

A formal framework for the study of the notion of undefined particle number in quantum mechanics

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Abstract

It is usually stated that quantum mechanics presents problems with the identity of particles, the most radical position -supported by E. Schrödinger- asserting that *elementary particles are not individuals*. But the subject goes deeper, and it is even possible to obtain states with an undefined particle number. In this work we present a set theoretical framework for the description of undefined particle number states in quantum mechanics which provides a precise logical meaning for this notion. This construction goes in the line of solving a problem posed by Y. Manin, namely, *to incorporate quantum mechanical notions at the foundations of mathematics*. We also show that our system is capable of representing quantum superpositions.

Key words: set theory-undefined particle number

1 Introduction

Quantum mechanics (QM) is considered as one of the most important physical theories of our time, giving rise to spectacular technological developments. Yet, interpretation of quantum mechanics still gives rise to difficult problems, which are far from finding a definitive solution. But this is, perhaps, one of the most interesting features of *QM*. The striking and radical interpretations which were posed by the foundation fathers of *QM* gave rise to a lot of interesting problems. In this work, we will concentrate in a generalization of a problem posed by Y. Manin. In his words [1, 2]

“We should consider the possibilities of developing a totally new language to speak about infinity. Set theory is also known as the theory of the ‘infinite’. Classical critics of Cantor (Brouwer *et al.*) argued that, say, the general choice axiom is an illicit extrapolation of the finite case.

I would like to point out that this is rather an extrapolation of common-place physics, where we can distinguish things, count them, put them in some order, etc. New quantum physics has shown us models of entities with quite different behavior. Even ‘sets’ of photons in a looking-glass box, or of electrons in a nickel piece are much less Cantorian than the ‘set’ of grains of sand. In general, a highly probabilistic ‘physical infinity’ looks considerably more complicated and interesting than a plain infinity of ‘things’.”[1]

Thus, Manin suggests the development of set theories [3, 4, 5, 6] incorporating the novel features of quantum entities, which depart radically from our every day concepts¹. In this line, many alternatives were developed, most of them grounded in non-reflexive logics [7, 8]. In particular, it is possible to incorporate in a Zermelo-Frenkel (*ZF*) set theory the notion of quantum non-individuality [9, 10, 11, 12, 13] and this was done by introducing indistinguishability “right at the start” [14]. According to the interpretation of E. Schrödinger (but we remark that there are different interpretations, see for example [12, 15, 16]) an elementary particle cannot be considered as an individual entity

“I mean this: that the elementary particle is not an individual; it cannot be identified, it lacks ‘sameness’. The fact is known to every physicist, but is rarely given any prominence in surveys readable by nonspecialists. In technical language it is covered by saying that the particles ‘obey’ a newfangled statistics, either Einstein-Bose or Fermi-Dirac statistics. [...] The implication, far from obvious, is that the unsuspected epithet ‘this’ is not quite properly applicable to, say, an electron, except with caution, in a restricted sense, and sometimes not at all.” E. Schrödinger ([17], p.197)

There is another important branch of formal developments induced by quantum mechanics, namely, a vast family of quantum logics. Since the seminal paper of Birkhoff and von Neumann [18], several investigations were motivated in the fields of logic, algebraic logic, and the foundations of physics. Besides these developments, some authors have claimed that according to the logical structure of QM, we should *abandon classical logic* (see for example [19]). On the other hand, the now days dominant interpretation of the quantum logical formalism developed by Birkhoff and von Neumann considers it as the study of algebraic structures linked to QM, and by no means is considered as an alternative to classical logic. Notwithstanding, it is important to remark that there are several examples of modifications of classical in the following sense. Even if it is a subtle matter to define exactly what classical logic is, it is possible to consider it as having two levels:

- 1) an *elementary level*, which is essentially first order predicate calculus, with or without identity, and
- 2) a *non elementary level*, which could be a set theory, a category theory, or a theory of logical types.

It is then possible to modify level 2 in order to develop a family of logics which can be considered non-classical. Indeed, the system presented in this paper is non-classical in the sense mentioned above. It is also possible to modify level 1, as shown in [20]. Of course, the existence of these possibilities does not suffice to settle the question about the adequacy or non adequacy of classical logic. Thought we will not discuss this subject in detail in this paper, we remark that

¹Although Manin has seemingly changed his position regarding this subject [3], the problem posed above still seems interesting to us and we will take it as a basis for our work.

it is a matter of fact that the influence of QM in the development of formal systems gave rise to a considerable proliferation of investigations [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40], including the development of “quantum set theories” [41, 42, 43].

