Quantum superposition justified in a new non-Aristotelian finitary logic

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Abstract

A new non-Aristotelian finitary logic (NAFL) is proposed in which it is postulated that the truth or falseness of an undecidable proposition in a theory T is meaningful only when asserted axiomatically; there is no truth other than axiomatic truth. It is shown that under this hypothesis, the law of the excluded middle and the law of non-contradiction for such undecidable propositions must fail to be theorems of T. The phenomenon of quantum superposition is thus explained in NAFL. It is also shown that infinite sets cannot exist in any consistent theory of NAFL, which makes it a very restrictive logic. Implications for some modern mathematical and physical theories are analyzed from the point of view of NAFL.

1 Introduction

Let Ψ be an undecidable proposition in a theory T formulated in classical first order predicate logic, i.e., neither Ψ nor $\neg\Psi$ is provable in T. If T is consistent (equivalently in first order logic, if models for T exist) then the truth or falseness of Ψ in T is defined semantically via model theory, that is, there exists a model for T in which Ψ is true and another model in which Ψ is false. Syntactically (or in a proof theoretic sense), however, the truth or falseness of Ψ in T is not defined *constructively*, and amounts to a Platonic belief expressed by the law of the excluded middle $\Psi \vee \neg\Psi$.

In this paper, a different, novel constructivist viewpoint is formulated that makes the syntactic and semantic definitions of the truth or falseness of Ψ in T equivalent. The main new postulate (and point of departure from classical logic), whose plausibility follows from elementary classical model theory, is that the truth or falseness of Ψ in T must be *axiomatic* in nature, if T is consistent; that is, Ψ can be true or false in T if and only if so asserted axiomatically, and the truth (falseness) of Ψ in T *is* its truth (falseness) in the theory T+Ψ (T+$\neg\Psi$). In the resulting new logic, it is meaningless to talk of the truth of a formal proposition *outside* of axiomatic theories. For example, the notion of truth is not definable in just the *language* of T, without reference to any formal

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system; in this sense, all *formalizable* truths are axiomatic. We argue later that the notions of *undecidability* in T of a proposition and the *consistency* of T are Platonic truths that cannot be formalized in any consistent theory T; this is another point of departure from classical logic.

It is demonstrated that this new postulate leads to the conclusion that the law of the excluded middle ($\Psi \vee \neg\Psi$) is not a theorem of T as in intuitionistic logic. However, in contrast to intuitionistic logic, it is shown that the law of non-contradiction ($\neg(\Psi \& \neg\Psi)$) also fails to be a theorem of T by the above postulate and therefore there must exist a non-classical model for T in which both Ψ and $\neg\Psi$ are asserted as axioms. It is argued that this does *not* make T inconsistent; inconsistency only follows if both Ψ and $\neg\Psi$ are *deducible* in T. Thus this new, non-Aristotelian logic provides a simple justification of the phenomenon of quantum superposition in which, for example, a single photon seemingly takes two different paths at the same time. However, the law of the excluded middle does indeed apply to *Platonic* notions such as consistency and undecidability. Thus T is either consistent or inconsistent and a proposition is either decidable or undecidable in T, although, in both cases, we may never know which.

It is shown that the proposed logic is also finitary. In a suitable set theory with classes formulated in this logic, infinite classes are permitted, but not infinite sets if the consistency constraint is to be satisfied. Thus consistency demands that all infinite classes must necessarily be proper classes (that is, they cannot belong to any class); in other words, the axiom of infinity for sets must be dropped. Therefore modern mathematics, theoretical physics and a whole lot of other scientific theories formulated using Cantorian assumptions cannot be justified in the framework of this new logic. Some other implications for quantum mechanics, elementary Peano arithmetic, special theory of relativity, theoretical computer science and non-Euclidean geometries are also considered.

2 The proposed non-Aristotelian finitary logic

In the rest of this paper, the proposed new non-Aristotelian logic is abbreviated as NAFL. The logic NAFL is basically a modified version of the classical first order predicate logic. The modification mainly affects the treatment of undecidable propositions in a theory T, in that the law of the excluded middle and the law of non-contradiction are not required to hold for such propositions. Instead, truth for propositions undecidable in T is governed by the main postulate of NAFL, to be described below. The constraint that T be consistent will then force additional points of departure from classical first order predicate logic, mainly in the form of restrictions. Some of these are stated in this section, in the remarks following the main postulate and the two metatheorems that follow from it. Additional restrictions are elaborated upon in the ensuing sections where Cantor's set theory and other well known theories are considered from the point of view of NAFL.

It should be emphasized that NAFL must be viewed as a series of metamathematical assertions *about* axiomatic theories. These assertions are to be viewed as Platonic truths; as noted earlier, the formal propositions of NAFL theories do not have any truth values in NAFL, which by itself is not a theory. NAFL is the sum total of all that is described progressively (and informally) in the rest of this paper. Throughout this paper, consistency of T is equated with the existence of models for T; this holds in any first order logic.

