# On Tensorial Concomitants and the Non-Existence of a Gravitational Stress-Energy Tensor

Erik Curiel [†]

November, 2010

**ABSTRACT**

The question of the existence of gravitational stress-energy in general relativity has exercised investigators in the field since the inception of the theory. Folklore has it that no adequate definition of a localized gravitational stress-energetic quantity can be given. Most arguments to that effect invoke one version or another of the Principle of Equivalence. I argue that not only are such arguments of necessity vague and hand-waving but, worse, are beside the point and do not address the heart of the issue. Based on an analysis of what it may mean for one tensor to depend in the proper way on another, I prove that, under certain natural conditions, there can be no tensor whose interpretation could be that it represents gravitational stress-energy in general relativity. It follows that gravitational energy, such as it is in general relativity, is necessarily non-local. Along the way, I prove a result of some interest in own right about the structure of the associated jet bundles of the bundle of Lorentz metrics over spacetime.

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[†]I thank Robert Geroch for many stimulating conversations in which the seeds of several of the paper's ideas were germinated and, in some cases, fully cultivated to fruition. I also thank David Malament for helpful conversations on the principle of equivalence and on gravitational energy. Finally, I am grateful to Ted Jacobson for pointing out a lacuna in an earlier version of the proof of theorem 5.3.

# 1 Gravitational Energy in General Relativity

There seems to be in general relativity no satisfactory, localized representation of a quantity whose natural interpretation would be "gravitational (stress-)energy". The only physically unquestionable expressions of energetic quantities associated solely with the "gravitational field" we know of in general relativity are quantities derived by integration over non-trivial volumes in spacetimes satisfying any of a number of special conditions. [1] These quantities, moreover, tend to be non-tensorial in character. In other words, these are strictly non-local quantities, in the precise sense that they are not represented by invariant geometrical objects defined at individual spacetime points (such as tensors or scalars).

This puzzle about the character and status of gravitational energy emerged simultaneously with the discovery of the theory itself. [2] The problems raised by the seeming non-localizability of gravitational energy had a profound, immediate effect on subsequent research. For example, it was directly responsible for Hilbert's request to Noether that she investigate conservation laws in a quite general setting, the work that led to her famous results relating symmetries and conservation laws. [3]

Almost all discussions of gravitational energy in general relativity, however, dating back to those earliest debates, have been plagued by vagueness and lack of precision. The main result of this paper addresses the issue head-on in a precise and rigorous way. Based on an analysis of what it may mean for one tensor to depend in the proper way on another, I prove that, under certain natural conditions, there can be no tensor whose interpretation could be that it represents gravitational stress-energy in general relativity. It follows that gravitational energy, such as it is in general relativity, is necessarily non-local. Along the way, I prove a result of some interest in own right about the structure of the associated first two jet bundles of the bundle of Lorentz metrics over spacetime.

# 2 The Principle of Equivalence: A Bad Argument

The most popular heuristic argument used to attempt to show that gravitational energy either does not exist at all or does exist but cannot be localized invokes the "Principle of Equivalence". Choquet-Bruhat (1983, p. 399), for example, puts the argument like this:

1. Weyl (1921, pp. 271–272) was perhaps the first to grasp this point with real clarity. See also Dirac (1962). Schrödinger (1950, pp. 104–105) gives a particularly clear, concise statement of the relation between the fact that the known energetic, gravitational quantities are non-tensorial and the fact that integration over them can be expected to yield integral conservation laws only under restricted conditions.
2. The first pseudo-tensorial entity proposed to represent gravitational stress-energy dates back to Einstein (1915), the paper in which he first proposed the final form of the theory.
3. See, *e.g.*, Brading (2005).

> This 'non local' character of gravitational energy is in fact obvious from a formulation of the equivalence principle which says that the gravitational field appears as non existent to one observer in free fall. It is, mathematically, a consequence of the fact that the pseudo-riemannian connexion which represents the gravitational field can always be made to vanish along a given curve by a change of coordinates.

Goldberg (1980, pp. 469-70) makes almost exactly the same argument, though he draws the conclusion in a slightly more explicit fashion: [4]

> [I]n Minkowski space any meaningful energy density should be zero. But a general space-time can be made to appear Minkowskian along an arbitrary geodesic. As a result, any nontensorial 'energy density' can be made to be zero along an arbitrary geodesic and, therefore, has no invariant meaning.

Trautman (1976, pp. 135-6) has also made essentially the same argument. In fact, the making of this argument seems to be something of a shared mannerism among physicists who discuss energy in general relativity; it is difficult to find an article on the topic in which it is not at least alluded to. [5]

The argument has a fundamental flaw. It assumes that, if there is such a thing as localized gravitational energy or stress-energy, it can depend only on "first derivatives of the metric"—that those first derivatives encode all information about the "gravitational field" relevant to stress-energy. But that seems wrong on the face of it. If there is such a thing as a localized gravitational energetic quantity, then surely it depends on the curvature of spacetime and not on the affine connection (or, more precisely, it depends on the affine connection at least in so far as it depends on the curvature), for any energy one can envision transferring from the gravitational field to another type of system in a different form (*e.g.*, as heat or a spray of fundamental particles) in general relativity must at bottom be based on geodesic deviation, [6] and so must be determined by the value of the Riemann tensor at a point, not by the value of the affine connection at a point or even along a curve. There is no solution to the Einstein field-equation that corresponds in any natural way to the intuitive Newtonian idea of a constant gravitational field, *i.e.*, one without geodesic deviation; that, however, would be the only sort of field that one could envision even being tempted to ascribe gravitational energy to in the absence of geodesic deviation, and that attribution is problematic even in Newtonian theory. Indeed, a spacetime has no geodesic deviation if and only if it is (locally) isometric to Minkowski spacetime, which we surely want to say is the unique spacetime to have vanishing gravitational energy, if one can make such a statement precise in the first place.

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4. Goldberg's formulation of the argument exhibits a feature common in the many instances of it I have found in the literature, the conclusion that a local gravitational energy *scalar* density does not exist and not that a gravitational stress-energy tensor does not exist. Perhaps one could imagine having a well-defined scalar energy density of a field in the absence of a well-defined stress-energy tensor for that field, though I cannot myself see any way to represent such an idea in general relativity. (Note that if one could, this would appear to be a violation of the thermodynamic principle that all energy is equivalent in character, in the sense that any one form can always in principle be tranformed into any other form, since all other forms have a stress-energy tensor as their fundamental representation.)

5. Bondi (1962), Penrose (1966) and Geroch (1973) are notable exceptions. I take their discussions as models of how one should discuss energetic phenomena in the presence of gravitational fields.

6. Penrose (1966) and Ashtekar and Penrose (1990) implicitly rely on the same idea to very fruitful effect.

An obvious criticism of my response to the standard line, related to a popular refinement of the argument given for the non-existence or non-locality of gravitational energetic quantities, is that it would make gravitational stress-energy depend on second-order partial derivatives of the field potential (the metric, so comprehended by analogy with the potential in Newtonian theory), whereas all other known forms of stress-energy depend only on terms quadratic in the first partial derivatives of the field potential. To be more precise, the argument runs like this:

> One can make precise the sense in which Newtonian gravitational theory is the 'weak-field' limit of general relativity.[7] In this limit, it is clear that the metric field plays roughly the role in general relativity that the scalar potential $\phi$ does in Newtonian theory. In Newtonian theory, bracketing certain technical questions about boundary conditions, there is a more or less well-defined energy density of the gravitational field, proportional to $(\nabla\phi)^2$. One might expect, therefore, based on some sort of continuity argument, or just on the strength of the analogy itself, that any local representation of gravitational energy in general relativity ought to be a "quadratic function of the first partials of the metric".[8] The stress-energy tensor of no other field, moreover, is higher than first-order in the partials of the field potential, so surely gravity cannot be different. No invariant quantity at a point can be constructed using only the first partials of the metric, however, so there can be no scalar or tensorial representation of gravitational energy in general relativity.

