Classical nonlocal models for quantum field states

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Abstract

Classical nonlocal field models consisting of probability density functionals over functions defined everywhere on Minkowski space are constructed directly from a quantum field state, using functional methods.

1 Introduction

This paper takes a relativistically local classical model for quantum field theory not to be possible. Obviously we then have the choice of abandoning classical models or considering what relativistically nonlocal classical models are possible. We will here construct classical probability density functionals over a classical field defined everywhere on Minkowski space that preserve relativistic signal locality and are relativistically covariant despite being relativistically nonlocal.

We will adopt an interpretation of quantum field theories as quantizations of field theories in the first instance, rather than as second quantized particle theories; the emergence of particles is taken as secondary. We will reproduce all field configuration observables of the quantum field at a single time and all combinations of such field observables at space-like separation, but we will not reproduce any field momentum observables or combinations of field configuration observables which do not commute because they are at time-like separation. The Kochen-Specker paradox prevents a classical model reproducing states over the quantum algebra of observables of a quantum field in every detail.

The approach of this paper is to construct, in section 2, a probability density functional over functions defined everywhere on Minkowski space, which describes a classical dynamics unconventionally through a description of 4-dimensional trajectories, by taking the inverse fourier transform of a c-number functional constructed as an expectation value from the quantum state. Everything else in this paper just tries to get some understanding of quantum field theory by pursuing the consequences of this construction.

This paper is offered only as **a** way of understanding quantum field theory in more-or-less classical terms. It offers some insight, perhaps particularly where particle oriented interpretations have found difficulties, but more empirical or even instrumentalist interpretations are in some ways preferable.

In section 3, classical models for states other than a vacuum are constructed, then section 4 takes a C-$*$ algebra approach to the construction of a classical probabilistic description from a quantum field. Turning to interpretation, section 5 discusses measurement, section 6 discusses the classical acceptability of the nonlocality as it appears in the models constructed here, then section 7 concludes.

2 Constructing a classical model

The starting point for this construction is to take the c-number functional

$$Q_\psi[f] = \langle\psi| \, e^{i\hat{\phi}_f} \, |\psi\rangle ,$$

where

$$\hat{\phi}_f = \int \hat{\phi}(x) f(x) \mathrm{d}^4 x$$

is a smeared field operator, to be the characteristic functional of a probability density functional $\rho_\psi[w]$. We can then construct $\rho_\psi[w]$ directly from the quantum state by taking the inverse fourier transform,

$$\rho_\psi[w] = \int \check{\mathcal{D}} f e^{-i\int f(x)w(x)\mathrm{d}^4 x} Q_\psi[f].$$

If this exists, there is a marginal probability density functional of $\rho_\psi[w]$ that corresponds to and is equal to each probability density functional that can be constructed from commuting sets of field observables — that is, can be constructed using mutually commuting $\hat{\phi}_f$, without using momentum observables $\hat{\pi}_f$.

A paradigm case of a set of mutually commuting field observables is obtained when we restrict functions f' to be defined on a space-like hyperplane S. Then,

$$\rho'_\psi[v] = \int \check{\mathcal{D}} f' e^{-i\int f'(x)v(x)\mathrm{d}^3x} \langle\psi| e^{i\hat{\phi}_{f'}} |\psi\rangle$$

is manifestly a probability density functional, since $\{\hat{\phi}_{f'}\}$ is effectively a set of classical commuting observables. It is also manifest in this case, by taking $f(x) = 0$ when $x \notin S$ in the fourier transform $Q_\psi[f]$, that the marginal probability density functional constructed for functions defined on S from $\rho_\psi[w]$ is $\rho'_\psi[v]$.

The results of an experiment can be described in terms of commuting field observables of a macroscopic apparatus without using field momentum observables (ultimately, as the position of ink on paper), so, for a macroscopic apparatus, $\rho_\psi[w]$ is as empirically adequate as a quantum field state.

