Generalizing Empirical Adequacy I: Multiplicity and Approximation

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Abstract

Based on a formalization of constructive empiricism's core concept of empirical adequacy, I show that some previous discussions rest on misunderstandings of empirical adequacy. Using one of the inspirations for constructive empiricism, I generalize the concept of a theory to avoid implausible presumptions about the relations of theoretical concepts and observations, and generalize empirical adequacy to allow for lack of knowledge, approximations, and successive gain of knowledge and precision. As a test case, I provide an application of the concepts to a simple interference phenomenon.

Keywords: constructive empiricism, empirical adequacy, approximation, vagueness, subtruth, subvaluation, received view, empirical substructure, empirical embedding, empirical relativized reduct

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1 Introduction

At the core of constructive empiricism lies the concept of empirical adequacy. For according to van Fraassen (1980, 12, emphasis removed), constructive empiricism is the view that

> [s]cience aims to give us theories which are empirically adequate; and acceptance of a theory involves as belief only that it is empirically adequate.

The model theoretic notion of empirical adequacy that van Fraassen uses is very strict: It does not allow for approximation, nor does it allow for lack of knowledge of the phenomena. What is more, the definition puts enormous restrictions on the structures of scientific theories (§2.2).

In this article, I will suggest a generalization of empirical adequacy that alleviates the restrictions on scientific theories (§4.1) and suggest generalizations of empirical adequacy that allow for lack of knowledge of the phenomena (§4.2) and for approximation (4.3). To show the viability of the generalizations, I apply them to a simple interference phenomenon. The basis for the generalizations is one of the inspirations for van Fraassen's definition of empirical adequacy, an unjustly overlooked book by Polish model theoretician Marian Przełęcki (§3). In further defense of the generalizations suggested here, I show in the companion piece to this article (Lutz 2011) that prominent previous generalizations are inadequate.

2 Empirical adequacy: Definitions and problems

Although van Fraassen (1980) defines empirical adequacy in terms of model theory, his formal exposition is rather light. I will thus rely on the standard notation as used by Chang and Keisler (1990, §1.3) and, more loosely, by Hodges (1993, §§1.2f). Hence a *structure* $\mathfrak{A}$ is a pair $\langle A, \mathscr{I} \rangle$ consisting of a domain A and a function $\mathscr{I}$ from a set of n_i-place relation symbols R_i, n_j-place function symbols F_j, and constant symbols c_k to, respectively, n_i-ary relations, n_j-ary functions, and constants on A. Unless stated otherwise, I will in the following always assume this set of symbols with the same arities. In the following, I will sometimes refer to symbols as 'terms' when this does not lead to ambiguity. Sometimes, I use indexed structures $\mathfrak{M}_i$ instead of $\mathfrak{A}$, $\mathfrak{B}$, etc. A will always be the domain $|\mathfrak{A}|$

of $\mathfrak{A}$, $B = |\mathfrak{B}|$ etc. If $\mathfrak{A} = \langle A, \mathscr{I} \rangle$, I write $R_i^{\mathfrak{A}}$ instead of $\mathscr{I}(R)$, and analogously for functions and constants. $R_i^{\mathfrak{B}}$ is the relation in $\mathfrak{B}$ that *corresponds* to relation $R_i^{\mathfrak{A}}$ in $\mathfrak{A}$, and analogous for functions and constants. In displayed form, I write a structure $\mathfrak{A}$ as $\langle A, R_1^{\mathfrak{A}}, \ldots, R_s^{\mathfrak{A}}, F_1^{\mathfrak{A}}, \ldots, F_t^{\mathfrak{A}}, c_1^{\mathfrak{A}}, \ldots, c_u^{\mathfrak{A}} \rangle$ or, for possibly infinite index sets, $\langle A, R_i^{\mathfrak{A}}, F_j^{\mathfrak{A}}, c_k^{\mathfrak{A}} \rangle_{i \in I, j \in J, k \in K}$.

In contradistinction to the above, Bell and Slomson (1974, 73) define a relational structure $\mathfrak{A}$ as a pair $\langle A, \{R_i\}_{i<\alpha} \rangle$ of a domain and a set of relations, where α is a cardinal. This difference is little more than notational, since in their definitions of further model theoretic concepts, the corresponding relations are determined by the index set $\{i : i < \alpha\}$, which therefore plays the role of the set of relation symbols $\{R_i : i \in I\}$ used by Chang and Keisler (1990) and Hodges (1993). For examples relevant in the following, see the definitions of reduct, isomorphism, and substructure by Chang and Keisler (1990, 20–23) and by Bell and Slomson (1974, 153, 73), respectively.[1] The reader who prefers the notation by Bell and Slomson (1974) will have no problems translating the following discussion.

2.1 Definitions

Within constructive empiricism, van Fraassen (1980, 64) states,

> [t]o present a theory is to specify a family of structures, its *models*; and secondly, to specify certain parts of those models (the *empirical substructures*) as candidates for the direct representation of observable phenomena.

Furthermore the models of the theory "are describable only up to structural isomorphism" (van Fraassen 2008, 238; cf. 2002, 22). More formally, this can be phrased as follows:

Definition 1. A *theory* is a family $\{\mathfrak{T}_n\}_{n \in N}$ of structures (the *models of the theory*) such that each of its members $\mathfrak{T}_n = \langle T_n, R_i^{\mathfrak{T}_n}, F_j^{\mathfrak{T}_n}, c_k^{\mathfrak{T}_n} \rangle_{i \in I_n, j \in J_n, k \in K_n}$ has a set $\mathbf{E}_n$ of *empirical substructures*, such that for each $\mathfrak{E} \in \mathbf{E}_n$, $\mathfrak{E} \subseteq \mathfrak{T}_n$. With each model, a theory also contains every isomorphic structure and its corresponding[2] empirical substructures.

Strictly distinguishing between the set O of observable objects and the unobservable objects, van Fraassen (1980, 64) suggests to describe observable phenomena by structures as well: "The structures which can be described in experimental

[1]For reasons that are not entirely clear, this notational convention has become a philosophical point of both contention and confusion. Van Fraassen (1989, 366, n. 4), for example, objects to structures being "yolked to a particular syntax", where 'syntax' seems to stand for 'set of symbols'. And French and Ladyman (1999, 115) see support for van Fraassen's position in the definition of 'structure' given by Hodges (1993), which, however, relies on symbols.

[2]To be precise: If $f : T_m \longrightarrow T_n$ is an isomorphism between $\mathfrak{T}_m$ and $\mathfrak{T}_n$, then the corresponding empirical substructures $\mathbf{E}_n$ are those structures for which f is an isomorphism to an element of $\mathbf{E}_m$.

and measurement reports we can call *appearances*" (van Fraassen 2008, 286). This suggests

Definition 2. *Appearances* are given by a set $\mathbf{P}$ of structures such that the domain of each $\mathfrak{P} \in \mathbf{P}$ is a subset of O. A structure $\mathfrak{P} \in \mathbf{P}$ is an appearance.

Note that the set of appearances does not have to be closed under isomorphism.

Van Fraassen (1980, 64) then defines a theory to be "empirically adequate if it has some model such that all appearances are isomorphic to empirical substructures of that model" (cf. van Fraassen 1991, 12):

Definition 3. A theory $\{\mathfrak{T}_n\}_{n\in N}$ is *empirically adequate* for appearances $\mathbf{P}$ if and only if there is some $n \in N$ such that for every $\mathfrak{P} \in \mathbf{P}$, there is an $\mathfrak{E} \in \mathbf{E}_n$ with $\mathfrak{E} \cong \mathfrak{P}$.

Definition 3 defines the empirical adequacy of a theory *relative* to a set of appearances. In contradistinction, empirical adequacy simpliciter is defined as empirical adequacy for the set of *all* appearances (cf. Monton and Mohler 2008, §1.5). Therefore any set $\mathbf{P}$ of appearances that does not contain all of them may allow the deductive inference that some theory is not empirically adequate; but the inference that a theory *is* empirically adequate will have to be in some way inductive. And it is for empirical adequacy simpliciter that van Fraassen (1980, 12, emphasis removed) claims that "acceptance of a theory involves as belief only that it is empirically adequate".

A note on terminology: Van Fraassen (1980, 66) and others (e. g., Turney 1990, 431; Suárez 2005, §4.1; Monton and Mohler 2008, §§1.5f) occasionally speak of the empirical adequacy of a theory as the embeddability of the appearances into a model of the theory. But the two are not equivalent: $\mathfrak{P} \in \mathbf{P}$ can be embedded in $\mathfrak{T}_n$ if and only if $\mathfrak{P}$ is isomorphic to *any* substructure of $\mathfrak{T}_n$ (Hodges 1993, 6). The substructure does not have to be an *empirical* substructure. In the following, I will call an isomorphic mapping to an empirical substructure an *empirical embedding*.

In a more puzzling oversight, some exponents of empirical adequacy, (e. g. Suárez 2005, 39), rely on

Definition 4. A theory $\{\mathfrak{T}_n\}_{n\in N}$ is *idiosyncratically empirically adequate* for appearances $\mathbf{P}$ if and only if for every $\mathfrak{P} \in \mathbf{P}$, there are an $n \in N$ and an $\mathfrak{E} \in \mathbf{E}_n$ such that $\mathfrak{E} \cong \mathfrak{P}$.

