# Soliton Coupling Driven by Phase Fluctuations in Auto-Parametric Resonance

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In this paper the interaction of sine-Gordon solitons and mediating linear waves is modelled by a special case of auto-parametric resonance, the Rayleigh-type self-excited non-linear autonomous system driven by a statistical phase gradient related to the soliton energy. Spherical symmetry can stimulate "whispering gallery modes" (WGM) with integral coupling number $M = 137$.

*Introduction.* For nonlinear field theory models in 1+1–dimensional space–time the equations of motion admit finite energy and finite width solutions called solitons [1]. The interest in such low-dimensional sine–Gordon (SG) models as an universal concept in nonlinear science arises from their integrability, duality properties, non-perturbative aspects, and electric-magnetic duality in gauge theories. The solitons retain their identity after collisions, can annihilate with anti–solitons, many–soliton solutions obey Pauli's exclusion principle. As pointed out by Skyrme [2] this can be interpreted as a fermion–like behavior. Vertex solitons are localized and non-singular solutions of a non-linear field theory, whereas the energy-momentum transfer between solitons can be assigned to linear bosons. In this paper we will try to determine the coupling strength of vertex soliton and linear waves. With the focus on low-dimensional autonomous behavior, the interaction can be approached by the equation of Lord Rayleigh with coupling and dissipation driven by phase fluctuations. The second order equation includes two terms that balance wave dissipation and regeneration as a special case of self–excited auto–parametric resonance [3]. This system can i.e. model simplified music instruments (in the original work of Rayleigh a clarinet reed).

*Preliminary measurements.* It is well known, that the Josephson junction qualifies as the proper nonlinear medium for the experimental study of quantum interaction between linear waves and vertex solitons. Since linear waves and vertex solitons oscillate and propagate in the same medium, the perturbed SG equation will have the same dissipation and regeneration terms as the linear wave differential equation. In recent experiments [4, 5] the excitation of linear electromagnetic modes by a topological SG soliton in a cylindrically curved Josephson junction waveguides has been reported. The Rayleigh terms show a small difference to the dissipation/regeneration terms in [4, 5] that describe their experimental setup using phenomenological electrode/junction impedance terms. The excited quantum modes that mediate circularly and couple to solitons refer to the classical acoustic phenomenon with name "whispering gallery modes" (WGM) [6]. The interaction results in the quantum current–voltage characteristic of the junction.

*Quantum field theory.* In quantum field theory, particles correspond to fluctuations of a space-time function that minimizes the action for some Lagrangian $L$ which is itself the space integral of a Lagrangian density $\mathcal{L}$. A low-dimensional (bosonic) SG field of a single hermitian scalar field $\theta$ in 1+1 dimensions with Lagrangian density $\mathcal{L} = \frac{\mu}{2}\partial_\nu\theta\partial^\nu\theta - V(\theta)$ is a function of one space dimension and time. The time independent field equations reads $\mu\partial_r^2\theta = \partial_\theta V$ which can also be written as

$$V(\theta) = \frac{\mu}{2}(\partial_r\theta)^2 \,. \tag{1}$$

In the quantum model, the expectation of the square of the phase gradient contributes to the soliton self-energy, where the phase gradient usually scales proportional to the wave number that is proportional to the frequency.

*Rayleigh–type perturbed SG.* The coupling between linear waves and non–linear solitons could be important for the understanding for scale-dependent electromagnetic coupling. The Rayleigh-type perturbed SG equation is given by

$$(\partial_{rr} - \partial_{tt})\,\theta - \sin\theta = \left[b^2(\partial_r\theta)^2 - a^2\right]\partial_t\theta \,, \tag{2}$$

where $\partial_{rr} - \partial_{tt}$ is the D'Alembert wave operator. On the left side are the usual SG terms, on the right side are with eq.(1) the typical Rayleigh wave dissipation and regeneration terms. These two terms balance wave dissipation and regeneration and can be found in self–excited auto-parametric systems [3]. The term $b^2(\partial_r\theta)^2 - a^2$ controls $\theta$ by weighting the dissipative first order term $\partial_t\theta$. The amplitude of the coupling wave can be assigned to a phase difference between two solitons.