Let T be a consistent theory formulated in classical first-order predicate logic (with the specific points of departure to be elaborated below in the definition of NAFL) and suppose that Ψ is an undecidable proposition in T. Note that the undecidability is established by the rules of inference and principles of proof in the classical first order predicate calculus. An important restriction in NAFL is that Ψ itself does not contain any component of the form $\phi \vee \neg\phi$ where ϕ is undecidable in T. As will be asserted shortly, propositions like "$\phi \vee \neg\phi$, but we do not know which" are not valid in NAFL. However, if ϕ is *not* undecidable in T, $\phi \vee \neg\phi$ may be *assumed* in a *proof* of either ϕ or of $\neg\phi$ in T. Further, if ψ and ϕ are both known to be undecidable in T, then it may happen (under certain conditions, but not always) that $\psi \Leftrightarrow \phi$, $\psi \Rightarrow \phi$ and $\phi \Rightarrow \psi$ are not legitimate propositions in T (see Remark 5 below for details). Note that these are disjunctions involving undecidable propositions. It is assumed that Ψ does not include any such component as well, when it is deemed illegal.

Define the theories T^+ and T^- by:

$$\mathrm{T}^+ \equiv \mathrm{T} + \Psi \quad \& \quad \mathrm{T}^- \equiv \mathrm{T} + \neg\Psi,$$

where addition in the above notation means addition as an axiom. Note that the undecidability of Ψ in T (assumed consistent) makes each of the theories T^+ and T^- consistent and distinct from T itself. The main postulate, which is the single assumption from which essentially all the other features of NAFL may be deduced, is stated as follows.

Main Postulate (NAFL). *Under the assumptions that* T *is consistent and that* Ψ *is undecidable in* T*, the truth or falseness of* Ψ *in* T *is indistinguishable from its truth in* T^+ *or falseness in* T^- *respectively. To say that* Ψ *is interpreted as true (false) in* T *is therefore the* same *as saying that* T *is to be interpreted as* T^+ *(*T^-*) respectively. Equivalently, the truth or falseness of* Ψ *in* T *has meaning* only *when asserted* axiomatically*, i.e., by an act of human will. In NAFL, it is not meaningful to talk of the truth or falseness of formal propositions outside of axiomatic theories.*

The meaning of the main postulate is summarized as follows. Given the axioms of a consistent T, the question "Is Ψ true (false) in T?" is answered by "Yes, if it has been so asserted axiomatically"; in other words, the truth (falseness) of Ψ in T is equated with its truth (falseness) in T^+ (T^-). It should be kept in mind that, unlike classical logic, there are *two* sets of axioms to be specified when T is being interpreted. These are the axioms of the theory being interpreted (T) and the axioms of the theory that interprets T (say, T*). If T*=T, then every undecidable proposition in T, including Ψ, is being interpreted as neither true nor false (as will become clear from metatheorem 2 below). If (and only if) $T^*=T^+$ (T^-), then Ψ is being interpreted as true (false) in T.

As a justification for the main postulate, we offer the following plausible argument. According to the modern *structuralist* ideology, a theory can be meaningfully interpreted only in *structures*, in particular, *models* [1]. Every valid metamathematical statement about a consistent theory is in fact a statement about the class of all of its models. In particular, two consistent theories T_1 and T_2 are identical if and only if every model for T_1 is a model for T_2 and conversely, every model for T_2 is a model for T_1. The said justification follows by noting that every model for T in which Ψ is true (false) is a model for T^+ (T^-) and conversely, every model for T^+ (T^-) is a model for T in which Ψ is true (false).

It should be emphasized that this argument is not necessarily a "proof" of the main postulate. The standard (classical) interpretation denies this postulate by maintaining a distinction between the truth or falseness of Ψ in T versus its truth in T^+ or falseness in T^-, respectively; however, the point of the main postulate is that such a distinction cannot be justified *constructively* (if one views a model as a construction) and so must be dropped in NAFL. It is extemely important to note that the main postulate essentially rejects the Platonistic philosophy inherent in classical logic. Every assertion like "Let Ψ be true (false) in T" (which is used to create models for T in classical logic) must necessarily be interpreted in NAFL as amounting to an axiomatic declaration of such truth (falseness) of Ψ and so T is being interpreted as T^+ (T^-). In other words, "Let Ψ be true (false) in T, but I am not axiomatically declaring it as true (false)" is a contradiction in NAFL; the second part of this assertion contradicts the first. In NAFL (unlike classical logic), there is no Platonic world in which such a truth holds independent of an axiomatic declaration (by the human mind that interprets T). The classical interpretation also asserts that the law of the excluded middle

$$\Psi \vee \neg\Psi \tag{1}$$

must hold in T despite the fact that Ψ is undecidable in T. This is denied in NAFL as a consequence of the main postulate, as stated below.

Metatheorem 1 (NAFL). *Suppose that Ψ is undecidable in a consistent theory* T*. Then if the main postulate holds, it is not valid to assert $\Psi \vee \neg\Psi$ as a theorem of* T*.*

The proof is trivial and is as follows.

