Inertial frames, special relativity and consistency

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Abstract

The theory of special relativity (SR) is considered in the framework of classical first-order logic. The four axioms of SR are: The existence of a flat four-dimensional spacetime continuum, the existence of global inertial frames of reference and Einstein's two postulates. The propositions permitted are those involving the kinematics and dynamics of SR as formulated from an inertial frame; these give a complete description of the evolution of the universe. Assuming SR as consistent, there must exist a model M for SR in which F is an inertial frame in a non-trivial universe $U(IBC)$ of material objects, with appropriate initial-boundary conditions IBC specified. Let P be defined as "F is an inertial frame in $U(IBC)$". Using Gödel's second incompleteness theorem, it is argued that P is undecidable in SR. Let Q be any proposition such that $(Q \Rightarrow P)$ is not a theorem of SR. If, in addition, $(P \Rightarrow Q)$ is a theorem of SR then Q must necessarily be true in M. It follows that there must exist a model N for SR in which Q is true and P is false, i.e., F is an accelerated frame in $U(IBC)$. The philosophical and mathematical implications of this result for the consistency of SR are discussed.

Introduction

It is well known that in the special theory of relativity the identification of a suitable inertial frame of reference is problematic. An excellent summary of the difficulties is available in [1], and one can do no better than quote extensively the relevant portions from this reference:

> In a sense, special relativity is analogous to "naive set theory" in mathematics. By this I mean that special relativity is based on certain plausible-sounding premises which actually are quite serviceable for treating a wide class of problems, but which on close examination are susceptible to self-referential antinomies. This is most evident with regard to the assumption of identifiability of inertial frames. As Einstein remarked, "in the special theory of relativity there is an inherent epistemological defect", namely, that the preferred class of reference frames on which the theory relies is circularly defined.

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> Special relativity asserts that the lapse of proper time between two (timelike-separated) events is greatest along the inertial worldline connecting these two events — a seemingly interesting and useful assertion — but if we ask which of the infinitely many paths connecting those two events is the "inertial" one, we can only answer that it is the one with the greatest lapse of proper time. If we simply accept this uncritically, and we are willing to naively rely on the testimony of accelerometers as unambiguous indicators of "inertia", we have a fairly solid basis on which to do physics, and we can certainly work out correct answers to many questions. However, the epistemological defect was worrisome to Einstein, and caused him (in a remarkably short time) to abandon special relativity and global Lorentz invariance as a suitable conceptual framework for the formulation of physics.
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> The naive reliance on accelerometers as unambiguous indicators of global inertia in the context of special relativity is immediately undermined by the equivalence principle, because we're then required to predicate *any* application of special relativity on the absence (or at least the negligibility) of irreducible gravitational fields, and this condition is simply not verifiable within special relativity itself, because of the circularity in the principle of inertia. This circularity genuinely troubled Einstein, and was one of the major motivations (along with the problem of reconciling mass-energy equivalence with the Equivalence Principle) that led him to abandon special relativity.

Elsewhere in [1], it is asserted that:

> Suppose we naively superimpose special relativity on Newtonian physics, and adopt a naive definition of "inertial worldline", such as a worldline with no locally sensible acceleration. On that basis we find that there can be multiple distinct "inertial" worldlines connecting two given events (e.g., intersecting elliptical orbits of different eccentricities), which conflicts with the special relativistic principle of unique inertial interval between any pair of timelike separated events. ... Of course, with special relativity (as with set theory) we can easily block such specific conundrums — once they are pointed out — by imposing one or more restrictions on the definition of "inertial" (or the definition of a "set"), and in so doing we make the theory somewhat less naive, but the experience raises legitimate questions about whether we can be sure we have blocked all possible escapes.

The author of [1] later comments:

> It might be argued that relativity is a purely formalistic theory, which simply assumes an inertial frame is specified, without telling

> how to identify one. Certainly we can completely insulate special relativity from any and all conflict by simply adopting this strategy, i.e., asserting that special relativity avers no mapping at all between it's elements and the objects of our experience. However, although this strategy effectively blocks conflict, it also renders the theory quite unfalsifiable and phenomenologically otiose.