Definitions 3 and 4 are equivalent if there is only one appearance, $\mathbf{P} = \{\mathfrak{P}\}$,[3] but not in general: Let the appearances be given by the set of the two structures $\{\langle\{1,2\},\{1,2\}\rangle, \langle\{3,4\},\{3\}\rangle\}$. Let the theory be given by the family with members $\mathfrak{T}_1 = \langle\{0,1,2,\},\{0,1,2\}\rangle$ and $\mathfrak{T}_2 = \langle\{3,4,5\},\{3,4\}\rangle$ as well as the singleton sets of empirical substructures $\mathbf{E}_1 = \{\langle\{1,2\},\{1,2\}\rangle\}$ and $\mathbf{E}_2 = \{\langle\{3,4\},\{3\}\rangle\}$. Let all other models of the theory be isomorphic to $\mathfrak{T}_1$ or $\mathfrak{T}_2$ and have the corresponding empirical substructures. Then the theory is idiosyncratically empirically adequate

[3] This is decidedly *not* what van Fraassen in general assumes (personal communication).

by virtue of the identity mapping on each of the appearances' domains, but it is not empirically adequate.

Since theories are closed under isomorphism, an appearance is empirically embeddable in a model of a theory if and only if it is an empirical substructure of that model (Hodges 1993, ex. 1.2.4b). Therefore a theory is idiosyncratically empirically adequate if and only if all appearances are empirical substructures of models of the theory (that is, in definition 4, $\mathfrak{E} \cong \mathfrak{P}$ could be exchanged for $\mathfrak{E} = \mathfrak{P}$). This is not the case for empirical adequacy: Let the appearances be given by the set of the two structures $\{\langle\{a,b\},\{a,b\}\rangle, \langle\{c,d\},\{c\}\rangle\}$, where a, b, c, and d are distinct objects. Let the theory be given by the family with the member $\mathfrak{T}_1 = \langle\{1,2,3\},\{1,2\}\rangle$ and the set of empirical substructures $\mathbf{E}_1 = \{\langle\{1,2\},\{1,2\}\rangle, \langle\{2,3\},\{2\}\rangle\}$. Let all other models of the theory be isomorphic to $\mathfrak{T}_1$ and have the corresponding empirical substructures. Then the theory is empirically adequate, but every bijection from $\{1,2,3\}$—and thus every isomorphism for $\mathfrak{T}_1$—maps 2, the object shared by the empirical substructures, to a single object. Since the domains of the appearances do not share an element, the appearances therefore can never be empirical substructures of the same model of the theory.[4]

2.2 Problems

Before listing problems of empirical adequacy, let me note one important virtue: While constructive empiricism crucially relies on empirical adequacy, the reverse is not true. Even a realist can rely on empirical adequacy as one property of a theory, for example by inferring that some theory is false from its empirical inadequacy. In this sense, empirical adequacy is metaphysically neutral, but can be used to define an anti-realist position like constructive empiricism.

While metaphysically neutral, empirical adequacy puts severe restrictions on the models $\{\mathfrak{T}_n\}_{n\in N}$ of a theory by relying on definition 1. It follows from the definition of a substructure that every constant of a model $\mathfrak{T}_n$ has to be in the domain E of *each* of its substructures $\mathfrak{E} \in \mathbf{E}_n$. Furthermore, every function of the model $\mathfrak{E} \in \mathbf{E}_n$ must map all (tuples of) elements of E to elements of E (Hodges 1993, lemma 1.2.2).

If now a theory $\{\mathfrak{T}_n\}_{n\in N}$ is empirically adequate, every appearance is a substructure of some $\mathfrak{T}_n$, so that $\mathfrak{T}_n$'s domain T_n contains observable objects, all constants of $\mathfrak{T}_n$ are observable objects, and all functions of $\mathfrak{T}_n$ map observable objects to other observable objects. If now the theory is about, say, elementary particles, the observable objects are, for example, the results shown on the measurement instruments. These then have to satisfy those formulas that the theory ascribes to elementary particles, and all constants in the theory have to be results shown on measurement instruments.[5]

[4]Note that the opposite would be true, that is, in definition 3, $\mathfrak{E} \cong \mathfrak{P}$ could be exchanged for $\mathfrak{E} = \mathfrak{P}$, if the appearances were closed under isomorphism.

[5]Assuming the right cardinality, this can *formally* always be arranged by defining the relations between the observable objects accordingly, although this would trivialize empirical adequacy (cf.

Besides these technical problems that stem from the reliance on substructures, there are two ways in which empirical adequacy is too strict to be used in most contexts of scientific research because it relies on an isomorphism between the appearances and the empirical substructures. For one, most theories' implications for the appearances are not true up to arbitrary precision. But the theories are still empirically adequate up to some precision, that is, *approximately* empirically adequate. Approximately empirically adequate theories like the ray theory of light, quantum field theory, and general relativity are clearly useful although not empirically adequate according to definition 3. For this reason, van Fraassen (1989, 366, n. 5) himself suggests, but does not explicate, the notion of 'approximate embedding'.

Second, scientists are seldom if ever in the situation that they know what the exact appearances are. Rather, their knowledge only restricts what appearances are (epistemically) possible, for example because it stems from measurements with a certain amount of error. Definition 3 does not allow for such lack of knowledge, since it refers solely to the set of appearances. This makes sense, since a theory that is empirically adequate as far as we know, or what I will call 'epistemically empirically adequate', is not always an empirically adequate theory. Still, definition 3 may define a concept that can never be applied in scientific research.

As an example, take the application of the ray theory of light to two light beams travelling in opposite directions. For the purpose of this example, I will assume that light behaves according to the wave theory of light. Then, while the ray theory asserts that the intensities of the beams will simply add up to some intensity I that is constant for all spatial positions x, the interference of the light waves in fact results in the pattern

$$\overline{\psi}(x) = 2I\cos^2\left(\frac{2\pi x}{\lambda}\right) = I + I\cos\left(\frac{4\pi x}{\lambda}\right) \tag{1}$$

for the time averaged intensity (cf. Batterman 2002, §6.2). Thus the ray theory is not empirically adequate, since the interference pattern cannot be empirically embedded in one of its models.[6]

However, it might be impossible for the scientist measuring the intensity to realize that there is no empirical embedding. A measuring device with a finite spatial resolution given by the normalized function b will blur the measurement of $\overline{\psi}$ to the convolution $\overline{\psi}_b(x) = \int_{-\infty}^{\infty} f(y)b(y-x)\,\mathrm{d}y$. Assuming, for ease of calculation, that b is a rectangular function of width p centered around 0, the convolution amounts to an averaging over the interval $[x - p/2, x + p/2]$. The

van Fraassen 2006); I am assuming here that the trivialization problem has been solved.

[6]Again assuming that the problem of the trivialization of empirical embedding has been solved. This will always be silently assumed in the following.

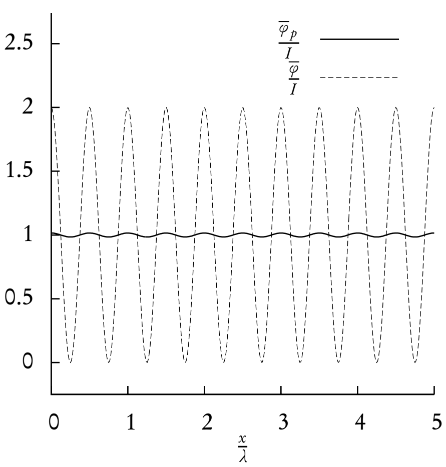

Figure 1: The real pattern of the intensity (dashed) and the blurred pattern (solid) for $p/\lambda = 10.25$.

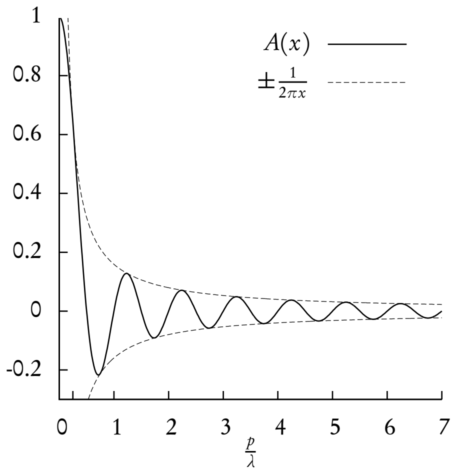

Figure 2: The deviation factor $D(x)$ (solid) enclosed between $\pm 1/(2\pi x)$ (dashed).

measurement of $\overline{\psi}$ then gives the result

$$\overline{\psi}_p(x) = \frac{1}{p}\int_{x-\frac{p}{2}}^{x+\frac{p}{2}} 2I\cos^2\left(\frac{2\pi y}{\lambda}\right)\,\mathrm{d}y = I + ID\left(\frac{p}{\lambda}\right)\cos\left(\frac{4\pi x}{\lambda}\right) . \tag{2}$$

The deviation factor

$$D(x) = \frac{\sin(2\pi x)}{2\pi x} \tag{3}$$

and the intensity I determine how far individual values of the measured intensity $\overline{\psi}_b$, can maximally deviate from I, the intensity's spatial average. The maximal deviation thus depends solely on I and the ratio of the precision of measurement and the wavelength, p/λ. As can be seen from both the graph of D (figure 2) and the formula (3) for D, the functions $1/(2\pi x)$ and $-1/(2\pi x)$ enclose D.[7]

The ray theory of light predicts a constant intensity $\overline{\psi^{\text{ray}}} = I$, so that

$$\overline{\psi^{\text{ray}}}_p(x) = \frac{1}{p}\int_{x_i-\frac{p}{2}}^{x_i+\frac{p}{2}} I = I \tag{4}$$

For intensities I and wavelengths λ small enough, as well as spatial resolutions p and intensity resolutions q low enough, $\overline{\psi}_p$ and $\overline{\psi^{\text{ray}}}_p$ will be hard or impossible to distinguish for the scientist. This also provides a very practical sense in which the ray theory is approximately empirically adequate: Up to certain wavelengths and intensities, the ray theory is empirically adequate up to certain resolutions. However, the ray theory is not empirically adequate *simpliciter*, since appearances

[7]Note that $|D(x)| \leq 1$ also for $x \in [0, 1/(2\pi)[$, unlike $1/(2\pi x)$.

given by the intensity pattern $\overline{\psi}$ are possible. As van Fraassen (1991, 12) puts it: "Empirical adequacy, like truth, admits of no degrees".