Proof. Suppose that (1) is a theorem of T. What (1) asserts is that "Ψ is either true or false in T, but we do not know which" (because of the undecidability of Ψ in T). But from the main postulate, this assertion is exactly equivalent to "T is to be interpreted as either T^+ or T^-, but we do not know which". This in turn means that "either Ψ or $\neg\Psi$ is present in T as an *axiom*, but we do not know which"; in other words, T has been defined as equivalent to the disjunction

$$T = T^+ \vee T^-.$$

But this definition of T is illegal! Since axioms are added to theories by human beings, we *ought* to know which of the two propositions Ψ or $\neg\Psi$ is present in T as an axiom! It is not legal to assert that one of these two propositions has been *axiomatically* declared to be true, but we do not know which one. In short, "$T^+ \vee T^-$" is not a legitimately defined theory, but the law of the excluded middle forces T to be equivalent to it, under the main postulate. Formally (or syntactically), this disjunction merely asserts the truth of the propositions belonging to (i.e., provable in) either T^+ or T^-. Clearly, this is how T has been defined; syntactically, (1) cannot mean that "(1) holds in all interpretations of T", for the formal definition of T cannot include the word "interpretation".

We conclude that from (1) and the main postulate, T becomes indistinguishable from the illegal definition $T^+ \vee T^-$ and is therefore a non-existent theory. The only possible conclusion that one can arrive at is that the assumption that (1) is a theorem of T, which forced this contradiction, must be false if T is consistent. □

Remark 1. If one insists on the validity of (1) in NAFL, then the conclusion from the main postulate would be that no consistent theory can admit undecidable propositions. However, this would render inconsistent even a theory T containing no axioms, for clearly, every proposition is undecidable with respect to the null set of axioms. This in turn means that consistent theories do not exist, which is not a very useful conclusion. It is tempting to say that this conclusion would violate Gödel's first incompleteness theorem [2], but Gödel's theorems and his notion of consistency are derived from different assumptions than those made in NAFL and so do not apply in this logic.

We next demonstrate that the main postulate implies that the law of non-contradiction must also fail to be a theorem of T.

Metatheorem 2 (NAFL). *Assuming that the main postulate holds, metatheorem 1 implies that there must exist a non-classical model for* T *in which both* Ψ *and* $\neg\Psi$ *are asserted as (non-classical) axioms.*

Obviously both T^+ and T^- would be invalid in such a non-classical model, which would give T the required distinct identity.

Proof. One can deduce the contradiction $\Psi \& \neg\Psi$ in such a model, but this is a deduction from precisely the above two (non-classical) axioms, neither of which is refutable in T. In NAFL, under the assumptions of the main postulate, the contradiction is in fact refutable in each of T^+ and T^-, but not in T. Note that the classical refutation of the contradiction in T is equivalent to the following argument: "If Ψ ($\neg\Psi$) is true in T, then $\neg\Psi$ (Ψ) must be false because (1) is a theorem of T". But this argument, by the main postulate in NAFL, really amounts to refuting $\Psi \& \neg\Psi$ in T^+ (T^-). The refutation works in classical logic because the truth or falseness of Ψ in T is postulated to be *not* axiomatic by the law of the excluded middle. Note that the truth of Ψ ($\neg\Psi$) in T is interpreted in classical logic as a Platonic (and *not* axiomatic) truth, when the refutation is executed semantically in models for T. In NAFL metatheorem 1 implies that the law of the excluded middle is not a theorem of T and secondly, the refutation of $\Psi\&\neg\Psi$ in T fails because it must necessarily appeal to axiomatic truths outside of T (both syntactically and semantically). The metatheorem follows from these two facts. $\square$

Remark 2 (Quantum superposition justified). Metatheorem 2 explicitly justifies the quantum superposition principle. What it asserts is that if Ψ is undecidable in a consistent theory T in NAFL, then $\Psi\&\neg\Psi$ must obtain in T if neither Ψ nor $\neg\Psi$ has been declared to be true. Here $\Psi\&\neg\Psi$ must be interpreted as not that "Ψ is true and false", that is, not as a contradiction, but in the sense that "neither Ψ nor $\neg\Psi$ is true in (or an axiom of) T". In other words, "Ψ" in $\Psi\&\neg\Psi$ asserts that "$\neg\Psi$ is not an axiom of T" whereas "$\neg\Psi$" asserts that "Ψ is not an axiom of T". Seen in this light, there is really no contradiction, because clearly this is how T has been defined. Note that there is no distinction between Ψ and $\neg\neg\Psi$ in NAFL (unlike intuitionism). Similarly, if in quantum theory it is undecidable as to whether a photon took path A or path B (but is restricted to these paths), then the proposition that "the photon took path A" (with the negation "the photon took path B") is undecidable when suitably formalized, and the superposition follows. According to NAFL, the superposition only means that the photon took neither path A nor path B. Thus NAFL is more in tune with the Copenhagen interpretation than the Many Worlds interpretation of quantum mechanics. The role of "measurement" in quantum theory must be equated with the axiomatic assertion of the truth of the measured result (e.g., "The photon took path A"). However, it is important to recognize that the probabilistic component of quantum mechanics, which arises in the measurement process, cannot be modelled consistently in NAFL. Thus the notion that "The photon will be detected either in detector C with a probability p or in detector D with probability $1-p$" can only be formulated in classical logic and not in NAFL, because the law of the excluded middle becomes indispensable for this notion.