*Rayleigh–type auto–parametric system.* Following [3] we characterize auto–parametric systems as a vibrating system which consists of at least two subsystems: an oscillator that is generally in a vibrating state and the excited system which is excited indirectly and coupled to the oscillator in a nonlinear way. A linear wave will be defined as a disturbance which transports energy from one location to another location without transporting matter evolving at light speed $c$ with $c\,\partial_r\theta = \partial_t\theta$. In the auto–parametric system the dissipation and excitement parameter will depend on the phase statistics. The Rayleigh–type system

$$\partial_{tt}\theta + \theta = a^2\partial_t\theta - b^2c^{-2}(\partial_t\theta)^3 \tag{3}$$

is called self–excited auto–parametric, and can be exactly treated by phase averaging methods providing for the semi–trivial solution [3]. The excitation is here also balanced by the (soliton vortex) regeneration term $V(\theta)$ of eq.(1) with parameter $b^2$ and (linear mode) damping term $\partial_t\theta$ with parameter $a^2$. So $a$, $b$, and $c$ control the amplitude $\theta$ of intermediating linear waves.

*The coupling mechanism.* The vertex solitons are assumed to be identical and not synchronized in phase, such that the excitement of coupling modes is based on the (statistical) phase difference. Let the SG solitons be coupled such, that the amplitude of the perturbed linear wave will be the phase difference $\theta$ of the perturbed nonlinear SG solitons controlled by the potential of the phase gradient $V(\theta)$. Usually the non-trivial equilibrium fluctuation amplitude $\theta_0$ can be found with given parameter $a$ and $b$ or ratio $q = b/a$, but in our case the scattering phase fluctuation amplitude $\theta_0$ is given by the relative phase fluctuation range of solitons. Regarding the kink-antikink solution there exists a characteristic time scale or oscillation frequency that can be related via coupling wave velocity to a characteristic length scale or wavelength $\lambda_\mu$. The fermion–like behavior pointed out by Skyrme [2] suggests to define an additive phase–fluctuation range by $2\times 2\pi = 4\pi$. Therefore, the coupling amplitude for the mediating wave will be given by

$$\theta_0 = 4\pi\lambda_\mu \,. \tag{4}$$

Later $\lambda_\mu$ will be reintroduced as the Compton wavelength of wave–soliton coupling.

*Phase averaging.* Statistically, there are phase oscillations or fluctuations randomly modulated in a special phase interval, so we can assume in the most simple case a maximum entropy phase modulation and fluctuation between two vertex solitons and find the most likely energy flow and coupling ratio from phase averaged solutions [7]. The phase–amplitude modulation with random $\tau$ as a solution to eq.(3) can be written as [3]

$$\theta = \theta_0 \cos(\tau + \psi_0)\,, \tag{5}$$

averaging over $\tau$ yields [3, 7] with constant $\psi_0$

$$\partial_t\theta_0 \propto \frac{1}{2}\left(a^2\theta_0 - \frac{3b^2}{4\lambda_\mu^2}\theta_0^3\right), \quad \partial_t\psi_0 = 0\,. \tag{6}$$

*1-dimensional coupling soliton energy.* With constant fluctuation strength $\partial_t\theta_0 = 0$, the averaged equations provide with eq.(4) and eq.(6) for a balanced non-trivial equilibrium

$$q = \frac{b}{a} = \frac{1}{4\pi\sqrt{\frac{3}{4}}}, \quad \frac{2\overline{V}}{E_{1d}} = \overline{(\partial_r\theta)^2} = q^{-2}\,. \tag{7}$$

