an arbitrary Hilbert space of functions. Notice also that the considered coordinate transformations are by definition linear isometries, i.e. they preserve the norm of strings. When restricted to coordinate transformations on L_2 they yield unitary transformations.

Example. Consider the Hilbert space H of functions as in example of section 4 of [1]. The space H in this case consists only of infinitely differentiable functions. The space H^* contains singular distributions. If $\widehat{G}$ is the metric on H, then the inner product on H^* is defined by

$$(f, g)_{H^*} = (\widehat{G}^{-1} f, g) = \int e^{-(x-y)^2 - x^2} e^{-(y-z)^2 - z^2} f(x) g(z) dy dx dz. \tag{3.20}$$

Let $\varphi \in H$ be such that $(\widehat{G}\varphi)(x) = \delta(x)$. Consider the bilinear expression

$$(\widehat{G}\varphi, x\psi), \tag{3.21}$$

where x is the operator of multiplication by the variable. Here unlike the case of $L_2(R)$-space we can choose φ in such a way that $\widehat{G}\varphi$ is a (generalized) eigenvector of x. With such a choice we have:

$$(\widehat{G}\varphi, x\psi) = \lambda_0 (\widehat{G}\varphi, \psi) = \lambda_0 (f, g)_{L_2}, \tag{3.22}$$

where as before f and g are L_2 coordinates of the strings Φ, Ψ and φ, ψ are the corresponding components in a basis e_H. By rewriting the original bilinear expression in $L_2(R)$ basis we also have:

$$(\widehat{G}\varphi, x\psi) = (f, A_{L_2} g)_{L_2}. \tag{3.23}$$

Here A_{L_2} is the coordinate representation in $L_2(R)$-basis of the operator $\mathbf{A}$ acting on $\mathbf{S}$, whose coordinate representation A_H in the H-basis is x.

Therefore, f will be an eigenvector of the operator A_{L_2} on $L_2(R)$. It follows in particular that A_{L_2} is not the operator of multiplication by a variable.

Notice also that no operator can be the operator of multiplication by the variable on both spaces at once: $L_2(R)$ and H above. This immediately follows from (3.22) and (3.23) as otherwise f in (3.23) would be an eigenvector of the operator x on $L_2(R)$.

This last fact was also verified in a different way in section 5 of [2]. There it has been established that the only coordinate transformations preserving the operator of multiplication by a variable are the trivial ones.

4 Concluding remarks

The coordinate formalism on Hilbert manifolds introduced in [1] permits us to relate different spaces of functions by considering them as coordinate representations of an invariant string space. Respectively, functional equations on different Hilbert spaces of functions are now classified by the equivalence classes of equations related by coordinate transformations.

Therefore the role played by the formalism is similar in spirit to the role played by vector algebra. The components of vectors become now ordinary or generalized functions on subsets of R^n. Likewise, the coordinate transformations become (local) diffeomorphisms of various Hilbert spaces of functions.

In the paper the notion of the string eigenbasis of a linear operator on the space $\mathbf{S}$ has been analyzed. We found its relationship to the classical notion of a basis of eigenvectors. The string eigenbasis of an Hermitian operator $\mathbf{A}$ with the nowhere constant eigenvalue function λ turned out to be an L_2-basis. That is, only in L_2-basis could a coordinate representation A_{L_2} of such an operator be the operator of multiplication by a function λ. In particular, if the coordinate space is $L_2(R)$, the generalized eigenfunctions of the operator $A_{L_2} = \lambda\cdot$ do not belong to the space itself. On the other hand, it is always possible to find a coordinate space which contains these eigenfunctions. It was shown that to preserve the Hermicity of A_{L_2} and to be able to include the generalized eigenfunctions of $A_H = \lambda\cdot$, the coordinate-transformed operator A_H must be a self-adjoint operator. The corresponding coordinate transformation preserves the norm of elements while changing the space itself. It is therefore an isometry. As a result, isometric transformations represent a natural generalization of the unitary transformations within the developed coordinate formalism.

References

[1] Kryukov A., Coordinate Formalism on Hilbert Manifolds, *math-ph/0201017*.

[2] Kryukov A., Coordinate Formalism on Abstract Hilbert Space, *math-ph/0201031*.

[3] I.M. Gel'fand and N.Y. Vilenkin, *Generalized Functions, Vol.4*, Academic Press, New York and London, (1964).