Remark 3. The (non-classical) model in metatheorem 2 is in fact a superposition of two or more classical models, at least one of which is a model for T^+ and another, a model for T^-. Note that in this non-classical model, we do permit the inferences

$$\Psi\&\neg\Psi \Rightarrow \Psi \quad \text{and} \quad \Psi\&\neg\Psi \Rightarrow \neg\Psi,$$

and conversely, $\Psi\&\neg\Psi$ follows from Ψ and $\neg\Psi$. But, as noted in the previous remark, "Ψ" and "$\neg\Psi$" in this non-classical model are *not* to be interpreted as corresponding to the truth or falseness of Ψ in T (as required by Tarski in classical logic). The classical assertion that "from a contradiction, any proposition can be deduced" fails in the non-classical model of NAFL. To classically deduce a proposition ϕ from $\Psi\&\neg\Psi$ in this non-classical model, one would proceed as follows:

$$\begin{aligned}
&\Psi\&\neg\Psi \Rightarrow \Psi,\\
&\Psi \Rightarrow \Psi \vee \phi,\\
&\Psi\&\neg\Psi \Rightarrow \neg\Psi,
\end{aligned}$$

$$\neg\Psi\&[\Psi \vee \phi] \Rightarrow \phi.$$

It is the last step that fails in NAFL by metatheorem 2 because it presumes the law of non-contradiction. A simple application of metatheorem 2 is to consider T as the null set of axioms. Clearly Ψ is undecidable in T and so there must exist a non-classical model for T in which $\Psi\&\neg\Psi$ obtains. That such a non-classical model indeed exists is obvious; $\Psi\&\neg\Psi$ does not contradict any axiom of T because T contains no axioms. This observation is important becasuse it illustrates that NAFL does permit undecdiable propositions, unlike some versions of intuitionism.

Remark 4. It is now clear why, in NAFL, the assertion "Ψ is undecidable in T" cannot be formalized in T. If T is consistent, this assertion cannot be proved in T because, by metatheorem 2, it is equivalent to $\Psi\&\neg\Psi$. Any such proof in T is also a proof of both Ψ and $\neg\Psi$, which would make T inconsistent and contradicts our hypothesis that neither proof exists in T. Hence if T is consistent, the formalized version of "Ψ is undecidable in T", if it exists, would be "true but undecidable" in T. The requirement of metatheorem 2 that a non-classical model should exist would then produce a contradiction. In other words, "true but undecidable propositions" are not permissible in NAFL; the "true" part has no meaning for a formal proposition unless asserted axiomatically. Similarly, in NAFL, the notion "T is consistent" cannot be formalized in T. Intuitively, one can see why the undecidability of Gödel's proposition con(T) follows from "T is consistent"; if T is indeed "really" consistent, the formalized version con(T) can only express this fact by referring to the class of all propositions in T, including itself. It is this self-reference that causes "true but undecidable" propositions, a contradiction in NAFL. Therefore the notions of undecidability and consistency must both remain Platonic in NAFL. It follows that Gödel's theorems do not apply to NAFL theories. A very important conclusion is that the classical equivalence of "T is consistent" and "Ψ is undecidable in T", which follows from Gödel's theorems for appropriate T, does not necessarily hold in NAFL.

Remark 5. Consider the proposition

$$\psi \Leftrightarrow \phi, \tag{2}$$

and suppose that (2) is a theorem of T. Then neither ψ nor ϕ can be undecidable in T.

To prove this assertion, suppose, to obtain a contradiction, that one of either ψ or ϕ were indeed undecidable in T; it immediately follows that the other too must be undecidable. Therefore both $\psi\&\neg\psi$ and $\phi\&\neg\phi$ would obtain in a (non-classical) model for T, which must exist by metatheorem 2. But then in this model one could infer $\psi\&\neg\phi$, which is the negation of (2) and a contradiction, because (2) is a theorem of T. This completes the proof. Indeed, (2) can be true in this non-classical model if and only if one could formalize in T the following sentence: "ψ is undecidable in T iff ϕ is undecidable in T". But we have seen that the notion of undecidability is *not* formalizable in any NAFL theory T. The important point to note here is that any sentence, such as, (2), can be a theorem of T in NAFL if and only if it were to be *classically* true. Here (2) can be classically true in NAFL iff ψ and ϕ were both either provable or refutable in T. To avoid confusion, we use the notation $\psi \& \neg\psi$ *obtains* (rather than "$\psi \& \neg\psi$ is true") in a non-classical model for T because "true" always means "classically true" in NAFL. Here $\psi \& \neg\psi$ is *not* classically true in the said non-classical model (and therefore can never be formally asserted as an axiom) because the notion of undecidability of ψ is not formalizable. Next consider