(No writer I know makes the argument exactly in this form; it is just the clearest, most concise version I can come up with myself.) As Pauli (1921, p. 178) forcefully argued, however, there can be *no* physical argument against the possibility that gravitational energy depends on second derivatives of the metric; the argument above certainly provides none. Just because the energy of all other known fields have the same form in no way implies that a localized gravitational energy in general relativity, if there is such a thing, ought to have that form as well. Gravity is too different a field from others for such a bare assertion to carry any weight. As I explain in footnote 28, moreover, a proper understanding of tensorial concomitants reveals that an expression linear in second partial derivatives is in the event equivalent in the relevant sense to one quadratic in first order partials. This illustrates how misleading the analogy with Newtonian gravity can be.

## 3 Geometric Fiber Bundles and Concomitants

> The introduction of a coordinate system to geometry is an act of violence.
>
> Hermann Weyl
>
> *Philosophy of Mathematics and Natural Science*

I have argued that, if there is an object that deserves to be thought of as the representation of gravitational stress-energy of in general relativity, then it ought to depend on the Riemann curvature

7. See, *e.g.*, Malament (1986).

8. In this light, it is interesting to note that gravitational energy pseudo-tensors do tend to be quadratic in the first-order partials of the metric.

tensor. Since there is no obvious mathematical sense in which a general mathematical structure can "depend" on a tensor, the first task is to say what exactly this could mean. I will call a mathematical structure on a manifold that depends in the appropriate fashion on another structure on the manifold, or set of others, a *concomitant* of it (or them).

The reason I am inquiring into the possibility of a concomitant in the first place, when the question is the possible existence of a representation of gravitational stress-energy tensor, is a simple one. What is wanted is an expression for gravitational energy that does not depend for its formulation on the particulars of the spacetime, just as the expression for the kinetic energy of a particle in classical physics does not depend on the particular interactions one imagines the particle to be experiencing with its environment, and just as the stress-energy tensor for a Maxwell field can be calculated in any spacetime in which there is a Maxwell field, irrespective of the particulars of the spacetime, in contradistinction to the definitions of all known expressions for gravitational energy in general relativity do (*e.g.*, the ADM mass, which can be defined only in asymptotically flat spacetimes). If there is a well-formed expression for gravitational stress-energy, then one should be able in principle to calculate it whenever there are gravitational phenomena, which is to say, in any spacetime whatsoever—it should be a *function* of some set of geometrical objects associated with the curvature in that spacetime, in some appropriately generalized sense of 'function'. This idea is what a concomitant is supposed to capture.

As near as I can make out, the term 'concomitant' and the general idea of the thing is due to Schouten. [9] The definition Schouten proposed—the only one I know of in the literature—is expressed in terms of coordinates: depending on what sort of concomitant one was dealing with, the components of the object had to satisfy various conditions of covariance under certain classes of coordinate transformations. This makes it not only unwieldy in practice and inelegant, but, more important, it

9. See Schouten (1954, p. 15), though of course he used the German *Komitant*. The specific idea of proving the uniqueness of a tensor that "depends" on another tensor, and satisfies a few collateral conditions, dates back at least to Weyl (1921, pp. 315-18) and Cartan (1922). In fact, Weyl proved that, in any spacetime, the only two-index symmetric covariant tensors one can construct at a point, using only algebraic combinations of the components of the metric and its first two partial derivatives in a coordinate system at that point, that are at most linear in the second derivatives of the metric, are linear combinations of the Ricci curvature tensor, the scalar curvature times the metric and the metric itself. In particular, the only such divergence-free tensors one can construct at a point are linear combinations of the Einstein tensor and the metric with constant coefficients. Using Schouten's definition of a concomitant, Lovelock (1972) proved the following theorem:

> *Let* $(\mathcal{M},\ g_{ab})$ *be a spacetime. In a coordinate neighborhood of a point* $p \in \mathcal{M}$*, let* $\Theta_{\alpha\beta}$ *be the components of a tensor concomitant of* $\{g_{\lambda\mu};\ g_{\lambda\mu,\nu};\ g_{\lambda\mu,\nu\rho}\}$ *such that*
>
> $$\nabla^n \Theta_{nb} = 0.$$
>
> *Then*
>
> $$\Theta_{ab} = rG_{ab} + qg_{ab},$$
>
> *where* $G_{ab}$ *is the Einstein tensor and* $q$ *and* $r$ *are constants.*

This is a much stronger result in several ways than Weyl and Cartan had been able to attain: one has a more generalized notion of concomitant than algebraic combination of coordinate components; one does not demand that $\Theta_{ab}$ be symmetric; and most strikingly, one does not demand that $\Theta_{ab}$ be at most linear in the second-order partial derivatives of the metric components.

makes it difficult to discern what of intrinsic physical significance is encoded in the relation of being a concomitant in particular cases. Schouten's covariance conditions translate into a set of partial differential equations in a particular coordinate system, which even in relatively straightforward cases turn out to be forbiddingly complicated.[10] It is almost impossible to determine anything of the general properties of the set of a particular kind of concomitant of a particular object by looking at these equations. I suspect that it is because these conditions are so complex, difficult and opaque that use is very rarely made of concomitants in arguments about spacetime structure in general relativity. This is a shame, for the idea is, I think, potentially rich, and so calls out for an invariant formulation.

I use the machinery of fiber bundles to characterize the idea of a concomitant in invariant terms. I give a (brief) explicit formulation of the machinery, because the one I rely on is non-standard. (We assume from hereon that all relevant structures, mappings, *etc.*, are smooth. Nothing is lost by the assumption and it will simplify exposition. All constructions and proofs can easily be generalized to the case of topological spaces and continuous structures.)

**Definition 3.1** *A* fiber bundle $\mathfrak{B}$ *is an ordered triplet,* $\mathfrak{B} \equiv (\mathcal{B}, \mathcal{M}, \pi)$*, such that:*

**FB1.** $\mathcal{B}$ *is a differential manifold*

**FB2.** $\mathcal{M}$ *is a differential manifold*

**FB3.** $\pi : \mathcal{B} \to \mathcal{M}$ *is smooth and onto*

**FB4.** *For every* $q, p \in \mathcal{M}$*,* $\pi^{-1}(q)$ *is diffeomorphic to* $\pi^{-1}(p)$ *(as submanifolds of* $\mathcal{B}$*)*

**FB5.** $\mathcal{B}$ *has a locally trivial product structure, in the sense that for each* $q \in \mathcal{M}$ *there is a neighborhood* $U \ni q$ *and a diffeomorphism* $\zeta : \pi^{-1}[U] \to U \times \pi^{-1}(q)$ *such that the action of* $\pi$ *commutes with the action of* $\zeta$ *followed by projection on the first factor.*

$\mathcal{B}$ is the *bundle space*, $\mathcal{M}$ the *base space*, $\pi$ the *projection* and $\pi^{-1}(q)$ the *fiber* over $q$. By a convenient, conventional abuse of terminology, I will sometimes call $\mathcal{B}$ itself 'the fiber bundle' (or 'the bundle' for short). A *cross-section* $\kappa$ is a continuous map from $\mathcal{M}$ into $\mathcal{B}$ such that $\pi(\kappa(q)) = q$, for all $q$ in $\mathcal{M}$.

This definition of a fiber bundle is non-standard in so far as no group action on the fibers is fixed from the start; this implies that no correlation between diffeomorphisms of the base space and diffeomorphisms of the bundle space is fixed.[11] One must fix that explicitly. On the view I advocate, the geometric character of the objects represented by the bundle arises arises not from the group action directly, but only after the explicit fixation of a correlation between diffeomorphisms on the base space with those on the bundle space—only after, that is, one fixes how a diffeomorphism on the base space induces one on the bundle. For example, depending on how one decides that a diffeomorphism on the base space ought to induce a diffeomorphism on the bundle over it whose fibers

10. For a good example of just how hairy these conditions can be, see du Plessis (1969, p. 350) for a complete set written out explicitly in the case of two covariant-index tensorial second-order differential concomitants of a spacetime metric.