For the quantized real Klein-Gordon field (called here QKG), the algebraic structure of the field is specified by the commutation relation $[a_f^\dagger, a_g] = \hbar(f,g)$, where $a_f^\dagger$ and a_f are creation and annihilation components of the QKG field, $\hat{\phi}_f = a_f^\dagger + a_f$, and (f,g) is a Lorentz covariant positive semi-definite inner product,

$$\begin{aligned}
(f,g) &= \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{\tilde{f}^*(k)\tilde{g}(k)}{2\sqrt{k^2+m^2}} \\
&= \int \frac{\mathrm{d}^4k}{(2\pi)^4} 2\pi\delta(k_\mu k^\mu - m^2)\theta(k_0)\tilde{f}^*(k)\tilde{g}(k).
\end{aligned}$$

A 3-dimensional inverse fourier transform for the QKG vacuum does exist,

$$\begin{aligned}
\rho'_0[v] &= \int \check{\mathcal{D}} f e^{-i\int f(x)v(x)\mathrm{d}^3x} \langle 0| e^{i\hat{\phi}_f} |0\rangle \\
&= \int \check{\mathcal{D}} f e^{-i\int f(x)v(x)\mathrm{d}^3x} \langle 0| e^{ia_f^\dagger} e^{-\frac{1}{2}\hbar(f,f)} e^{ia_f} |0\rangle \\
&= \int \check{\mathcal{D}} f e^{-i\int f(x)w(x)\mathrm{d}^3x} e^{-\frac{1}{2}\hbar(f,f)}
\end{aligned}$$

$$\stackrel{\mathrm{N}}{=} \quad \exp\left[-\frac{1}{\hbar}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\tilde{v}^*(k)\sqrt{k^2+m^2}\tilde{v}(k)\right],$$

where $\stackrel{\mathrm{N}}{=}$ represents equality up to normalization. The fourier-mode kernel $\sqrt{k^2+m^2}$ is nonlocal; $\rho_0'[v]$ can be converted to a nonlocal real-space description,

$$\rho_0'[v] \stackrel{\mathrm{N}}{=} \exp\left[-\frac{1}{\hbar}\iint d^3x d^3y v(x)\frac{m^2K_2(m|x-y|)}{\sqrt{\frac{\pi}{2}}|x-y|^2}v(y)\right],$$

where $K_2(m|x-y|)$ is a modified Bessel function. Unfortunately, a 4-dimensional inverse fourier transform for the QKG vacuum is **not** obviously well-defined,

$$\begin{aligned}\int \check{\mathcal{D}}f e^{-i\int f(x)w(x)\mathrm{d}^4x}\langle 0|\, e^{i\hat{\phi}_f}\,|0\rangle &= \int \check{\mathcal{D}}f e^{-i\int f(x)w(x)\mathrm{d}^4x}e^{-\frac{1}{2}\hbar(f,f)} \\ &\stackrel{\mathrm{N}}{=} \exp\left[-\frac{1}{2\hbar}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\tilde{w}^*(k)\tilde{w}(k)}{2\pi\delta(k_\mu k^\mu - m^2)\theta(k_0)}\right]!\end{aligned}$$

For a modified quantized real Klein-Gordon field (mQKG), however, with the Lorentz covariant inner product

$$(f,g) = \int\frac{\mathrm{d}^4k}{(2\pi)^4}2\pi F(k_\mu k^\mu)\theta(k_0)\tilde{f}^*(k)\tilde{g}(k),$$

where $F(\cdot)$ is a positive semi-definite function (that is, no longer a distribution) of measure 1, and $F(x) > 0$ only if $x \geq 0$, we obtain

$$\rho_0[w] \stackrel{\mathrm{N}}{=} \exp\left[-\frac{1}{2\hbar}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\tilde{w}^*(k)\tilde{w}(k)}{2\pi F(k_\mu k^\mu)\theta(k_0)}\right]$$

for the mQKG vacuum, which is well-defined (or, rather, see Appendix A for how it can be made well-defined).