Suppes (1962) famously noted that measurement results differ from phenomena like $\overline{\psi}$ beyond having a finite resolution: No measurement would result in the continuous function $\overline{\psi}_p$, simply because measurements are discontinuous sets of values. Again simplifying the situation for easier tractability, the measurement of the intensity pattern $\overline{\psi}$ can thus be taken to result in some finite set

$$\{\langle x_1, y_1\rangle, \ldots, \langle x_s, y_s\rangle\} \tag{5}$$

with

$$y_i \in \left[\overline{\psi}_p(x_i) + \frac{q}{2}, \overline{\psi}_p(x_i) + \frac{q}{2}\right] \text{ for all } 1 \leq i \leq s \tag{6}$$

of pairs of spatial values x_i and intensity values y_i. The values restrict each other because they are related by the blurred intensity pattern $\overline{\psi}_p$ and the intensity resolution q. Accordingly, van Fraassen (2008, 166–172) makes a distinction between *surface structures*, $\overline{\psi}_p$, and *data structures*, the measurement results,[8] and demands that "the data or surface [structures] must ideally be isomorphically embeddable in theoretical models" (van Fraassen 2008, 168). In their "cardinality objection", Bueno et al. (2002, 503) argue that this demand is problematic because the domains of data structures "in general are finite". The implicit assumption of the cardinality objection is that empirical substructures always have infinite domains, but this is not necessarily so. To follow van Fraassen and allow an empirical embedding of both infinite and finite appearances, theories only need both infinite and finite empirical substructures. This is easily achieved, since for any model $\mathfrak{T}_n$ of a theory, the set of empirical substructures $\mathbf{E}_n$ can be closed under the substructure relation, that is, if $\mathfrak{E} \in \mathbf{E}_n$ and $\mathfrak{E}' \subseteq \mathfrak{E}$, then $\mathfrak{E}' \in \mathbf{E}_n$, so that with every empirical substructure $\mathfrak{E}$, all of $\mathfrak{E}$'s finite substructures are empirical substructures as well. This response to the cardinality objections is restricted only by the problems stemming from the technical aspects of substructures discussed above. Thus a solution to these problems will also provide a response to the cardinality objection.

As to the problems, one may object for two reasons. For one, in early works van Fraassen (1970, §3) relied on "elementary statements" and a "satisfaction function" to give the relation between a theory and observations, so that one could argue that the model theoretic formalization above does not capture van Fraassen's position. However, van Fraassen (1989, 365, n. 34) himself states that he soon "found it much more advantageous to concentrate on the propositions expressible by elementary statements, rather than on the statements themselves". Thus van

[8] Van Fraassen (2008, 166) actually speaks of "surface models" and "data models", but since it is at least for surface models not clear what they are models *of*, I will speak of structures.

Fraassen had abandoned the reliance on elementary statements and satisfaction functions by the time he defined empirical adequacy. More importantly, empirical adequacy is defined without reference to either of the two concepts, and thus an analysis of empirical adequacy does not have to take them into account either.

One may also object that the terms 'embedding', 'substructure', and 'isomorphism' are not meant literally, but refer to relations between theories and phenomena given by either satisfaction functions or something completely different. Possible support comes from van Fraassen's standard example of embedding, the seven point geometry (1980, §3.1; 1989, §9.1), which is not an embedding in the sense of model theory (Turney 1990, 441–443). But this looks more like an oversight than a conscious decision. More generally, the terms are well-defined within, but not outside of model theory, where they also in general do not occur together. And the objection makes van Fraassen use these terms in a different, undefined way *without pointing this out*. It also renders downright nonsensical passages in which he uses technical results from model theory. For example, van Fraassen (1980, 43) discusses cases in which "every model of T_1 can be embedded in (identified with a substructure of) a model of T_2." The parenthetical equivalence claim relies on the model theoretic definition of 'embedding' and 'substructure', on the closure of the set of models of a theory under isomorphism, and the equivalence of embedding and the substructure relation for isomorphically closed sets of structures (Hodges 1993, ex. 1.2.4b). If the terms were not meant in the model theoretic sense, there would be no reason at all for this equivalence claim.

Thus, the purported problems are indeed problems. But they can be solved by further developing empirical adequacy, which, it turns out, is easiest by looking at its origins.

3 The prehistory of empirical adequacy

Van Fraassen (1980, 64; 1989, 227) traces the notion empirical substructure back to a monograph by Przełęcki (1969) and other works on the application of model- and set theory in the philosophy of science, only noting that "some of these formulations were still more language-oriented than [he] liked" (1989, 227). However, the connection between Przełęcki's monograph and constructive empiricism is not obvious, to put it mildly: Van Fraassen (1980, §3.6) famously declared that the received view on scientific theories, as developed by Carnap, Hempel, and others within logical positivism, is in principle unable to describe the correct relation between theory and phenomena. Przełęcki (1975, 284), on the other hand, thought of himself as "positivistically-minded" and of the monograph as an introduction to the received view (Przełęcki 1974, 402).[9] Przełęcki's discussion differs from

[9]It is hence fascinating to see Przełęcki's work cited as a precursor or even an elaboration of the semantic view (da Costa and French 1990, 249; Volpe 1995, 566), even though the semantic view, of which constructive empiricism is one variety, is usually considered to be diametrically opposed to the received, or "syntactic", view.

previous expositions (e. g. Carnap 1939, §24) mainly in that he explicitly develops the model theoretic implications of the received view and discusses vague relations. This discussion, and its elaboration that Przełęcki (1976) published after Fine (1975, n. 13) had developed a similar idea, provide the relation to constructive empiricism.

Przełęcki (1969, §4) argues in some detail that an empirical language cannot be interpreted by structures that are determined only up to isomorphism. Thus there has to be a set $\mathbf{M}$ of "intended" structures (1969, §§3f), which, because empirical languages are not arbitrarily precise, is not a singleton set even when there is exhaustive empirical information (Przełęcki 1969, §§5f). Following the terminology of Fine (1975, §3), one can then give

Definition 5. A sentence α is *supertrue* in a set $\mathbf{M}$ of intended interpretations if and only if α is true in every $\mathfrak{A} \in \mathbf{M}$, and *superfalse* if it is false in every $\mathfrak{A} \in \mathbf{M}$.

Przełęcki (1969, 20f) then suggests that supertruth should be a sufficient condition for truth in $\mathbf{M}$, and superfalsity a sufficient condition for falsity in $\mathbf{M}$.[10]

Following the terminology of Hyde (1997), one can also give

Definition 6. A sentence α is *subtrue* if and only if α is true in *at least one* of the structures $\mathfrak{M} \in \mathbf{M}$.

Trivially, α is subtrue if and only if α is not superfalse.

Przełęcki (1976) uses the idea of a set of intended structures to interpret vague languages. The denotation of a relation symbol R_i that is vague over some domain A tripartitions the product domain A^{m_i} into a set R_i^+ of definite instances (the positive extension of R_i), a set R_i^- of definite non-instances (the negative extension), and a set of borderline cases of R_i, which I will call R° (the neutral extension). The denotation of a function symbol F_j that is vague over A does not assign a single element $b \in A$ to an n_j-tuple $(a_1, \dots, a_{n_j}) \in A^{n_j}$, but rather a set $F_j^{+\circ}(a_1, \dots, a_{n_j}) = B \subseteq A$ (Przełęcki 1976, 375).[11] B can be seen as the set of possible values of the function named by F_j for the arguments $a_1, \dots, a_{n_j}$, and I will refer to the set $\{(a_1, \dots, a_{n_j}, b) : a_1, \dots, a_{n_j} \in A, b \in F_j^{+\circ}(a_1, \dots, a_{n_j})\}$ as the non-negative extension $F^{+\circ}$ of F_j.[12] If $F_j^{+\circ}(a_1, \dots, a_{n_j})$ is a singleton set, I will say that F_j has a positive extension for $(a_1, \dots, a_{n_j})$. Considering constant symbols 0-place function symbols, this means that the denotation of a constant symbol c_k that is vague over A is a set $c_k^{+\circ} \subseteq A$.

Przełęcki (1976, 376) notes that for a function symbol F_j, $F_j^{+\circ}$ may contain unintended functions. For example, unless F_j has a positive extension over the

[10] Przełęcki (1969, 19–21) also gives an interesting discussion of possible necessary conditions for truth and falsity that, however, will not be relevant in the following (cf. Przełęcki 1976, 377f).

[11] This is a slight generalization of Przełęcki's account, who assumes that B is an interval of reals, which would therefore have to be in A.

[12] $F_j^{+\circ}$ is the union of a vague relation symbol's positive and neutral extension.

whole domain, $F_j^{+\circ}$ contains discontinuous functions, which may go against the intended denotation of F_j. Przełęcki therefore allows the denotation of a function symbol F_j to be further determined by a set of "additional conditions" $W(F_j)$, which all members of $\mathbf{M}$ have to fulfill as well. Similarly to Przełęcki's additional conditions are what Fine (1975, 124) calls "penumbral connections", sentences that have to be true in any $\mathfrak{M} \in \mathbf{M}$. However, Fine assumes that these connections are given in the object language, not in the meta-language determining the denotations, and he does not restrict the penumbral connections to functions only. I will follow Przełęcki in assuming that the penumbral connections are given in the meta-language, but I will follow Fine in allowing penumbral connections for all terms, $W(R_i, F_j, c_k)_{i\in I, j\in J, k\in K}$.