11. See, *e.g.*, Steenrod (1951) for the traditional definition and the way that a fixed group action on the fibers induces a correlation between diffeomorphisms on the bundle space and those on the base space.

consist of 1-dimensional vector spaces, one will ascribe to the objects of the bundle the character either of ordinary scalars or of $n$-forms (where $n$ is the dimension of the base space). The idea is that the diffeomorphisms induced on the bundle space then implicitly define the group action on the fibers appropriate for the required sort of object. [12]

I call an appropriate mapping of diffeomorphisms on the base space to those on the bundle space an *induction*. (I give a precise definition in a moment.) In this scheme, therefore, the induction comes first conceptually, and the relation between diffeomorphisms on the base space and those they induce on the bundle serves to fix the fibers as spaces of *geometric objects*, *viz.*, those whose transformative properties are tied directly and intimately to those of the ambient base space. This way of thinking of fiber bundles is perhaps not well suited to the traditional task of classifying bundles, but it turns out to be just the thing on which to base a perspicuous and useful definition of concomitant. Although a diffeomorphism on a base space will naturally induce a unique one on certain types of fiber bundles over it, such as tensor bundles, in general it will not. There is not known, for instance, any natural way to single out a map of diffeomorphisms of the base space into those of a bundle over it whose fibers consist only of spinorial objects. [13] Inductions neatly handle such problematic cases.

I turn now to making this intuitive discussion more precise. A diffeomorphism $\phi^\sharp$ of a bundle space $\mathcal{B}$ is *consistent* with $\phi$, a diffeomorphism of the base space $\mathcal{M}$, if, for all $u \in \mathcal{B}$,

$$\pi(\phi^\sharp(u)) = \phi(\pi(u))$$

For a general bundle, there will be scads of diffeomorphisms consistent with a given diffeomorphism on the base space. A way is needed to fix a unique $\phi^\sharp$ consistent with a $\phi$ so that a few obvious conditions are met. For example, the identity diffeomorphism on $\mathcal{M}$ ought to pick out the identity diffeomorphism on $\mathcal{B}$. More generally, if $\phi$ is a diffeomorphism on $\mathcal{M}$ that is the identity on an open set $O \subset \mathcal{M}$ and differs from the identity outside $O$, it ought to be the case that the mapping picks out a $\phi^\sharp$ that is the identity on $\pi^{-1}[O]$. If this holds, we say that that $\phi^\sharp$ is *strongly consistent* with $\phi$.

Let $\mathfrak{D}_\mathcal{M}$ and $\mathfrak{D}_\mathcal{B}$ be, respectively, the groups of diffeomorphisms on $\mathcal{M}$ and $\mathcal{B}$ to themselves, respectively. Define the set

$$\mathfrak{D}^\sharp_\mathcal{B} = \{\phi^\sharp \in \mathfrak{D}_\mathcal{B} : \exists \phi \in \mathfrak{D}_\mathcal{M} \text{ such that } \phi^\sharp \text{ is strongly consistent with } \phi\}$$

It is simple to show that $\mathfrak{D}^\sharp_\mathcal{B}$ forms a subgroup of $\mathfrak{D}_\mathcal{B}$. This suggests

**Definition 3.2** *An* induction *is an injective homomorphism* $\iota : \mathfrak{D}_\mathcal{M} \to \mathfrak{D}^\sharp_\mathcal{B}$.

$\phi$ will be said to *induce* $\phi^\sharp$ (under $\iota$) if $\iota(\phi) = \phi^\sharp$. [14]

**Definition 3.3** *A* geometric fiber bundle *is an ordered quadruplet* $(\mathcal{B}, \mathcal{M}, \pi, \iota)$ *where*

12. I will not work out the details of how this comes about here, as they are not needed for the arguments of the paper; see Curiel (2009).
13. See, *e.g.*, Penrose and Rindler (1984).
14. In a more thorough treatment, one would characterize the way that the induction fixes a group action on the fibers, but we do not need to go into that for our purposes. Again, see Curiel (2009).

**GFB1**. $(\mathcal{B}, \mathcal{M}, \pi)$ *satisfies FB1-FB5*

**GFB2**. $\iota$ *is an induction*

Geometric fiber bundles are the appropriate spaces to serve as the domains and ranges of concomitant mappings.

Most of the fiber bundles one works with in physics are geometric fiber bundles. A tensor bundle $\mathcal{B}$, for example, is a fiber bundle over a manifold $\mathcal{M}$ each of whose fibers is diffeomorphic to the vector space of tensors of a particular index structure over any point of the manifold; a basis for an atlas is provided by the charts on $\mathcal{B}$ naturally induced from those on $\mathcal{M}$ by the representation of tensors on $\mathcal{M}$ as collections of components in $\mathcal{M}$'s coordinate systems. There is a natural induction in this case, $\iota : \mathfrak{D}_{\mathcal{M}} \to \mathfrak{D}^{\sharp}_{\mathfrak{B}}$, fixed by the pull-back action of a diffeomorphism $\phi$ of tensors on $\mathcal{M}$. It is straightforward to show that $\iota$ so defined is in fact an induction. Spinor bundles provide interesting examples of physically important bundles that have no natural, unique inductions, though there are classes of them.

We are finally in a position to define concomitants. Let $(\mathcal{B}_1, \mathcal{M}, \pi_1, \iota_1)$ and $(\mathcal{B}_2, \mathcal{M}, \pi_2, \iota_2)$ be two geometrical bundles with the same base space. [15]

**Definition 3.4** *A mapping* $\chi : \mathcal{B}_1 \to \mathcal{B}_2$ *is a* concomitant *if*

$$\chi(\iota_1[\phi](u_1)) = \iota_2[\phi](\chi(u_1))$$

*for all* $u_1 \in \mathcal{B}_1$ *and all* $\phi \in \mathfrak{D}_{\mathcal{M}}$.

In intuitive terms, a concomitant is a mapping between bundles that commutes with the action of the induced diffeomorphisms that lend the objects of the bundles their respective geometric characters. It is easy to see that $\chi$ must be fiber-preserving, in the sense that it maps fibers of $\mathcal{B}_1$ to fibers of $\mathcal{B}_2$. This captures the idea that the dependence of the one type of object on the other is strictly local; the respecting of the actions of diffeomorphisms captures the idea that the mapping encodes an invariant relation.

# 4 Jet Bundles and Higher-Order Concomitants

Just as with ordinary functions from one Euclidean space to another, it seems plausible that the dependence encoded in a concomitant from one geometric bundle to another may take into account not only the value of the first geometrical structure at a point of the base space, but also "how that value is changing" in a neighborhood of that point, something like a generalized derivative of a geometrical structure on a manifold. The following construction is meant to capture in a precise sense the idea of a generalized derivative in such a way so as to make it easy to generalize the idea of a concomitant to account for it.

Fix a geometric fibre bundle $(\mathcal{B}, \mathcal{M}, \pi, \iota)$, and the space of its sections $\Gamma[\mathcal{B}]$. Two sections $\gamma, \eta : \mathcal{M} \to \mathcal{B}$ *osculate to first-order* at $p \in \mathcal{M}$ if $T\gamma$ and $T\eta$ (the differentials of the mappings) agree in

15. One can generalize the definition of concomitants to cover the case of bundles over different base spaces, but we do not need this here.

their action on $T_p\mathcal{M}$. (They osculate to zeroth-order at $p$ if they map $p$ to the same point in the domain.) If $(x^i,\, v^\alpha)$ are coordinates at the point $\gamma(q)$ adapted to the bundle structure (as defined by the induction), then a coordinate representation of this relation is:

$$\left.\frac{\partial(v^\alpha \circ \gamma)}{\partial x^i}\right|_q = \left.\frac{\partial(v^\alpha \circ \eta)}{\partial x^i}\right|_q$$

for all $i \leq \dim(\mathcal{M})$ and $\alpha \leq \dim(\pi^{-1}[q])$. This defines an equivalence relation on $\Gamma[\mathcal{B}]$. A *1-jet* with source $q$ and target $\gamma(q)$, written '$j^1_q[\gamma]$', is such an equivalence class. The set of all 1-jets,

$$J^1\mathcal{B} \equiv \bigcup_{q\in\mathcal{M},\gamma\in\Gamma[\mathcal{B}]} j^1_q[\gamma]$$

naturally inherits the structure of a differentiable manifold. Let $(\phi,\, U)$ be an adapted coordinate chart of $\mathcal{B}$ around $\gamma(q)$, with the coordinate functions $(x^i,\, v^\alpha)$. Then the induced coordinate chart on $J^1\mathcal{B}$ is $(\phi^1,\, U^1)$ where

$$U^1 \equiv \{j^1_q[\gamma] \mid \gamma(q) \in U\} \tag{4.1}$$

and the coordinate functions associated with $\phi^1$ are $(x^i,\, v^\alpha,\, v^\alpha_i)$, where

$$\begin{aligned} x^i(j^1_q[\gamma]) &\equiv x^i(q) \\ v^\alpha(j^1_q[\gamma]) &\equiv v^\alpha(\gamma(q)) \\ v^\alpha_i(j^1_q[\gamma]) &\equiv \left.\frac{\partial(v^\alpha \circ \gamma)}{\partial x^i}\right|_q \end{aligned} \tag{4.2}$$

where $\gamma$ is any member of $j^1_q[\gamma]$; this is well defined since all members of $j^1_q[\gamma]$ agree on $\gamma(q)$ and $\left.\frac{\partial(v^\alpha \circ \gamma)}{\partial x^i}\right|_q$ by definition.

