Pseudospherical Junctions or

Josephson Effect, Bäcklund Transformations, and Fine Structure Coupling

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It is shown, that the geometric phase evolution within M circularly and toroidally arranged virtual Josephson junctions (coupled discrete impedance system) can be described by the integrable case of Bäcklund transformations. The phase gradient of a junction is induced by a pseudospherical curvature. The internal phase difference and external bias is mediated by sine-Gordon solitons that provide for internal and external coupling. The idealized soliton resonance or feedback condition corresponds to an oscillator potential (Long Josephson Junction LJJ condition) that can be mapped by projective geometry to Coulomb coupling. The effective coupling strength is a generalized fine structure constant that can be iteratively determined, for $M = 137$ extremely close to measured values of the Sommerfeld fine structure.

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Introduction. There are some interesting relations regarding Baecklund and Darboux transformations and their applications in the theory of integrable systems, also known as soliton theory [1]. The sine-Gordon equation (SG) as the only Lorentz-invariant equation with auto-Baecklund transformations is intimately related to i.e. the Josephson equations describing the Long Josephson Junction (LJJ), see i.e. the work and seminars of Ustinov or [2, 3]. In the previous papers quantum coupling has been defined by geometric phase relationships [4] comparing the evolution of an input to an output phase defining a non-linear dynamic impedance, where the phase is evolving on a curved (pseudospherical) surface [5]. In the classical picture the phase is evolving according to the mechanics of the coupled pendula chain. Any junction, barrier, or boundary can be related to a fundamental geometric origin in terms of curvature. In this paper we will analyze the geometric phase evolution within a circularly arranged system of Josephson junctions, where junction means simply a scalar field phase gradient. Therefore, the "virtual Josephson junction" will be defined by curvature and geometric phase gradient, a synonym for an electromagnetic (gauge) field, characteristic oscillations will be related to a bias voltage.

Topological phase fields. In non-linear optics, the Bloch equation can be used to describe dipole spin precession [7, 8]. In this context a scalar field of precession can be described by a nonlinear and simplified system of Maxwell-Bloch equations that can be reduced to a sine-Gordon soliton equation and related topological phase fields evolving on a pseudospherical surface. In this paper we will define a special Josephson junction arrangement allowing for a toroidal configuration of (subloop) spin currents with precession along the (loop) trajectory, see fig.1. Topological solitons or skyrmions naturally emerge with loop and subloob currents (torque strength) related by Baecklund transformations. The scalar field phase variable θ will allow for a pure geometrical interpretation as a precession cone angle of spin (or subloop/loop combination) or generalized pendulum amplitude. The Maxwell equations determine the strength of the torque vector, electromagnetic fields are treated as generalized precession phase gradients, where a non–trivial Berry phase and effective Yang-Mills field can emerge from the toroidal spin currents. Using Baecklund transformations in this context has some advantages such as

- path integrals are not involved defining geometric phases on curved surfaces in R^3,
- starting from one solution you can get infinitely many more,
- with integrability (spinning top equations are single–valued [9]), Lorentz invariance, and stereographic projection naturally included.

Loop/sub-loop coupling. The curved pseudospherical surface as a nonlinear impedance will represent in this paper a "virtual Josephson junction". The Josephson junctions will be arranged on a subloop in M-gonal symmetry (say M-SQUID), M units of M-SQUIDs will be arranged in a closed loop geometry on the torus, where the global phase evolution of two neighboring M-SQUIDs will be given by Θ, see fig.1. With $\widetilde{\theta}$ as the input phase (the reference) and θ the output phase including geometric phase shift we have within the virtual Josephson junction the two interference terms

- $\Theta = (\theta + \widetilde{\theta})/2$: loop (global phase),
- $\triangle\theta = (\theta - \widetilde{\theta})/2$: subloop (local phase),

where the (geometric) phase evolution related to precession is induced by "parallel transport". Note, that the usual Berry phase [6] φ is related to the precession semicone angle θ on (pseudo)spherical surfaces by $\varphi \propto 1 - \cos\theta$. The coupling function g at the junctions leading to a tunnelling current $I_J(\xi) \propto g(\xi)$ has the following properties:

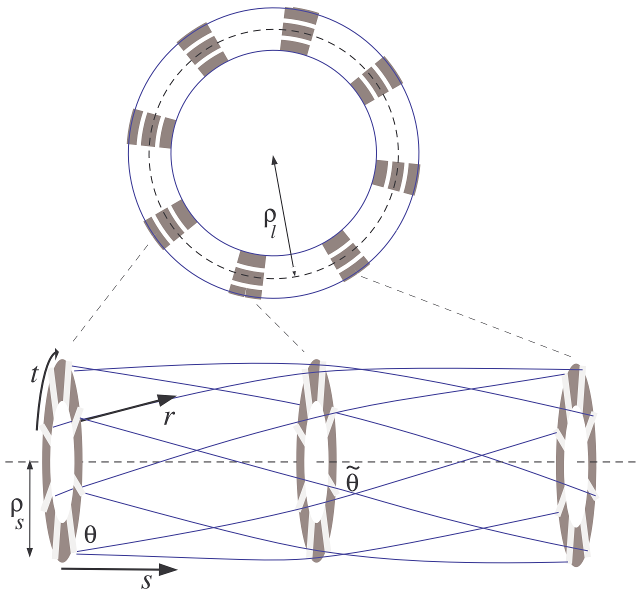

FIG. 1: A coupled chain of M-SQUIDS for $M = 8$, analogue to the coupled pendula chain. Coupling by helicity is given by the scattering resonance condition eq.(5). The junction width (white) depends as usual on frequency and voltage, the tunnelling current is expected to evolve on chiral pseudospherical hasimoto surfaces [1]. The local phase evolution $\triangle\theta_M$ and the global phase evolution Θ_M are related via $\partial_r \triangle\theta_M = -iM^2\partial_t\Theta_M$.

- without phase difference $\xi = 0$ there is no Josephson current $I_J = 0$,
- the current is periodic $I_J(\xi) = I_J(\xi + 2\pi)$,
- the current has symmetry $I_J(-\xi) = -I_J(\xi)$.

There are two types of coupling terms relevant to describe a closed system arranged in regular M-gonal symmetry:

- the angular momentum or circular flux proportional to the spatial $f_r(\xi) = \partial_r(\xi/\pi)$ or temporal evolution $f_t(\xi) = i\partial_t(\xi/\pi)$ of θ,
- and the typical spin-spin vector or pendula coupling term $g(\xi) = M_g \sin(\xi)$ (in sine-Gordon soliton theory given by the pseudospherical curvature).

ξ can either be the phase evolution on the loop with phase Θ or on the subloop with phase $\triangle\theta$. One 2π subloop rotation (within M-SQUID junctions) corresponds to a $2\pi/M$ loop rotation. The combination that provides for a closed–loop momentum exchange of orthogonal spatial (r, loop) and temporal (t, subloop) types of couplings ($r \perp it$) shows two degrees of freedom:

$$\begin{aligned} (r) \quad & Mg(\triangle\theta) = f_r(\Theta), \\ (t) \quad & Mf_t(\triangle\theta) = ig(\Theta). \end{aligned} \tag{1}$$

(r): On the loop the coupling of M units $Mg(\triangle\theta)$ (M twisted lines) provides for the loop phase evolution $f_r(\Theta)$.

(t): The total subloop phase evolution in serial circular arrangement given by $-iMf_t(\triangle\theta)$ provides for the coupling of neighboring SQUIDS on the loop torus given by $g(\Theta)$.

The resulting self-consistent and integrable coupling system are the Bäcklund transformations (BT)

$$\begin{aligned} (\partial_r\widetilde{\theta} + \partial_r\theta)/M &= 2\pi M_g \sin[(\widetilde{\theta} - \theta)/2], \\ -i(\partial_t\widetilde{\theta} - \partial_t\theta)M &= 2\pi M_g \sin[(\widetilde{\theta} + \theta)/2], \end{aligned} \tag{2}$$

a manifestation of integrability [1, 10]. This has two important consequences:

- With i.e. $\partial_t(\partial_r\widetilde{\theta}) = \partial_r(\partial_t\widetilde{\theta})$ the mediator or coupling quanta are soliton solutions of the SG $\partial_{rt}\theta = -R\sin(\theta)/2$,
- the internal and external precession phase offset or potential bias can now be exactly determined.