The denotations of vague terms over A and the penumbral connections immediately lead to a set of of structures:

Definition 7. Let the terms $\{R_i, F_j, c_k\}_{i\in I, j\in J, k\in K}$ be vague over domain A with positive, negative, and non-negative extensions $\{R_i^+, R_i^-, F_j^{+\circ}, c_k^{+\circ}\}_{i\in I, j\in J, k\in K}$, and penumbral connections $W(R_i, F_j, c_k)_{i\in I, j\in J, k\in K}$. Then the terms' *vagueness set* $\mathbf{M}$ for A contains all and only structures $\mathfrak{M}$ that fulfill the penumbral connections and for which

$$M = A, \tag{7}$$

$$R_i^+ \subseteq R_i^{\mathfrak{M}} \subseteq A^{m_i} - R_i^- \text{ for all } i \in I, \tag{8}$$

$$F_j^{\mathfrak{M}} \subseteq F_j^{+\circ} \text{ for all } j \in J, \text{ and} \tag{9}$$

$$c_k^{\mathfrak{M}} \in c_k^{+\circ} \text{ for all } k \in K\ . \tag{10}$$

In the following, I will always assume that the vagueness set for the terms and penumbral connections is never empty, that is, the penumbral connections are not in conflict with the positive, negative, and non-negative extensions of the terms. Furthermore, I will assume that the penumbral connections are only used to exclude those structures from vagueness sets that cannot be excluded with the help of positive, negative, and non-negative extensions. This entails that for any vagueness set $\mathbf{M}$ over domain A for terms with $\{R_i^+, R_i^-, F_j^{+\circ}, c_k^{+\circ}\}_{i\in I, j\in J, k\in K}$ and $W(R_i, F_j, c_k)_{i\in I, j\in J, k\in K}$,

$$R_i^+ = \bigcap \left\{ R_i^{\mathfrak{M}} : \mathfrak{M} \in \mathbf{M} \right\}\ , \tag{11}$$

$$R_i^- = \bigcap \left\{ A^{m_i+1} - R_i^{\mathfrak{M}} : \mathfrak{M} \in \mathbf{M} \right\}\ , \tag{12}$$

$$F_j^{+\circ} = \bigcup \left\{ F_j^{\mathfrak{M}} : \mathfrak{M} \in \mathbf{M} \right\}\ , \text{ and} \tag{13}$$

$$c_k^{+\circ} = \bigcup \left\{ c_k^{\mathfrak{M}} : \mathfrak{M} \in \mathbf{M} \right\} \tag{14}$$

for all $i \in I, j \in J, k \in K$.

Przełęcki (1976, 378) further suggests to treat approximation in the same formalism as vagueness, with a vagueness set **M** representing those structures that approximate the right structure (or set of structures). Approximate truth is then truth in a structure approximating the right structure (or set of structures).

Definition 8. Let **M** be a vagueness set. A *sentence* is *approximately true* in **M** if and only if it is subtrue in **M**. A *set of sentences* is *approximately true* if and only if in at least one $\mathfrak{M} \in \mathbf{M}$ all its elements are true.

The definition of approximate truth for sets of sentences avoids inconsistent approximately true sets. In this way, a finite set of sentences is approximately true if and only if the conjunction of its elements is approximately true, that is, subtrue.

An application of definition 8 to a scientific theory is only possible if the theory is given as a set of sentences H,[13] which illustrates van Fraassen's remark that some of his inspirations were more language-oriented than he liked. But this language-orientation can be avoided with a slight generalization. It follows from definition 8 that H is approximately true if and only if a model of H is in **M**. So if a theory is given as a family of models, one can give

Definition 9. Let **M** be a vagueness set. A family of structures $\{\mathfrak{T}_n\}_{n \in N}$ is *approximately true* in **M** if and only if for some $n \in N$, $\mathfrak{T}_n \in \mathbf{M}$.

Przełęcki (1969, ch. 4) argues that in science, at least some symbols $\mathcal{O} = \{R_i, F_j, c_k\}_{i \in I, j \in J, k \in K}$,[14] which he calls 'observational terms', have to be interpreted by ostension. Because of the psychological fact that ostensively classified paradigmatic examples allow to classify newly encountered objects as well if they are similar enough to the paradigmatic examples, such an ostensive interpretation leads to a vagueness set $\mathbf{M}_O$ over the domain of observable objects O. Przełęcki (1969, 38–41) argues further that all unobservable objects are in the neutral or non-negative extension of all observational terms, leading to the set $\mathbf{M}^*_{\mathcal{O}}$, which contains all extensions of each element of $\mathbf{M}_O$ that fulfill the penumbral connections.[15] $\mathbf{M}^*_{\mathcal{O}}$ is the set of intended interpretations of the observation terms.

Since not all elements of $\mathbf{M}^*_{\mathcal{O}}$ share the same domain, $\mathbf{M}^*_{\mathcal{O}}$ is not a vagueness set. But it is something similar:

Definition 10. $\mathfrak{M}$ is in the *generalized vagueness set* for positive, negative, and non-negative extensions $\{R^+_i, R^-_i, F^{+\circ}_j, c^{+\circ}_k\}_{i \in I, j \in J, k \in K}$ and $W(R_i, F_j, c_k)_{i \in I, j \in J, k \in K}$, if

[13] If a theory is primarily given as a *set* $\mathbf{T}$ (e. g., $\mathbf{T} = \{\mathfrak{T} : \mathfrak{T} \vDash H\}$) rather than a family, I will implicitly rely on the family $\{\mathfrak{T}_{\mathfrak{T}}\}_{\mathfrak{T} \in \mathbf{T}}$ associated with the set.

[14] Using the same symbols as before will be convenient for back-reference in the following.

[15] An extension $\mathfrak{A}$ of $\mathfrak{M}$ is a structures that has $\mathfrak{M}$ as a substructure, so that $\mathfrak{A}$'s restriction to the domain of $\mathfrak{M}$ is $\mathfrak{M}$, $\mathfrak{A}|M = \mathfrak{M}$.

and only if there is some A such that

$$M = A, \tag{15}$$

$$R_i^+ \subseteq R_i^{\mathfrak{M}} \subseteq A^{m_i} - R_i^- \text{ and } R_i^- \subseteq A^{m_i} \text{ for all } i \in I, \tag{16}$$

$$F_j^{\mathfrak{M}} a_1 \dots a_{n_j} \in F_j^{+\circ} a_1 \dots a_{n_j} \text{ for all } (a_1, \dots, a_{n_j}, b) \in F^{+\circ}, j \in J, \text{ and} \tag{17}$$

$$c_k^{\mathfrak{M}} \in c_k^{+\circ} \text{ for all } k \in K\,. \tag{18}$$

The relation between definitions 7 and 10 is given by

Claim 1. *Let the penumbral connections be* $W(R_i, F_j, c_k)_{i\in I, j\in J, k\in K}$, *let* $\mathbf{M}_A$ *be the vagueness set for the terms* $\{R_i, F_j, c_k\}_{i\in I, j\in J, k\in K}$ *over* A *with* $\{R_i^+, R_i^-, F_j^{+\circ}, c_k^{+\circ}\}_{i\in I, j\in J, k\in K}$, *and* $\mathbf{N}$ *be the generalized vagueness set for* $\{R_i^+, R_i^-, F_j^{+\circ}, c_k^{+\circ}\}_{i\in I, j\in J, k\in K}$. *Then* $\mathbf{N} = \bigcup\{\mathbf{M}_A : R_i^+, R_i^- \subseteq A^{m_i}, F^{+\circ} \subseteq A^{n_j}, c_k^{+\circ} \subseteq A\}$.

Proof. Immediately from the definitions. □

In analogy to definition 9, a generalized vagueness set immediately leads to generalized approximate truth for families of structures:

Definition 11. Let $\mathbf{M}$ be a generalized vagueness set. A family of structures $\{\mathfrak{T}_n\}_{n\in N}$ is *generalized approximately true* in $\mathbf{M}$ if and only if for some $n \in N$, $\mathfrak{T}_n \in \mathbf{M}$.

The notion of generalized approximate truth now provides the connection to the notion of an empirical substructure:

Claim 2. *Let theory* $\{\mathfrak{T}_n\}_{n\in N}$ *be such that* $\mathfrak{T}_n|O \in \mathbf{E}_n$ *whenever* $O \subseteq T_n$ *and let* $\{\mathfrak{T}_n\}_{n\in N}$*'s elements fulfill the penumbral connections.*[16] *Then, if* $\mathbf{P} = \mathbf{M}_O$ *and* $|\mathbf{M}_O| = 1$, $\{\mathfrak{T}_n\}_{n\in N}$ *is empirically adequate if and only if* $\{\mathfrak{T}_n\}_{n\in N}$ *is generalized approximately true in* $\mathbf{M}^*_{\mathcal{O}}$.