One can naturally fibre $J^1\mathcal{B}$ over $\mathcal{M}$. The *source projection* $\sigma^1 : J^1\mathcal{B} \to \mathcal{M}$, defined by

$$\sigma^1(j^1_q[\gamma]) = q$$

gives $J^1\mathcal{B}$ the structure of a bundle space over the base space $\mathcal{M}$, and in this case we write the bundle $(J^1\mathcal{B},\, \mathcal{M},\, \sigma^1)$. A section $\gamma$ of $\mathcal{B}$ naturally gives rise to a section $j^1[\gamma]$ of $J^1\mathcal{B}$, the *first-order prolongation* of that section:

$$j^1[\gamma] : \mathcal{M} \to \bigcup_{q\in\mathcal{M}} j^1_q[\gamma]$$

such that $\sigma_1(j^1[\gamma](q)) = q$. (We assume for the sake of simplicity that global cross-sections exist; the modifications required to treat local cross-sections are trivial.)

The points of $J^1\mathcal{B}$ may be thought of as coordinate-free representations of first-order Taylor expansions of sections of $\mathcal{B}$. To see this, consider the example of the trivial bundle $(\mathcal{B},\, \mathbb{R}^2,\, \pi)$ where $\mathcal{B} \equiv \mathbb{R}^2 \times \mathbb{R}$ and $\pi$ is projection onto the first factor. Fix global coordinates $(x^1,\, x^2,\, v^1)$ on $\mathcal{B}$, so that the induced (global) coordinates on $J^1\mathcal{B}$ are $(x^1,\, x^2,\, v^1,\, v^1_1,\, v^1_2)$. Then for any 1-jet $j^1_q[\gamma]$, define the inhomogenous linear function $\hat{\gamma} : \mathbb{R}^2 \to \mathbb{R}$ by

$$\hat{\gamma}(p) = v^1(\gamma(p)) + v^1_1(j^1_q[\gamma])(p_1 - q_1) + v^1_2(j^1_q[\gamma])(p_2 - q_2)$$

where $\gamma \in j^1_q[\gamma]$, and $p, q \in \mathbb{R}^2$ with respective components $(p_1, p_2)$ and $(q_1, q_2)$. Clearly $\hat{\gamma}$ defines a cross-section of $J^1\mathcal{B}$ first-order osculant to $\gamma$ at $p$ and so is a member of $j^1_q[\gamma]$; indeed, it is the unique globally defined, linear inhomogeneous map with this property.

A *2-jet* is defined similarly, as an equivalence class of sections under the relation of having the same first and second partial-derivatives at a point. More precisely, $\gamma, \eta \in \Gamma[\mathcal{B}]$ *osculate to second order* at $q \in \mathcal{M}$ if $\gamma(q) = \eta(q)$ and

$$\begin{aligned} \left.\frac{\partial(v^\alpha \circ \gamma)}{\partial x^i}\right|_q &= \left.\frac{\partial(v^\alpha \circ \eta)}{\partial x^i}\right|_q \\ \left.\frac{\partial^2(v^\alpha \circ \gamma)}{\partial x^i \partial x^j}\right|_q &= \left.\frac{\partial^2(v^\alpha \circ \eta)}{\partial x^i \partial x^j}\right|_q \end{aligned} \tag{4.3}$$

One then defines $J^2\mathcal{B}$, *et al.*, in the analogous ways. There is a natural projection from $J^2\mathcal{B}$ to $J^1\mathcal{B}$, the *truncation* $\theta^{2,1}$, characterized by "dropping the second-order terms in the Taylor expansion". In general, one has the natural truncation $\theta^{n,m} : J^n\mathcal{B} \to J^m\mathcal{B}$ for all $0 < m < n$.

An important fact for the present goal of defining concomitants is that the jet bundles of a geometric bundle are themselves naturally geometric bundles. Fix a geometric bundle $(\mathcal{B}, \mathcal{M}, \pi, \iota)$ and a diffeomorphism $\phi$ on $\mathcal{M}$. Then $\iota[\phi]$ not only defines an action on points of $\mathcal{B}$, but, as a diffeomorphism itself on $\mathcal{B}$, it naturally defines an action on the cross-sections of $\mathcal{B}$ and thus on the 1-jets. The action can be characterized by the appropriate coordinate transformations of the formulæ (4.2) and (4.3). It is easy to show that the mapping $\iota^1$ so specified from $\mathfrak{D}_\mathcal{M}$ to $\mathfrak{D}^\sharp_{J^1\mathcal{B}}$ is an injective homomorphism and thus itself an induction; therefore, $(J^1\mathcal{B}, \mathcal{M}, \sigma^1, \iota^1)$ is a geometric fiber bundle. One defines inductions for higher-order jet bundles in the same way.

We can now generalize our definition of concomitants. Let $(\mathcal{B}_1, \mathcal{M}, \pi_1, \iota)$ and $(\mathcal{B}_2, \mathcal{M}, \pi_2, \jmath)$ be two geometric fiber bundles over the manifold $\mathcal{M}$.

**Definition 4.1** *An* $n^{\text{th}}$-order concomitant *(n a strictly positive integer) from $\mathcal{B}_1$ to $\mathcal{B}_2$ is a smooth mapping $\chi : J^n\mathcal{B}_1 \to \mathcal{B}_2$ such that*

1. $(\forall u \in J^n\mathcal{B}_1)(\forall \phi \in \mathfrak{A}_\mathcal{M})\ \jmath(\phi)(\chi(u)) = \chi(\iota^n(\phi)(u))$
2. *there is no $(n-1)^{th}$ concomitant $\chi' : J^{n-1}\mathcal{B}_1 \to \mathcal{B}_2$ satisfying*

$$(\forall u \in J^n\mathcal{B}_1)\ \chi(u) = \chi'(\theta^{n,n-1}(u))$$

A zeroth-order concomitant (or just 'concomitant' for short, when no confusion will arise), is defined by 3.4. By another convenient abuse of terminology, I will often refer to the range of the concomitant mapping itself as 'the concomitant' of the domain. It will be of physical interest in §6 to consider the way that concomitants interact with multiplication by a scalar field. (Since we consider in this paper only concomitants of linear and affine objects, multiplication of the object by a scalar field is always defined.) In particular, let us say that a concomitant is *homogeneous of weight w* if for any constant scalar field $\xi$

$$\chi(\iota_1[\phi](\xi u_1)) = \xi^w \iota_2[\phi](\chi(u_1))$$

An important property of concomitants is that, in a limited sense, they are transitive.

**Proposition 4.2** *If $\chi_1 : J^n\mathcal{B}_1 \to \mathcal{B}_2$ is an $n^{th}$-order concomitant and $\chi_2 : \mathcal{B}_2 \to \mathcal{B}_3$ is a smooth mapping, where $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ are geometric bundles over the same base space, then $\chi_2 \circ \chi_1$ is an $n^{th}$-order concomitant if and only if $\chi_2$ is a zeroth-order concomitant.*

This follows easily from the fact that inductions are injective homomorphisms and concomitants respect the fibers.