Coming back to the problem of quantum indistinguishability, Michael Readhead and Paul Teller claim [44, 45] that:

“Interpreters of quantum mechanics largely agree that classical concepts do not apply without alteration or restriction to quantum objects. In Bohr’s formulation this means that one cannot simultaneously apply complementary concepts, such as position and momentum, without restriction. In particular, this means that one cannot attribute classical, well defined trajectories to quantum systems. But in a more fundamental respect it would seem that physicists, including Bohr, continue to think of quantum objects classically as individual things, capable, at least conceptually, of bearing labels. It is this presumption and its implications which we need to understand and critically examine.” M. Readhead and P. Teller ([45], p.202)

The last quote clearly points in the direction of “critically examining” the assertion about the individuality of quantum entities. In a similar vein, the usual assumption that a *definite particle number* can be always obtained was also criticized. Thus, a new turn of the Manin’s problem was presented in [46, 47, 48, 49, 50]. In this work we will follow the interpretation of QM which denies that quantum systems can be always considered as singular unities (a quantum system as a “one”), or collections of them (a quantum system as a “many”).

It is important to remark here that there are other interpretations which deny the existence of systems with undefined particle number. In such interpretations, states which involve superpositions with different particle number are interpreted as ordinary mixtures. Our interest in this work is not to give arguments in favor or against each interpretation, but to provide a framework for studying the consequences of assuming that undefined particle number states actually exist.

We will face the problems linked to *undefined particle number* and -going in the line of the Manin’s problem- we will develop a formal set theoretical framework capable of incorporating such a quantum mechanical feature. This is the reason why our system could be considered as a solution to a generalization of the problem posed by Manin. We believe that the formal setting presented in this work could be a concrete step —for interpretational purposes— to give a precise logical meaning to what is meant by “undefined particle number” by incorporating this notion into a set theoretical framework. And also that it constitutes in itself an interesting structure for the possible development of new non-standard mathematics, which in turn, could be the basis for new formal frameworks with potential applications to physics. As an example of this procedure see [51].

The article is organized as follows. In section 2, we discuss the meaning of superpositions of particle number eigenstates in Fock-space, introducing the interpretation which supports the existence of undefined particle number states. In section 3 we present the preliminary notions of our set theoretical framework by introducing its specific axioms. After doing this, we are ready to show how our framework solves the problem of incorporating undefined particle number in section 4, and also that it is capable of describing quantum superpositions. We will also present in this Section some special features of our axiomatic and general remarks about our construction, which could be useful for further developments. Finally, we pose our conclusions in 5.

2 Undefined particle number: an overview

In order that a superposition of states with different particle number occur, it is necessary to have a space which includes states with different particle number. This is provided by the Fock-

Space formalism (*FSF*). The *FSF* is used, for example, in the *second quantization formalism*, and we find a version of it both in relativistic and non-relativistic quantum mechanics. In the following subsection we make a brief review of the non-relativistic case, which will suffice for our purposes. After that, we show how to construct a superposition in particle number in the quantized theory of the electromagnetic field, namely, a coherent state capable of being produced in the laboratory. There are other more involved examples of undetermined particle number, as is the case of Rindler quanta [52] or the BCS state of Bose-Einstein condensates [53], but we will not treat them here. We will concentrate on coherent states of the electromagnetic field in order to make the exposition simpler.

2.1 Fock-space formalism

It can be shown that the *FSF* may be used as an alternative approach to non relativistic quantum mechanics [54]. Let us review how this is done in order to understand second quantization. For the sake of simplicity, we will do this using the heuristic approach presented in elementary expositions like [53, 54] (but see for example [52], [55] and [56] for a mathematically rigorous presentation). Remember first that the standard wave mechanics approach presupposes a kinetic energy

$$T_1(r) = -\left(\frac{\hbar^2 \nabla^2}{2m}\right) \quad (1)$$

which for the n particles case takes the form

$$T_n = \sum_{i=1}^n T_1(r_i) \quad (2)$$

with a similar equation for the external potential. Suppose that we have a pairwise interaction potential

$$V_n = \sum_{i>j=1}^n V_2(\mathbf{r}_i, \mathbf{r}_j) \quad (3)$$

The total hamiltonian operator is thus given by

$$H_n = \sum_{i=1}^n \left[-\frac{\hbar^2 \nabla_i^2}{2m} \right] + V_1(\mathbf{r}_i) + \sum_{i>j=1}^n V_2(\mathbf{r}_i, \mathbf{r}_j) \quad (4)$$