$$\psi \Rightarrow \phi. \tag{3}$$

If both ψ and ϕ are undecidable in T, then (3) cannot be a theorem of T, by an identical argument to that given for (2). Indeed, if both ψ and ϕ are undecidable in T, neither (3) nor (2) can even be valid propositions of T under certain conditions. We illustrate this for the case of (3). Suppose ϕ and $\neg\psi$ were to be theorems of T+ψ and T+$\neg\phi$ respectively; both of these must happen simultaneously by the classical rules of inference, which apply in NAFL. Then (3) is not a valid proposition of T. This is so because under this restriction (3) can only be either "true" or "neither true nor false" in T and can never be (classically) false. To see this, note that for (3) to be false in T, we need ψ to be true and ϕ to be false, but this can never happen. The moment we assert the truth of ψ ($\neg\phi$), we are interpreting T as T+ψ (T+$\neg\phi$) by the main postulate, and ϕ ($\neg\psi$) cannot be false because it is a theorem of T+ψ (T+$\neg\phi$) by hypothesis. However, if neither ψ nor ϕ have been asserted as axioms, then by metatheorem 2, both $\psi \& \neg\psi$ and $\phi \& \neg\phi$ obtain in T. But then we may infer both (3) and its negation in such a non-classical model for T, which implies (3) is neither true nor false. Hence the conclusion that (3) is not a valid proposition of T if ϕ and $\neg\psi$ are theorems of T+ψ and T+$\neg\phi$ respectively. However, it is legitimate to use (3) as a rule of inference in order to prove ϕ ($\neg\psi$) in T+ψ (T+$\neg\phi$). If ϕ and $\neg\psi$ are *not* theorems of T+ψ and T+$\neg\phi$ respectively, then (3) is indeed a valid proposition of T. The conclusion is that disjunctions involving undecidable propositions must be handled very carefully in NAFL. Observe that if we replace ϕ by ψ then (3) reduces to $\psi \vee \neg\psi$ which, as we have already observed, is an illegitimate proposition for ψ undecidable in T. Indeed, ψ ($\neg\psi$) are trivially theorems of T+ψ (T+$\neg\psi$) and so the criterion for such illegitimacy is satisfied. A similar analysis for the legality of the proposition (2) in T applies. Note that if ψ were to be undecidable and ϕ provable in T, then (3) is provable in T.

For another example, consider

$$\psi \Rightarrow \phi \& \neg\phi, \tag{4}$$

and suppose that ϕ is known to be undecidable in T. If (4) were a theorem of T, it would then follow even in NAFL that ψ is provably false in T. This is despite the fact that $\phi \& \neg\phi$ *does* obtain in T in a non-classical model required by metatheorem 2. Indeed, under the hypothesis that ψ were true in T one could *deduce* $\phi \& \neg\phi$ in T from (4), which *must* be interpreted as a contradiction because the notion of undecidability is not formalizable in T. This automatically falsifies the said hypothesis and one obtains a proof by contradiction of $\neg\psi$. It should be emphasized that the rules of inference and the principles of proof in NAFL are the same as in classical first order predicate logic (except for certain special restrictions that will be mentioned in the ensuing sections).

Remark 6. An objection may be raised that $\Psi \vee \neg\Psi$ itself is an undecidable proposition in T, but apparently the law of the excluded middle has been applied to it; for there exist only the classical models in which $\Psi \vee \neg\Psi$ is true, and the non-classical models in which $\Psi \vee \neg\Psi$ is false. Careful thought will show that it is only "$\Psi \vee \neg\Psi$, but we do not know which" that is undecidable in this sense. But we have, at the beginning of this section, banned such propositions as illegal in NAFL. Indeed, if the truth or falseness of Ψ in T is axiomatic, this proposition can never be legal. Further, the negation of this proposition is precisely $\Psi \& \neg\Psi$ which is also not formalizable in NAFL, as noted earlier.

Remark 7. Finally, we highlight an important distinction between classical logic, intuitionistic/constructivistic logic and NAFL. Consider the rhetorical question "Which came first, the chicken or the egg?". Of course, in NAFL, this question can only be meaningfully answered in some theory T. Let Ψ formalize "The chicken came first", and let $\neg\Psi$ formalize its negation "The egg came first". Suppose Ψ were undecidable in T. The classical logician would then give the answer as "Either the chicken came first or the egg came first, but we do not know which". The intuitionist/constructivist, however, would consider the question itself meaningless and would assert "you tell me which chicken and which egg, and I will tell you which of these came first". In other words, the non-constructive assertion of existence of chicken and egg will not be permissible in intuitionism/constructivism. In NAFL, we *do* admit Ψ as a proposition despite its non-constructiveness, *provided* there is no axiom of T that implies $\Psi \vee \neg\Psi$. The answer in NAFL would be "if T is consistent, then $\Psi \& \neg\Psi$ obtains, provided we have not asserted either Ψ or $\neg\Psi$ axiomatically". It is extremely important to note that the assumption of undecidability of Ψ automatically makes T consistent in classical logic, but not in NAFL, as observed earlier.