# 5 Concomitants of the Metric

As a specific example that will be of use in what follows, consider the geometric fiber bundle $(\mathcal{B}_\mathrm{g}, \mathcal{M}, \pi_\mathrm{g}, \iota_\mathrm{g})$, with $\mathcal{M}$ a 4-dimensional, Hausdorff, paracompact, connected, smooth manifold (*i.e.*, a candidate spacetime manifold), the fibers of $\mathcal{B}_\mathrm{g}$ diffeomorphic to the set of Lorentz metrics at each point of $\mathcal{M}$, all of the same signature $(+, -, -, -)$, and $\iota_\mathrm{g}$ the induction defined by the natural pull-back. Since the set of Lorentz metrics in the tangent plane over a point of a 4-dimensional manifold, all of the same signature, is a 10-dimensional manifold,[16] the bundle space $\mathcal{B}_\mathrm{g}$ is a 14-dimensional manifold. A cross-section of this bundle defines a Lorentz metric field on the manifold.

The following proposition precisely captures the statement one sometimes hears that there is no scalar or tensorial quantity one can form depending only on the metric and its first-order partial derivatives at a point of a manifold.

**Proposition 5.1** *There is no first-order concomitant from $\mathcal{B}_g$ to any tensor bundle over $\mathcal{M}$.*

To prove this, it suffices to remark that, given any spacetime $(\mathcal{M}, g_{ab})$ and any two points $p, p' \in \mathcal{M}$, there are coordinate neighborhoods $U$ of $p$ and $U'$ of $p'$ and a diffeomorphism $\phi : \mathcal{M} \to \mathcal{M}$, such that $\phi(p) = p'$, $\phi^\sharp(g'_{ab}) = g_{ab}$ at $p$, and $\phi^\sharp(\partial'_a g_{bc}) = \partial_a g_{bc}$ at $p$, where $\partial_a$ $(\partial'_a)$ is the ordinary derivative operator associated with the coordinate system on $U$ $(U')$, and $\phi^\sharp$ is the map naturally induced by the pull-back action of $\phi$.

This is not to say, however, that no information of interest is contained in $J^1\mathcal{B}_\mathrm{g}$. Indeed, two metrics $g_{ab}$ and $h_{ab}$ are first-order osculant at a point if and only if they have the same associated covariant derivative operator at that point. To see this, first note that, if they osculate to first order at that point, then $\hat{\nabla}_a(g_{bc} - h_{bc}) = 0$ at that point for all derivative operators. Thus, for the derivative operator $\nabla_a$ associated with, say, $g_{ab}$, $\nabla_a(g_{bc} - h_{bc}) = 0$, but $\nabla_a g_{bc} = 0$, so $\nabla_a h_{bc} = 0$ at that point as well. Similarly, if the two metrics are equal and share the same associated derivative operator $\nabla_a$ at a point, then $\hat{\nabla}_a(g_{bc} - h_{bc}) = 0$ at that point for all derivative operators, since their difference will be identically annihilated by $\nabla_a$, and $g_{ab} = h_{ab}$ at the point by assumption. Thus they are first-order osculant at that point and so in the same 1-jet. This proves that all and only geometrically relevant information contained in the 1-jets of Lorentz metrics on $\mathcal{M}$ is encoded in the fiber bundle over spacetime the values of the fibers of which are ordered pairs consisting of a metric and the metric's associated derivative operator at a spacetime point.

The second jet bundle over $\mathcal{B}_\mathrm{g}$ has a similarly interesting structure. Clearly, if two metrics are in the same 2-jet, then they have the same Riemann tensor at the point associated with the 2-jet,

16. In fact, it is diffeomorphic to a connected, convex, open subset—an open cone with vertex at the origin—in $\mathbb{R}^{10}$.

since they have the same partial-derivatives up to second order at the point. Assume now that two metrics are in the same 1-jet and have the same Riemann tensor at the associated spacetime point. If it follows that they are in the same 2-jet, then essentially all and only geometrically relevant information contained in the 2-jets of Lorentz metrics on $\mathcal{M}$ is encoded in the fiber bundle over spacetime the points of the fibers of which are ordered triplets consisting of a metric, the metric's associated derivative operator and the metric's Riemann tensor at a spacetime point. To demonstrate this, it suffices to show that if two Levi-Civita connections agree on their respective Riemann tensors at a point, then the two associated derivative operators are in the same 1-jet of the bundle whose base-space is $\mathcal{M}$ and whose fibers consist of the affine spaces of derivative operators at the points of $\mathcal{M}$ (because they will then agree on the values of first-partial derivatives of their Christoffel symbols at that point in any coordinate system as well as agreeing in the values of the Christoffel symbols themselves, and thus will be in the 2-jet of the same metric at that point).

Assume that, at a point $p$ of spacetime, $g_{ab} = \tilde{g}_{ab}$, $\nabla_a = \tilde{\nabla}_a$ (the respective derivative operators), and $R^a{}_{bcd} = \tilde{R}^a{}_{bcd}$ (the respective Riemann tensors). Let $C^a{}_{bc}$ be the symmetric difference-tensor between $\nabla_a$ and $\tilde{\nabla}_a$, which is itself 0 at $p$ by assumption. Then by definition $\nabla_{[b}\nabla_{c]}\xi^a = R^a{}_{bcn}\xi^n$ for any vector $\xi^a$, and so at $p$ by assumption

$$\begin{aligned} R^c{}_{abn}\xi^n &= \nabla_{[a}\tilde{\nabla}_{b]}\xi^c \\ &= \nabla_a(\nabla_b\xi^c + C^c{}_{bn}\xi^n) - \tilde{\nabla}_b\nabla_a\xi^c \\ &= \nabla_a\nabla_b\xi^c + \nabla_a(C^c{}_{bn}\xi^n) - \nabla_b\nabla_a\xi^c - C^c{}_{bn}\nabla_a\xi^n + C^n{}_{ba}\nabla_n\xi^c \end{aligned}$$

but $\nabla_a\nabla_b\xi^c - \nabla_b\nabla_a\xi^c = R^a{}_{bcn}\xi^n$ and $C^a{}_{bc} = 0$, so expanding the only remaining term gives

$$\xi^n\nabla_a C^c{}_{bn} = 0$$

for arbitrary $\xi^a$ and thus $\nabla_a C^b{}_{cd} = 0$ at $p$; by the analogous computation, $\tilde{\nabla}_a C^b{}_{cd} = 0$ as well. It follows immediately that $\nabla_a$ and $\tilde{\nabla}_a$ are in the same 1-jet over $p$ of the affine bundle of derivative operators over $\mathcal{M}$. We have proved

**Theorem 5.2** *$J^1\mathcal{B}_g$ is naturally diffeomorphic to the geometric fiber bundle over $\mathcal{M}$ whose fibers consist of pairs $(g_{ab},\, \nabla_a)$, where $g_{ab}$ is the value of a Lorentz metric field at a point of $\mathcal{M}$, and $\nabla_a$ is the value of the covariant derivative operator associated with $g_{ab}$ at that point, the induction being defined by the natural pull-back. $J^2\mathcal{B}_g$ is naturally diffeomorphic to the geometric fiber bundle over $\mathcal{M}$ whose fibers consist of triplets $(g_{ab},\, \nabla_a,\, R_{abc}{}^d)$, where $g_{ab}$ is the value of a Lorentz metric field at a point of $\mathcal{M}$, and $\nabla_a$ and $R_{abc}{}^d$ are respectively the covariant derivative operator and the Riemann tensor associated with $g_{ab}$ at that point, the induction being defined by the natural pull-back.*

It follows immediately that there is a first-order concomitant from $\mathcal{B}_g$ to the geometric bundle $(\mathcal{B}_\nabla,\, \mathcal{M},\, \pi_\nabla,\, \iota_\nabla)$ of derivative operators, *viz.*, the mapping that takes each Lorentz metric to its associated derivative operator; likewise, there is a second-order concomitant from $\mathcal{B}_g$ to the geometric bundle $(\mathcal{B}_{\text{Riem}},\, \mathcal{M},\, \pi_{\text{Riem}},\, \iota_{\text{Riem}})$ of tensors with the same index structure and symmetries as the Riemann tensor, *viz.*, the mapping that takes each Lorentz metric to its associated Riemann tensor. (This is the precise sense in which the Riemann tensor associated with a given Lorentz metric is "a function of the metric and its partial derivatives up to second order".) It is easy to see, moreover, that both concomitants are homegeneous of degree 0.

