Coordinate formalism on abstract Hilbert space: Kinematics of a quantum measurement

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Abstract

Coordinate form of tensor analysis on an abstract (infinite-dimensional) Hilbert space is presented. The developed formalism permits one to naturally include the improper states in the apparatus of quantum theory. In the formalism the observables are represented by the self-adjoint extensions of Hermitian operators. The unitary operators become linear isometries. Quantum measurement and collapse are interpreted as isometric functional transformations. Several experiments including the two-slit experiment are analyzed in the new context.

1 Introduction

The improper (i.e. non-square integrable) states play the essential role in quantum theory (QT): they serve the building blocks of the theory and simultaneously provide a bridge between the quantum and the classical worlds. In particular, a possibility (at least in principle) to measure precise positions of particles permits one to define the state function in the non-relativistic quantum mechanics (QM). A possibility to measure precise momenta of free particles is essential to define the scattering amplitude in quantum field theory (QFT). Simultaneously, such measurements create improper states thereby furnishing quantum particles with the classical-mechanical properties and providing the foundation of the familiar classical world.

The mathematical apparatus for working with the improper states seems to be well developed, yet somewhat unsatisfactory. This becomes especially clear in the QFT. In fact, the improper states are described by continuous linear functionals (generalized functions) defined on an appropriate space of functions. The passage to QFT involves working with the products of these functionals - the operation which up until now is not defined satisfactory, leading to divergences in QFT.

The famous measurement problem in QM also seems to be related to the lack of the appropriate treatment of generalized functions. In fact, whenever the spectrum of an observable is continuous, the complete system of projectors needed to represent mathematically the von-Neumann reduction is not available. This fact together with the success of the rigged Hilbert space approach to the spectral analysis of operators (see [6]) makes one wonder if the mathematics and, possibly, the physics of von Neumann reduction must be re-examined.

It seems therefore that the appropriate treatment of generalized functions is the key to resolving the main difficulties in QT from explanation of collapse of a state to divergences and inconsistencies in QFT.

In the paper it is shown that the generalized functions and the square-integrable functions *must be* and *can be* included on equal footing into the formalism of QT. The "must be" part is shown by using two simple thought-experiments in which the evolution operators relate different Hilbert spaces of functions. The "can be" part is verified by developing a functional tensor analysis on an abstract infinite-dimensional Hilbert space. In this we follow the earlier work [1]-[3], however, the material is presented in a self-contained way.

The developed formalism leads to a necessary extension of the usual QM observables to the self-adjoint operators acting on spaces of generalized functions. The formalism also suggests that the unitary evolution of quantum states must be replaced with the more general *isometric* evolution. That is, the evolution operator while preserving the norm of a state is represented in general by a map between *different* functional Hilbert spaces. Moreover, this conclusion permits one to interpret both the unitary evolution and the non-unitary collapse of a state as a single isometric process.

The invariance of the formalism under functional transformations of coordinates leads to a further extension of the principle of relativity. This in turn leads to a powerful "relativistic" interpretation of measurement and collapse in QT.

Here is the plan of the paper. We start with a mathematical description of two thought experiments: In section 2 the work of a simple spectrometer is described in terms of the Fourier transform relating two spaces of generalized functions.

In section 3 a thought experiment with a heavy quantum harmonic oscillator is considered. It is shown that the states of such an oscillator form a basis of the space of generalized functions concentrated at a point and, in particular, cannot be reduced to the δ-function alone. To make this space of states into a Hilbert space requires a different type of metric than the metric on the space of square-integrable functions.

We conclude that the language of QM should include a simultaneous consideration of different Hilbert spaces of functions and of morphisms of these spaces. This leads to the functional coordinate formalism (string formalism) on the abstract Hilbert space developed in section 4.

Several examples of Hilbert spaces containing improper states are given in section 5.

In section 6 and 7 we explore the notion of a string basis of eigenvectors of a linear operator. The string and the ordinary bases of eigenvectors are compared and the generalized eigenvalue problem for a linear operator is discussed.

In section 8 the evolution of quantum states is investigated. We argue that in the process of evolution the functional representation of a state can move from one Hilbert space of functions to another. In particular, the evolution is not always unitary but is, more generally, an isometry. We also conclude that the states have an important property of relativity. That is, while the state itself is coordinate independent, a particular functional form of the state depends on a (physically chosen) system of functional coordinates. More generally, we put forward a *principle of functional relativity* that asserts that the equations of physics must be independent of a choice of functional coordinates.

In section 9 a standard mathematical model of quantum measurement is reviewed. This model is used in sections 10 and 11 where the notions of quantum measurement and collapse are analyzed in the context of the string formalism. We show in particular that decoherence and collapse can be both considered as results of a non-unitary (but isometric, i.e. norm preserving!) evolution of the states under a measurement.

The results and the open questions are briefly summarized in the conclusion.

2 The "Fourier transform" experiment

Let us begin with a mathematical description of a simple experiment with a magnetic spectrometer.

A free electron enters a constant magnetic field of magnitude B perpendicular to the electron's momentum vector p. Due to the Lorentz force the electron will move in a circle of radius $r = |p|/B$, where $|p|$ is the magnitude of p (here we neglect the effects related to the spin as well as the emission of photons due to acceleration of the electron).

By putting an absorbing scintillating screen on the electron path we will observe a single flash indicating the location of the electron at the moment of absorption. As electrons with the different momenta will move in circles of different radii we conclude that position of the electron at the moment of absorption is uniquely determined by p. Notice also that if we change the direction of motion of the electron to the opposite, the Lorentz force will change the sign as well, showing that such an electron will be absorbed at the opposite side of the screen.

Long enough before the electron enters the field its wave function is the eigenfunction of the momentum operator, i.e. it is proportional to e^{ipx}, where x is the coordinate along the electron path. At the moment of absorption the state function of the electron can be assumed to be an eigenfunction of the position operator, i.e. it is proportional to $\delta(y-p)$. Here y is the coordinate along the screen and the scale is chosen in such a way that the electron of momentum p is absorbed at the point with $y = p$.

On the other hand, the (inverse) Fourier transform F of the generalized function e^{ipx} is:

$$F[e^{ipx}](y) = \frac{1}{2\pi}\int e^{ipx}e^{-ixy}dx = \frac{1}{2\pi}\int e^{i(p-y)x}dx = \delta(y-p). \tag{2.1}$$

It follows that the state of the electron after it hits the screen is the Fourier transformation of the original state of the electron. Linearity of QM ensures then that any superposition of the free electrons will be "Fourier transformed" by the spectrometer into the corresponding superposition of the localized electrons.

As a result, the Fourier transform F is the mathematical model of the spectrometer. The operator F can be considered as the evolution operator $U(-\infty, t)$ that transforms the eigenstates of the momentum operator at $t \longrightarrow -\infty$ into the eigenstates of the position operator. We will return to this model in the following sections.

3 The heavy quantum harmonic oscillator

For the second thought-experiment consider a quantum harmonic oscillator described by the Hamiltonian

$$\widehat{H} = \frac{\widehat{p}^2}{2m} + \frac{m\widehat{x}^2}{2}. \tag{3.1}$$

Here $\widehat{p}$ and $\widehat{x}$ are the momentum and position operators respectively, m is the mass and we assume that the frequency of oscillations ω is equal to one.

In this section we want to answer the following question: What happens to the states of such an oscillator as m increases without bound?

From the correspondence principle we know that when m is large the oscillator behaves classically. In particular, at any moment of time we expect a fixed position of the oscillating particle.

This conclusions is verified by finding the exact eigenfunctions of (3.1). In fact, for any mass m the n-th state ψ_n of the oscillator is given by

$$\psi_n(x) = N_n e^{-\frac{mx^2}{2}} H_n(x\sqrt{m}), \tag{3.2}$$

where

$$H_n(\xi) = (-1)^n e^{\xi^2} \frac{d^n e^{-\xi^2}}{d\xi^n} \tag{3.3}$$

is the Hermite polynomial of order n and N_n is a normalization constant. As m increases we observe the localization of states due to the exponential factor $e^{-\frac{mx^2}{2}}$.

It is important that the relativistic effects do not prevent such a localization. In fact, the indeterminacy Δx of a measurement of the coordinates of a particle in QFT is estimated by

$$\Delta x \approx \frac{h}{mc}. \tag{3.4}$$

Therefore a particle of a large enough mass can be localized as well as one wishes.

As localized at a point in space states considered in QM are eigenfunctions of the position operator we expect that for any fixed n the n-th state of a heavy oscillator can be described by the δ-function.

To verify this conclusion mathematically we need to show that for any given n the sequence of states $\psi_n(x) = \psi_n(m,x)$ parametrized by m is a delta-convergent sequence in the following sense (see also [4]):

Definition. A sequence of smooth (for simplicity) functions $f_\alpha(x)$ on the set of real numbers R is a delta-convergent sequence if for any fixed $a < 0, b > 0$

$$\lim_{\alpha \longrightarrow \infty} \int_a^b f_\alpha(x)dx = 1, \tag{3.5}$$

and for any fixed $a > 0, b > 0$ or $a < 0, b < 0$, the limit in (3.5) is zero. We will denote this fact by writing

$$f_\alpha(x) \longrightarrow \delta(x). \tag{3.6}$$

First of all, if the constants N_n in (3.2) are determined from the unit norm condition, the sequence $\psi_n(m,x)$ cannot be a delta-convergent sequence. In fact, if $f_\alpha(x) \longrightarrow \delta(x)$ then the sequence of integrals $\int f_\alpha^2(x)dx$ is not bounded.

However, if the final state is to be given by the δ-function, the unit norm condition is not possible anyway. Let us therefore drop the unit norm condition and assume an arbitrary choice of the constants N_n.

It is known that the sequence

$$f_\alpha(x) = \frac{\alpha}{\sqrt{\pi}} e^{-\alpha^2 x^2} \tag{3.7}$$

is a delta-convergent sequence as $\alpha \longrightarrow \infty$. Therefore, assuming in (3.2) that $N_0 = \sqrt{\frac{m}{2\pi}}$ we have for the ground state ψ_0:

$$\psi_0(x) = N_0 e^{-\frac{m^2x^2}{2}} \longrightarrow \delta(x) \tag{3.8}$$

as $m \longrightarrow \infty$. That is, the ground state converges to the δ-function (the so-called weak convergence).

However, the state ψ_1 is given by an odd function and thus cannot form a delta-convergent sequence. Therefore, not every state of the oscillator will yield the δ-state as $m \longrightarrow \infty$.

To find the limit of the sequence of states ψ_n (with a given n) as $m \longrightarrow \infty$, let us make the following interesting observation. Notice that the sequence obtained by the p-fold differentiation of the terms of a delta-convergent sequence converges to the p-th derivative $\delta^{(p)}(x)$ of the δ-function. Together with the fact that (3.7) is a δ-convergent sequence this means that

$$\frac{\alpha^{n+1}}{\sqrt{\pi}} \frac{d^p e^{-\alpha^2 x^2}}{d(\alpha x)^p} \longrightarrow \delta^{(p)}(x) \tag{3.9}$$

as $\alpha \longrightarrow \infty$. Formula (3.3) gives:

$$\frac{\alpha^{n+1}}{\sqrt{\pi}} \frac{d^p e^{-\alpha^2 x^2}}{d(\alpha x)^p} = \frac{(-1)^n \alpha^{n+1}}{\sqrt{\pi}} H_n(\alpha x) e^{-\alpha^2 x^2}. \tag{3.10}$$

On another hand, by (3.2) the n-th state of the oscillator is given by:

$$\psi_n(x) = N_n e^{-\alpha^2 x^2} H_n(\sqrt{2}\alpha x), \tag{3.11}$$

where $\alpha = \sqrt{\frac{m}{2}}$.