Proof. Given the assumptions, $\{\mathfrak{T}_n\}_{n\in N}$ is empirically adequate if and only if $\mathfrak{P}$, the sole element of $\mathbf{P}$, is isomorphic to an empirical substructure of $\{\mathfrak{T}_n\}_{n\in N}$. Since $\{\mathfrak{T}_n\}_{n\in N}$ is closed under isomorphism, this holds if and only if $\mathbf{P}$ is an empirical substructure of some $\mathfrak{T}_n$ (Hodges 1993, ex. 1.2.4b). Since $\mathfrak{T}_n|O \in \mathbf{E}_n$ whenever $O \subseteq T_n$, this is the case if and only if $\mathfrak{T}_n$ is an extension of $\mathfrak{P}$. Since $\mathbf{M}^*_{\mathcal{O}}$ is a generalized vagueness set for $\{R_i^+, R_i^-, F_j^{+\circ}, c_k^{+\circ}\}_{i\in I, j\in J, k\in K}$ with $R_i^+ = R_i^{\mathfrak{P}}$, $R_i^- = P^{n_i} - R_i^{\mathfrak{P}}$, $F_j^{+\circ} = F_j^{\mathfrak{P}}$, and $c_k^{+\circ} = c_k^{\mathfrak{P}}$ for all $i \in I, j \in J, k \in K$, and for every n, $\mathfrak{T}_n$ fulfills the penumbral connections, this hold if and only if $\{\mathfrak{T}_n\}_{n\in N}$ is generalized approximately truth in $\mathbf{M}^*_{\mathcal{O}}$. □

[16] $\mathfrak{T}_n|O$, the *restriction* of $\mathfrak{T}_n$ to O, is the substructure of $\mathfrak{T}_n$ with domain O.

Claim 2 rests on the condition that the unobservable objects are in the neutral or non-negative extensions of all observation terms, which fits well with van Fraassen's position, and so is the condition that the observational terms are not vague over O. That $\mathfrak{T}_n|O$ is an empirical substructure of every $\mathfrak{T}_n$ with $O \subseteq T$ is plausible, since the elements of the empirical substructures' domains are meant to represent observable objects. The problematic assumption is that all observations have to form one big structure $\mathfrak{P}$, whose domain contains all observable objects.[17] In this last respect, van Fraassen's account goes beyond Przełęcki's.

As is constitutive of the received view (cf. Carnap 1939, §24; Hempel 1952, §5), Przełęcki (1969, chs. 5f) also assumes that there is a set $\mathscr{T} = \{R_i, F_j, c_k\}_{i\in I', j\in J', k\in K'}$ of theoretical terms, where $I \cap I' = J \cap J' = K \cap K' = \emptyset$. Theoretical terms are not ostensively interpreted. However, any structure $\mathfrak{M}$ for all terms $\mathscr{O} \cup \mathscr{T}$ has to be such that its *reduct* to $\mathscr{O}$, the structure $\mathfrak{M}|_{\mathscr{O}}$ that differs from $\mathfrak{M}$ only in that it provides no interpretation for theoretical terms (Hodges 1993, 9), is in $\mathbf{M}^*_{\mathscr{O}}$.[18] In its use of theoretical terms, Przełęcki's account differs from van Fraassen's, since an empirical substructure $\mathfrak{E} \in \mathbf{E}_n$ of a structure $\mathfrak{T}_n$ contains all and only the terms of $\mathfrak{T}_n$ (Hodges 1993, §1.2; cf. Suppes 2002, 62), as does a structure $\mathfrak{P}$ isomorphic to $\mathfrak{E}$ (Hodges 1993, §1.2; cf. Suppes 2002, 56). In this respect, Przełęcki's account goes beyond van Fraassen's.

4 Generalizations

While it is of interest that under specific conditions, Przełęcki's account can capture van Fraassen's, I will use it in the following to *generalize* van Fraassen's account. As long as the assumption that $\mathbf{P}$ is a singleton set can be avoided, this has the immediate advantage of relying only on conceptual assumptions that van Fraassen could accept, and in fact was inspired by. To be adequate, however, the generalizations of empirical adequacy will have to fulfill some further conditions.

One condition for a generalization is that it should, like definition 3 of empirical adequacy, be metaphysically neutral, while allowing a strict empiricism. This is just a corollary of the demand that epistemic empirical adequacy has to be a generalization of empirical adequacy. That is, whenever everything that is possible to know about the appearances is known, epistemic empirical adequacy should be equivalent to empirical adequacy. When not everything is known, epistemic empirical adequacy should be properly weaker than empirical adequacy.

Further, the example of the interference pattern $\overline{\psi}$ suggests that it is very fruitful to develop approximate empirical adequacy as a special case of epistemic empirical adequacy. Specifically, an approximate value of a quantity should be treated as a lack of knowledge about the precise value of the quantity. In this way, a theory $\{\mathfrak{T}_n\}_{n\in N}$ that was epistemically empirically adequate given the resolution of the best measurements can still be valued for its approximate empirical adequacy,

[17] As noted, this is not what van Fraassen assumes.

[18] $\mathfrak{A}$ is an *expansion* of $\mathfrak{B}$ if and only if $\mathfrak{B}$ is a reduct of $\mathfrak{A}$.

even if upon more precise measurements, $\{\mathfrak{T}_n\}_{n\in N}$ ceases to be epistemically empirically adequate. Given the two concepts relation, any conditions of adequacy for epistemic empirical adequacy also hold for approximate empirical adequacy.

For practical purposes, results about the generalizations of empirical adequacy should transfer sensibly from finite to infinite domains. Specifically, the generalizations of empirical adequacy for appearances with an infinite domain should be limiting cases of the generalizations for appearances with a finite domain. Otherwise it would, for example, be impossible to infer anything about the generalized empirical adequacy for appearances of a continuum with the help of any finite number of measurements.

Finally, both epistemic and approximate empirical adequacy should have a comparative version, since a typical investigation into some phenomenon will involve the accumulation of knowledge, and specifically more and more precise measurements. Hence, it should be possible to speak of one theory being more epistemically or approximately empirically adequate than another.

4.1 Empirical relativized reducts

In van Fraassen's definition 1 of a theory, the components of a theory that are connected to the appearances are given by empirical substructures. This leads to a variety of problems, since all the terms that occur in the theory also have to occur in the substructures. Przełęcki, on the other hand, allows theories to contain non-observational terms, which themselves do not have to be directly connected to the appearances. This suggests the following generalization of definition 1:

Definition 12. A *theory* is a family $\{\mathfrak{T}_n\}_{n\in N}$ of structures (the *models of the theory*) such that each of its members $\mathfrak{T}_n = \langle T_n, R_i^{\mathfrak{T}_n}, F_j^{\mathfrak{T}_n}, c_k^{\mathfrak{T}_n}\rangle_{i\in I_n, j\in J_n, k\in K_n}$ has a set $\mathbf{E}_n$ of *empirical relativized reducts*, such that for each $\mathfrak{E}\in\mathbf{E}_n$, there is a set $\mathscr{A}_{\mathfrak{E}}$ of terms such that $\mathfrak{E}\subseteq\mathfrak{T}_n|_{\mathscr{A}_{\mathfrak{E}}}$. With each model, a theory also contains every isomorphic model and its corresponding empirical substructures.

This means that every empirical relativized reduct $\mathfrak{E}\in\mathbf{E}_n$ is a *relativized reduct* of $\mathfrak{T}_n$, $\mathfrak{E}=\mathfrak{T}_n|E_{\mathscr{A}_{\mathfrak{E}}}$, which is a standard notion in model theory (cf. Hodges 1993, §5.1),[19] and is already implicitly used by Suárez (2005, 38) instead of the notion of a substructure in his discussion of empirical adequacy.

This definition solves the problems connected with the use of substructures: There can be unobservable constants, and functions from observable objects to unobservable ones, since the constants' and functions' symbols may not be in the vocabulary $\mathscr{A}_{\mathfrak{E}}$. The definition may also alleviate the problem that, to be empirically adequate, a theory has to describe observational relations between observable objects: A theory may, for instance, already contain or be extended to contain the term 'being part of' and terms for observable objects and relations.

[19] Hodges (1993, 202f) defines relativized reducts as those substructures of a reduct that have the extension of some one place predicate as their domain. I use a slight generalization.

This would allow, say, elementary particles to be treated as part of observable objects, rather than as observable objects themselves (cf. Przełęcki 1969, 86f). Both the elementary particles and the everyday objects would then be in the (extended) theory's domain, but the observational vocabulary of the theory might then refer only to observable objects, not elementary particles.

Although Przełęcki uses reducts to capture the received view, the use of relativized reducts is not a relapse into Carnap's or Hempel's conception of a theory, since the notion of a substructure has not been abandoned. What is more, definition 12 of a theory contains definition 1 as a special case, namely when for each $\mathfrak{E} \in \mathbf{E}_n$, $\mathscr{A}_{\mathfrak{E}}$ contains all symbols of $\mathfrak{T}_n$, so that $\mathfrak{T}_n|_{\mathscr{A}_{\mathfrak{E}}} = \mathfrak{T}_n$ and thus $\mathfrak{E} \subseteq \mathfrak{T}_n$. Then a theory can be formalized as before.

Furthermore, one could even avoid the use of relativized reducts in the formalization of a theory by first treating functions and constants as special kinds of relations, thereby reformulating the theory to contain only relations. For each model $\mathfrak{T}_n$ of the theory, define then a new model $\mathfrak{T}'_n$ with at least one object $t \in T'_n - \bigcup\{E : \mathfrak{E} \in \mathbf{E}_n\}$ such that $\mathfrak{T}_n|_{\mathscr{O}} = \mathfrak{T}'_n|O_{\mathscr{O}}$ and for each m_i-ary theoretical relation $R_i^{\mathfrak{T}_n}$, $\mathfrak{T}'_n$ contains an m_i+1-ary relation $\{\langle x_1, \ldots, x_{m_i}, t\rangle : \langle x_1, \ldots, x_{m_i}\rangle \in R_i^{\mathfrak{T}_n}\}$. Then for every $\mathfrak{E} \in \mathbf{E}_n$, $\mathfrak{T}'_n|E$ differs from $\mathfrak{E} = \mathfrak{T}_n|E$ only by empty relations, so that the empirical relativized reducts of the original theory can be considered the empirical substructures of the reformulated theory.[20] Given the possibility of this reformulation, the use of relativized reducts cannot in principle be a problem. Since reformulating theories exclusively in relation terms needlessly complicates their application (Hodges 1993, 2), and redefining some of these relations to always contain an unobservable object is not a paragon of simplicity either, the use of relativized reducts is preferable in the following.