Before moving one, I record a final result about concomitants of the metric. One can form from algebraic combinations of the Riemann tensor and the metric 14 independent scalar invariants,[17] which manifestly are themselves concomitants of the metric of order at least second. (This follows directly from theorem 5.2 and proposition 4.2.) The important point about these scalar invariants for this argument is that the result of taking the variation of any of them with respect to the metric yields a two covariant-index, symmetric, divergence-free tensor that is itself a concomitant of the metric of the same order as the scalar invariant; moreover, all two covariant-index, symmetric, divergence-free, concomitants of the metric, of all orders second and higher, can be derived in this fashion.

To make this claim precise before proving it, let $\mathfrak{S}_{\mathrm{g}}$ be the space of scalar invariants of $R_{abc}{}^{d}$ on $\mathcal{M}$ with associated metric $g_{ab}$, and $g(\lambda)_{ab}$ be a smooth one-parameter family of metrics defined for $\lambda \in (-1, 1)$ such that $g(0)_{ab} = g_{ab}$. I will be interested in the inverse metric $g^{ab}$ and the associated smooth one-parameter family $g(\lambda)^{ab}$; denote $\left.\frac{dg(\lambda)^{ab}}{d\lambda}\right|_{\lambda=0}$ by '$\delta g^{ab}$'. Define the functional $S : \mathfrak{S}_{\mathrm{g}} \to \mathbb{R}$ by

$$S[\Sigma] = \int_{\mathcal{M}} \Sigma\sqrt{-g}\; e_{abcd}$$

for $\Sigma \in \mathfrak{S}_{\mathrm{g}}$, where $e_{abcd}$ is a fixed 4-form (not necessarily the one associated with $g_{ab}$) and $\sqrt{-g}$ is the determinant of $g_{ab}$ in a coordinate system in which the only components of $e_{abcd}$ are 0, 1 or $-1$.[18] Clearly $\left.\frac{dS}{d\lambda}\right|_{\lambda=0}$ exists for all smooth one-parameter families $g(\lambda)_{ab}$, and is independent of the choice of one-parameter family; denote it by '$\delta S$'. Then the *variation of* $\Sigma$ *with respect to* $g^{ab}$ is the unique symmetric tensor field $\Sigma_{ab} \equiv \frac{\delta\Sigma}{\delta g^{ab}}$ such that

$$\delta S = \int_{\mathcal{M}} \Sigma_{mn}\, \delta g^{mn} \sqrt{-g}\; e_{abcd} \tag{5.1}$$

for all smooth one-parameter families $g(\lambda)_{ab}$.

We can now state the first part of the desired result.

**Theorem 5.3** *Fix a metric $g_{ab}$, with its associated derivative operator $\nabla_a$ and Riemann tensor $R^a{}_{bcd}$. All and only two covariant-index, symmetric tensors that are divergence-free with respect to $\nabla_a$ can be derived as the result of taking the variation of a scalar invariant of $R^a{}_{bcd}$ with respect to $g^{ab}$.*

A straightforward calculation shows that the variation of any scalar curvature invariant has vanishing covariant divergence with respect to the derivative operator associated with the given metric. Thus, the variation is a two covariant-index, symmetric tensor divergence-free with respect to the fixed metric. The linearity of the functional guarantees that the tensor fields resulting from varying all

17. *Cf.* Haskins (1902).

18. If $\mathcal{M}$ is not compact, or if the integral does not converge for some other reason, then it ought to be taken over some appropriate compact subset of $\mathcal{M}$, on the boundaries of which $\delta g^{ab}$ is to be set to zero.

scalar curvature invariants with respect to the fixed metric form a real vector space over each point of spacetime. Showing that this vector space has dimension six and that the linear map defined by $\delta S$ is onto will complete the proof, for six is the dimension of the vector space of all two covariant-index, symmetric tensors divergence-free with respect to the fixed metric at a point.

One starts with the 14-dimensional vector space of scalar curvature invariants. A simple computation (Schrödinger 1950, pp. 93*ff.*) shows that, when expressed in an arbitrary coordinate system $x_\mu$, the general covariance of the integral (5.1) implies the existence of four identities among the components of $\Sigma_{ab}$,

$$\frac{\partial \Sigma^\mu{}_\nu}{\partial x_\nu} - \frac{1}{2}\Sigma_{\kappa\lambda}\frac{\partial g_{\kappa\lambda}}{\partial x_\mu} = 0 \tag{5.2}$$

which already restrict the dimension of the space by four. (If one likes, these identities encode the demand that the value of the integral remain invariant under arbitrary changes of the four coordinate-functions, *i.e.*, they encode its diffeomorphism invariance, though they also express the fact that the tensor is identically divergence-free—the two are intimately related.)

Now, we need to determine when the variations of two invariants $\Sigma$ and $\Sigma'$ yield the same tensor. Fix $\Sigma$; if the class of $\Sigma'$ yielding the same tensor as $\Sigma$ under variation forms a four-dimensional space, then it will follow that the space of tensors constructed by such variations has dimension six and that the linear map defined by $\delta S$ is onto. By linearity of $\delta S$, the variation of $\Sigma - \Sigma'$ is just $\Sigma_{ab} - \Sigma'_{ab}$, the difference of the results of varying each independently, and thus itself must identically satisfy equation (5.2), *viz.*,

$$\frac{\partial(\Sigma^\mu{}_\nu - \Sigma'^\mu{}_\nu)}{\partial x_\nu} - \frac{1}{2}(\Sigma_{\kappa\lambda} - \Sigma'_{\kappa\lambda})\frac{\partial g_{\kappa\lambda}}{\partial x_\mu} = 0 \tag{5.3}$$

When $\Sigma_{ab} = \Sigma'_{ab}$, then it follows that

$$\frac{\partial(\Sigma^\mu{}_\nu - \Sigma'^\mu{}_\nu)}{\partial x_\nu} = 0$$

One can thus solve for four components of $\Sigma'_{ab}$, resulting in a reduction of four degrees of freedom. Conversely, if $\Sigma_{ab} - \Sigma'_{ab}$ identically satisfies (5.2), then it follows that $\Sigma_{ab} = \Sigma'_{ab}$. In consequence, all and only two covariant-index, symmetric, divergence-free tensors can be derived by taking the variation of a scalar curvature invariant with respect to the metric, completing the proof.

Now, it is easy to see, moreover, by the linearity of the functional, that the tensor is a concomitant of the associated metric of the same order as the scalar curvature invariant itself. In particular, it follows from theorem 5.2 and proposition 4.2 that a concomitant of the metric of that form will be second order if and only if the scalar curvature invariant one varies is linear in the Riemann tensor (*i.e.*, is a zeroth-order concomitant of the Riemann tensor), and there is essentiallyt only one of those, constant multiples of the Gaussian scalar curvature invariant. All other such concomitants of the metric, therefore, will be of order higher than the second. This completes the desired result:

**Corollary 5.4** *All and only covariant two-index, symmetric, second-order and higher concomitants of a metric, covariantly divergence-free with respect to that metric, can be formed by taking the variation, with respect to the metric, of an associated scalar curvature invariant.*

Thus, the only second-order concomitants of the metric of that form are constant multiples of the Einstein tensor.

## 6 Conditions on a Possible Gravitational Stress-Energy Tensor

We are almost in a position to state and prove the main result of the paper, the nonexistence of a gravitational stress-energy tensor. In order to formulate and prove a result having that proposition as its natural interpretation, one must first lay down some natural conditions on the proposed object, to show that no such object exists satisfying the conditions. In general relativity, the invariant representation of energetic quantities is always in the form of a stress-energy tensor, *viz.*, a two-index, symmetric, divergence-free tensor. Not just any such tensor will do, however, for that gives only the baldest of formal characterizations of it. From a physical point of view, at a minimum the object must have the physical dimension of stress-energy for it to count as a stress-energy tensor. That it have the dimension of stress-energy is what allows one to add two of them together in a physically meaningful way to derive the physical sum of total stress-energy from two different sources. In classical mechanics, for instance, both velocity and spatial position have the form of a three-dimensional vector, and so their formal sum is well defined, but it makes no physical sense to add a velocity to a position because the one has dimension of `length/time` and the other the dimension of `length`. (I will give a precise characterization of "physical dimension" below.)