From (3.9), (3.10), and (3.11) it follows that under an appropriate choice of the normalization constants N_n, as $m \longrightarrow \infty$, ψ_n converges to a linear combination of δ-function and its derivatives. For any given n the coefficients of this combination can be obtained by expressing the polynomials $H_n(\sqrt{2}\alpha x)$ in terms of the polynomials $H_k(\alpha x)$.

In particular, for the first states we have:

$$\psi_0(x) \longrightarrow \delta(x) \tag{3.12}$$
$$\psi_1(x) \longrightarrow \delta'(x) \tag{3.13}$$
$$\psi_2(x) \longrightarrow \delta^{(2)}(x) + \delta(x) \tag{3.14}$$
$$\psi_3(x) \longrightarrow \delta^{(3)}(x) + 3\delta'(x) \tag{3.15}$$
$$\psi_4(x) \longrightarrow \delta^{(4)}(x) + 6\delta^{(2)}(x) + 3\delta(x), etc. \tag{3.16}$$

The obtained limits are continuous functionals concentrated at the origin, i.e. equal to zero on a neighborhood of any point but the origin. Furthermore, any generalized function concentrated at a point is a finite linear combination of the δ-function and its derivatives [5].

The similarity of all functionals concentrated at a point makes one wonder why QM recognizes the δ-states but discriminates against other localized states. It is true that the generalized functions concentrated at a point are not square-integrable and therefore cannot be treated within the formalism of the Hilbert spaces L_2. This is, however, no reason to tolerate only one such function and forbid all others. Rather, it is an indication that the metric of the L_2-spaces must be generalized to allow for singular generalized functions. Examples of such a generalization will be given in section 6. In the following section we develop a coordinate formalism that permits working equally well with spaces of ordinary and generalized functions.

4 Coordinate formalism on abstract Hilbert space

To proceed we need to formulate clearly the guiding principles of the developing formalism. First of all, we want to include in the formalism a variety of functional spaces rather than just the spaces L_2 of square-integrable functions. In particular, we want to consider spaces containing singular generalized functions to accommodate the situations similar to the ones in the previous two sections.

It is often assumed that the δ-function and the free particle state functions are only idealizations of the "real" state functions which are square-integrable. It is important to understand, however, that the entire quantum theory is based on the notion of such functions, similar to the fact that the classical physics is based on the notion of a point. It is therefore essential to include in the formalism different Hilbert spaces of functions, in particular, the spaces of singular generalized functions on equal footing.

At this point we will assume that all such spaces are infinite-dimensional separable Hilbert spaces. In what follows such spaces will be called *separable Hilbert* or simply *Hilbert* spaces. As any two separable Hilbert spaces are isomorphic, it follows that the difference between them will be in the choice of Hilbert metric and elements.

It is also important to include functional transformations relating these spaces. We will assume that these transformations are isomorphisms of Hilbert spaces. The evolution operators in both thought experiments of section 2 and 3 shall provide examples of such transformations.

The isomorphisms of Hilbert spaces of functions shall be considered to be functional analogues of coordinate transformations on the Euclidean space R^n. That is, a functional version of the tensor analysis in coordinates must be developed.

The physics must be coordinate independent. This together with the possibility to physically change the coordinates shall lead to a functional version of the principle of relativity.

The coordinate transformations are not going to be *unitary* in general as they will relate different Hilbert spaces of functions. Instead, such transformations will be *isometries*, that is, they will still preserve the norms of the elements while relating different Hilbert spaces of functions.

Let us now develop the details of such a formalism. What follows in this section and section 6 is an abbreviated version of [1] and [3]. The most important definitions and results of these papers are presented in a self-contained form.

As just discussed, we want to include in the coordinate formalism the arbitrary Hilbert spaces of functions. As all separable Hilbert spaces are isomorphic, it is possible to think that there is a single abstract Hilbert space **S**. Different Hilbert spaces of functions would then *represent* **S**. We accept the following definitions:

Definition. A *string space* **S** is an abstract infinite-dimensional separable Hilbert space.

Definition. The elements of **S** are called strings and will be denoted by the capital Greek letters $\Phi, \Psi, \ldots$.

Definition. A *Hilbert space of functions* is either a Hilbert space H, elements of which are equivalence classes of maps between two given subsets of R^n or the Hilbert space H^* dual to H. Two elements $f, g \in H$ are called *equivalent* if the norm of $f-g$ in H is zero.

Definition. Any Hilbert space of functions H is called a *coordinate space*. Elements of H are called *coordinates* (or *functional coordinates*) of strings and will be denoted by the small Greek letters $\varphi, \psi, \ldots$.

To identify strings with their coordinates we need a linear map $e_H : H \longrightarrow \mathbf{S}$ from a Hilbert space of functions H into the string space **S**. The action of e_H on $\varphi \in H$ will be written in one of the following ways:

$$e_H(\varphi) = (e_H, \varphi) = \int e_H(k)\varphi(k)dk = e_{Hk}\varphi^k. \tag{4.1}$$

The integral sign is used as a notation for the action of e_H on elements of H and in general does not refer to an actual integration. We also use an obvious and convenient generalization of the Einstein's summation convention over the repeated indices k one of which is above and one below. Once again, only in special cases does this notation refer to an actual summation or integration over k.

Definition. A linear isomorphism e_H from a Hilbert space H of functions onto **S** is called a *string basis* on **S**. The inverse map $e_H^{-1} : \mathbf{S} \longrightarrow H$ is called a *linear coordinate system on* **S** (or a *linear functional coordinate system*).

It is clear that any string Φ is the image of a unique element $a \in H$, i.e.

$$\Phi = e_H(a) \tag{4.2}$$

for a unique $a \in H$. Also, if

$$e_H(a) = 0, \tag{4.3}$$

then $a = 0$. The similarity of (4.2), (4.3) with the corresponding properties of the ordinary basis justifies the definition above.

It is worth noticing that the basis e_H defines the space H itself. In fact, it acquires the meaning only as a map on H.

Let $\mathbf{S}^*$ be the dual string space. That is, $\mathbf{S}^*$ is the space of all linear continuous functionals on strings.

Definition. A linear isomorphism of H^* onto $\mathbf{S}^*$ is called a *string basis on* $\mathbf{S}^*$.

We will denote such a basis by e_{H^*}. Decomposition of an element $F \in \mathbf{S}^*$ with respect to the basis will be written in one of the following ways:

$$F = e_{H^*}(f) = (e_{H^*}, f) = \int e_{H^*}(k) f(k) dk = e^k_{H^*} f_k. \tag{4.4}$$

Definition. The basis e_{H^*} is *dual* to the basis e_H if for any string $\Phi = e_{Hk}\varphi^k$ and for any functional $F = e^k_{H^*} f_k$ the following is true:

$$F(\Phi) = f(\varphi). \tag{4.5}$$

In general case we have

$$F(\Phi) = e_{H^*} f(e_H \varphi) = e^*_H e_{H^*} f(\varphi), \tag{4.6}$$

where $e^*_H : \mathbf{S}^* \longrightarrow H^*$ is the adjoint of e_H. Therefore, e_{H^*} is the dual string basis if $e^*_H e_{H^*} : H^* \longrightarrow H^*$ is the identity operator. In this case we will also write

$$e_{Hl}{}^* e^k_{H^*} = \delta^k_l. \tag{4.7}$$

In special cases δ^k_l is the usual Krœnecker symbol or the δ- function.

Notice also that the action of F on Φ in any bases e_H on $\mathbf{S}$ and e_{H^*} on $\mathbf{S}^*$ can be written in the following way:

$$F(\Phi) = e^k_{H^*} f_k e_{Hl} \varphi^l = G(f, \varphi) = g^k_l f_k \varphi^l, \tag{4.8}$$

where G is a non-degenerate bilinear functional on $H^* \times H$.

By definition the string space $\mathbf{S}$ is a separable Hilbert space H. Any linear isomorphism $\pi_H : \mathbf{S} \longrightarrow H$ can be thought of as inducing the Hilbert structure on H. Respectively, we will assume that the string bases $e_H = {\pi_H}^{-1}$ are isomorphisms of Hilbert spaces. That is, a Hilbert structure on any coordinate space H must be induced by a choice of string basis. In particular, two Hilbert spaces with the same elements can have different inner products in which case they represent different coordinate spaces.

Let us assume that H is a real Hilbert space (generalization to the case of a complex Hilbert space will be obvious). We have:

$$(\Phi, \Psi)_S = \mathbf{G}(\Phi, \Psi) = G(\varphi, \psi) = g_{kl} \varphi^k \psi^l. \tag{4.9}$$

Here $\mathbf{G} : \mathbf{S} \times \mathbf{S} \longrightarrow R$ is a bilinear form defining the inner product on $\mathbf{S}$ and $G : H \times H \longrightarrow R$ is the induced bilinear form. The expression on the right is a convenient form of writing the action of G on $H \times H$.

Theorem. The choice of a coordinate Hilbert space determines the corresponding string basis up to a unitary transformation.

Proof. Let e_H and $\widetilde{e}_H$ be two string bases on $\mathbf{S}$ with the same coordinate space H. Then $(\Phi, \Psi)_S = G(\varphi, \psi) = \widetilde{G}(\widetilde{\varphi}, \widetilde{\psi}) = \widetilde{G}(U\varphi, U\psi)$. As $\widetilde{G} = G$ we have $G(\varphi, \psi) = G(U\varphi, U\psi)$, that is, U is a unitary transformation. Therefore $e_H = \widetilde{e}_H U$, i.e. the basis e_H is determined up to a unitary transformation on H.

Definition. A string basis e_H in $\mathbf{S}$ will be called *orthonormal* if for any $\Phi, \Psi \in \mathbf{S}$

$$(\Phi, \Psi)_S = f_\varphi(\psi), \tag{4.10}$$

where $f_\varphi = (\varphi, \cdot)$ is a *regular* functional and $\Phi = e_H\varphi$, $\Psi = e_H\psi$ as before. That is,

$$(\Phi, \Psi)_S = f_\varphi(\psi) = \int \varphi(x)\psi(x)d\mu(x), \tag{4.11}$$

where $\int$ here denotes an actual integral over the μ-measurable set $D \in R^n$ which is the domain of definition of functions in H.

It will be assumed in the sequel that $\mu(x) = x$ or $\mu(x) = \sum_k \theta(x-k)$, where $\theta(x)$ is the θ-function equal to 1 if $x \geq 0$ and 0 if $x < 0$. In this case the integral in (4.11) is the usual Lebesgue integral or the sum over k.

The bilinear form $\mathbf{G} : \mathbf{S} \times \mathbf{S} \longrightarrow R$ generates a linear isomorphism $\widehat{\mathbf{G}} : \mathbf{S} \longrightarrow \mathbf{S}^*$ by $\mathbf{G}(\Phi, \Psi) = (\widehat{\mathbf{G}}\Phi, \Psi)$. In any basis e_H we have

$$(\Phi, \Psi)_S = (\widehat{\mathbf{G}}e_H\varphi, e_H\psi) = e_H^*\widehat{\mathbf{G}}e_H\varphi(\psi), \tag{4.12}$$

where $e_H^*\widehat{\mathbf{G}}e_H$ maps H onto H^*. If e_H is orthonormal, then $e_H^*\widehat{\mathbf{G}}e_H\varphi = f_\varphi$. In this case we will also write

$$(\Phi, \Psi)_S = \varphi^k\psi^k = \delta_{kl}\varphi^k\psi^l. \tag{4.13}$$

In a special case δ_{kl} can be the Krœnecker symbol or Dirac δ-function $\delta(k-l)$.