3 Infinite sets cannot exist in theories of NAFL

A *class* is any collection of sets; here we are considering set theory with classes and assume that the only mathematical objects permitted to belong to classes are sets. A class is specified as an *extension* of a definite property. Thus if $P(x)$ is one such property of sets, in which all the bound variables range only over sets, then we may denote the class C associated with it as $\forall x[x \in C \Leftrightarrow P(x)]$ or more concisely, $C = \{x : P(x)\}$. Set theory tells us that certain classes are sets which may belong to other classes. A *proper class* is a collection that is deemed to be not a set and therefore cannot belong to any class. A necessary and sufficient condition for a class to exist is that all of its elements should exist as sets. It is therefore valid to speak of infinite classes even in a theory that permits only finite sets. The *universe* of a model for a theory will be taken here as a nonempty class U over which all the variables of the theory range.

Consider Zermelo's set theory, **Z**, in which classes are permitted. For an elementary presentation of the axioms of **Z** that uses classes, see Machover [3], Ch. 1 and Ch. 5. A rigorous justification of the notion of class is available, for example, in Gödel-Bernays set theory (**GB**) as presented in Cohen [4], pp. 73–78 and we will assume that the axioms for the existence of classes discussed therein, on page 75, are present in **Z**. Following Cohen, we will regard our system as having only class variables and a unary relation, "X is a set", which will be used whenever we wish to indicate that the class in question, in upper case notation, is a set; lower case notation, say x, is used only for classes which are also sets and "x is a set" is assumed to be understood. The negation of "X is a set" will be stated as "X is a proper class". The only other nonlogical symbol in our language is the membership relation, $\in$, between sets and classes. The other notable features of **Z** are that there is an axiom which asserts that only sets can be members of classes, and that the axiom of extensionality (AE) is stated for classes, as follows: any two classes having the same members are identical. Upon dropping the axiom of infinity (AI) from **Z**, we obtain the theory of finite sets **F**. In what follows, we will study this theory in NAFL with the goal of probing the existence of infinite sets.

That every property expressible in the language determines a class is a theorem scheme of **GB** and in particular, of **F** and **Z**. We will denote this as the theorem of comprehension for classes (theorem TC):

Theorem TC. *If $P(x)$ is a valid property (expressible in the language) of sets in which all bound variables range only over sets, then the class $\{x : P(x)\}$ exists.*

See Cohen [4], pp. 76–77, for a proof of this theorem in **GB**. The natural numbers in **F** are defined in the usual set-theoretic sense, with 0 as the empty set $\emptyset$ and $n+1 = n \cup \{n\}$. The class $\mathbb{N}$ of all natural numbers is defined by

$$\mathbb{N} = \{x : x \text{ is a natural number}\}.$$

Note that we have admitted "x is a natural number" (which we will henceforth abbreviate by "$x \in \mathbb{N}$") as a valid property in our language. This guarantees the existence, via theorem TC, of $\mathbb{N}$ as an infinite class.

Consider the class D, defined as:

$$D = \{x : \forall n[n \in \mathbb{N} \Rightarrow x \notin \wp^{(n)}(\emptyset)]\}. \tag{5}$$

Here $\wp^{(n)}(\emptyset)$ is the power set operation performed n times on the null set $\emptyset$; we let $\wp^{(0)}(\emptyset) = \emptyset$. Note that D is guaranteed to exist by theorem TC. Consider the following proposition:

$$D = \emptyset. \tag{6}$$

According to the standard interpretation, (6) is undecidable in **F**, for the following reason. Define a model $\mathfrak{B}$ for **F** in which only finite sets exist; the universe B of $\mathfrak{B}$ is given by

$$B = \{x : \exists n[n \in \mathbb{N} \,\&\, x \in \wp^{(n)}(\emptyset)]\}. \tag{7}$$

In the interpretation of $\mathfrak{B}$, $D = \emptyset$ holds and hence (6) is true. But in a model in which infinite sets do exist, (6) is false. Note that the undecidability of (6) in **F** is a direct consequence of the fact that in the absence of AI, the question of the existence of infinite sets is undecidable according to the conventional wisdom which holds that AI is indeed a valid axiom. Of course, according to the finitists (of whom Kronecker was certainly the most famous and uncompromising), AI *is* unambiguously false and hence (6) ought to be provably true in **F**. In what follows, we demonstrate that Kronecker's vision is indeed justified in NAFL.

Uniqueness of D is deducible from AE, which implies:

$$D = \emptyset \quad \vee \quad D \neq \emptyset. \tag{8}$$

The key point is that AE *requires (8) to be a theorem of* **F**, *even in NAFL. But by metatheorem 1 and metatheorem 2, (8) cannot be a theorem of* **F**, *given that* **F** *is consistent and that (6) is undecidable in* **F**. We are forced to the conclusion that one of these two "givens" must be false in NAFL. Either (6) is undecidable in **F**, which is therefore inconsistent, or else (6) *is*, after all, decidable in **F**. To see why AE demands (8), see the discussion of (2), with ψ interpreted as $D = \emptyset$ and ϕ as the negation of AI (i.e., infinite sets do not exist); this interpretation makes (2) equivalent to an instance of AE. Hence $D = \emptyset$ & $D \neq \emptyset$ does imply a violation of AE, even in NAFL.