QKG is in this approach a singular, and not obviously well-defined, limit of mQKG. If we regard QKG as only an effective field theory, however, we can equally effectively describe a system using mQKG, provided $F(\cdot)$ is as small off mass-shell as is necessary to reproduce results of experiments. In general, quantum field theories which are delta-function concentrated to on mass-shell will be singular limits of quantum field theories like mQKG.

3 Models for other mQKG states

We can construct probability density functionals straightforwardly for arbitrary mQKG states in a Fock space generated from the vacuum. For the mQKG states $a_g^\dagger\left|0\right\rangle$, $a_g^\dagger a_g^\dagger\left|0\right\rangle$ and $a_g^\dagger a_g^\dagger a_g^\dagger\left|0\right\rangle$, for example, we obtain

$$
\begin{aligned}
\rho_1[w] &= \int \check{\mathcal{D}}f e^{-i\int f(x)w(x)\mathrm{d}^4x}\left\langle 0\right| a_g e^{i\hat{\phi}_f} a_g^\dagger \left|0\right\rangle \\
&= \int \check{\mathcal{D}}f e^{-i\int f(x)w(x)\mathrm{d}^4x}\left[1-\frac{\hbar|(g,f)|^2}{(g,g)}\right] e^{-\frac{1}{2}\hbar(f,f)} \\
&\stackrel{\mathrm{N}}{=} \left[\int g(x)w(x)\mathrm{d}^4x\right]^2 \rho_0[w], \\
\rho_2[w] &= \int \check{\mathcal{D}}f e^{-i\int f(x)w(x)\mathrm{d}^4x}\left[1-2\frac{\hbar|(g,f)|^2}{(g,g)}+\frac{\hbar^2|(g,f)|^4}{2(g,g)^2}\right] e^{-\frac{1}{2}\hbar(f,f)} \\
&\stackrel{\mathrm{N}}{=} \left[\left[\int g(x)w(x)\mathrm{d}^4x\right]^2-\hbar(g,g)\right]^2 \rho_0[w], \\
\rho_3[w] &\stackrel{\mathrm{N}}{=} \left[\int g(x)w(x)\mathrm{d}^4x\right]^2\left[\left[\int g(x)w(x)\mathrm{d}^4x\right]^2-3\hbar(g,g)\right]^2 \rho_0[w];
\end{aligned}
$$

for the coherent state $\exp(a_g^\dagger)\left|0\right\rangle$ we obtain

$$\rho_c[w] \stackrel{\mathrm{N}}{=} \exp\left[\int g(x)w(x)\mathrm{d}^4x\right]\rho_0[w];$$

and for the superposition $(v+ua_g^\dagger)\left|0\right\rangle$ we obtain

$$\rho_s[w] \stackrel{\mathrm{N}}{=} \left|v+u\int g(x)w(x)\mathrm{d}^4x\right|^2\rho_0[w].$$

In general, quantum states in the Fock space of mQKG will result in the vacuum probability density $\rho_0[w]$ multiplied by a positive multinomial in terms $\int g_i(x)w(x)d^3x$, for a finite set of functions g_i (or, more generally, the closure of such multinomials that is induced by closure in the Fock space norm). The exponential quadratic term $\rho_0[w]$ will dominate the functions which multiply $\rho_0[w]$. Thermal and other states not in the Fock space will include terms that may not necessarily be dominated by $\rho_0[w]$. Note that the constructed multinomials for straightforward quantum field states are independent of the mass distribution function $F(\cdot)$, which appears only in $\rho_0[w]$.

There are no particles as such in this field approach, but there is a countable basis for the Fock space, which can lead to the conventional particle interpretation. The set of all probability density functionals, including thermal states, for example, with different boundary conditions at infinity, does not have a countable basis associated with it, however. A particle interpretation for quantum field theory is not possible in general, when not only Fock space representations are considered. The Unruh effect, which in the approach of this paper is a straightforward consequence of a non-Lorentz transformation of the exponent in $\rho_0[w]$, is typically considered especially problematic for a particle interpretation of quantum field theory.