The form of writing in (4.13) makes it especially clear why the basis in the definition above is called orthonormal. In fact, in this case the kernel of the metric functional is a generalization of the Krœnecker symbol, which represents the Euclidean metric in an orthogonal basis.

It is important to realize that not every coordinate Hilbert space H can produce an orthonormal string basis e_H. Assume, for example, that H contains the δ-function as a coordinate φ of a string $\Phi \in \mathbf{S}$ (examples of such spaces will be given below in section 6). Then $\delta_{kl}\varphi^k\varphi^l$ is not defined, that is, δ-function is not a coordinate function of a string in orthonormal basis. It follows from the definition that if e_H is orthonormal, then H is a space $L_2(D, \mu)$ of square integrable functions on a μ-measurable set $D \in R^n$. Thus, Hilbert spaces l_2 and $L_2(R)$ are examples of coordinate spaces that admit an orthonormal string basis.

This result does not contradict the well known existence of an orthonormal basis in any separable Hilbert space. In fact, the meaning of a string basis is quite different from the meaning of a classical basis on a Hilbert space. Namely, a string basis on $\mathbf{S}$ permits us to represent an invariant with respect to functional transformations object (string) in terms of a function, which is an element of a Hilbert space of functions. A basis on the latter space in turn permits us to represent this function in terms of numbers, components of the function in the basis.

Notice that formula (4.13) suggests that if H possesses an orhonormal basis e_H, then the chart $(\mathbf{S}, e_H^{-1})$ is analogous to a rectangular Cartesian coordinate systems in Euclidean space. More general formula (4.9) shows that other coordinate Hilbert spaces produce analogues of oblique Cartesian coordinate systems in Euclidean space.

Definition. A linear coordinate transformation on $\mathbf{S}$ is an isomorphism $\omega : \widetilde{H} \longrightarrow H$ of Hilbert spaces which defines a new string basis $e_{\widetilde{H}} : \widetilde{H} \longrightarrow \mathbf{S}$ by $e_{\widetilde{H}} = e_H \circ \omega$.

Let φ be coordinate of a string Φ in the basis e_H and $\widetilde{\varphi}$ its coordinate in the basis $e_{\widetilde{H}}$. Then $\Phi = e_H\varphi = e_{\widetilde{H}}\widetilde{\varphi} = e_H\omega\widetilde{\varphi}$. That is, $\varphi = \omega\widetilde{\varphi}$ by the uniqueness of the decomposition. This provides the transformation law of string coordinates.

Let now $\Phi, \Psi \in \mathbf{S}$ and let $\mathbf{A}$ be a linear operator on $\mathbf{S}$. Let $\Phi = e_H\varphi, \Psi = e_H\psi$ with $\varphi, \psi \in H$. The scalar product $(\Phi, \mathbf{A}\Psi)_S$ is independent of a basis and in a basis e_H reduces to

$$(\varphi, A\psi)_H = (\widehat{G}\varphi, A\psi), \tag{4.14}$$

where $\widehat{G} : H \longrightarrow H^*$ defines the metric on H.

If $\omega : \widetilde{H} \longrightarrow H$ is a linear coordinate transformation and $\varphi = \omega\widetilde{\varphi}, \psi = \omega\widetilde{\psi}$, then

$$(\widehat{G}\varphi, A\psi) = (\widehat{G}\omega\widetilde{\varphi}, A\widetilde{\psi}) = (\omega^*\widehat{G}\omega\widetilde{\varphi}, \omega^{-1}A\omega\widetilde{\psi}) = (\widehat{G}_{\widetilde{H}}\widetilde{\varphi}, A_{\widetilde{H}}\widetilde{\psi}). \tag{4.15}$$

Therefore we have the following transformation laws:

$$\varphi = \omega\widetilde{\varphi} \tag{4.16}$$
$$\psi = \omega\widetilde{\psi} \tag{4.17}$$
$$\widehat{G}_{\widetilde{H}} = \omega^*\widehat{G}\omega \tag{4.18}$$
$$A_{\widetilde{H}} = \omega^{-1}A\omega. \tag{4.19}$$

More generally, we can consider $\mathbf{S}$ as a Hilbert manifold (modelled on itself). Let then (U_α, π_α) be an atlas on $\mathbf{S}$.

Definition. A collection of quadruples $(U_\alpha, \pi_\alpha, \omega_\alpha, H_\alpha)$, where each H_α is a Hilbert space of functions and ω_α is an isomorphism of $\mathbf{S}$ onto H_α is called a *functional atlas* on $\mathbf{S}$. A collection of all compatible functional atlaces on $\mathbf{S}$ is called a *coordinate structure* on $\mathbf{S}$.

Definition. Let (U_α, π_α) be a chart on $\mathbf{S}$. If $p \in U_\alpha$, then $\omega_\alpha \circ \pi_\alpha(p)$ is called the *coordinate* of p. The map $\omega_\alpha \circ \pi_\alpha : U_\alpha \longrightarrow H_\alpha$ is called a *coordinate system.* The isomorphisms $\omega_\beta \circ \pi_\beta \circ (\omega_\alpha \circ \pi_\alpha)^{-1} : \omega_\alpha \circ \pi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \omega_\beta \circ \pi_\beta(U_\alpha \cap U_\beta)$ are called *functional (or string) coordinate transformations.*

As $\mathbf{S}$ is a differentiable manifold one can also introduce the tangent bundle structure $\tau : T\mathbf{S} \longrightarrow \mathbf{S}$ and the bundle $\tau^r_s : T^r_s\mathbf{S} \longrightarrow \mathbf{S}$ of tensors of rank (r, s).

Coordinate structure on the string space permits one to obtain a functional description of any tensor. Namely, let $\mathbf{G}_p(F_1, ..., F_r, \Phi_1, ..., \Phi_s)$ be an (r, s)-tensor on $\mathbf{S}$.

Definition. The coordinate map $\omega_\alpha \circ \pi_\alpha : U_\alpha \longrightarrow H_\alpha$ for each $p \in U_\alpha$ yelds the linear map of tangent spaces $d\rho_\alpha : T_{\omega_\alpha\circ\pi_\alpha(p)}H_\alpha \longrightarrow T_p\mathbf{S}$, where $\rho_\alpha = \pi_\alpha^{-1} \circ \omega_\alpha^{-1}$. This map is called a *local coordinate string basis on* $\mathbf{S}$.

Let $e_{H_\alpha} = e_{H_\alpha}(p)$ be such a basis and $e_{H^*_\alpha} = e_{H^*_\alpha}(p)$ be the corresponding dual basis. Notice that for each p the map e_{H_α} is a string basis as defined earlier. Therefore, the local dual basis is defined for each p as before and is a function of p.

We now have $F_i = e_{H^*_\alpha} f_i$, and $\Phi_j = e_{H_\alpha}\varphi_j$ for any $F_i \in T^*_p\mathbf{S}$, $\Phi_j \in T_p\mathbf{S}$ and some $f_i \in H^*_\alpha, \varphi_j \in H_\alpha$. Therefore $\mathbf{G}_p(F_1, ..., F_r, \Phi_1, ..., \Phi_s) = G_p(f_1, ..., f_r, \varphi_1, ..., \varphi_s)$ defining component functions of the (r, s)-tensor $\mathbf{G}_p$ in the local coordinate basis e_{H_α}.

5 Hilbert spaces of generalized functions

So, where are we now? First of all, the class of Hilbert spaces of square-integrable functions used in QT is replaced with a larger class of all Hilbert spaces of functions (as defined in the previous section).

All such spaces are isomorpic and this usually creates a wrong impression that a single Hilbert space is sufficient. However, no particular Hilbert space of functions is large enough to contain all functions useful in applications. In particular, the singular generalized functions should not be left aside.

Secondly, the coordinate formalism on an abstract Hilbert space is developed, which relates different spaces of functions (coordinate spaces). This formalism can be extended to an arbitrary abstract

Hilbert manifold (see [1]). In a nutshell the coordinate formalism is nothing but the tensor analysis on infinite-dimensional manifolds in a coordinate form.

In recent years the idea of describing mathematics in coordinates was not very popular and the "coordinate free" approach has been used. In case of manifolds of a finite dimension this is well motivated as coordinates are "all alike" representing the points of a manifold by columns of numbers. However, functions representing the points of infinite-dimensional manifolds can have very different analytic properties. As a result, the coordinate approach becomes again important.

One of the main motivations for developing the presented formalism was to include singular generalized functions in the apparatus of QT. The physical consequences of this will be discussed in the following sections. Here we present several examples of Hilbert spaces of generalized functions.

Example 1. Consider first the space $H = \left\{\varphi | \varphi = \sum_{k=-\infty}^{\infty} c_k \delta(x-k), c_k \in R\right\}$ of linear combinations of δ-functions of a single variable with the inner product given by

$$(\varphi, \psi)_H = \int e^{-(x-y)^2-(y-z)^2} \varphi(x)\psi(z)dxdydz. \tag{5.1}$$

Integration with respect to x and z is understood here in the sense of section 4, i.e. as the action of a bilinear functional on $H \times H$.

As

$$\int e^{-(a-y)^2} e^{-(y-b)^2} dy = \sqrt{\frac{\pi}{2}} e^{-\frac{(a-b)^2}{2}}, \tag{5.2}$$

we have for the inner product of $\varphi = \sum_{k=-\infty}^{\infty} c_k \delta(x-k)$ and $\psi = \sum_{l=-\infty}^{\infty} d_l \delta(x-l)$:

$$(\varphi, \psi)_H = \sqrt{\frac{\pi}{2}} \sum_{k,l=-\infty}^{\infty} c_k d_l e^{-\frac{1}{2}(k-l)^2}. \tag{5.3}$$

Theorem. The bilinear functional (5.1) is symmetric and positive definite, i.e. it does indeed define the inner product on H.

Proof. The functional is obviously symmetric. Observe now that

$$(\varphi, \psi)_H = (\rho\varphi, \rho\psi)_{L_2}, \tag{5.4}$$

where

$$(\rho\varphi)(y) = \int e^{-(x-y)^2} \varphi(x)dx \tag{5.5}$$

is a linear map from H into $L_2(R)$. This map is injective as the set of functions

$$\left\{e^{-(x-k)^2} | k \in Z\right\}, \tag{5.6}$$

where Z is the set of integers, is linearly independent. The positive definiteness follows now from injectivity of ρ and (5.4).

Theorem. The space H with the inner product (5.1) is Hilbert.

Proof. We only need to check that H is complete. Assume therefore that

$$\varphi^N(x) = \sum_{k=-\infty}^{\infty} c_k^N \delta(x-k) \tag{5.7}$$

is a fundamental sequence, i.e. $\left\|\varphi^N - \varphi^M\right\|_H \longrightarrow 0$ as $N, M \longrightarrow \infty$. Here $\|\varphi\|_H^2 = (\varphi, \varphi)_H$.

By (5.3) we then have

$$\|\varphi^N - \varphi^M\|_H^2 = \sqrt{\frac{\pi}{2}} \sum_{k,m=-\infty}^{\infty} (c_k^N - c_k^M)(c_m^N - c_m^M) e^{-\frac{1}{2}(k-m)^2} \longrightarrow 0 \tag{5.8}$$

as $N, M \longrightarrow \infty$. It follows that $c_k^N - c_k^M \longrightarrow 0$ for any k as $N, M \longrightarrow \infty$.

By completeness of the set of real numbers R for any k there exists c_k such that $c_k^N \longrightarrow c_k$. Consider then $\varphi = \sum_{k=-\infty}^{\infty} c_k \delta(x-k)$. The standard argument shows now that $\varphi \in H$ and $\|\varphi^N - \varphi\|_H \longrightarrow 0$ proving the theorem.