An essential, defining characteristic of energy in classical physics is its obeying some formulation of the First Law of Thermodynamics. The formulation of the First Law I rely on is somewhat unorthodox: that all forms of stress-energy are in principle ultimately fungible—any form of energy can in principle be transformed into any other form [19]—not necessarily that there is some absolute measure of the total energy contained in a system or set of systems that is constant over time. In more precise terms, this means that all forms of energy must be represented by mathematical structures that allow one to define appropriate operations of addition and subtraction among them, which the canonical form of the stress-energy does allow for. [20] I prefer this formulation of the First Law in general relativity because there will not be in a general cosmological context any well-defined global energetic quantity that one can try to formulate a conservation principle for. In so far as one wants to hold on to some principle like the classical First Law in a relativistic context, therefore, I see no other way of doing it besides formulating it in terms of fungibility. (If one likes, one can take the fungibility condition as a necessary criterion for any more traditional conservation law.) This idea is what the demand that *all* stress-energy tensors, no matter the source, have the same physical dimension is intended to capture. [21]

19. Maxwell (1877, ch. v, §97) makes this point especially clearly.

20. Note that this is a requirement even if one takes a more traditional view of the First Law as making a statement about global conservation of a magnitude measuring a physical quantity.

21. For what it's worth, this conception has strong historical warrant—Einstein (implicitly) used a very similar formulation in one of his first papers laying out and justifying the general theory (Einstein 1916, p. 149):

To sum up, the stress-energy tensor encodes in general relativity all there is to know of ponderable (*i.e.*, non-gravitational) energetic phenomena at a spacetime point:

1. it has 10 components representing with respect to a fixed pseudo-orthonormal frame, say, the 6 components of the classical stress-tensor, the 3 components of linear momentum and the scalar energy density of the ponderable field at that point
2. that it has two covariant indices represents the fact that it defines a linear mapping from timelike vectors at the point ("worldline of an observer") to covectors at that point ("4-momentum covector of the field as measured by that observer"), and so defines a bi-linear mapping from pairs of timelike vectors to a scalar density at that point ("scalar energy density of the field as measured by that observer"), because energetic phenomena, crudely speaking, are marked by the fact that they are quadratic in velocity and momental phenomena linear in velocity
3. that it is symmetric represents, "in the limit of the infinitesimal", the classical principle of the conservation of angular momentum; it also encodes part of the relativistic equivalence of momentum-density flux and scalar energy density
4. that it is covariantly divergence-free represents the fact that, "in the limit of the infinitesimal", the classical principles of energy and linear momentum conservation are obeyed; it also encodes part of the relativistic equivalence of momentum-density flux and scalar energy density
5. the localization of ponderable stress-energy and its invariance as a physical quantity are embodied in the fact that the object representing it is a *tensor*, a multi-linear map acting only on the tangent plane of the point it is associated with
6. finally, the thermodynamic fungibility of energetic phenomena is represented by the fact that the set of stress-energy tensors forms a vector space—the sum and difference of any two is itself a possible stress-energy tensor—all having the same physical dimension

Consequently, the appropriate mathematical representation of localized gravitational stress-energy, if there is such a thing, is a two covariant-index, symmetric, covariantly divergence-free tensor having the physical dimension of stress-energy. (That we demand it be covariantly divergence-free is a delicate matter requiring special treatment, which I give at the end of this section.)

Now, in order to make precise the idea of having the physical dimension of stress-energy, recall that in general relativity all the fundamental units one uses to define stress-energy, namely time, length and mass, can themselves be defined using only the unit of time; these are so-called geometrized units. For time, this is trivially true: stipulate, say, that a time-unit is the time it takes a certain kind of atom to vibrate a certain number of times under certain conditions. A unit of

> It must be admitted that this introduction of the energy-tensor of matter is not justified by the relativity postulate alone. For this reason we have here deduced it from the requirement that the energy of the gravitational field shall act gravitatively in the same way as any other kind of energy.

Møller (1962) also stresses the fact that the formulation of integral conservation laws in general relativity based on pseudo-tensorial quantities depends crucially on the assumption that gravitational energy, such as it is, shares as many properties as possible with the energy of ponderable (*i.e.*, non-gravitational) matter.

length is then defined as that in which light travels *in vacuo* in one time-unit. A unit of mass is defined as that of which two, placed one length-unit apart, will induce in each other by dint of their mutual gravitation alone an acceleration towards each other of one length-unit per time-unit per time-unit.[22] These definitions of the units of mass and length guarantee that they scale in precisely the same manner as the time-unit when new units of time are chosen by multiplying the time-unit by some fixed real number $\lambda^{-\frac{1}{2}}$. (The reason for the inverse square-root will become clear in a moment). Thus, a duration of $t$ time-units would become $t\lambda^{-\frac{1}{2}}$ of the new units; an interval of $d$ units of length would likewise become $d\lambda^{-\frac{1}{2}}$ in the new units, and $m$ units of mass would become $m\lambda^{-\frac{1}{2}}$ of the new units. This justifies treating all three of these units as "the same", and so expressing acceleration, say, in inverse time-units. To multiply the length of all timelike vectors representing an interval of time by $\lambda^{-\frac{1}{2}}$, however, is equivalent to multiplying the metric by $\lambda$ (and so the inverse metric by $\lambda^{-1}$), and indeed such a multiplication is the standard way one represents a change of units in general relativity. This makes physical sense as the way to capture the idea of physical dimension: all physical units, the ones composing the dimension of any physical quantity, are geometrized in general relativity, in the most natural formulation, and so depend only on the scale of the metric itself.

Now, the proper dimension of a stress-energy tensor can be determined by the demand that the Einstein field-equation, $G_{ab} = \gamma T_{ab}$, where $\gamma$ is Newton's gravitational constant, remain satisfied when one rescales the metric by a constant factor. $\gamma$ has dimension $\frac{(\text{length})^3}{(\text{mass})(\text{time})^2}$, and so in geometrized units does not change under a constant rescaling of the metric. Thus $T_{ab}$ ought to transform exactly as $G_{ab}$ under a constant rescaling of the metric. A simple calculation shows that $G_{ab}$ $(= R_{ab} - \frac{1}{2}Rg_{ab})$ remains unchanged under such a rescaling. Thus, a necessary condition for a tensor to represent stress-energy is that it remain unchanged under a constant rescaling of the metric. It follows that the concomitant at issue must be homogeneous of weight 0 in the metric, whatever order it may be.

We must still determine the order of the required concomitant, for it is not *a priori* obvious. In fact, the way a homogeneous concomitant of the metric transforms when the metric is multiplied by a constant factor suffices to fix the differential order of that concomitant.[23] This can be seen as follows, as exemplified by the case of a two covariant-index, homogeneous concomitant $S_{ab}$ of the metric. A simple calculation based on definition 4.1 and on the fact that the concomitant must be homogeneous shows that the value at a point $p \in \mathcal{M}$ of an $n^{\text{th}}$-order concomitant $S_{ab}$ can be written

22. This definition may appear circular, in that it would seem to require a unit of mass in the first place before one could say that bodies were of the *same* mass. I think the circularity can be mitigated by using two bodies for which there are strong prior grounds for positing that they are of equal mass, *e.g.*, two fundamental particles of the same type. It also suffers from a fundamental lack of rigor that the definition of length does not suffer from. In order to make the definition rigorous, one would have to show, *e.g.*, that there exists a solution of the Einstein field-equation (approximately) representing two particles in otherwise empty space (as defined by the form of $T_{ab}$)—*viz.*, two timelike geodesics—such that, if on a spacelike hypersurface at which they both intersect 1 unit of length apart (as defined on the hypersurface with respect to either) they accelerate towards each other (as defined by relative acceleration of the geodesics) one unit length per unit time squared, then the product of the masses of the particles is 1. I will just assume, for the purposes of this paper, that such solutions exist.