What we have demonstrated is that if **F** is to be a consistent theory of NAFL, then we need a principle of proof for either $D = \emptyset$ or for $D \neq \emptyset$. Of course, the question then arises, which one? We will first argue the case for $D = \emptyset$. In the following paragraph, we assert that infinite sets can only be defined impredicatively, i.e., by self-reference via the universal class (which must again be defined impredicatively by reference to infinite sets). The appropriate prinicple of proof in NAFL for $D = \emptyset$ is that such impredicative definitions must be banned if they indeed cause a contradiction.

The axioms of **F** tacitly presume that a definite class U is specified as the universe, because these formulas contain the universal quantifiers $\forall$ and $\exists$ which may be thought of as referring to the universe. However, to define U we need to consider the formulas of **F** as meaningful because these assert the existence of sets that must necessarily belong to U. Thus both U and all the sets of **F** are impredicatively defined. However, finite sets may also be defined predicatively, at least in principle, merely by listing all of their elements; but this is not possible for infinite sets. Note that an equivalent method for deducing a contradiction is to consider the proposition

$$U = B, \tag{9}$$

where $U = \{x : x = x\}$ and B is defined by (7). Here U is the universal class which is guaranteed to exist by theorem TC and must be uniquely defined by AE. In NAFL, the undecidability of (9) in **F** contradicts AE in exactly the same manner as before.

We now examine the question of whether there is a case for deducing a principle of proof in NAFL for $D \neq \emptyset$ instead. We assert that this is not so. Firstly, any such principle in **F** would amount to requiring an absolute infinity.

The problem is that even if one does manage to work out such a principle, there will always exist other propositions in NAFL theories permitting infinite sets whose undecidability will produce a conflict with AE. An example is considered in the ensuing paragraph.

To kill two birds with one stone, revert to set theory with properties instead of classes; the objective is to demonstrate that the power set axiom leads to a contradicion similar to that described earlier even in standard set theory, with the power set playing the role of the universe. Of course, this contradiction is trivially demonstrable in set theory with classes as well. Consider conventional Zermelo-Fraenkel set theory **ZF**. Denote the axiom of constructibility [4] by ACN. It is known from Gödel's work that if **ZF** is consistent, then the theory **ZF** + ACN is consistent. Define the set c by

$$c = \{x \in \wp(\mathbb{N}) : x \text{ is not constructible}\},$$

where $\mathbb{N}$ is the set of all natural numbers and the property "x is constructible" may be taken as short-hand for the usual definition of constructibility (see Cohen [4]). Consider the following proposition:

$$c = \emptyset. \tag{10}$$

It is easy to see that (10) must be undecidable in **ZF**. If the universe is constructible (i.e., if ACN holds), then $c = \emptyset$ and not otherwise. Given that the axiom of subsets in **ZF** affirms the existence of c, we may conclude that in NAFL, the undecidability of (10) in **ZF** constitutes a violation of AE, now stated for sets. The arguments are identical to those given earlier. If either (10) or its negation is taken to be provable in **ZF**, then the contradiction would be that either ACN or its negation would also have to be provable in **ZF** and this is known to be impossible. The reason for this contradiction can be traced to the impredicative nature of the power set axiom (when applied to infinite sets), which causes the definition of the power set to be dependent on the definition of the universe. The existence of infinite sets via AI also plays a crucial role in the sense that the contradiction can only be stated in terms of power sets of infinite sets.

4 Is Peano arithmetic consistent in NAFL?

First note that Peano arithmetic (**PA**) is strongly equivalent to the theory **F** (when restricted to statements about sets only). In order to argue for the consistency of either **PA** or **F** in NAFL, one needs to assert the axioms of these theories as *tautologies*, i.e., they cannot be denied; this includes the induction axiom of **PA**. Otherwise it is possible to deduce contradictions in these theories in the same manner as in the previous section. A proposition that is *known* to be undecidable in **F** and **PA** is the celebrated Robertson-Seymour graph minor theorem (see Ch. 12 of Diestel [5]). In NAFL however, the undecidability of the graph minor theorem does cause a contradiction in **F** and **PA**, via metatheorem 2; again, this can be demonstrated by the methods of the previous section. One can escape this predicament in **F**, for example, by noting that the graph minor theorem requires quantification over infinitely many infinite (proper) classes of finite graphs. Thus one has to assert that such quantification should be banned in the NAFL version of **F**, so that it would not be possible to formulate the graph minor theorem. In **PA**, the corresponding principle would have to be the banning of the nesting of universal quantifiers that is required to formulate the graph minor theorem. Quantification over infinitely many infinite proper classes should be interpreted in NAFL as speaking of these infinite classes as "completed objects", i.e., as infinite sets, which are illegal in NAFL. An extremely important point to note, however, is that the classical proof of the undecidability of the graph minor theorem in **F** and **PA** is presumably executed in a meta-theory that permits infinite sets and other techniques that are not acceptable (and so would not carry conviction) in NAFL. A convincing meta-theory for establishing undecidability results in NAFL remains to be developed. An obvious starting point for such a meta-theory is the observation that every proposition is undecidable with respect to the null set of axioms.