Notice in particular that the space H can be identified with the space of sequences c_k such that

$$\sum_{k=-\infty}^{\infty} c_k c_m e^{-\frac{1}{2}(k-m)^2} < \infty. \tag{5.9}$$

Example 2. As a next example consider the space $H = \left\{\varphi | \varphi = \sum_{k=0}^{\infty} c_k \delta^{(k)}(x), c_k \in R\right\}$, where $\delta^{(k)}$ denotes the derivative of order k of the δ-function of $x \in R$.

For the metric on H let us choose the following bilinear functional:

$$(\varphi, \psi)_H = \int e^{-(x-y)^2} e^{y^2} e^{-(y-z)^2} \varphi(x)\psi(z) dx dy dz. \tag{5.10}$$

The factor e^{y^2} is chosen for simplicity of the final expression.

Theorem. The space H with the metric (5.10) is Hilbert.

Proof. For the elements $\varphi = \sum_{k=0}^{\infty} c_k \delta^{(k)}(x)$, $\psi = \sum_{m=0}^{\infty} d_m \delta^{(m)}(x)$, using definition (5.10) and "integration by parts" we have:

$$(\varphi, \psi)_H = \sum_{k,m=0}^{\infty} c_k c_m (-1)^k (-1)^m \int \left(\frac{d^k}{dx^k} e^{-(x-y)^2}\right) e^{y^2} \left(\frac{d^m}{dz^m} e^{-(y-z)^2}\right) \delta(x)\delta(z) dx dy dz. \tag{5.11}$$

Also, from (3.3) we have:

$$\frac{d^k}{dx^k} e^{-(x-y)^2}|_{x=0} = \frac{d^k}{d\xi^k} e^{-\xi^2}|_{\xi=-y} = (-1)^k H_k(\xi) e^{-\xi^2}|_{\xi=-y} = H_k(y) e^{-y^2}. \tag{5.12}$$

Similarly,

$$\frac{d^m}{dz^m} e^{-(y-z)^2}|_{z=0} = H_m(y) e^{-y^2}. \tag{5.13}$$

Let us use the orthogonal property of Hermite polynomials:

$$\int e^{-y^2} H_k(y) H_m(y) dy = N_{km} \delta_{km}, \tag{5.14}$$

where N_{km} are constants. Formulas (5.12), (5.13) and (5.14) then give

$$(\varphi, \psi)_H = \sum_{k=0}^{\infty} N_k^2 c_k^2, \tag{5.15}$$

where $N_k = N_{kk}$. It follows that the space H can be identified with the space l_2 of square-summable sequences. In particular, H is a Hilbert space.

Notice also that only when the number of terms in $\sum_{k=0}^{\infty} c_k \delta^{(k)}(x)$ is finite does the sum represent a functional concentrated at $x = 0$.

Example 3. The example from section 4 of [1] gives a Hilbert space H^* of functions which in particular includes the space dual to the Schwartz space of infinitely differentiable rapidly decreasing functions. The metric on H^* is given by

$$(\varphi, \psi)_H = \int e^{-(x-y)^2 - x^2} e^{-(y-z)^2 - z^2} \varphi(x)\psi(z) dy dx dz. \tag{5.16}$$

More generally, it is easy to see that by "smoothing the metric" we extend the class of (generalized) functions for which the norm defined by this metric is finite. The same is true if we "improve" the behavior of the metric for large $|x|$. Conversely, by "spoiling the metric" we make the corresponding Hilbert space "poor" in terms of a variety of the elements of the space.

This shows once again that the developed coordinate formalism deals with a different "quality" of elements than its finite dimensional counterpart. By changing a string basis we change such properties of coordinates of a string as smoothness, convergence, etc. These properties make no sense for the finite dimensional coordinates of a point which simply are columns of numbers.

6 Generalized eigenvalue problem

Let A be a linear operator on a linear topological space V. If V is finite-dimensional and A is, say, Hermitian, then a basis in V exists, such that each vector of it is an eigenvector of A. If V is infinite-dimensional, this statement is no longer true. Yet quite often there exists a complete system of "generalized eigenfunctions" of A in the sense of the definition below (see [6]):

Definition. A linear functional F on V, such that

$$F(A\Phi) = \lambda F(\Phi) \tag{6.1}$$

for every $\Phi \in V$, is called a *generalized eigenfunction of A corresponding to the eigenvalue λ.*

It is assumed here that V is a space of test functions, i.e. a linear topological space of infinitely differentiable (and "good behaving" at infinity) functions used in the theory of generalized functions.

Notice however, that the definition above makes sense in a more general setting when F and Φ are elements of a pair of dual spaces. Assume then that V is the string space $\mathbf{S}$ and e_H is a string basis on $\mathbf{S}$. Assume F is a generalized eigenfunction of a linear operator $\mathbf{A}$ on $\mathbf{S}$. Then

$$F(\mathbf{A}\Phi) = \lambda F(\Phi), \tag{6.2}$$

that is,

$$e_H^* F(e_H^{-1} \mathbf{A} e_H \varphi) = \lambda e_H^* F(\varphi), \tag{6.3}$$

where $e_H \varphi = \Phi$ and $e_H^{-1} \mathbf{A} e_H$ is the representation of $\mathbf{A}$ in the basis e_H.

By defining $e_H^* F = f$ we have

$$f(e_H^{-1} \mathbf{A} e_H \varphi) = \lambda f(\varphi), \tag{6.4}$$

or:

$$f(A\varphi) = \lambda f(\varphi), \tag{6.5}$$

where $A = e_H^{-1} \mathbf{A} e_H$.

Notice that the last equation describes not just one eigenvalue problem, but a family of such problems, one for each string basis e_H. As we change e_H, the operator A in general changes as well, as do the eigenfunctions f.

Example. Let H be the Hilbert space of functions dual to the space H^* of functionals $f(x) = \sum_{p=-\infty}^{\infty} c_p e^{ipx}$ with the metric to be defined later (see(8.5)). Consider the operator of differentiation $A = i\frac{d}{dx}$ on H. The generalized eigenvalue problem for A is

$$f(i\frac{d}{dx}\varphi) = pf(\varphi). \tag{6.6}$$

The equation (6.6) must be satisfied for every φ in H. The functionals

$$f(x) = e^{-ipx} \tag{6.7}$$

are the eigenvectors of A. Let us now consider a coordinate change $\rho : H \longrightarrow \widetilde{H}$ given by the Fourier transform. Here $\widetilde{H}$ is dual to the Hilbert space of functionals $g(x) = \sum_{k=-\infty}^{\infty} c_k \delta(x-k)$ considered in example 1 of section 5. We have:

$$\psi(k) = (\rho\varphi)(k) = \int \varphi(x)e^{ikx}dx. \tag{6.8}$$

The Fourier transform is an isomorphism of the Hilbert spaces $\widetilde{H}$ and H. The inverse transform is given by

$$(\omega\psi)(x) = \frac{1}{2\pi}\int \psi(k)e^{-ikx}dk. \tag{6.9}$$

According to (6.4) the eigenvalue problem in new coordinates is

$$\omega^* f(\rho A \omega\psi) = p\omega^* f(\psi). \tag{6.10}$$

We have

$$A\omega\psi = i\frac{d}{dx}\frac{1}{2\pi}\int \psi(k)e^{-ikx}dk = \frac{1}{2\pi}\int k\psi(k)e^{-ikx}dk. \tag{6.11}$$

Therefore,

$$(\rho A\omega\psi)(k) = k\psi(k). \tag{6.12}$$

So, the eigenvalue problem in new coordinates is as follows:

$$g(k\psi) = pg(\psi). \tag{6.13}$$

Thus, we have the eigenvalue problem for the operator of multiplication by the variable. The eigenfunctions here are given by

$$g(k) = \delta(k-p). \tag{6.14}$$

Notice that $g = \omega^* f$ is as it should be. Indeed,

$$(\omega^* f)(k) = \frac{1}{2\pi}\int f(x)e^{ikx}dx = \frac{1}{2\pi}\int e^{-ipx}e^{ikx}dx = \delta(k-p). \tag{6.15}$$

As a result, the eigenvalue problems (6.6), and (6.13) can be considered as two coordinate expressions of a single eigenvalue problem

$$F(\mathbf{A}\Phi) = \lambda F(\Phi) \tag{6.16}$$

for an operator $\mathbf{A}$ on $\mathbf{S}$.

Notice that the spaces H and $\widetilde{H}$ in the example are the ones we have used in the experiment with the spectrometer in section 2. We will return to the physical interpretation of this fact in section 8.

7 The string basis of eigenvectors of a linear operator

Having introduced generalized eigenvectors it is natural to ask whether we can make a basis out of them. For this we introduce the following

Definition 1. A string basis e_H is a *basis of eigenvectors* of a linear operator $\mathbf{A}$ on $\mathbf{S}$ with eigenvalues (eigenvalue function) $\lambda = \lambda(k)$, if

$$\mathbf{A}e_H(\varphi) = e_H(\lambda\varphi) \tag{7.1}$$

for any $\varphi \in H$.

As any string basis, a string basis of eigenvectors of $\mathbf{A}$ is a linear map from H onto $\mathbf{S}$ and λ is a function defined on the same set as functions $\varphi \in H$.

By rewriting (7.1) as

$$e_H{}^{-1}\mathbf{A}e_H(\varphi) = \lambda\varphi \tag{7.2}$$

we see that the problem of finding a basis of eigenvectors of $\mathbf{A}$ is equivalent to the problem of finding such a string basis e_H in which the action of $\mathbf{A}$ reduces to multiplication by a function λ. In the particular case of an l_2-basis this yields the classical problem of finding a basis of eigenvectors of a linear operator.

The operator of multiplication by a function does not always map a Hilbert space H into itself. It is then useful to generalize the definition above.

Notice, that for any $F \in \mathbf{S}^*, \Phi \in \mathbf{S}$ and $\mathbf{A} : \mathbf{S} \longrightarrow \mathbf{S}$ we have

$$F(\mathbf{A}\Phi) = F(\mathbf{A}e_H\varphi) = F(e_{\widetilde{H}}e_{\widetilde{H}}^{-1}\mathbf{A}e_H\varphi) = e_{\widetilde{H}}^{*}F(e_{\widetilde{H}}^{-1}\mathbf{A}e_H\varphi) = f(A\varphi), \tag{7.3}$$

where $f = e_{\widetilde{H}}^{*}F \in \widetilde{H}^*$ and $A : H \longrightarrow \widetilde{H}$ is given by $A = e_{\widetilde{H}}^{-1}\mathbf{A}e_H$.

We then have the following

Definition 2. The representation $A = e_{\widetilde{H}}^{-1}\mathbf{A}e_H$ of $\mathbf{A}$ is called *proper* (or eigen-representation) if

$$f(A\varphi) = f(\lambda\varphi) \tag{7.4}$$

for *any* $f \in \widetilde{H}^*, \varphi \in H$ and some function λ.

In most cases, however, definition 1 above is sufficient.

Let us compare the notions of a string basis of eigenvectors and of the ordinary basis of eigenvectors of the operator $\mathbf{A}$ in more detail. For this assume first that $H = l_2$. Let e_{l_2} be a string basis of eigenvectors of $\mathbf{A}$. Let $\varphi = (\varphi_1, \varphi_2, ...) \in l_2$. The action of e_{l_2} on φ reduces to the matrix multiplication of the matrix $e_{l_2} = (e_1, e_2, ...)$ of linearly independent vectors by the column of components of φ. That is, $e_{l_2}\varphi = e_1\varphi_1 + e_2\varphi_2 + ...$.