and the n -particles wave function

$$\Psi_n(\mathbf{r}_1, \dots, \mathbf{r}_n, t) \quad (5)$$

is a solution of the Schrödinger's equation

$$H_n \Psi_n = i\hbar \frac{\partial}{\partial t} \Psi_n \quad (6)$$

The second quantization approach to QM has its roots in considering equation (6) as a *classical field equation*, and its solution $\Psi_n(\mathbf{r}_1, \dots, \mathbf{r}_n)$ as a *classical field to be quantized*. This alternative view was originally adopted by P. Jordan [57, 58], one of the foundation fathers of quantum mechanics, and spread worldwide after the Dirac's paper [59]. And it is a standard way of

dealing with relativistic quantum mechanics (canonical quantization). The space in which these quantized fields operate is the Fock-space. The standard Fock-space is built up from the one particle Hilbert spaces. Let \mathcal{H} be a Hilbert space and define:

$$\begin{aligned}\mathcal{H}^0 &= \mathcal{C} \\ \mathcal{H}^1 &= \mathcal{H} \\ \mathcal{H}^2 &= \mathcal{H} \otimes \mathcal{H} \\ &\vdots \\ \mathcal{H}^n &= \mathcal{H} \otimes \cdots \otimes \mathcal{H}\end{aligned}\tag{7}$$

The Fock-space is thus constructed as the direct sum of n particle Hilbert spaces:

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{H}^n\tag{8}$$

When dealing with bosons or fermions, the symmetrization postulate (SP) must be imposed. In order to do so, given a vector $v = v_1 \otimes \cdots \otimes v_n \in \mathcal{H}^n$, define:

$$\sigma^n(v) = \left(\frac{1}{n!}\right) \sum_P P(v_1 \otimes \cdots \otimes v_n)\tag{9}$$

and:

$$\tau^n(v) = \left(\frac{1}{n!}\right) \sum_P s^p P(v_1 \otimes \cdots \otimes v_n)\tag{10}$$

where:

$$s^p = \begin{cases} 1 & \text{if } p \text{ is even,} \\ -1 & \text{if } p \text{ is odd.} \end{cases}$$

Calling

$$\mathcal{H}_\sigma^n = \{\sigma^n(v) : v \in \mathcal{H}^n\}\tag{11}$$

and:

$$\mathcal{H}_\tau^n = \{\tau^n(v) : v \in \mathcal{H}^n\}\tag{12}$$

we have the Fock-space

$$\mathcal{F}_\sigma = \bigoplus_{n=0}^{\infty} \mathcal{H}_\sigma^n\tag{13}$$

for bosons and

$$\mathcal{F}_\tau = \bigoplus_{n=0}^{\infty} \mathcal{H}_\tau^n\tag{14}$$

for fermions. Now the second quantization procedure considers the one particle wave function $\psi(\mathbf{r})$ and its hermitian conjugate $\psi(\mathbf{r})^\dagger$ as operators acting on the Fock-space and satisfying:

$$\begin{aligned} [\psi(\mathbf{r}), \psi(\mathbf{r}')]_{\mp} &= 0 \\ [\psi(\mathbf{r})^\dagger, \psi(\mathbf{r}')^\dagger]_{\mp} &= 0 \\ [\psi(\mathbf{r}), \psi(\mathbf{r}')^\dagger]_{\mp} &= \delta_{\mathbf{r}-\mathbf{r}'} \end{aligned} \quad (15)$$

where $\delta(\mathbf{r} - \mathbf{r}')$ is the Dirac delta function. If A and B are operators, the brackets are defined by:

$$[A, B]_{\mp} = AB \mp BA \quad (16)$$

The n particle wave function $\Psi_n(\mathbf{r}_1, \dots, \mathbf{r}_n)$ of the standard formulation is now written as

$$|\psi_n\rangle = (n!)^{-\frac{1}{2}} \int d^3r_1 \cdots \int d^3r_n \psi(\mathbf{r}_1)^\dagger \cdots \psi(\mathbf{r}_n)^\dagger |0\rangle \Psi_n(\mathbf{r}_1, \dots, \mathbf{r}_n) \quad (17)$$

and it is an eigenvector (with eigenvalue n) of the particle number operator:

$$N := \int d^3r \psi(\mathbf{r})^\dagger \psi(\mathbf{r}) \quad (18)$$

and we find the relation

$$\Psi_n(\mathbf{r}_1, \dots, \mathbf{r}_n) = (n!)^{-\frac{1}{2}} \langle 0 | \psi(\mathbf{r}_1) \cdots \psi(\mathbf{r}_n) | \Psi_n \rangle \quad (19)$$

An arbitrary vector of the Fock-space will be a superposition of states with different particle number of the form

$$|\Psi\rangle = \sum_{n=0}^{\infty} |\Psi_n\rangle \quad (20)$$

and will not be in general an eigenstate of the particle number operator. Thus, according to the standard interpretation of QM, its particle number will be undetermined. This is very important, because in the presence of particle interactions, the states may evolve into an undefined particle state like (20).