The last two probability density functionals, $\rho_c[w]$ and $\rho_s[w]$, give a classical understanding of a quantum superposition, even when a state is not an eigenstate of the number operator. The interference which arises for the state $(v + ua_g^\dagger)\left|0\right\rangle$, for example, is a result of the linear term in the positive semi-definite quadratic form

$$\frac{\rho_s[w]}{\rho_0[w]} \stackrel{\mathrm{N}}{=} |v|^2 + \underline{(v^*u + u^*v)\int g(x)w(x)\mathrm{d}^4x} + |u|^2\left[\int g(x)w(x)\mathrm{d}^4x\right]^2,$$

which can be understood without an appeal to an intrinsic complex structure.

$\rho_\psi[w]$ constructed in this way will always be a probability density functional (see, for example, Cohen[1, 2], extending a result of Khinchin). We have explicitly constructed $\rho_1[w]$, $\rho_2[w]$, $\rho_3[w]$, $\rho_c[w]$, and $\rho_s[w]$ and found them to be positive definite. Note, however, that the construction we have given for $\rho_\psi[w]$, although a natural choice in the coordinate structure of fields that we have used implicitly to describe the inverse fourier transform, is not unique (again, see Cohen[1, 2], and also section 4 below).

Note that the perturbation theory of this classical model will be identical to the perturbation theory of mQKG, since the correlation functions of the classical vacuum are identical to the Feynman propagator of mQKG, giving rise to the same Feynman diagram rules.

4 A C-$*$ algebra approach

The construction above can be discussed in terms of C-$*$ algebras. We can generate a C-$*$ algebra from a set of bounded operators constructed using smeared quantum field operators,

$$\mathcal{A}_Q = \text{the C-}{*}\text{ algebra generated by } e^{i\hat{\phi}_f},$$

and generate a second C-$*$ algebra from a set of bounded operators constructed using a set of classical observables,

$$\mathcal{A}_C = \text{the C-}{*}\text{ algebra generated by } e^{i\check{\phi}_f},$$

where $\check{\phi}_f$ is a classical operator valued distribution smeared by the test function f. $\check{\phi}_f$ commutes with $\check{\phi}_g$ for all test functions f and g, in contrast to the nontrivial commutation relations for $\hat{\phi}_f$.

There is a natural 1-1 correspondence between the generating elements of $\mathcal{A}_C$ and the generating elements of $\mathcal{A}_Q$, which generates a 1-1 correspondence $\Xi : \mathcal{A}_C \to \mathcal{A}_Q; e^{i\check{\phi}_f} \mapsto e^{i\hat{\phi}_f}$, as vector spaces. The probability density $\rho_\omega[w]$ is the extension of the state over $\mathcal{A}_C$ generated by a state over $\mathcal{A}_Q$

$$\omega_C(\check{O}) = \omega_Q(\hat{O}), \qquad \forall \check{O} \in \mathcal{A}_C, \quad \hat{O} = \Xi(\check{O}), \tag{1}$$

to the full algebra of (unbounded) classical observables generated by $\check{\phi}_f$. The nonuniqueness of the extension is apparent in the C-$*$ algebra formalism, in contrast to the seeming uniqueness of the inverse fourier transform. Equally, however, if we generate a quantum field state as an extension of a state over $\mathcal{A}_Q$ generated by a state over $\mathcal{A}_C$, we would take the quantum field state to be nonunique.

Both quantum field states and probability density functionals over functions defined on Minkowski space go far beyond the empirical evidence we can accumulate, so we should not take either too seriously, except as particular intuitively and empirically effective models. If we nonetheless decide to take quantum field states as fundamental, we can only work with the classical models of this paper if we gloss the nonuniqueness of the probability density functionals we generate.

5 Measurement

The difference between classical measurement and quantum measurement is that classical measurement is **non**-disturbing, whereas quantum measurement is **disturbing**. Despite the difference in units and associated functional forms, Planck's constant of action plays a very similar role in $\rho_0[w]$ to the role played by the Boltzmann energy $\mathsf{k}T$ in a Gibbs probability density $\exp[-H[v]/\mathsf{k}T]$. Both determine the amplitude of fluctuations. We have to be careful to remember the difference between the Euclidean symmetry of an equilibrium state and the Poincaré symmetry of the quantum field theory vacuum, but the Boltzmann energy and Planck's constant are nonetheless closely analogous in their effect.