Therefore,

$$\mathbf{A}e_{l_2}\varphi = e_{l_2}(\lambda\varphi) = e_{l_2}(\lambda_1\varphi_1, \lambda_2\varphi_2, ...) = e_1\lambda_1\varphi_1 + e_2\lambda_2\varphi_2 + ... \quad . \tag{7.5}$$

Let now $e_{l_2} = (e_1, e_2, ...)$ be the ordinary basis of eigenvectors of $\mathbf{A}$ instead, and $\varphi = e_1\varphi_1 + e_2\varphi_2 + ...$. Then

$$\mathbf{A}(e_1\varphi_1 + e_2\varphi_2 + ...) = \lambda_1 e_1\varphi_1 + \lambda_2 e_2\varphi_2 + ... = e_1\lambda_1\varphi_1 + e_2\lambda_2\varphi_2 + ... \quad . \tag{7.6}$$

Comparison of (7.5) and (7.6) shows that when $H = l_2$, the string basis of eigenvectors of $\mathbf{A}$ can be identified with the basis of eigenvectors of $\mathbf{A}$ in the ordinary sense. In particular, if $\mathbf{S}$ is identified with the space $L_2(R)$ and $H = l_2$, the string basis is the ordinary basis on $L_2(R)$.

Notice that in the equation

$$(\varphi, \lambda\psi)_{l_2} = \sum_{i=1}^{\infty} \lambda_i \varphi_i \psi_i \tag{7.7}$$

φ can be chosen to be an eigenvector of the operator of multiplication by the function λ. In particular, the process of measurement of $\mathbf{A}$ can be described as a two step process. First, a string basis of eigenvectors of $\mathbf{A}$ is chosen. This reduces $\mathbf{A}$ to the operator $A = \lambda\cdot$ of multiplication by a function λ. Then, an eigenvector of A is chosen giving the final "collapsed" state.

Assume now that A is an Hermitian operator on $L_2(R)$. Then

$$(f, Ag)_{L_2} = (Af, g)_{L_2} \tag{7.8}$$

for any $f, g \in L_2(R)$. Therefore,

$$(\Phi, \mathbf{A}\Psi)_S = (\mathbf{A}\Phi, \Psi)_S, \tag{7.9}$$

where $\Phi = e_{L_2} f, \Psi = e_{L_2} g, \mathbf{A} = e_{L_2} A {e_{L_2}}^{-1}$, and e_{L_2} is an L_2-string basis on $\mathbf{S}$.

It follows that the same will be true in any string basis e_H, that is,

$$(\varphi, A_H \psi)_H = (A_H \varphi, \psi)_H, \tag{7.10}$$

where $\Phi = e_H \varphi, \Psi = e_H \psi, \mathbf{A} = e_H A_H {e_H}^{-1}$, and e_H is any string basis on $\mathbf{S}$.

Hermicity of $\mathbf{A}$ leads to a severe restriction on possible spaces H on which a string basis of eigenvectors of $\mathbf{A}$ is defined. In fact, if e_H is a string basis of eigenvectors of an Hermitian operator $\mathbf{A}$ on $\mathbf{S}$, then

$$(\Phi, \mathbf{A}\Psi)_S = (e_H \varphi, \mathbf{A} e_H \psi)_S = (\varphi, \lambda\psi)_H = \int g(x, y)\varphi(x)\lambda(y)\psi(y)dxdy. \tag{7.11}$$

In agreement with section 4, the integral symbol is used here for the action of the bilinear metric functional G with the kernel g. Hermicity gives then

$$\int g(x, y)(\lambda(x) - \lambda(y))\varphi(x)\psi(y)dxdy = 0 \tag{7.12}$$

for any $\varphi, \psi \in H$. It follows that the bilinear functional defined by $g(x, y)(\lambda(x) - \lambda(y))$ is equal to zero.

Assume that H is a Hilbert space of functions on R and λ is injective (i.e. the spectrum of $\mathbf{A}$ is simple). Then $g(x, y) = a(x)\delta(x - y)$, i.e. H must coincide with the space $L_2(R)$ with the weight function $a(x)$.

Assume instead that H is a Hilbert space of sequences and the spectrum of $\mathbf{A}$ is simple. Then $g(x, y) = a_x \delta_{y,x}$, where $\delta_{y,x}$ is the Krœnecker δ-symbol. It follows that the coordinate space H must be the space l_2 with the weight a_x.

Therefore, the string bases of eigenvectors are orthogonal in these cases (as defined in section 4).

In particular case of $H = l_2$ this simply means orthogonality of the eigenvectors corresponding to different eigenvalues of an Hermitian operator on l_2.

The result can be generalized to conclude that whenever the spectrum of $\mathbf{A}$ is simple, hermicity of $\mathbf{A}$ leads to orthogonality of the string basis of eigenvectors of $\mathbf{A}$.

This conclusion seems to favor the use of L_2-spaces in QT. Notice however, that in an orthogonal string basis of eigenvectors of $\mathbf{A}$ with $H = L_2(R)$ we have

$$(\Phi, \mathbf{A}\Psi)_S = (f, \lambda g)_{L_2(R)} = \int f(x)\lambda(x)g(x)dx, \tag{7.13}$$

where $f, g \in L_2(R)$. Therefore, it is impossible to complete the second step of the measurement process as described earlier in this section. This means that it is impossible for $f(x)$ to be a (generalized) eigenvector of the operator A of multiplication by the function $\lambda(x)$. Indeed, the eigenvectors of such an operator would be δ-functions and the latter ones do not belong to $L_2(R)$.

From section 6 we recall that the eigenvectors of $A = \lambda\cdot$ acting on $L_2(R)$ can be defined as functionals F on a space V of functions, such that

$$F(A\varphi) = \lambda_0 F(\varphi) \tag{7.14}$$

for any $\varphi \in V$ and some $\lambda_0 \in R$.

To include such functionals in the formalism we need to consider Hilbert spaces containing "more" functions than $L_2(R)$. By the above this in general requires consideration of non-Hermitian operators.

Example. We know that $A_H = x\cdot$ (operator of multiplication by the variable) has no eigenvectors on $L_2(R)$. Consider then a different Hilbert space H with a metric G. We have:

$$(G\varphi, A_H\psi) = \int g(k,m)\varphi(k)m\psi(m)dkdm. \tag{7.15}$$

By (7.12)

$$\int g(k,m)\varphi(k)m\psi(m)dkdm \neq \int g(k,m)k\varphi(k)\psi(m)dkdm, \tag{7.16}$$

that is, A_H is not Hermitian unless H is an L_2-space. On the other hand, if $g(k,m)$ is an ordinary function which defines a regular functional, then obviously

$$\int g(k,m)\varphi(k)m\psi(m)dkdm = \int mg(k,m)\varphi(k)\psi(m)dkdm. \tag{7.17}$$

So by taking

$$F = \int g(k,m)\varphi(k)dk, \tag{7.18}$$

we have:

$$F(m\psi) = (mF)(\psi). \tag{7.19}$$

That is, A_H, although not Hermitian, is self-adjoint. We point out here that when a single Hilbert space is used it is customary to identify Hermitian and self-adjoint operators. However, this identification breaks up under coordinate transformations (see (7.21),(7.22) below).

If the functional $F = \int g(k,m)\varphi(k)dk$ above is not regular, we can define the action of A_H on it by the standard in the theory of generalized functions procedure, i.e. by taking

$$(mF)(\varphi) = F(m\varphi). \tag{7.20}$$

The resulting operator of multiplication by m is defined on H^* and is self-adjoint.

Notice now that the L_2-spaces are special in that the dual to any such space consists of the regular functionals defined by functions $f \in L_2$. The Hermitian operators on L_2 will be then identified with the corresponding self-adjoint operators. Namely, let A^+ be the Hermitian conjugate operator and let

A^* be the adjoint operator of a given operator A on a Hilbert space H. Let $\widehat{G} : H \longrightarrow H^*$ define a metric on H. Then

$$(\widehat{G}\varphi, A\psi) = (A^*\widehat{G}\varphi, \psi) = (\widehat{G}A^+\varphi, \psi) \tag{7.21}$$

and the relationship between the operators is as follows:

$$A^+ = \widehat{G}^{-1}A^*\widehat{G}. \tag{7.22}$$

On L_2-spaces the kernel of $\widehat{G}$ is the δ-function. Therefore it becomes natural to identify A^+ and A^*. However, this identification is not preserved under a general transformation of coordinates. In particular, on coordinate spaces different than L_2, A^+ and A^* will not be the same.

We can therefore accommodate both the Hermicity of operators on L_2 and the inclusion of generalized eigenvectors of operators on H into the dual space H^*. For this we first apply a coordinate transformation to obtain a Hilbert space H the dual of which contains the eigenvectors of the given Hermitian operator A on L_2. Then we extend (if necessary) the action of A to the action of the corresponding self-adjoint operator on H^*. This extension by conjugation is well known in the theory of generalized functions and has been already used here in (7.20) (see also section 10). In particular, the derivative of generalized functions is defined in such a way.

Example. Consider the Hilbert space H of functions as in example of section 4 of [1] (see also example 3 of section 5 in this paper). The space H in this case consists of infinitely differentiable functions only. The space H^* contains singular distributions. If $\widehat{G}$ is the metric on H, then the inner product on H^* is defined by

$$(f,g)_{H^*} = (\widehat{G}^{-1}f, g) = \int e^{-(x-y)^2-x^2}e^{-(y-z)^2-z^2}f(x)g(z)dydxdz. \tag{7.23}$$

Consider the bilinear expression

$$(\widehat{G}\varphi, x\psi), \tag{7.24}$$

where x is the operator of multiplication by the variable. Here unlike the case of $L_2(R)$-space we can choose $\varphi \in H$ in such a way that $\widehat{G}\varphi \in H^*$ is a (generalized) eigenvector of x. We can take, for example, $(\widehat{G}\varphi)(x) = \delta(x - \lambda_0)$. With such a choice we have:

$$(\widehat{G}\varphi, x\psi) = \lambda_0(\widehat{G}\varphi, \psi) = \lambda_0(f,g)_{L_2}, \tag{7.25}$$

where as before f and g are L_2-components of the strings Φ, Ψ and φ, ψ are the corresponding components in a basis e_H. By rewriting the original bilinear expression in $L_2(R)$-basis we also have:

$$(\widehat{G}\varphi, x\psi) = (f, A_{L_2}g)_{L_2}. \tag{7.26}$$

Here A_{L_2} is the representation in $L_2(R)$-basis of the operator $\mathbf{A}$ acting on $\mathbf{S}$, whose representation in the e_H-basis is x.

Therefore, f will be an eigenvector of the operator A_{L_2} on $L_2(R)$. It follows in particular that A_{L_2} is *not* the operator of multiplication by the variable.

Notice also that no operator can be represented by the operator of multiplication by the variable on both spaces at once: $L_2(R)$ and H above. This immediately follows from (7.25) and (7.26) as otherwise f in (7.26) would be an eigenvector of the operator x on $L_2(R)$.

This last fact was also verified in a different way in section 5 of [2]. There it has been established that the only coordinate transformation preserving the operator of multiplication by a variable are the transformations of multiplication by a function.

As a corollary to this observation we have the following uniqueness result:

Corollary. If e_H is a string basis of eigenvectors of an operator $\mathbf{A}$ with the eigenvalue function $\lambda(x) = x$, then any other string eigenbasis of $\mathbf{A}$ is given by the composition of maps $e_H \circ a$, where $a : \widetilde{H} \longrightarrow H$ is the operator of multiplication by a function.