Finally, one might object to definition 12 because through its reference to a set of symbols, it appears much more language dependent than van Fraassen's definition 1. This appearance is misleading, as discussed at the beginning of §2: Depending on the notation, either definition 1 implicitly contains a reference to a set of symbols, or definition 12 can be reformulated without reference to a set of symbols, using an index set instead.

4.2 Epistemic empirical adequacy

Relying on Przełęcki's use of reducts into the definition of a theory has solved the problems connected with empirical adequacy's exclusive reliance on substructures. Przełęcki's use of sets of intended structures will solve empirical adequacy's problems with incomplete knowledge. Przełęcki uses the formalism developed for the semantic notion of vagueness to formalize the epistemic notion of approximation. Generalizing this idea, one can use the formalism developed for multiple intended structures to formalize lack of knowledge. Thus, one can liberalize the assumption that an appearance is given by a single structure to the assumption that it is given

[20] I thank Albert Visser for this point.

by a set thereof. This set can be seen as the multiplicity of epistemically possible appearances, with a singleton set representing epistemic certainty. Definition 2 thus becomes

Definition 13. Epistemic appearances are given by a set $\mathbf{P}$ of sets of structures such that the domain of each $\mathfrak{P} \in \bigcup \mathbf{P}$ is a subset of O. A set $\mathbf{Q} \in \mathbf{P}$ is an epistemic appearance.

In the case of approximation, truth in one structure becomes truth in one of a multiplicity of structures; analogously, for empirical adequacy the isomorphy of an empirical substructure to one structure becomes the isomorphy of an empirical substructure to one of a multiplicity of structures. Thus Definition 3 becomes

Definition 14. Given the epistemic appearances $\mathbf{P}$, a theory $\{\mathfrak{T}_n\}_{n \in N}$ is *epistemically empirically adequate* for $\mathbf{P}$ if and only if there is some $n \in N$ such that for every $\mathbf{Q} \in \mathbf{P}$, there are a $\mathfrak{P} \in \mathbf{Q}$ and an $\mathfrak{E} \in \mathbf{E}_n$ with $\mathfrak{E} \cong \mathfrak{P}$.

Going back to the example of the two light beams, it is now possible to investigate under what circumstances the ray theory of light is epistemically empirically adequate. Let, as in the example, the measurement apparatus have the known spatial resolution p and intensity resolution q. If the intensity pattern is measured, giving a finite set of pairs of values (5), the resolutions of the apparatus restricts the possible actual values of the intensity $\overline{\psi}$. The measurements thus result in a set

$$\left\{ \left\langle \mathbb{R} \times \mathbb{R}^{\geq 0}, \varphi \right\rangle \quad : \quad \frac{1}{p} \int_{x_i - \frac{p}{2}}^{x_i + \frac{p}{2}} \varphi(x)\,\mathrm{d}x \in \left[y_i - \frac{q}{2}, y_i + \frac{q}{2} \right], 1 \leq i \leq s \right\} \tag{19}$$

of structures, one of which contains the actual graph $\overline{\psi}$ of the intensity. Informally, each structure contains the set of pairs of a location and an intensity as its domain, and a function for the intensity whose values are restricted by the measurement results and the resolution. Note that the epistemic appearances are already given as a set of structures with infinite domains, thereby avoiding the cardinality objection from the start.

Since $\overline{\psi^{\mathrm{ray}}}_p = I$, for the ray theory to be epistemically empirically adequate given measurements with resolutions p and q, the measured intensities y_i must not deviate more than $q/2$ from I. Since by assumption each measurement result y_i can lie anywhere in the interval $\left[\overline{\psi}_p(x_i) - q/2, \overline{\psi}_p(x_i) + q/2\right]$, this is not necessary, but becomes increasingly unlikely for higher wave-lengths and lower intensity. For a crude estimate of the upper bound of the probability, assume that the measurement results are randomly distributed over the range q. The spatial average over the probability for a single measurement result to lie outside of $[I - q/2, I + q/2]$ is then

$$\frac{1}{\lambda} \int_0^{\lambda} \frac{\left|\overline{\psi}_p(x) - I\right|}{q} \mathrm{d}x = \frac{8I}{\lambda q} \left| D\left(\frac{p}{\lambda}\right) \right| \int_0^{\frac{\lambda}{8}} \cos\left(\frac{4\pi x}{\lambda}\right) \mathrm{d}x = \frac{2I}{q\pi} \left| D\left(\frac{p}{\lambda}\right) \right| \tag{20}$$

Since the extremal values for A lie on $1/(2\pi x)$, the maximal probability is thus $\pi^{-2}\cdot\lambda/p\cdot I/q$, which, as was to be expected, decreases with p and q and increases with λ and I. A more realistic scenario could provide a proper statistical analysis of the probability of discovering that the ray theory is not epistemically empirically adequate.

The connection between epistemic empirical adequacy and empirical adequacy can be formulated with the help of

Definition 15. Given epistemic appearances $\mathbf{P}$, $\mathbf{P}'$ are *epistemically possible appearances* if and only if $\mathbf{P}' = \{e(\mathbf{Q}) : \mathbf{Q} \in \mathbf{P}\}$, where e is any function from $\mathbf{P}$ to $\bigcup \mathbf{P}$ with $e(\mathbf{Q}) \in \mathbf{Q}$.

Claim 3. *A theory* $\{\mathfrak{T}_n\}_{n\in N}$ *is epistemically empirically adequate for epistemic appearances* $\mathbf{P}$ *if and only if there are epistemically possible appearances* $\mathbf{P}'$ *such that* $\{\mathfrak{T}_n\}_{n\in N}$ *is empirically adequate for* $\mathbf{P}'$*.*

Proof. '$\Rightarrow$': $\{\mathfrak{T}_n\}_{n\in N}$ is epistemically empirically adequate for $\mathbf{P}$ if and only if there is some $n \in N$ such that for every $\mathbf{Q} \in \mathbf{P}$, there are $\mathfrak{P} \in \mathbf{Q}$ and $\mathfrak{E} \in \mathbf{E}_n$ with $\mathfrak{E} \cong \mathfrak{P}$. For each $\mathbf{Q}$, choose $e(\mathbf{Q}) = \mathfrak{P}$. Then there is some $n \in N$ such that for every $\mathfrak{P} \in \mathbf{P}'$, there is an $\mathfrak{E} \in \mathbf{E}_n$ with $\mathfrak{E} \cong \mathfrak{P}$, so that $\{\mathfrak{T}_n\}_{n\in N}$ is empirically adequate for $\mathbf{P}'$.

'$\Leftarrow$': Similar. □

Thus the epistemic appearances can indeed be considered the set of epistemically possible appearances. Specifically, whenever the appearances are completely known (without any uncertainty as expressed by a multiplicity of structures), epistemic empirical adequacy is equivalent to empirical adequacy:

Claim 4. *Let* $\mathbf{P}$ *be appearances, and* $\mathbf{P}' = \{\{\mathfrak{P}\} : \mathfrak{P} \in \mathbf{P}\}$ *be epistemic appearances. Then* $\{\mathfrak{T}_n\}_{n\in N}$ *is empirically adequate for* $\mathbf{P}$ *if and only if* $\{\mathfrak{T}_n\}_{n\in N}$ *is epistemically empirically adequate for* $\mathbf{P}'$*.*

Proof. Given the epistemic appearances $\mathbf{P}' = \{\{\mathfrak{P}\} : \mathfrak{P} \in \mathbf{P}\}$, the only epistemically possible appearances are given by $\mathbf{P}$. Claim 4 now follows from claim 3. □

Thus, as required, definition 14 generalizes van Fraassen's definition 3 of empirical adequacy.

Definitions 13 and 14 rely only on concepts that are defined both for finite and infinite domains, and that are known to behave well in the transition from one to the other. Since furthermore the definitions themselves do not refer to the cardinality of the domains, it can be expected that they behave well in the transition from finite to infinite domains.

Finally, the definitions to not add any metaphysical assumptions. The only additional assumption, that lack of knowledge can be represented as multiplicity, is epistemic.

4.3 Approximate empirical adequacy

As demanded by the conditions of adequacy, one can now define approximation as a special case of lack of knowledge:

Definition 16. Approximate appearances are given by epistemic appearances $\mathbf{P}$ where each $\mathbf{Q} \in \mathbf{P}$ is a vagueness set.

Accordingly, approximate empirical adequacy is a special case of epistemic empirical adequacy:

Definition 17. A theory $\{\mathfrak{T}_n\}_{n\in N}$ is *approximately empirically adequate* for approximate appearances $\mathbf{P}$ if and only if $\{\mathfrak{T}_n\}_{n\in N}$ is epistemically empirically adequate for $\mathbf{P}$.

Since singleton sets of structures are vagueness sets, it follows immediately from claim 4 that definition 17 generalizes van Fraassen's definition 3 of empirical adequacy.