Consequently, the 1-dimensional coupling strength $q$ will enter the energy definitions via mean unit energy $E_{1d}$ of 1-dimensional coupling

$$E_{1d} = q^2\overline{(\partial_r\theta)^2} = 1\mu c^2\,, \tag{8}$$

where $\mu$ is a unit mass. To compare our theoretical soliton coupling model to real existing couplings, mass/energy has to be quantified. Let the unit mass be the 1d coupling mass–energy of a soliton with $E_{1d} = 1\mu c^2$. The mutual 1-d coupling to a photon with amplitude/wavelenght fluctuation $\lambda_\mu$ can be regarded as a permanent Compton scattering process with mass–energy value related to $\lambda_\mu$ via Compton relation

$$E_{1d} = 2q^2\overline{V} = \frac{hc}{\lambda_\mu}\,. \tag{9}$$

*3-dimensional coupling.* The Gauss relation can connect the 1-d coupling strength to a 3-d coupling strength with a spherical symmetric potential $\phi_{3d}(r)$ such, that the radial coupling energy is defined by

$$E_{3d}(r) = \frac{q}{\epsilon_0}\,\phi_{3d}(r)\,, \tag{10}$$

with

$$E_{3d}(r) = -\frac{1}{\epsilon_0}\int_\infty^r \phi_{3d}^2 4\pi dr' = \frac{q^2}{4\pi\epsilon_0 r}, \quad \phi_{3d} = \frac{q}{4\pi r}\,. \tag{11}$$

Let's define the fine structure by

$$\alpha = \frac{q^2}{4\pi\epsilon_0\hbar c} = \frac{E_{3d}(\lambda_\mu)}{E_{1d}}\,, \tag{12}$$

where the relations at the special reference distance $\lambda_\mu$ given by dimensionless Planck units $h = c = \lambda_\mu = 1$ must obey the unit condition

$$E_{3d}(\alpha\lambda_\mu) = E_{1d} = \phi_{3d}(\alpha\lambda_\mu) = \phi_{1d} \equiv 1\,, \tag{13}$$

that provides for

$$\alpha = \frac{q}{4\pi} = \frac{1}{136,75725...}\,. \tag{14}$$

eq.(7)-eq.(14) correspond to a flux density reduction given by an 3-dimensional integral flux factor $M$

$$M = \left[\frac{1}{\alpha}\right] = 137\,, \tag{15}$$

where [ ] means next higher integral value. Why integral? When the round-trip path fits integer numbers of the wavelength (single–valuedness), WGM are formed.

*Virtual photons.* In [8, 9, 10] it was possible to describe coupling vertex solitons in (auto)resonance by a circularly closed system of Bäcklund transformations. With projective geometry and symmetry the local soliton coupling can be be assigned via Bohlin transformation to a non-local Coulomb potential, and leads to an iterative determination of generalized fine structure constants, provided the coupling strength is known. It turned out that $M = 137$ with $q = e$ corresponds to the Sommerfeld fine structure constant $\alpha = e^2/(4\pi\epsilon_0\hbar c)$ according to the Dirac theory of magnetic monopoles [11] (a generator of the topological Berry phase [12]). The meaning behind $M = 137$ now becomes clearer: virtual photons (the mediator of the electromagnetic force) could act similarly to Lord Rayleighs WGM in a circular quantum billiard-type orbit. The projective resonance condition that provides for generalized fine structure constants in [8, 9, 10] is given by $\partial_r\theta \propto r$, the energy expectation in eq.(7) is a typical spatial variance $\overline{(r - r_0)^2}$ driven by phase fluctuations.

*Conclusion.* The interaction of linear waves and nonlinear solitons realizes a self-excited and integrable autonomous system driven by phase fluctuations defined in eq.(4) and eq.(5). The non-linear regeneration term can be decomposed into the product of a vertex soliton energy factor $(\partial_r\theta)^2$ and a linear dissipation factor $\partial_t\theta$. The balancing factor $q^{-2} = 12\pi^2$ and also $M = 137$ indicate, that the averaged phase fluctuations providing for the soliton energy are considerably reduced with respect to the linear wave or photon amplitude of eq.(4). Regarding the space variance of the soliton compared to the photon wavelength this would mean, that the interaction is driven by a thermodynamic phase space compression of strength $q^{-2}$.

*Outlook.* In the successive paper [13] the parameters of dissipative terms determine not only the most likely quantum coupling between solitons and linear waves but also the most likely mass $\mu$ of the solitons in SI-units. With proper scaling relations in length and velocity, $\mu$ shows a highly interesting sequence: the is about 1.00138 times the neutron and 1.00276 times the proton mass.


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