More generally, the result holds true whenever the spectrum of $\mathbf{A}$ is simple.

Remark. Using the spectral theorem for self-adjoint operators in a rigged space (see [6]) one can prove that any self-adjoint operator on $\mathbf{S}$ possesses a string eigenbasis. We therefore have the existence and uniqueness (in the above sense) of the string eigenbasis of an arbitrary self-adjoint operator on $\mathbf{S}$ with a simple spectrum.

Notice once again that whenever the spectrum is discrete, the string eigenbasis can be identified with the ordinary eigenbasis of $\mathbf{A}$. In this case the existence and uniqueness (up to normalization) of the eigenbasis of an Hermitian operator $\mathbf{A}$ with a simple spectrum are well known.

We now turn to the physical interpretation of the developed formalism.

8 The evolution of quantum states and transformation of Hilbert metric

In the conventional QM the (unitary) evolution of a quantum system can be described by the evolution operator which acts on the initial state (or observables) of the system given at time t_1 and produces the final state (observables) at a later time t_2.

The state and the observables of a quantum system can be given in different representations. In the string formalism a *representation* of the state is a specific functional form of the state in a given string basis. At the same time the state itself is a "vector", i.e. an element of the string space, and as such, it is independent of any particular representation. The same is true about the observables. More specifically, we have the following definitions:

Definition. A *linear system of reference* on the string space $\mathbf{S}$ (or simply a *system of reference*) is an arbitrary linear coordinate system on $\mathbf{S}$ (as defined in section 4) used to describe the state and the observables of a quantum system.

Definition. The *state* (or *state-vector*) of a quantum system is an element of $\mathbf{S}$ whose specific *representation* in a given system of reference is the *state function* of the system.

Mathematically the state-vector is therefore the equivalence class of state functions related by coordinate transformations.

Definition. An *observable* (or *invariant observable*) is a self-adjoint operator on the string space whose representation in a given system of reference is a usual quantum-mechanical observable of a quantum system.

Again, mathematically an (invariant) observable is the equivalence class of operators obtained from a given QM-observable by coordinate transformations.

Definition. A *change of representation* of a quantum system is a simultaneous and covariant (i.e. according to (4.16)-(4.19)) functional transformation (active or passive) of the state, metric, observables and other string-tensors describing the system.

By an *active* change of representation we understand here the change due to the *actual* transformation of the string-tensors relative to a given system of reference rather than the mere transformation of the system of reference itself. The latter case would describe a *passive* change of representation.

Notice that the usual notion of change of representation in QM is a particular case of the above defined passive change of representation.

The "Fourier transform" experiment in section 2 is an example of the active change of representation. In the experiment the spectrometer "Fourier transforms" the space

$$H = \left\{ \varphi | \varphi = \sum_{k=-\infty}^{\infty} c_k e^{ikx}, c_k \in R \right\} \tag{8.1}$$

into the space

$$\widetilde{H} = \left\{ \psi | \psi = \sum_{k=-\infty}^{\infty} c_k \delta(x-k), c_k \in R \right\}. \tag{8.2}$$

Both spaces H and $\widetilde{H}$ can be assumed to be Hilbert subspaces of a larger Hilbert space H_0 (as the one given in example 3 of section 5). The space H_0 together with the isomorphism $\pi : \mathbf{S} \longrightarrow H_0$ can be taken as a reference system appropriate for the Fourier-transform experiment. By applying the Fourier transform F to the elements of $\widetilde{H}$ one has:

$$(F\psi)(y) = \sum_{k=-\infty}^{\infty} c_k \int \delta(x-k) e^{ixy} dx = \sum_{k=-\infty}^{\infty} c_k e^{iky}. \tag{8.3}$$

From section 5 we know that the space $\widetilde{H}$ with the metric (5.1) is a Hilbert space. The Fourier transform F establishes a one-to-one correspondence between the spaces H and $\widetilde{H}$. In particular, it induces a Hilbert structure on H by setting

$$(F\psi_1, F\psi_2)_H = (\psi_1, \psi_2)_{\widetilde{H}} \tag{8.4}$$

for any $\psi_1, \psi_2 \in \widetilde{H}$. If H is furnished with the induced Hilbert structure, then F becomes an isomorphism of Hilbert spaces $\widetilde{H}$ and H. Namely, by (4.18), the action of $F : \widetilde{H} \longrightarrow H$ induces on H the following metric:

$$g_H(k,z) = \left(F^{-1*} \widehat{G}_{\widetilde{H}} F^{-1} \right)(k,z) = \frac{1}{(2\pi)^2} \int e^{ikx} e^{-(x-y)^2} e^{-iyz} dx dy = \frac{1}{2\sqrt{\pi}} \delta(k-z) e^{-\frac{k^2}{4}}, \tag{8.5}$$

where the metric $\widehat{G}_{\widetilde{H}}$ given by (5.1) has been used.

We see in particular that the evolution operator in the given coordinates is an isomorphism of the Hilbert spaces H and $\widetilde{H}$ above.

Notice also that the measured observable in the experiment changes as well: the magnetic field measures the electron's momentum while the scintillating screen measures its position.

By accepting the transformation law (8.5) we conclude then that the spectrometer in the experiment Fourier-transforms the state of the electron, the metric and the observable. That is, the spectrometer performs an active change of representation as defined above.

Notice that the evolution operator in the experiment is not rigorously speaking unitary. In fact, it is an isomorphism of *different* Hilbert spaces of functions. Nevertheless, the norm of a state is preserved under the evolution, i.e. the evolution operator is an *isometry.* The isometries represent the most obvious generalization of the unitary operators. Whenever an isometry preserves the Hilbert structure of its space of definition it becomes a unitary transformation.

The treatment in section 6 of the generalized eigenvalue problem for the Fourier transform experiment shows that not only the state-functions, but the equations for these functions themselves transform in a "covariant" fashion under the active change of representation.

As an active change of representation is an isomorphism of Hilbert spaces, it can be thought of as a change of coordinates (i.e. a passive change of representation). For this we simply assume that during the evolution the state, the metric and the observables do not change but the coordinates do.

Thus, in the "Fourier transform" experiment we would say that the observable in the final state is the momentum but observed in the Fourier-transformed coordinates. This could be objected as unnatural as we "know" that the electron's position rather than momentum is determined by the flash on the screen. However, mathematically both approaches are completely equivalent. In the sequel we will use this equivalence interchanging active changes of representation and coordinate transformations whenever necessary.

The above mentioned tensor-like properties of equations in QT lead to a very important conclusion about the relativity of states and observables. This means first of all the covariant dependency of their functional representation on a chosen reference system. In particular, the same state can be localized in one reference system and represented by a smooth function in another.

By itself this relativity can be considered as a purely mathematical property, as "just a change of representation". However, the possibility to change the representation *actively*, i.e. *physically* as in the Fourier-transform experiment together with the equivalence of the active and passive changes of representation make this relativity an important physical principle. The principle permits, in particular, to compare observations made in different reference systems. It can be formulated as follows:

Hypothesis A (The principle of functional relativity). *The laws of quantum physics must be the same in any functional system of reference. That is, the equations interpreting these laws must be functional tensor equations.*

The eigenvalue problem (6.16) provides an example of such an equation.

Naturally related to the principle of functional relativity is the following interpretation of the process of measurement of a quantum system:

Hypothesis B. *Relative to a fixed system of reference the measurement performed on a quantum system is represented mathematically by a functional transformation which is an isomorphism of Hilbert spaces of functions. This transformation changes in a covariant fashion the state function of the system and the Hilbert metric on the coordinate space to which the state function belongs.*

In particular, measurement of the electron's position in the Fourier-transform experiment is completely described by the Fourier transformation. The state function and the metric in the experiment change by (4.16), (4.18). Notice also that, according to section 6, the appearance of the final "collapsed" state of the electron in the experiment can be interpreted as follows. The spectrometer "Fourier transforms" the eigenvalue problem for the momentum operator into the eigenvalue problem for the position operator. In particular, the solutions to these problems are transformed into each other producing the collapsed (given by the δ-function) final state of the electron.

In general we would then expect that collapse can be interpreted as a transformation of the eigenstate of an operator into an eigenstate of the measured observable. More precisely we accept the following hypothesis:

Hypothesis C. *Relative to a given reference system collapse can be considered as a functional transformation induced by the measurement which isometrically transforms the state of a system into one of the eigenstates of the observable.*

Such interpretation of collapse would also resolve the dualism in QM dynamics, i.e. the question of why QM needs two different dynamical processes: the unitary evolution and the non-unitary state reduction. In the described approach both processes can be combined into a single isometric evolution. If the evolution does not alter the original Hilbert space structure, the isometry becomes a unitary transformation. If the evolution produces a collapsed state, then all but one component of the state function in the string basis become negligibly small. Therefore, to preserve the norm of the state the metric on the coordinate space must change. Consequently, the evolution cannot be unitary. However, it remains an isometry, thereby preserving all information about the state.

This interpretation of collapse, although very attractive, finds an immediate objection. In fact, the initial state would then always determine the collapsed state of the electron in a deterministic way, contradicting the observed probabilistic pattern.

More generally, it needs to be explained how exactly the proposed interpretation of measurement and collapse is related to the conventional one and to the known properties of quantum states. We will return to these questions in sections 10 and 11 after briefly reviewing a generally accepted mathematical approach to quantum measurement.

9 A mathematical model of quantum measurement

To see how to treat a quantum measurement in the string formalism and to understand better the hypotheses **B** and **C** let us start with a simple description of the main mathematical model of quantum measurement. In this we will closely follow [7] (see also references therein).

A quantum measurement in continuous spectrum can be described with the use of a natural generalization of von Neumann projectors. Namely, a fuzzy (i.e. with a finite resolution) measurement with a continuous set of alternatives is represented mathematically by a multiplication of the state function $\psi(a)$ of the system in A-representation by a characteristic function $R_{a'}(a)$ of the measurement. The functions $R_{a'}$ satisfy the "decomposition of unity" condition

$$\int da' R^*_{a'} R_{a'} = 1. \tag{9.1}$$

Most typically, $R_{a'}(a) = R(a - a')$, where $R(a)$ is the Gaussian function, i.e.

$$R(a) = N e^{-\frac{a^2}{4\lambda^2}}, \tag{9.2}$$

where N is a normalization constant and λ is the resolution of the measurement.

The measurement is therefore represented by the scheme

$$\psi \longrightarrow \psi_{a'} = \widehat{R}_{a'} \psi, \tag{9.3}$$

where $\widehat{R}_{a'}$ is the operator of multiplication by the function $R_{a'}$ (which will be called the fuzzy projector). The probability density is given by

$$p_{a'} = \|\psi_{a'}\|^2 . \tag{9.4}$$

Notice that such a description can be used for discrete spectra as well. In fact, assume A is an observable with a spectrum a_i. Then

$$\psi(a) = \sum_i c_i \psi_i(a) \tag{9.5}$$

where ψ_i form an orthonormal basis of eigenfunctions of A. The measurement is then described by

$$\widehat{R}_{a'} \psi = \sum_i R(a_i - a') c_i \psi_i . \tag{9.6}$$

For a sharp enough measurement the characteristic function $R(a_i - a')$ can be taken to be 1 at a_i and 0 everywhere else in which case $\widehat{R}_{a'}$ becomes the von Neumann projector on ψ_i.

More realistically the model above with the Gaussian characteristic function describes a fuzzy measurement (i.e. the fuzzy projection) rather than a projection. However, the final result of a sequence of repeated instantaneous fuzzy measurements (9.6) is represented by the von Neumann projection.