In this paper, we adopt the strategy suggested in the above quote, that is, we treat special relativity as a "purely formalisitic theory ...". We then argue that the above conclusion of [1], namely, "Certainly we can completely insulate special relativity from any and all conflict by adopting this strategy ...", is questionable. The problem is caused by the fact that in a formalisation in first-order predicate logic of the theory of special relativity (which treats only inertial frames), the self-referential nature of any proposition like "F is an inertial frame in universe U" causes it to be formally undecidable in the theory, assumed consistent. Therefore the consistency of the theory demands the existence of a model in which the negation of the above assertion holds, i.e.,"F is an accelerated frame in universe U". In fact in this model, one can require a broad class of logical consequences of "F is an inertial frame in universe U" to also hold. We conclude with a discussion of the philosophical and mathematical implications of this result for the consistency of the theory of special relativity.

The postulates of special relativity

The emphasis in this paper is on presenting the basic ideas without burying them in a detailed mathematical treatment (which is reserved for future work). The four postulates of the formal theory of special relativity (SR) that will be considered here are adapted from [2], as follows:

P1. The geometry of spacetime.
Statement: "Space and time form an infinite four-dimensional continuum".

P2. The existence of globally inertial frames.
Statement: "There exist global spacetime frames with respect to which unaccelerated objects move in straight lines at constant velocities".

P3. The speed of light is constant.
Statement: "The speed of light c is a universal constant, the same in any inertial frame".

P4. The principle of special relativity.
Statement: "The laws of physics are the same in any inertial frame, regardless of position or velocity".

Note that P2 implies that the geometry of spacetime is Euclidean (flat), "light" in P3 means electromagnetic waves in general (propagating in a vacuum) and "laws" in P4 refer to (a re-formulation of) Newton's laws of mechanics and Maxwell's laws of electromagnetism. We consider a suitable formalization of the theory SR in the classical first-order predicate logic. A non-trivial universe $U(IBC)$ containing a large number of material objects is specified; here the argument IBC represents well-posed initial-boundary conditions consistent with P1–P4 and sufficient for a complete determination of the evolution of $U(IBC)$ in spacetime. One could take $U(IBC)$ to be that part of the "real" universe that is sufficiently far away from massive celestial objects, so that it is essentially free from gravitational fields and spacetime is flat for all practical purposes; the objects in $U(IBC)$ would have to be sufficiently small for this approximation to be valid. Of course, in a purely formal treatment of SR, one need not necessarily have to deal with reality and this is the viewpoint adopted here (as noted in the Introduction). The propositions permitted in SR are all (and only) those relating to the kinematics and dynamics of special relativity as determined by the above postulates and involve both mechanics and electromagnetism. Thus if SR is consistent, a model for SR will assign truth values to all these propositions which will provide a complete and detailed description of the evolution of $U(IBC)$ in spacetime. Note that SR is sufficiently strong for Gödel's incompleteness theorems [3] to apply.

Main results

It is extremely important to observe that P2 not only affirms the existence of globally inertial frames of reference *non-constructively* (i.e., without specifying how to identify an inertial frame); in fact P2 *does not even give a complete, unambiguous definition of an inertial frame.* The reason for this ambiguity is that the term "unaccelerated" in P2 has meaning only *after* the concept of "inertial frames" is in place. So it is not possible to *define* inertial frames by first trying to identify unaccelerated objects. This is also precisely why accelerometers cannot serve the purpose of defining inertial frames from a formal point of view.