There are many other well-known conjectures, such as, Fermat's last theorem, the twin prime conjecture, Goldbach's conjecture, etc., whose undecidability in **F** and **PA** (not established at present) would cause contradictions in NAFL which may be routinely established as in the previous section; the requirement of the existence of a non-classical model via metatheorem 2 is the cause for such contradictions. The intuitive reason for these contradictions is as follows. All these conjectures (including the graph minor theorem), if indeed undecidable in **F** and **PA**, would have to be "true". But in NAFL, "true" has no meaning independent of axiomatic theories. Hence our intuition that these curiously "true but undecidable" propositions are paradoxical is confirmed in NAFL. However, in the absence of any metamathematical proof that these conjectures are indeed undecidable, their status as valid/invalid propositions in the NAFL versions of **F** and **PA** will remain unclear. The safe strategy would be to admit these as valid propositions if and only if proofs or refutations are found in **F** and **PA**. The conclusion is that NAFL is a very severely restricted logic, with only ultra-finitism being more restrictive.

5 Implications for other theories

First consider Euclid's geometry. It is easily shown that the undecidability of Euclid's fifth postulate (with respect to the other four postulates) produces a contradiction in an NAFL theory comprising the first four postulates; the non-classical model required by metatheorem 2 cannot exist. Therefore all five axioms of Euclid must be asserted as *tautologies* in NAFL and non-Euclidean geometries would become inconsistent. The reason we must abandon non-Euclidean geometries in NAFL is that they can be formulated only by appeal to self-reference. Thus space is "curved" in non-Euclidean geometries, but curvature itself is a notion that cannot be formulated independent of space. Similarly, in the general theory of relativity, curvature of spacetime is defined in terms of gravity and gravity is defined in terms of the curvature of spacetime. This self-reference causes unavoidable contradictions (via undecidable propositions) in NAFL theories that cannot be fixed, just as in the case of infinite sets in the previous section. Secondly, note that Turing's undecidability of the halting problem cannot be formalized in NAFL, for the same reason that the graph minor theorem could not be formalized – quantification over infinitely many infinite (proper) classes, required to establish such undecidability, must be banned in NAFL. A Turing machine, *by definition*, must either halt or not halt and undecidability of the halting problem in NAFL would contradict this (Aristotelian) assumption. Once again, NAFL rejects the "true but undecidable" assertion that certain Turing machines do not halt. The diagonalization principle used by Turing in his proof (and by Gödel in proving his incompleteness theorems [2]) must therefore be banned in NAFL. Hence Gödel's "true but undecidable" propositions are illegitimate in NAFL and his incompleteness theorems do not apply in this logic.

Finally, consider an interesting implication for the special theory of relativity (SR). Consider a hypothetical universe in which there are only two material objects A and B, and suppose that both A and B measure a relative acceleration with respect to each other. First of all, note that SR requires either A or B to be a inertial frame of reference because (a) the existence of an inertial frame is an axiom of SR and (b) if neither A nor B is an inertial frame, then one must point to empty space as the location of the origin of the inertial frame; but this would amount to postulating the existence of absolute space, which is denied in SR. Let ψ denote "A is an inertial frame" and $\neg\psi$ denote its negation "B is an inertial frame". Clearly, ψ is undecidable in any classical formulation of SR. In NAFL, such undecidability means that there must exist a non-classical model for SR in which $\psi \& \neg\psi$ must obtain (by metatheorem 2). But such a non-classical model cannot exist. By an axiom of SR, both A and B cannot be inertial frames because they are accelerating with respect to each other. One concludes that SR cannot be formalized in any consistent theory of NAFL. The non-existence of infinite sets in NAFL would inevitably force this conclusion anyway.

6 Concluding remarks

The non-Aristotelian finitary logic (NAFL) developed in this paper could perhaps be described as corresponding to the best dreams of Kronecker and the worst nightmares of Cantor and Hilbert. Clearly, NAFL is *not* the logic in which Hilbert hoped to justify classical infinitary reasoning (in his celebrated "Hilbert's program"). It has been conjectured that primitive recursion arithmetic is more likely to be the finitary logic in which Hilbert hoped to implement his program. The positive aspect of NAFL, however, is that it justifies and demystifies the phenomenon of quantum superposition. In conclusion, the main achievement of NAFL is that the question of validity of the vast jungle of complex infinitary reasoning has been reduced to the question of validity of the main postulate, without which NAFL collapses back to classical logic. An interesting idea would be to introduce the notion of para-consistency into NAFL with the goal of enabling it to handle infinite sets. A second possibility is to consider a radically new formulation of set theory in NAFL, in which, say, the axiom of extensionality is either dropped or suitably modified to permit infinite sets.

Dedication

The author dedicates this research to his son R. Anand and wife R. Jayanti.

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