Let us reproduce this result following [7]. Assume for simplicity that the observable A has two eigenvalues a_1, a_2 and ψ_1, ψ_2 are the corresponding eigenvectors. If the initial state of the system is given by

$$\psi = c_1\psi_1 + c_2\psi_2, \tag{9.7}$$

then after a single measurement (9.3) giving the result a' the state of the system is

$$\psi_{a'} = R(a_1 - a')c_1\psi_1 + R(a_2 - a')c_2\psi_2. \tag{9.8}$$

According to (9.4) the probability density of this result is given by

$$\|\psi_{a'}\|^2 = |\alpha_1|^2 |c_1|^2 + |\beta_1|^2 |c_2|^2, \tag{9.9}$$

where $\alpha_1 = R(a_1 - a')$ and $\beta_1 = R(a_2 - a')$. The sequence of such measurements yields the sequence of states

$$c_1\psi_1 + c_2\psi_2 \longrightarrow \alpha_1 c_1\psi_1 + \beta_1 c_2\psi_2 \longrightarrow ... \longrightarrow \alpha_n ...\alpha_1 c_1\psi_1 + \beta_n ...\beta_1 c_2\psi_2 = \widetilde{c}_1\psi_1 + \widetilde{c}_2\psi_2. \tag{9.10}$$

It is important that only ψ_1 and ψ_2 are stable under the repeated measurements. In fact, assume for example that at some step of the process the coefficient of ψ_1 is larger than the coefficient of ψ_2. Then the probability is high that it will increase even more in the next step. As a result, the component ψ_2 dies out.

Finally, under the assumption that the Hamiltonian of interaction between the system and the measuring device dominates the free Hamiltonians of the system and the device, the above model is quite generally realizable [7]. In the next section it will be also shown that the model naturally fits in the presented string formalism.

10 Interpretation of measurement in the string formalism

According to hypothesis **B** of section 8 a measurement of a quantum system in the string formalism can be described by a functional transformation which is an isomorphism of Hilbert spaces of functions. This transformation is due to the interaction between the system and the measuring device. We do not study here the specifics of such an interaction and therefore the dynamics of a quantum measurement. However, the change in state as a result of this (unknown) dynamics can be described by a functional transformation. By studying these transformations we study the *kinematics* of a quantum measurement.

Let us return to the "Fourier transform" experiment of section 2. In the process of measurement the spectrometer Fourier-transforms the state of the electron as well as the observables of the system. In fact, the entire eigenvalue problem for the momentum operator is transformed into the eigenvalue problem for the position operator.

Using the "passive" approach we can say that the operators $\widehat{p}$ and $\widehat{x}$ represent the same string operator in two different functional coordinate systems: before and after the measurement by the spectrometer.

Moreover, the mere fact of non-commutativity of $\widehat{p}$ and $\widehat{x}$ can be interpreted as a non-commutativity of the position operator $\widehat{x}$ with a transformation of functional coordinates. In fact, if $F : H \longrightarrow \widetilde{H}$ is the Fourier transform, then

$$[\widehat{x}, \widehat{p}]\psi = \widehat{x}F^{-1}\widehat{x}F\psi - F^{-1}\widehat{x}F\widehat{x}\psi \tag{10.1}$$

for any $\psi \in H$. That is, suppose that the state ψ is first Fourier-transformed, then the operator $\widehat{x}$ acts on the state, then the state is Fourier-transformed back to H and $\widehat{x}$ is applied again. The resulting state is different than the state obtained by first applying $\widehat{x}$, then Fourier transforming the resulting state, then using $\widehat{x}$ again and then returning to H. This is a (passive) interpretation of the non-commutativity of $\widehat{p}$ and $\widehat{x}$ in the string formalism. A similar interpretation in terms of the momentum operator is obvious.

In the string formalism an observable $\mathbf{A}$ is a self-adjoint operator acting on the string space $\mathbf{S}$. We assume for simplicity that $\mathbf{A}$ is defined on the entire space $\mathbf{S}$. By identifying $\mathbf{S}$ with the space $L_2(R)$ for example, we have

$$(A^{*}\widehat{G}_{L_2}\varphi, \psi) = (\widehat{G}_{L_2}\varphi, A\psi) \tag{10.2}$$

for any $\varphi, \psi \in L_2(R)$. In particular, A here can be considered Hermitian (see section 7).

By fixing functional coordinates on $\mathbf{S}$ we obtain a representation of $\mathbf{A}$ by an operator $A : H \longrightarrow \widetilde{H}$ defined on a Hilbert space H. The following definitions will be accepted:

Definition. The *proper* basis of a quantum measurement performed on a system is the string eigenbasis of the measured observable. The corresponding reference system will be called *proper* as well.

Definition. If the state Φ of a quantum system is an eigenstate of the observable $\mathbf{A}$, then we say that the system is *at rest* with respect to $\mathbf{A}$.

In the latter case there exists a string basis (namely, the string eigenbasis of $\mathbf{A}$, see section 7) in which $\mathbf{A}$ is represented by the operator $A = \lambda\cdot$ of multiplication by a function and Φ is represented by an eigenfunction of A.

As a result of a quantum measurement the state of the system becomes an eigenstate of the measured observable $\mathbf{A}$, that is, the measured system is at rest with respect to $\mathbf{A}$. Notice that this property can be similarly expressed in arbitrary coordinates.

When the system is at rest relative to A, the outcome of measurement of the observable A can be predicted with certainty. In particular, the exact location of the final electron in the spectrometer experiment can be determined in advance. On the other hand, when a system is not at rest, the result of measurement of A can never be predicted with certainty. In fact, the system in this case is in a superposition of eigenstates of the observable and the usual probability rules apply.

We conclude that the probabilistic interpretation in QT appears when we measure the quantum system from a "moving" string basis. The collapse of a state in turn can be interpreted as a transition of the state to the "rest" system.

More specifically, if the system is at rest with respect to an observable A_H acting on a Hilbert space H, then

$$(\varphi, A_H\psi)_H = \lambda_0(\varphi, \psi)_H, \tag{10.3}$$

where φ is the state of the system and $\psi \in H$ is any. If now $\rho : H \longrightarrow \widetilde{H}$ is an arbitrary change of functional coordinates and $A_{\widetilde{H}} = \rho^{-1}A_H\rho$ is the representation of $\mathbf{A}$ in new coordinates, then

$$(f, A_{\widetilde{H}}g)_{\widetilde{H}} = \lambda_0(f, g)_{\widetilde{H}}, \tag{10.4}$$

where $f = \rho\varphi, g = \rho\psi$. In particular, f is an eigenstate of $A_{\widetilde{H}}$, i.e. the transformed system is at rest with respect to $A_{\widetilde{H}}$ and the measurement of $A_{\widetilde{H}}$ gives the eigenvalue λ_0 with certainty.

Assume instead that φ is a generalized eigenstate of A_H and that we measure a different observable B_H acting on H and not commuting with A_H. Assume also that $B_H = \rho^{-1}A_{\widetilde{H}}\rho$, where $\rho : H \longrightarrow \widetilde{H}$ is a functional coordinate transformation and $A_{\widetilde{H}}$ is the operator A_H extended (if necessary) to act on $\widetilde{H}$. Then measuring of $A_{\widetilde{H}}$ can be thought of as measuring the observable A_H from a "moving" system. As $[A_H, B_H] \neq 0$ the result of measurement cannot be predicted with certainty.

In such a "relativistic" treatment of measurement it was assumed that any self-adjoint operator A acting on a Hilbert space H can be extended to act on an arbitrary Hilbert space of functions defined on the same set as functions in H. This is assured by the "dualization" procedure which was used already in section 7. The technique is well known in the theory of generalized functions. First, the action of A_H is defined on a small enough topological subspace $\widetilde{H}$ of H. Then the adjoint action on the space $\widetilde{H}^*$ of functionals on $\widetilde{H}$ is identified with the action of A on $\widetilde{H}^*$ (see example in section 4 of [1]) . In particular, the derivative of singular generalized functions is defined this way.

Notice that the description of a quantum measurement from a chosen reference system does not always make sense. In particular, the norm of a state given by the δ-function is infinite in the $L_2(R)$-reference system. This is not an indication that the state itself is singular, rather, that the reference system is not appropriate to represent the measurement. Analogous examples are well known in the theory of gravity. Thus, the Schwarzschild metric on the space-time is singular on the Schwarzschild sphere of the black hole. This is known to be a coordinate effect only. An appropriate choice of coordinates eliminates the singularity of the metric.

The discussion so far was rather general but mostly concerned with the infinite-dimensional representation of states. It is well known that the state function can be often represented by a finite dimensional vector. In this case the state is still an element of the infinite-dimensional Hilbert space $\mathbf{S}$. However, $\mathbf{S}$ is represented by a finite direct sum $\oplus H_i$ of Hilbert spaces H_i and the action of operators of interest on each H_i is proportional to the identity. That is, the operators are represented by finite matrices.

To see how the string formalism works in case of a finite number of dimensions let us analyze the measurement of the electron's position in the two-slit experiment. The experiment is also called "washing interference out" or "which way" experiment and the modern interpretation of it can be found in [7].

Example. The initial state of the electron in the experiment is taken to be a superposition of two localized states each corresponding to the electron going through each of the slits. Assume that

$$\psi(q) = c_1 N e^{-\frac{(q+a)^2}{4\varepsilon^2}} + c_2 N e^{-\frac{(q-a)^2}{4\varepsilon^2}} = c_1\psi_a(q) + c_2\psi_{-a}(q), \tag{10.5}$$

where $\pm a$ are coordinates of the slits, $\psi_{\pm a}(q) = N e^{-\frac{(q\pm a)^2}{4\epsilon^2}}$, N is a normalization constant (so that $\psi_{\pm a}(q)$ are unit-normalized) and $\varepsilon \ll a$.

Assume that a measurement or a sequence of fuzzy measurements is performed with the goal to find out which slit the particle went through. Each fuzzy measurement consists, for example, in a scattering of a photon off the electron.

Mathematically, as in section 9, each measurement will be described by the operator $\widehat{R}_x(q)$ of multiplication of ψ by the characteristic function $R_x(q)$ of the measurement. Although the exact form of $R_x(q)$ will not be important, we will assume that, as in (9.2), it is the Gaussian function i.e.

$$R_x(q) = M e^{-\frac{(q-x)^2}{4\lambda^2}}, \tag{10.6}$$

where M and λ are constants.

We then have after a single measurement

$$\widehat{R}_x(q)\psi(q) = M\left(c_1 e^{-\frac{(q-x)^2}{4\lambda^2}}\psi_a(q) + c_2 e^{-\frac{(q+x)^2}{4\lambda^2}}\psi_{-a}(q)\right). \tag{10.7}$$

Respectively, the probability density to obtain such a state is given by

$$p(x) = M^2\left[|c_1|^2 e^{-\frac{(q-x)^2}{2\lambda^2}} + |c_2|^2 e^{-\frac{(q+x)^2}{2\lambda^2}}\right]. \tag{10.8}$$

The action of $\widehat{R}_x(q)$ can be represented by the matrix

$$\rho = \begin{pmatrix} Me^{-\frac{(q-x)^2}{4\lambda^2}} & 0 \\ 0 & Me^{-\frac{(q+x)^2}{4\lambda^2}} \end{pmatrix}. \tag{10.9}$$

In the string formalism the operator $\widehat{R}_x(q)$ given by the matrix ρ is a functional transformation which changes not only the state ψ, but also the metric. Assume that the original metric in the experiment is represented by the 2×2 identity matrix. Then, by (4.18) after the measurement the metric becomes

$$\widehat{G} = (\rho^* \rho)^{-1} = \begin{pmatrix} \frac{1}{M^2} e^{\frac{(q-x)^2}{2\lambda^2}} & 0 \\ 0 & \frac{1}{M^2} e^{\frac{(q+x)^2}{2\lambda^2}} \end{pmatrix}. \tag{10.10}$$

Assume now that the measurement is precise enough to resolve between the slits, i.e. $\lambda \ll a$. Then the probability density in (10.8) is not negligible only for $x = \pm a$. This means that only one of the two diagonal entries in (10.9) survives. The second, although different than zero, is much smaller.