The example of the two light beams can be treated in terms of approximate empirical adequacy as well. Since there are no restrictions on the functions in the set (19) beyond that their integral over some spatial range has to fall in a specific interval, no value $\varphi(x)$ is impossible for any x. Thus, to arrive at a non-trivial vagueness set given the finite set of measurement results (5), one can phrase the approximate appearances in terms of blurred intensity functions rather than the intensity functions themselves:

$$\left\{ \left\langle \mathbb{R} \times \mathbb{R}^{\geq 0}, \varphi \right\rangle \quad : \quad \varphi(x_i) \in \left[y_i - \frac{p}{2}, y_i + \frac{p}{2} \right], 1 \leq i \leq s, \right. \\ \left. \exists \chi : \chi(x) \geq 0, \varphi(x) = \frac{1}{p} \int_{x_i - \frac{p}{2}}^{x_i + \frac{p}{2}} \chi(x) \mathrm{d}x \text{ for all } x \right\} \tag{21}$$

This is a vagueness set with the non-negative extension $F^{+\circ} = \{\langle x_i, y\rangle : y \in [y_i - q/2, y_i + q/2], 1 \leq i \leq s\}$ and a penumbral connection demanding that φ be the integral of a non-negative function. For the ray theory to be approximately empirically adequate, there must again be no measurement result y_i that deviates more than $q/2$ from I.

However, even if at some point the interference pattern $\overline{\psi}$ is measured with greater and for all purposes arbitrary accuracy, the ray theory can still be approximately true in the sense that, for some spatial resolution p, $\overline{\psi^{\text{ray}}}_p$ never deviates more than some value d from the actual blurred intensity $\overline{\psi}_p$. This leads to the approximate appearances

$$\left\{ \left\langle \mathbb{R} \times \mathbb{R}^{\geq 0}, \varphi \right\rangle \quad : \quad \left| \frac{1}{p} \int_{x - \frac{p}{2}}^{x + \frac{p}{2}} \varphi(y) \mathrm{d}y - \overline{\psi}_p(x) \right| \leq d \text{ for all } x \right\}, \tag{22}$$

where, to be precise, φ would again have to be the integral of a non-negative function. Since the extremal values for $\left|\overline{\psi}_p - I\right|$ lie on $I\lambda/(2\pi p)$, $\overline{\psi^{\text{ray}}}_p$ is a function in the approximate appearances if $I\lambda/(2\pi p) \leq d$. Thus even if better resolutions than p and q become possible, the ray theory stays approximately empirically adequate in this sense. Note that in contradistinction to recent discussions of the relation between wave- and ray theory of light (e. g., Batterman 2002, §6.2), there has been no need to take any singular limits into account.

The connection between approximate empirical adequacy and empirical adequacy is given by definition 15 and claim 3. When the epistemic appearances are approximate appearances, I will call the epistemically possible appearances sometimes *approximately possible appearances*.

Using the notion of a generalized approximate vagueness set instead of the notion of an approximate vagueness set in definitions 16 and 17 leads to the notions of *generalized approximate appearances* and *generalized approximate empirical adequacy*, which provide yet another response to the cardinality objection. The initial measurement results (5) and the intensity resolution q lead to the non-negative extension

$$\left\{\left\langle x_i, \left[y_i - \frac{p}{2}, y_i + \frac{p}{2}\right]\right\rangle\right\}_{1\leq i\leq s} \tag{23}$$

for the blurred intensity function, which contains only a finite number of spatial positions. The generalized approximate appearances, however, contain structures with domains of any cardinality greater or equal n. Since the notion of generalized approximate empirical adequacy is a special case of epistemic empirical adequacy and is related to the notion of approximate empirical adequacy by claim 1, I will not discuss it further.[21]

4.4 Hierarchies of empirical adequacy

With epistemic and approximate empirical adequacy defined relative to epistemic and approximate appearances, it would be very helpful to formalize the development of the appearances, specifically, the increase of knowledge about the phenomena. As Bueno (1997, 603) puts it, it would be helpful to have a hierarchy of appearances "built in such a way that, at each level, there is a gain of information regarding the phenomena being modeled". To this effect, I suggest

Definition 18. A *hierarchy of epistemic appearances* is a sequence $\langle \mathbf{P}_l \rangle_{l\in L}$ of epistemic appearances such that for any $m \geq l$ with $l, m \in L$, there is an injection $b : \mathbf{P}_l \longrightarrow \mathbf{P}_m$ such that for all $\mathbf{Q} \in \mathbf{P}_l$, $b(\mathbf{Q}) \subseteq \mathbf{Q}$.

A hierarchy of epistemic appearances allows the growth of knowledge in two respects. First, since b only has to be an injection, $\mathbf{P}_m$ can contain more epistemic appearances than $\mathbf{P}_l$. Second, since $b(\mathbf{Q}) \subseteq \mathbf{Q}$, the knowledge about specific

[21] Note also that generalized approximate empirical adequacy does not contain empirical adequacy as a special case, since generalized approximate appearances are never singleton sets.

appearances can grow as well. In this way, definition 18 captures the gain of knowledge from one step of the series to the next, or better, since the index set L does not have to be countable,[22] the gain of knowledge going from one point in the hierarchy (as indicated by $l \in L$) to any higher point.

As a special case of definition 18, there is

Definition 19. A *restricted hierarchy of epistemic appearances* is a hierarchy of epistemic appearance in which the injections between epistemic appearances are bijections.

A restricted hierarchy captures the idea that the increase of knowledge may be restricted to describing specific appearances more precisely, rather than considering new appearances.

A theory that is found out not to be epistemically empirically adequate at some point should not become epistemically empirically adequate when the knowledge about the phenomena increases, and vice versa, a theory epistemically empirically adequate at one point should also be epistemically empirically adequate when less is known about the appearances. This is brought out by

Claim 5. *Let $\langle \mathbf{P}_l \rangle_{l \in L}$ be a hierarchy of epistemic appearances. If theory $\{\mathfrak{T}_n\}_{n \in N}$ is epistemically empirically adequate for $\mathbf{P}_l, l \in L$, then $\{\mathfrak{T}_n\}_{n \in N}$ is epistemically empirically adequate for any $\mathbf{P}_k, k \in L, k \leq l$. If theory $\{\mathfrak{T}_n\}_{n \in N}$ is not epistemically empirically adequate for $\mathbf{P}_l, l \in L$, then $\{\mathfrak{T}_n\}_{n \in N}$ is not epistemically empirically adequate for any $\mathbf{P}_m, m \in L, m \geq l$.*

Proof. By definition 18, for any $k \leq l$ and any $\mathbf{Q} \in \mathbf{P}_k$, there is a b such that $b(\mathbf{Q}) \in \mathbf{P}_l$ and $b(\mathbf{Q}) \subseteq \mathbf{Q}$. Thus, if $\{\mathfrak{T}_n\}_{n \in N}$ is epistemically empirically adequate for $\mathbf{P}_l$, there is some $n \in N$ such that for every $\mathbf{Q} \in \mathbf{P}_k$, there are a $\mathfrak{P} \in b(\mathbf{Q}) \subseteq \mathbf{Q}$ and an $\mathfrak{E} \in \mathbf{E}_n$ with $\mathfrak{E} \cong \mathfrak{P}$. The proof of the claim's second conjunct is similar. □

Relative to such a hierarchy of epistemic appearances, it is now possible to define what it means for one theory to be more epistemically empirically adequate than another.

Definition 20. Given a (restricted) hierarchy of epistemic appearances $\langle \mathbf{P}_l \rangle_{l \in L}$, theory $\{\mathfrak{T}_n\}_{n \in N}$ is at least as (restrictedly) epistemically empirically adequate for $\langle \mathbf{P}_l \rangle_{l \in L}$ as theory $\{\mathfrak{T}_s\}_{s \in S}$ if and only if for any $l \in L$ it holds that $\{\mathfrak{T}_s\}_{s \in S}$ is epistemically empirically adequate for $\mathbf{P}_l$ only if $\{\mathfrak{T}_n\}_{n \in N}$ is epistemically empirically adequate for $\mathbf{P}_l$.

Given claim 5, definition 20 entails that if theory $\{\mathfrak{T}_n\}_{n \in N}$ is at least as epistemically empirically adequate as theory $\mathbf{S}$ and $\{\mathfrak{T}_n\}_{n \in N}$ ceases to be empirically adequate due to knowledge gain, so does $\mathbf{S}$.

[22] I thank Leszek Wroński for the suggestion to allow any series, not only finite ones.

In the example of the two light beams, the measurement results (5) lead to the epistemic appearances (19), which are determined by the spatial resolution p and the intensity resolution q. For intensity resolutions $q' \leq q$, the epistemic appearance is a subset of that for q, so that an increase in resolution (i. e., a decrease of q) leads naturally to a hierarchy of epistemic appearances. A decrease of p does not lead to such a hierarchy. Rather, changing p leads to *different* hierarchies depending on q: For $p = n\lambda/2$ and for $p \to \infty$, the deviation factor $D(p/\lambda) = 0$, so that $\overline{\psi}_p = \overline{\psi^{\text{ray}}}_p = I$. Other values $0 \leq p < \lambda$ lead to sinusoidal patterns.

The definition of a hierarchy for approximations is straightforward:[23]

Definition 21. A (restricted) hierarchy of approximate appearances is a (restricted) hierarchy of epistemic appearances in which all members are vagueness sets.