Note, that the hierarchical loop-subloop system can be easily extended to a fractal tori junction geometry.

Sine-Gordon soliton. Fixing $\widetilde{\theta}$ to a special reference phase evolution given by $\widetilde{\theta} = 4\pi(\frac{1}{2} + n)$ for simplicity, the quantum gauge (or spin) dependent winding number $n = 0, 1, 2, ...$ provides in eq.(2) for the simplification $\partial_r = -iM^2\partial_t$. Parametrizing $\partial_s^2 = \partial_r^2 - \partial_t^2 = (1 + 1/M^4)\partial_r^2$ we have

$$\mp \sqrt{M^2 + 1/M^2}\,\partial_s\theta = 2\pi M_g \sin(\theta/2). \tag{3}$$

The nonlinear SG phase field evolves with a pseudospherical curvature constraint. This property is found with generalized Chebyshev coordinates ($x = r, y = it$) on a plane $\mathcal{S}$ embedded in $\mathbb{R}^3$

$$ds^2 = (dr)^2 - (dt)^2 + 2i\cos\theta drdt \tag{4}$$

with scalar curvature $R = 2R_{1212}/\det(g_{ij})$ [10] of the generalized Chebyshev metric, $\theta = \pi/2$ is the special case of Minkowskian spacetime. There is a clear geometrical interpretation: the coordinate vector field is parallel transported along the signal/soliton vector field with respect to the Levi–Civita connection. A "privileged" surface of scalar curvature $R = -2$ is given i.e. by the Lobachevskian plane. Topological solitons as solutions to the SG are spatially confined (localized), non-dispersive and non-singular solution of a non–linear field theory. In 2+1-dimensional gauge vortex scattering it follows from purely geometric considerations that the head-on scattering of M topological solitons (like monopoles, vortices, skyrmions, ...) distributed symmetrically around the point of scattering (relative angular separations $2\pi/M$) is by an angle π/M, independent of various details of the scattering [13]. In this case the initial configuration has the symmetry group of a regular M-gon, the "moduli space" of M vortices, $\mathcal{M}_M$[14].

Resonance condition. An ideal LJJ Josephson junction showing sine-Gordon dynamics can support periodic motion of magnetic flux quanta. Singularities in the current-voltage characteristic of the junction emit very narrow line width electromagnetic radiation whose frequency is proportional to the geometric length of the junction, see i.e. the work of Ustinov, Grauer, or [2, 3, 11, 12]. This means, that the energy of coupling induced by the offset bias is inversely proportional to the distance between start- and endpoint of resonant transmission on the junction. When an external signal is applied to the junction a new singularity in the current-voltage curve appears, known as phase-locking step: the junction is said to be in resonance with the external signal [12], essentially the phase-locking of a long Josephson junction (LJJ) to an external (virtual) photon. This transmission behavior can effectively be treated by a stereographic projection of the stationary dynamics on the two-dimensional (pseudo)sphere with $PSL(2, \mathbb{R})$ symmetry that connects angular variable and proper distance s [15]. A parabolic potential $V \propto s^2$ is obtained with eq.(3) and the general projective (or rather scattering) condition

$$s = \mp 2\rho \sin(\theta/2). \tag{5}$$

In fig.1 the helical phase evolution on the torus surface is shown. Eq.(5) is the projection that maps the local oscillator potential to the non–local Coulomb potential under stereographic projection given by the conventional Bohlin transformation $s_c \rightarrow s^2$ [15]. This relates the Coulomb system (coupling energy inversely proportional to distance) to the resonance oscillator potential and maps with eq.(3) and eq.(5) the square of the phase gradient to the potential

$$2\pi s_c = \rho^2 \partial_s \theta \rightarrow 4\pi^2 s^2 = \rho^4 (\partial_s \theta)^2. \tag{6}$$