Assuming for example that $x = a$, the transformation matrix ρ in (10.9) can be written as

$$\rho = \begin{pmatrix} 1-\epsilon & 0 \\ 0 & \epsilon \end{pmatrix}, \tag{10.11}$$

where $\epsilon \ll 1$. For the transformation of $\psi = \begin{pmatrix} \psi_a \\ \psi_{-a} \end{pmatrix}$ we have then

$$\rho\psi = \begin{pmatrix} 1-\epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \begin{pmatrix} \psi_a \\ \psi_{-a} \end{pmatrix} = \begin{pmatrix} (1-\epsilon)\psi_a \\ \epsilon\psi_{-a} \end{pmatrix}. \tag{10.12}$$

As $\epsilon \ll 1$ we see that only ψ_a component is not negligible. Notice also that, as discussed in section 9, under the repeated measurements the second component becomes smaller and smaller approaching zero. The resulting transformation then looks like a von Neumann projector and the resulting state $\widetilde{\psi}$ looks like the eigenstate of the position operator.

On the other hand, it is very important that the determinant of ρ is never zero, that is, in the "passive" approach ρ can be considered as a coordinate transformation.

As always, the transformation given by ρ is an isometry. That is, the norm of ψ as well as the inner product of any two vectors are preserved.

The presented analysis of the two-slit experiment can be now summarized as follows. In trying to determine which slit the electron went through we scatter the photons off the electron. Mathematically each scattering event can be described as a multiplication of the state function of the electron by the Gaussian function.

In the conventional approach to QT each scattering event is therefore a fuzzy projector. The sequence of scattering events leads to the gradual decoherence. The final result is represented by the von Neumann projection.

In the string formalism each scattering event is a (reversible) functional transformation which changes the state and the metric. In the new metric the norm of the new state is the same as the norm of the original state in the original metric. That is, the transformation is an isometry.

However, the electron's position is obtained by measuring the already scattered free photons. That is, the measurement is described in a "free" reference system which can be assumed to be the same as the reference system used to describe the initial state. We will call such a reference system the *lab* system. By (10.12) only one of the components of the state function in the lab system survives. The second one is negligibly small in the lab metric.

The fact that the second component of the state function is not exactly equal to zero is a consequence of the finite dimensional setting of the problem. We could instead take the initial state function ψ to be an element of a more general space H of linear combinations of functions supported on a small enough neighborhood of the point $x = \pm a$. Alternatively, H could be a space of linear combinations of $\delta^{(p)}(x \pm a)$ functions. This would be in fact a more realistic approach as the function $\psi_{\pm a}$ or the δ-function alone cannot describe the variety of such *localized* states. The latter fact becomes especially clear in light of the "heavy oscillator" experiment of section 3. The QFT approach to the two-slit experiment confirms this result. In fact, the high energy localization of the electron leads to creation of the electron-positron pairs making it necessary to consider more general state functions.

Assume then that the initial state ψ in the experiment is an element of such a Hilbert space H. The state after measuring the electron's position would be then an element of the Hilbert space $\widetilde{H}$ of functions localized on a small neighborhood of one of the points $x = a$ or $x = -a$ (or concentrated at $x = \pm a$. In this case the measurement itself is represented by an isomorphism

$$M : H \longrightarrow \widetilde{H}. \tag{10.13}$$

In particular, the final state is exactly zero at one of the slits.

It is nevertheless interesting that even such a simplistic model of measurement as the one described in section 9 nicely fits into the string fomalism.

Notice also, that a sequence of fuzzy measurements was necessary to ensure that one of the components of the state function in the example can be always assumed negligible. On the other hand, a single measurement is enough in the treatment of the two-slit experiment according to (10.13). As a trade off, the spaces of states in (10.13) are infinite-dimensional. So we can either consider a sequence of scattering events with the electron represented by a finite dimensional state function, or a single scattering event with the electron represented by an infinite-dimensional state function.

11 So, what is the collapse?

We have seen that the inclusion of generalized functions in the formalism of QT lead in a natural way to the non-Hermitian (but self-adjoint) operators and to the non-unitary (but isometric) evolutions. We have also seen that these non-unitary evolutions can produce localized states, in particular, the states described by the δ-function. The collapse then obtains a natural interpretation as a result of such an evolution in support of the hypothesis **C** of section 8.

What needs to be explained, however, is where the probability interpretation of the state comes from and what is the mechanism of the functional transformation by the measurement.

The complete answer to these questions depends on the interaction between the measuring device and the quantum system and has yet to be obtained. Let us, however, summarize the observations based on the kinematics of quantum measurements considered in the paper.

For this, let us return to the two-slit experiment considered in the example of section 10. As in the example, to measure a position of the electron we consider a sequence of scattering events where the incoming photons are scattered off the electron.

The sequence of scattering events is represented by (9.10) with the resulting state $\widetilde{\psi}$ given by

$$\widetilde{\psi} = \alpha_n ... \alpha_1 c_1 \psi_1 + \beta_n ... \beta_1 c_2 \psi_2. \tag{11.1}$$

The following assumptions are being used:

1. The initial state of the electron is $\psi = c_1\psi_1 + c_2\psi_2$, with the standard probabilistic interpretation of the coefficients c_1, c_2. This probabilistic interpretation of the coefficients of ψ_1 and ψ_2 is assumed to hold at each intermediate step of the chain (9.10).

2. A single measurement of the electron position by a photon is described by multiplication of ψ by the Gaussian function $R_x(N, \lambda, q) = Ne^{-\frac{(q-x)^2}{4\lambda^2}}$. Notice here that, as mentioned in section 9, such an assumption is realistic if the interaction between the electron and the photon is sufficiently strong. The fact that the *form* of the function $R_x(N, \lambda, q)$ does not depend on x can be interpreted as identity of the photons scattered off the electron.

3. The probability distributions of different values of the products $\alpha_n ... \alpha_1$ and $\beta_n ... \beta_1$ in (11.1) are identical. This can be interpreted as homogeneity of distribution of the measuring photons within the region of interaction.

The conclusion is two-sided. First of all, according to the results of section 9, only one of the components of the state function ψ will survive. Another one will be negligibly small destroying the interference. This effect is a sort of *spontaneous symmetry breaking* and it is based on the probabilistic interpretation of the coefficients of ψ after each measurement. In particular, even if $|c_1|^2 = |c_2|^2$, the system is predicted to fall in one of the stable states ψ_1 or ψ_2.

Secondly, because of the assumption 3 above, the probability to obtain the final state of ψ_1 or ψ_2 turns out to be proportional to the square of the modulus of the original coefficients c_1, c_2 in ψ.

This last fact does not explain the probabilistic interpretation of the coefficients of ψ. However, it shows that such an interpretation is consistent with the used model of measurement.

The above analysis can be generalized without difficulties to the case of a continuous spectrum as well.

As a result, the model of fuzzy measurements is more than just a model of decoherence of a quantum system by interaction with the environment (photons). The model also predicts the collapse by a spontaneous symmetry breaking. Yet, the model falls short of "explaining" collapse as the probability interpretation of the coefficients has not been obtained. In addition, the realization of the model involves the yet unexplained collapse of the state of the measuring device (see [7]). Finally, using the model, it is difficult in general to understand the collapse by scattering of a single photon.

The operator of multiplication by the Gaussian function used in the model represents a very special functional coordinate transformation. We saw that this transformation is invertible and, on the other hand, the sequence of such transformations yields a von Neumann projector.

However, the transformation is not always realistic and as we saw, it cannot serve the model of collapse. A more realistic model as in (10.13) must be used in this case. This, however, is not enough. To fully succeed in describing the collapse within the developed formalism we need a (functional) tensor theory of interactions between a quantum system and a measuring device. Notice here that the considered model of measurement suggests that such a theory must be nonlinear. In fact, in the formalism the collapse of a quantum system is described by a linear functional coordinate transformation. However, the final collapsed state function (and therefore a specific form of the transformation) depends on the state of the system through the probabilistic interpretation of the coefficients. The resulting theory must be therefore nonlinear.

12 Concluding remarks

The birth of the quantum theory signified the arrival of a new era of infinite-dimensional physics. In quantum theory not only the fields, but the particles themselves become functional objects. The classical world becomes an emergent phenomenon behind which the whole new quantum world is still awaiting our discoveries.

Up until now the functionality in quantum physics has not been used consistently. That is, there was no guiding principle in the infinite-dimensional world. It is advocated here that such a principle must be a further extension of the familiar principle of relativity. Namely, the equations of physics must be invariant with respect to the choice of a functional system of reference.

This principle, when accepted, leads to several important consequences.

First of all, it permits us to describe physics equally well in arbitrary functional coordinates including the coordinates represented by singular generalized functions.

Second, it unifies different pictures of the quantum world. In particular, we saw in section 6 that the eigenvalue problems for the position and the momentum operators are but two representations of a single (tensor) eigenvalue problem on the string space. Also, a state which is concentrated at a point in one coordinate system, can be represented by a smooth function in another one. In particular, the "collapsed" states look collapsed only when a particular coordinate system is used.

This stresses one more time that the significant part (if not all!) of the physics that we observe is produced by a special choice of coordinates that we use.

The principle also leads to the conclusion that the observables in physics must be defined on arbitrary Hilbert spaces of functions and are in general self-adjoint but not necessarily Hermitian operators. Similarly, the evolution operator is an isometry but not necessarily a unitary transformation. In this interpretation, both the unitary evolution and the non-unitary collapse are examples of such an isometric evolution.

The principle gives a geometric interpretation of quantum measurements and the collapse by relating the process of measurement to the choice of a reference system (in the passive approach). In particular, the determination of the slit the electron went through in the two-slit experiment of section 10 can be interpreted as such a change of coordinates that makes all but one component of the state negligibly small.

The principle also interprets at least some of the infinities in the QT as the effect related to the choice of the reference system (see section 10). A very intriguing possibility is to interpret *all* infinities in the QFT in such a way.

The described formalism represents first of all the kinematics of the quantum world rather than its dynamics. In particular, it describes the collapse as a functional transformation but does not provide details of the collapse. Such details are related to the yet to be developed functional tensor theory of quantum interactions.

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References

[1] A. Kryukov, *Coordinate formalism on Hilbert manifolds*, arXiv: math-ph/0201017 v1, 2002.

[2] A. Kryukov, *Coordinate formalism on abstract Hilbert space*, arXiv: math-ph/0201031 v1, 2002.

[3] A. Kryukov, *Coordinate formalism on Hilbert manifolds: String bases of eigenvectors*, PITT-PHIL-SCI00000580, 2002.

[4] I.M. Gel'fand and G.E. Shilov, *Generalized Functions, Vol.1*, Academic Press, New York and London, (1964).

[5] I.M. Gel'fand and G.E. Shilov, *Generalized Functions, Vol.2*, Academic Press, New York and London, (1968).

[6] I.M. Gel'fand and N.Y. Vilenkin, *Generalized Functions, Vol.4*, Academic Press, New York and London, (1964).

[7] M.B. Mensky, *Quantum Measurements and Decoherence*, Kluwer Academic Publishers, (2000).