Our contention is that the true, constructive definition of inertial frames is in P3 and P4. Note that one could merely state P2 as "There exist globally inertial spacetime frames as defined in P3 and P4"; the straight line motion of unaccelerated objects with respect to inertial frames is Newton's first law and is part of P4. Observe that P1–P4 say nothing at all about the laws of physics in accelerated frames. So an inertial frame cannot be identified non-constructively in SR, i.e., by an argument of the form: "Suppose F were an accelerated frame. Then such-and-such contradiction would arise (such as, an accelerometer giving a non-zero reading). Therefore F is an inertial frame".

Definition (Inertial frame). *The only way to confirm that a given frame F can be inertial in $U(IBC)$ is to first* assume *it to be one and then verify that* P3 *and* P4 *are satisfied globally in spacetime. In other words the proposition P,* *defined as "F is an inertial frame in $U(IBC)$" can be true only if the theory* SR+P *is consistent.*

Note that the above definition asserts the implication

$$P \Rightarrow \text{con}(\text{SR} + P), \tag{1}$$

where con(SR+P) is Gödel's formalization of the proposition "SR+P is consistent". To see why our definition of inertial frame demands (1), note that *any* inconsistency in SR+P would automatically translate into a proof (by contradiction) of $\neg P$ in SR, because such inconsistency would contradict P3 and/or P4. "Inconsistency" here means that SR+P proves both T and $\neg T$ for some legitimate proposition T of SR. If one is able to deduce such an inconsistency in SR+P, the laws of physics certainly do not hold as seen from F. A non-zero reading of an accelerometer from F would imply that P4 does not hold and so would rule out F as an inertial frame; but a zero reading is only one instance of P4 holding from F, and does not rule out the possibility of some other contradiction arising elsewhere in spacetime. So a zero reading of an accelerometer from F does not confirm it as an inertial frame, by our definition.

It is important to recognize that the reverse implication in (1) need not hold — there is nothing in the *formal* theory SR that implies uniqueness of possible inertial frames. In fact if SR+P is consistent then the reverse implication *cannot* hold, as the following metatheorem asserts:

Metatheorem. *If* SR *is consistent, then P is not a theorem of* SR*.*

Proof. Since $\text{con}(\text{SR} + P) \Rightarrow \text{con}(\text{SR})$, it follows from (1) that $P \Rightarrow \text{con}(\text{SR})$. Gödel's second incompleteness theorem then implies that if SR is consistent, P is unprovable in SR. One concludes that if SR is consistent, then there must exist a model for SR in which P is false, i.e., F is an accelerated frame in $U(IBC)$. The metatheorem follows. □

Remark 1. Suppose SR+P is consistent. By our definition of inertial frame, P can be true in SR and there must exist a model M for SR in which F is an inertial frame in $U(IBC)$. By the metatheorem, it follows that one *must also* be able to take P as false, in a model T for SR. One concludes that if SR is consistent, it *cannot* uniquely identify a given frame F as inertial in $U(IBC)$. Of course, one could argue that the evolution of $U(IBC)$ in the models M and T need not be identical and so this result does not necessarily imply that SR is inconsistent. But a stronger result in Remark 2 below seems to increase the probability of an inconsistency in SR. It is interesting to note that in the non-Aristotelian finitary logic (NAFL) proposed by the author [4], this non-uniqueness immediately translates into an inconsistency in SR. So the theory of special relativity cannot be formalized in any consistent NAFL theory.

Remark 2. Suppose SR+P is consistent. Let Q be such that $(Q \Rightarrow P)$ is *not* a theorem of SR; this is possible because P is not a theorem of SR, by the metatheorem. If, in addition, Q is a logical consequence of P, i.e., if $(P \Rightarrow Q)$

is a theorem of SR, then Q must necessarily hold in model M of Remark 1. For example, Q cannot be identical to P, but Q could stand for the null result of an accelerometer experiment performed from F. Q could also be the conjunction of an arbitrarily large (but finite) number of propositions provable in SR+P, subject only to the above restriction. It follows that there must exist a model N for SR in which Q is true and P is false.