Applying the Euler-Lagrange equation to the SG soliton Lagrangian and correspondent Hamiltonian, the potential for a stationary solution is with $\partial_s^2 \theta \propto \partial_\theta V$ and $V \propto (\partial_s \theta)^2$ in accordance with eq.(6) having parabolic shape providing for resonant coupling by oscillations.

a. Self-energy and coupling energy. The potential is given by

$$\begin{aligned} 2V(\theta) &= (\partial_s \theta / M)^2 &(7)\\ &= 2\pi^2 M_g^2 (1 - \cos\theta), &(8) \end{aligned}$$

where $\cos\theta = 1 - 2\sin^2(\theta/2)$. From eq.(8) the θ-independent self-energy term can be identified as a constant Riemann curvature scalar $R = -2/\rho^2$, with the SG $\pi M_g \rho = 1$. Therefore, it is plausible to decompose energy in eq.(8) into at least two terms: a self-energy term $\pi^2 M_g^2$ and a dynamic coupling term $\pi^2 M_g^2 \cos\theta$ that accounts for the field evolution based on the BT. Integrating eq.(6) provides for $\theta \propto s^2 + c$, where the integration constant c can be obtained by comparing the correspondent parts in eq.(8) and eq.(5). These relations provide for a dynamic coupling term $\mp \pi M_g \sqrt{M^2 + 1/M^2}\theta$ that can be combined with a self-energy term and integration constant to

$$V(\theta) = \pi^2 M_g^2 \mp \pi M_g \theta \sqrt{M^2 + 1/M^2}. \tag{9}$$

With eq.(8) and eq.(9) we immediately obtain an iterative equation for the external phase offset θ_M in resonance

$$\theta_M \sqrt{M^2 + 1/M^2} = \pm \pi M_g \cos\theta_M, \quad \theta_M = \pi M_g \alpha_M \tag{10}$$

where the coupling allows for two possible signs. Consequently, a mutual resonant coupling of two identical junction systems via Coulomb mapping [4] is proportional to θ and α. The iteration eq.(10) is invariant with respect to the inversion and duality $M \leftrightarrow 1/M$. The relations between the electric and magnetic monopole charge $(2ge)^2 = 1$ with $(2g/e)^2 = M^2$, and also between group and phase velocity of a wave packet in the ground state $v_g v_p = 1$ with $v_p/v_g = M^2 = \partial \triangle \theta_M / \partial \Theta_M$ could explain the role of M. The coupling constant and special θ-value or oscillation range is iteratively obtained in eq.(10), where $M = 137$ or $M = 1/137$ from the Dirac theory of magnetic monopoles [16] (generater of the Berry phase [6]) provides with $M_g = 1$ for $1/\alpha = 137.03600960$ that fits within some ppb's to the Sommerfeld fine structure constant obtained from frequency-voltage relations of a Josephson junction or in neutron interferometry. The meaning of the number 137 remains unclear. Eq.(10) is an chaotic algorithm, bifurcation starts above a special values of M_g, the coupling strength of the external field.

Experimental tests. The author suggests to investigate local Josephson currents induced in junctions arranged in regular M-gonal symmetry and eventually the toroidal coupling of M devices. To excite the projective resonance the hierarchial (fractal) parameter relating loop and subloop radii should be given by $\rho_l/\rho_s = M$, see fig.1. α_M can be obtained by comparing the oscillator frequency of the subloop to the bias voltage and frequency equivalent given by θ_M on the loop, similar to a typical Josephson junction fine-structure constant measurement. $M_g \alpha_M \leq 1/M$ includes relativistic and geometric phase effects already included in the Bäcklund transformations. A positive outcome (resonant coupling and lossless energy transfer) could model and enhance energy propagation by SIT (self-induced transparency) and other supra effects depending on the solid state symmetry and geometry, more illustration can be found i.e. in [4, 5]. Applications of such a resonant device could be on a broad scale starting with energy storage, signal processing (i.e. ring–oscillators for dynamic carrier frequencies and antennas), or fullerene design for special nano-electronic purposes.

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