23. I thank Robert Geroch for pointing this out to me.

in the general form

$$S_{ab} = \sum_{\alpha} k_{\alpha} g^{qx} \dots g^{xr} \left( \widetilde{\nabla}_x^{(n_1)} g_{qx} \right) \dots \left( \widetilde{\nabla}_x^{(n_i)} g_{xr} \right) \tag{6.1}$$

where: $\widetilde{\nabla}_a$ is any derivative operator at $p$ *other* than the one naturally associated with $g_{ab}$; '$x$' is a dummy abstract index; '$\widetilde{\nabla}_x^{(n_i)}$' stands for $n_i$ iterations of that derivative operator (obviously each with a different abstract index); $\alpha$ takes its values in the set of all permutations of all sets of positive integers $\{n_1, \dots, n_i\}$ that sum to $n$, so $i$ can range in value from 1 to $n$; the exponents of the derivative operators in each summand themselves take their values from $\alpha$, *i.e.*, they are such that $n_1 + \dots + n_i = n$; there is exactly one summand for which $n_1 = n$ (which makes it an $n^{\text{th}}$-order concomitant); for each $\alpha$, $k_\alpha$ is a constant; and there are just enough of the inverse metrics in each summand to contract all the covariant indices but $a$ and $b$.

Now, a combinatorial calculation shows

**Proposition 6.1** *If, for $n \geq 2$, $S_{ab}$ is an $n^{th}$-order homogeneous concomitant of $g_{ab}$, then to rescale the metric by the constant real number $\lambda$ multiplies $S_{ab}$ by $\lambda^{n-2}$.*

In other words, the only such homogeneous $n^{\text{th}}$-order concomitants must be of weight $\lambda - 2$.[24] So if one knew that $S_{ab}$ were multiplied by, say, $\lambda^4$ when the metric was rescaled by $\lambda$, one would know that it had to be a sixth-order concomitant. In particular, $S_{ab}$ does not rescale when $g_{ab} \to \lambda g_{ab}$ only if it is a second-order homogeneous concomitant of $g_{ab}$, *i.e.*, (by theorem 5.2) a zeroth-order concomitant of the Riemann tensor. There follows from proposition 4.2 and corollary 5.4

**Lemma 6.2** *A 2-covariant index concomitant of the Riemann tensor is homogeneous of weight zero if and only if it is a zeroth-order concomitant.*

Thus, such a tensor has the physical dimension of stress-energy if and only if it is a zeroth-order concomitant of the Riemann tensor.

We now address the issue whether it is appropriate to demand of a potential gravitational stress-energy tensor that it be covariantly divergence-free. In general, I think it is not, even though that is one of the defining characteristics of the stress-energy tensor of ponderable matter in the ordinary formulation of general relativity.[25] To see this, let $T_{ab}$ represent the aggregate stress-energy of all ponderable matter fields. Let $S_{ab}$ be the gravitational stress-energy tensor, which we assume for the sake of argument to exist. Now, we ask: can the "gravitational field" interact with ponderable matter fields in such a way that stress-energy is exchanged? If it could, then, presumably, there could be interaction states characterized (in part) jointly by these conditions:

1. $\nabla^n (T_{na} + S_{na}) = 0$

24. Note that the exponent $(n - 2)$ in this result depends crucially on the fact that $S_{ab}$ has only two indices, both covariant. One can generalize the result for tensor concomitants of the metric of any index structure. A slight variation of the argument, moreover, shows that there does not in exist in general a homogeneous concomitant of a given order from a tensor of a given index structure to one of another structure—one may not be able to get the number and type of the indices right by contraction and tensor multiplication alone.

25. I thank David Malament for helping me get straight on this point. The following argument is in part paraphrastically based on a question he posed to me.

2. $\nabla^n T_{na} \neq 0$
3. $\nabla^n S_{na} \neq 0$

The most one can say, therefore, without wading into some very deep and speculative waters about the way that a gravitational stress-energy tensor (if there were such a thing) might enter into the righthand side of the Einstein field-equation, is that we expect such a thing would have vanishing covariant divergence when the aggregate stress-energy tensor of ponderable matter vanishes, *i.e.*, that gravitational stress-energy on its own, when not interacting with ponderable matter, be conserved. This weaker statement will suffice for our purposes, so we can safely avoid those deep waters.

Finally, it seems reasonable to require one more condition: were there a gravitational stress-energy tensor, it should not be zero in any spacetime with non-trivial curvature, for one can always envision the construction of a device to extract energy in the presence of curvature by the use of tidal forces and geodesic deviation. [26]

To sum up, we have the following necessary condition:

**Condition 6.3** *The only viable candidates for a gravitational stress-energy tensor are two covariant-index, symmetric, second-order, zero-weight homogeneous concomitants of the metric that are not zero when the Riemann tensor is not zero and that have vanishing covariant divergence when the stress-energy tensor of ponderable matter vanishes.*

# 7 No Gravitational Stress-Energy Tensor Exists

I now state and prove the main result.

**Theorem 7.1** *The only two covariant-index, symmetric, divergence-free, second-order, zero-weight homogeneous concomitants of the metric are constant multiples of the Einstein tensor.*

The theorem does bear the required natural interpretation, for the Einstein tensor is not an appropriate candidate for the representation of gravitational stress-energy: the Einstein tensor will be zero in a spacetime having a vanishing Ricci tensor but a non-trivial Weyl tensor; such spacetimes, however, can manifest phenomena, *e.g.*, pure gravitational radiation in the absence of ponderable matter, that one naturally wants to say possess gravitational energy in some (necessarily non-localized) form or other. [27] There immediately follows the corollary that precisely captures the condition stated at the end of §6.

**Corollary 7.2** *There are no two covariant-index, symmetric, divergence-free, second-order, homogeneous concomitants of the metric that are not zero when the Riemann tensor is not zero.*

26. See, *e.g.*, Bondi (1962).
27. As an historical aside, it is interesting to note that early in the debate on gravitational energy in general relativity Lorentz (1916) and Levi-Civita (1917) proposed that the Einstein tensor be thought of as the gravitational stress-energy tensor. Einstein criticized the proposal on the grounds that this would result in attributing zero total energy to any closed system.

Now, to prove the theorem, note first that it follows from corollary 5.4 and lemma 6.2, in conjunction with condition 6.3, that any candidate gravitational stress-energy tensor must be a zeroth-order concomitant of $\mathcal{B}_{\text{Riem}}$, the geometric bundle of Riemann tensors over spacetime. (One can take this as a precise statement of the fact that any gravitational stress-energy tensor ought to "depend on the curvature", as I argued in §2.) It then follows from theorem 5.3 that any candidate gravitational stress-energy tensor must be the variation with respect to the metric of a scalar curvature invariant. According to the definition of a homogeneous concomitant, if a scalar curvature invariant that is, *e.g.*, quadratic in the Riemann tensor, that is to say, quadratic in second derivatives of the metric, yields a homogeneous concomitant when its variation is taken, then it is a fourth-order concomitant of the metric. The rule is that the order of a homogeneous concomitant is the sum of the exponents of the derivative operators when the concomitant is represented in the form of equation (6.1).[28] Consequently a scalar curvature invariant is a second-order concomitant of the metric (and thus by lemma 6.2 homogeneous of weight 0)—and so its variation with respect to the metric yields a second-order concomitant of the metric—if and only if it is linear in the Riemann tensor. The only scalar curvature invariants that are linear in the Riemann tensor are constant multiples of the Gaussian scalar curvature itself, $R$, *viz.*, the result of contracting the two indices of the Ricci tensor. The tensor that results from taking the variation of $R$ with respect to the metric is the Einstein tensor, proving the theorem.

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28. This remark, by the way, obviates the criticism of the claim that gravitational stress-energy depend on the curvature, *viz.*, that this would make gravitational stress-energy depend on second-order partial derivatives of the field potential whereas all other known forms of stress-energy depend only on terms quadratic in the first partial derivatives of the field potential. It is exactly second-order, homogeneous concomitants that possess terms quadratic in the first partials.

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