In a (restricted) hierarchy of approximate appearances, the denotations of the terms at each point in the hierarchy is at least as vague as at any higher point in the hierarchy:

Claim 6. *$\langle \mathbf{P}_l \rangle_{l\in L}$ is a (restricted) hierarchy of approximate appearances if and only if the following holds: For any $l \leq m$ with $l, m \in L$, there is an injection (bijection) $b : \mathbf{P}_l \longrightarrow \mathbf{P}_m$ such that for all $\mathbf{Q} \in \mathbf{P}_l$ with $\{R_i^+, R_i^-, F_j^{+\circ}, c_k^{+\circ}\}_{i\in I, j\in J, k\in K}$ and for $b(\mathbf{Q}) \in \mathbf{P}_m$ with $\{\tilde{R}_i^+, \tilde{R}_i^-, \tilde{F}_j^{+\circ}, \tilde{c}_k^{+\circ}\}_{i\in I, j\in J, k\in K}$ it holds that $R_i^+ \subseteq \tilde{R}_i^+$, $R_i^- \subseteq \tilde{R}_i^-$, $\tilde{F}_j^{+\circ} \subseteq F_j^{+\circ}$, and $\tilde{c}_j^{+\circ} \subseteq c_j^{+\circ}$ for all $i \in I, j \in J, k \in K$.*

Proof. '⇒': By definition 18, there is an injection (bijection) $b : \mathbf{P}_l \longrightarrow \mathbf{P}_m$ such that for all $\mathbf{Q}$, $b(\mathbf{Q}) \subseteq \mathbf{Q}$. The claim follows from (11)–(14).

'⇐': Immediate. □

In the example of the interference pattern, the approximate appearances for the intensity resolution $q' \leq q$ are again a subset of the approximate appearances (22) for q, so that an increase in resolution leads to a hierarchy of approximate appearances. Accordingly, the non-negative extensions for $q' \leq q$ are subsets of those for q.

The notion of an hierarchy of approximate appearances leads directly to a comparative version of approximate empirical adequacy:

Definition 22. Given a (restricted) hierarchy of approximate appearances $\langle \mathbf{P}_l \rangle_{l\in L}$, theory $\{\mathfrak{T}_n\}_{n\in N}$ is at least as (restrictedly) approximately empirically adequate for $\langle \mathbf{P}_l \rangle_{l\in L}$ as theory $\{\mathfrak{T}_s\}_{s\in S}$ if and only if $\{\mathfrak{T}_n\}_{n\in N}$ is at least as (restrictedly) epistemically empirically adequate for $\langle \mathbf{P}_l \rangle_{l\in L}$ as $\{\mathfrak{T}_s\}_{s\in S}$.

The hierarchies of epistemic and approximate appearances are meant to formalize specific routes of the increase of knowledge about the phenomena, that is, different series of experiments will lead to different hierarchies. According to

[23] In definitions and claims here and in the following, texts in brackets has to be either systematically included or omitted, thus leading to two different definitions and claims.

claims 3 and 5, a theory that at one point in the hierarchy is not epistemically empirically adequate is not empirically adequate. But it is also of interest at which point of a hierarchy one can be sure that a theory *is* empirically adequate.

Claim 7. *$\{\mathfrak{T}_n\}_{n\in N}$ is epistemically/approximately empirically adequate at all points of all restricted hierarchies of epistemic/approximate appearances with the initial sequence $\langle \mathbf{P}_l\rangle_{l\leq m}$ if and only if $\{\mathfrak{T}_n\}_{n\in N}$ is empirically adequate for all appearances that are epistemically possible given $\mathbf{P}_l$.*

Proof. '⇒': Assume $\{\mathfrak{T}_n\}_{n\in N}$ is restrictedly epistemically/approximately adequate at all points of all hierarchies of epistemic appearances with the initial sequence $\langle \mathbf{P}_l\rangle_{l\leq m}$. For all appearances $\mathbf{P}$ that are epistemically possible given $\mathbf{P}_l$, the sequence $\langle \mathbf{P}_m, \mathbf{P}'\rangle_{l\leq m}$ with $\mathbf{P}' = \{\{\mathfrak{P}\} : \mathfrak{P} \in \mathbf{P}\}$ as its last element is a hierarchy of epistemic/approximate appearances. Therefore $\{\mathfrak{T}_n\}_{n\in N}$ is epistemically empirically adequate for $\mathbf{P}'$, and thus, by claim 4, $\{\mathfrak{T}_n\}_{n\in N}$ is empirically adequate for $\mathbf{P}$.

'⇐': For any point $\mathbf{P}_n$ of any hierarchy with initial sequence $\langle \mathbf{P}_l\rangle_{l\leq m}$, there is a bijection $b : \mathbf{P}_m \longrightarrow \mathbf{P}_n$ with $b(\mathbf{Q}) \subseteq \mathbf{Q}$. By assumption, there is therefore a function e from $\mathbf{P}_n$ to $\bigcup \mathbf{P}_n$ with $e(\mathbf{Q}) \in \mathbf{Q} \subseteq b^{-1}(\mathbf{Q})$ such that $\{\mathfrak{T}_n\}_{n\in N}$ is empirically adequate for $\{e(\mathbf{Q}) : \mathbf{Q} \in \mathbf{P}_m\}$. Since $\{e(\mathbf{Q}) : \mathbf{Q} \in \mathbf{P}_m\}$ is epistemically possible given $\mathbf{P}_m$, by claim 3, $\{\mathfrak{T}_n\}_{n\in N}$ is epistemically/approximately empirically adequate for $\mathbf{P}_n$. □

One can also compare the epistemic and approximate empirical adequacy adequacy of theories independently of specific hierarchies of epistemic of approximate appearances:[24]

Claim 8. *$\{\mathfrak{T}_n\}_{n\in N}$ is at least as (restrictedly) epistemically/approximately empirically adequate as $\{\mathfrak{T}_s\}_{s\in S}$ for any (restricted) hierarchy of epistemic approximate appearances if and only if $\{\mathfrak{T}_n\}_{n\in N}$ is empirically adequate for all appearances $\mathbf{P}$ for which $\{\mathfrak{T}_s\}_{s\in S}$ is empirically adequate.*

Proof. '⇒': Choose the trivial (restricted) hierarchy of epistemic/approximate appearances $\langle \mathbf{P}\rangle$ with $\mathbf{P}$ containing all appearances. Then all appearances are epistemically possible appearances given $\mathbf{P}$, and by claim 3, the claim follows.

'⇐': Immediate from the definitions and claim 3. □

Claim 8 connects the preceding discussion directly with a definition by van Fraassen (1980, 67): "If for every model M of T there is a model M' of T' such that all empirical substructures of M are isomorphic to empirical substructures of M', then T is empirically at least as strong as T'". The relation is given by

Claim 9. *$\{\mathfrak{T}_s\}_{s\in S}$ is empirically adequate for all appearances $\mathbf{P}$ for which $\{\mathfrak{T}_n\}_{n\in N}$ is empirically adequate if and only if for every $n \in N$, there is an $s \in S$ such that all empirical substructures of $\mathfrak{T}_n$ are isomorphic to empirical substructures of $\mathfrak{T}_s$.*

[24]Here and in the following definitions, claims, and proofs, either the left or the right side of a slash is to be included, leading to two different definitions, claims, or proofs.

Proof. '⇒': For each $\mathbf{E}_n$, choose $\mathbf{P} = \mathbf{E}_n$. Then $\{\mathfrak{T}_n\}_{n\in N}$ is empirically adequate for $\mathbf{P}$, and thus $\{\mathfrak{T}_s\}_{s\in S}$ is empirically adequate for $\mathbf{P}$. Therefore for every $\mathfrak{E} \in \mathbf{E}_n = \mathbf{P}$ there is an $s \in S$ and an $\mathfrak{E}' \in \mathbf{E}_s$ such that $\mathfrak{E} \cong \mathfrak{E}'$.

'⇐': Assume that for some $\mathbf{P}$, $\{\mathfrak{T}_s\}_{s\in S}$ but not $\{\mathfrak{T}_n\}_{n\in N}$ is empirically adequate. Then there is an $s \in S$ such that for all $\mathfrak{P} \in \mathbf{P}$, there is an $\mathfrak{E}' \in \mathbf{E}_s$ with $\mathfrak{P} \cong \mathfrak{E}'$. Since there is no such $n \in N$ and isomorphy is transitive, there is no n such that for all $\mathfrak{E}' \in \mathbf{E}_s$, there is an $\mathfrak{E} \in \mathbf{E}_n$ with $\mathfrak{E}' \cong \mathfrak{E}$. □

Thus, by claims 7 and 9, if $\{\mathfrak{T}_n\}_{n\in N}$ is at least as (restrictedly) epistemically/approximately adequate as $\{\mathfrak{T}_s\}_{s\in S}$ for any (restricted) hierarchy of epistemic/approximate appearances, then $\{\mathfrak{T}_s\}_{s\in S}$ is empirically at least as strong as $\{\mathfrak{T}_n\}_{n\in N}$.

5 Conclusion

Looking at the inspiration for empirical adequacy has provided a way to generalize empirical adequacy to avoid implausible presumptions about the relations of theoretical concepts and observations, and to include lack of knowledge and approximations. As the discussion of the ray theory of light has shown, these generalizations are applicable to real life examples. The generalizations also connect fruitfully to each other and to the concepts developed by van Fraassen.

If the generalizations may seem more complicated than one would expect from previous generalizations, this has two reasons. For one, previous generalizations have misconstrued van Fraassen's concept of empirical adequacy, which led to a simplification. Furthermore, and this will be one result of the companion piece to this article, the previous generalizations are inadequate even for this simplification.

On the other hand, one may object that the generalizations given are still too simplistic, especially in that they do not take statistical methods into account. I can only agree. The concepts suggested here are first steps towards even more general accounts.

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