From a classical point of view, a real measurement device, as part of the quantum world, inescapably has "q-temperature" $\hbar$, so it does disturb the measured system. We have no way to "q-refrigerate" a measurement device. This doesn't prevent us from imagining and discussing an ideal classical measurement of a system, however. Our construction of a classical probability density obtains the same classical measurement result on any hyperplane as would be obtained by a quantum measurement, but without disturbing the system, so that we can discuss probabilities of joint measurements at time-like separation. It is best to remember that we can only imagine and discuss an ideal quantum measurement, particularly in the context of quantum field theory, so the empirical credentials of quantum theory should not be taken too seriously. The ideal measurements of a theory serve as starting points for a description, never entirely accurate, of a real measurement.

Historically, many physicists thought in terms of this kind of classical measurement model for quantum theory, until the Bohr-Einstein debate focussed on the EPR experiment and it was insisted that relativistic locality is necessary in classical physics. If that insistence is relaxed a little, to require only signal locality and relativistic covariance, we can return to something close to the old understanding, albeit a little wiser for the intervening years.

On a naïve view of probability, we need an ensemble of Minkowski spaces for our 4-dimensional construction of $\rho_\psi[w]$ to make sense, which is a point of view very close to Everettian interpretations of quantum theory. Similar worries have never stopped us from using classical statistical fields as effective models, however. We can calculate interesting properties of simple models, which we then relate to much more complex experimental apparatuses and measured systems in nontrivial ways, without ever modelling the experimental apparatus precisely. We can insist that the world is really a model of quantum theory if we want, perhaps including a many-worlds interpretation of probability, but we don't have to, and on our past experience of physical theory we would be wrong to.

6 Nonlocality

The dynamical nonlocality of the classical models we have constructed is manifest in the nonlocal properties of the fourier mode operator $\tilde{f}(k) \rightarrow \sqrt{k^2+m^2}\tilde{f}(k)$, which extend to mQKG (further to the real-space description given in section 2, the nonlocal properties of $\tilde{f}(k) \rightarrow \sqrt{k^2+m^2}\tilde{f}(k)$ are also described by Segal and Goodman[3]). This nonlocality, however, is qualitatively the same as the nonlocality of the heat equation in classical physics, in that it has exponentially reducing effects at increasing distance, so it is broadly acceptable as pre-relativistic classical physics. Signal locality holds for the classical nonlocal models we have constructed, because of the signal locality of states of QKG, and the classical nonlocal models we have constructed are also described in a relativistically covariant way, so the nonlocality should also be acceptable as post-relativistic classical physics.

The violation of Bell inequalities is rather different. A classical model constructed from an mQKG model that describes an apparatus which exhibits violations of a Bell inequality would essentially be a local beables model, in Bell's terminology[4, 5], despite the above paragraph, insofar as only time-like fourier modes have non-zero probability. In such models, consequently, the classical "explanation" for the violation has to be taken to be one of a "conspiracy" of initial conditions, as Bell pejoratively describes it, but we can more equably describe it as kinematical nonlocality in contrast to dynamical nonlocality. The step from an mQKG state to a classical state is mathematically so direct that if an mQKG description of an experiment is deemed acceptable, then so, it would seem, should be the classical equivalent. As an interaction-free theory, mQKG is not adequate to describe a classical apparatus, so discussion of Bell inequalities from the classical perspective of this paper is not yet properly possible. Taking the inverse fourier transform of $Q_\psi[f]$ works as a general method, however, straightforwardly for bosonic fields and without major difficulty for fermion fields (see [6] for an approach to fermion fields).