Remark 3. Some care would be needed in reformulating IBC and Q so as to hold in the models T and N of Remarks 1 and 2. This is best illustrated with a simple example. Suppose A and B are two bodies undergoing an initial relative acceleration with respect to each other. Now IBC for the model M of Remark 1 might be that A is inertial and B is accelerating with respect to it, i.e., there is a force on B. When reformulating IBC to hold in T or N, where, for instance, B is an inertial frame, obviously the force will now act on A. Note that force is simply an imposed acceleration and this will obviously change if inertial frames are switched; this does not mean that IBC has been invalidated in these models. What should not change, if one wishes to retain the same IBC, is the *relative* acceleration between A and B. To summarize, IBC should be specified in such a way that $(IBC \Rightarrow P)$ is not be provable in SR (as in the case of Q). Of course, similar considerations apply when specifying Q in model N.

Discussion

For appropriate P, Q and $U(IBC)$, any proof in SR that the model N of Remark 2 cannot exist implies that SR is inconsistent. In the model N, let the appropriate inertial frame be G, which must be different from F. Not only must IBC be satisfied in N; the requirement that Q must also hold heavily over-determines the governing differential equations of the system and the genuine question that arises is whether N can exist at all, given that Q can be chosen to be an arbitrary logical consequence of P. This is a mathematical issue that may be difficult to settle. As an example, suppose that the "real" world we live in is free from gravitational fields, so that SR applies. No matter how many facts we collect in support of the conjecture P, the conjunction of all these facts (i.e., Q) must hold in a model N for SR in which P is false, assuming SR to be consistent. In other words, no matter how many experiments (numerical simulations) support our conjecture that F is an inertial frame in our universe, the proposition that "there exists some experiment (numerical simulation) whose results will violate this conjecture" cannot be refuted in SR (assumed consistent). Note that this proposition asserts the indicated existence non-constructively; this is permitted in classical logic. This is very similar to Gödel's assertion that con(SR) is unprovable in SR (assumed consistent), and so there must exist a model for SR in which con(SR) is false.

Philosophically, one could ask whether classical logic is indeed the appropriate logic in which to formulate SR. Is it possible to avoid the above difficulty by formulating SR in intuitionistic or constructivistic logic, where non-constructive existence would be banned? Note that the paradox is caused by the self-referential nature of the definition of "inertial frames" and "acceleration". Is it valid to assert that an object is "really" accelerating with respect to the "truly" inertial frame, when the theory SR fails to prove such an assertion? In the logic NAFL proposed by the author [4], the answer is an emphatic "no"; the law of the excluded middle does not apply to a proposition undecidable in the theory in which it is formulated, and any "reality" independent of the human mind does not exist.

A particularly interesting consequence of this paper is for the well-known twin paradox [5]. The twins A and B are together at some particular instant and are of identical age; B then takes off on a round trip consisting of straight-line motion away from A and back. Let P denote "A is attached to an inertial frame in $U(IBC)$", and let Q stand for "B is younger than A when they meet again after the round trip by B". Then by Remark 2, there must exist a model N for SR (assumed consistent) in which P is false and Q is true for an *identical* round trip by B in (observably) identical conditions. What would make this paradox even more interesting is if we posit that the two twins are the *only* objects in $U(IBC)$. Of course, the first issue would be whether Gödel's second incompleteness theorem, used in our result, would apply in this case. Regardless of the answer, it is our contention that SR still fails to identify an inertial frame uniquely; the model N must still exist, if SR is consistent. If A and B are the only two identical material objects in $U(IBC)$, it will be impossible *in principle* to identify which of the two is "really" accelerating. Neither A nor B can be attached to an inertial frame in N, and we would have to point to empty space to posit an inertial frame. The question is, does this not amount to postulating absolute space, denied in SR? It should be emphasized that these paradoxes arise only in the theory SR as formulated in this paper; they may possibly be satisfactorily resolved in other formulations of special relativity in which accelerated frames of reference are treated.

Dedication

The author dedicates this research to his son R. Anand and wife R. Jayanti.

References

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