There is a relationship between the models constructed here and de Bröglie-Bohm models for quantum field theory, simply because for both the configuration space is the degrees of freedom of a classical field, which in principle leads to a probability density over trajectories of the de Bröglie-Bohm field analogous to $\rho_\psi[w]$. The more-or-less thermal nonlocality of the classical statistical field theory adopted here, however, seems a preferable description to the classically unusual nonlocality of the quantum potential in de Bröglie-Bohm approaches.

7 Conclusion

We can understand mQKG moderately well in terms of classical fields, and we can understand QKG, rather less well, as a singular limit of mQKG. Much more detail is required before we can claim an understanding in these terms of fermion fields or of gauge fields, but for at least some quantum fields our classical intuition need not be perplexed.

The classicality of the models in this paper will be relatively weak for some tastes. The models we have constructed are rather beyond conventional classical mechanics, particularly because the probability density functionals $\rho_\psi[w]$ we have constructed over 4-dimensional functions are not equivalent to probability density functionals over a classical phase space. The specification of $\rho_0[w]$ is Lorentz invariant, but it is not Lagrangian. It should not be a surprise, however, that classical physics has to be extended a little to equal the descriptive power of quantum field theory; although these models do go beyond conventional classical mechanics, it does not require a very liberal view to accept them as classical, simply because they are just sophisticated probabilities applied to classical fields. Note that the extension of classical physics that is introduced here is different from the extension that is introduced in the construction of Wigner functions: Wigner functions *are* defined over a classical phase space, but as a consequence they are *not* probability densities.

The approach of this paper is effective only for a field theory. The reduction from a relativistic continuum to a non-relativistic finite-dimensional system introduces a nonlocality additional to the dynamical nonlocality of the field; this results in the descriptions of nonlocality given, for example, by the quantum potential of the de Bröglie-Bohm approach in finite dimensions and by the non-Markovian evolution of Nelson's approach, which are difficult to accept from a classical point of view. Quantum field theory is more open to a classical interpretation, when it is taken to be about fields, than is the quantum mechanics of particles.

A probability density functional can be transformed to an accelerating frame of reference, under which the vacuum state becomes a thermal state, or transformed by an arbitrary diffeomorphism. A formalism of probability density functionals is therefore more appropriate for quantum gravity than a Fock space formalism (but not necessarily more appropriate than a treatment of quantum gravity in terms of a type III von Neumann algebra of observables). We can immediately write down an example of a conceptually straightforward generally covariant quantum gravity vacuum:

$$\rho_g[g,w] \stackrel{\mathrm{N}}{=} \delta[G_{\mu\nu}[g] + 8\pi T_{\mu\nu}[g,w]] \exp\left[-\frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{|w(k)|^2}{2\pi F(-\lambda^2(k))\theta(k_0)}\right],$$

where k indexes eigenfunctions of the linear operator $g^{\mu\nu}\nabla_\mu\nabla_\nu$, with eigenvalues $\lambda^2(k)$, and $\delta[...]$ is a delta functional, which selects solutions of the Einstein equation. Making this well-defined, which is beyond the scope of this paper, will require additional restrictions, and it may be intractable, but at least we avoid the unhappy combination of the concepts of general relativity with the usual concepts of quantum theory.

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A Inverse fourier transform of a positive semi-definite Gaussian

In a finite dimensional case, it is well-defined to take the inverse fourier transform of a Gaussian $e^{-q(x)}$, where $q(x)$ is a positive semi-definite quadratic form, since $q(x)$ splits the space $X \ni x$ into orthogonal subspaces $X_0, q(x_0) = 0$, and $X_1, q(x_1) > 0$. For the inverse fourier transform we have

$$\int_X e^{-iy.x} e^{-q(x)} = \int_{X_0} e^{-iy_0.x_0} \int_{X_1} e^{-iy_1.x_1} e^{-q(x_1)} = \delta(y_0) e^{-q^{-1}(y_1)},$$

where the inverse quadratic form q^{-1} exists on X_1. This simple method extends to mQKG, but, given only a definition of $\delta(x)$ as a distribution, it does not extend to QKG. If we define $\delta(x)$ as a Colombeau generalized function[7, 8], this simple method may possibly extend to QKG.

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