The kinetic energy operator (1) can now be written as

$$T = \int d^3r \psi^\dagger(\mathbf{r}) T_1(r) \psi(\mathbf{r}) \quad (21)$$

It can also be shown that:

$$T|\Psi_n\rangle = (n!)^{-\frac{1}{2}} \int d^3r_1 \cdots \int d^3r_n \Psi_n^\dagger(\mathbf{r}_1) \cdots \Psi_n^\dagger(\mathbf{r}_n) |0\rangle \sum_{i=1}^n T_1(\mathbf{r}_i) \Psi_n(\mathbf{r}_1, \dots, \mathbf{r}_n) \quad (22)$$

The pairwise interaction potential $V_2(\mathbf{r}, \mathbf{r}')$ can be written as

$$V = \frac{1}{2} \int d^3r \int d^3r' \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}') V_2(\mathbf{r}, \mathbf{r}') \psi(\mathbf{r}') \psi(\mathbf{r}) \quad (23)$$

Its action on $|\Psi_n\rangle$ is given by:

$$V|\Psi_n\rangle = (n!)^{-\frac{1}{2}} \int d^3r_1 \cdots \int d^3r_n [V, \psi^\dagger(\mathbf{r}_n) \cdots \psi^\dagger(\mathbf{r}_1)] |\mathbf{0}\rangle \Psi_n(\mathbf{r}_1 \cdots \mathbf{r}_n) \quad (24)$$

and it follows that:

$$\begin{aligned} V|\Psi_n\rangle &= (n!)^{-\frac{1}{2}} \int d^3r_1 \cdots \int d^3r_n \psi^\dagger(\mathbf{r}_n) \cdots \psi^\dagger(\mathbf{r}_1) |\mathbf{0}\rangle \\ &\times \sum_{i=1}^n \sum_{j=1}^{i-1} V_2(\mathbf{r}_i, \mathbf{r}_j) \Psi_n(\mathbf{r}_1, \dots, \mathbf{r}_n) \end{aligned} \quad (25)$$

It can be shown that that the following equations holds:

$$T_n \Psi_n(\mathbf{r}_1, \dots, \mathbf{r}_n) = (n!)^{-\frac{1}{2}} \langle 0 | \Psi(\mathbf{r}_1) \cdots \Psi(\mathbf{r}_n) T | \Psi_n \rangle \quad (26)$$

$$V_n \Psi_n(\mathbf{r}_1, \dots, \mathbf{r}_n) = (n!)^{-\frac{1}{2}} \langle 0 | \Psi(\mathbf{r}_1) \cdots \Psi(\mathbf{r}_n) V | \Psi_n \rangle \quad (27)$$

where:

$$T_n = \sum_{i=1}^n T_1(\mathbf{r}_i) \quad (28)$$

$$V_n = \sum_{i>j=1}^n V_2(\mathbf{r}_i, \mathbf{r}_j) \quad (29)$$

The equivalence with wave mechanics can now be established as follows. If $\Psi_n(\mathbf{r}_1, \dots, \mathbf{r}_n)$ satisfies the n particle Schrödinger wave equation with Hamiltonian (4), it follows that in the Fock-space formulation $|\Psi_n\rangle$ must satisfy the Fock-space Schrödinger equation:

$$[i\hbar \left(\frac{\partial}{\partial t}\right) - H] |\Psi_n\rangle = 0 \quad (30)$$

with $H = T + V$: given by:

$$\begin{aligned} H &= \int d^3r \psi^\dagger(\mathbf{r}) \left[\left(-\frac{\hbar^2 \nabla^2}{2m}\right) + V_1(\mathbf{r}) \right] \psi(\mathbf{r}) \\ &+ \frac{1}{2} \int d^3r \int d^3r' \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}') V_2(\mathbf{r}, \mathbf{r}') \psi(\mathbf{r}') \psi(\mathbf{r}) \end{aligned} \quad (31)$$

It is important to remark that the n particle Schrödinger wave equation is not completely equivalent to its analogue in the Fock-space formalism. Only solutions of the Fock-space equation which are eigenvectors of the particle number operator with particle number n can be solutions of the corresponding n particle Schrödinger wave equation. And the other way around, not all the solutions of the n particle Schrödinger wave equation can be solutions of the Fock equation, only those which are symmetrized do. *Then, both conditions, definite particle number and symmetrization, must hold in order that both formalisms yield equivalent predictions.*

Observables are usually expressed in terms of creation and annihilation operators. The quantized field may be expanded as

$$\psi(\mathbf{r}) = \sum_k a_k u_k(\mathbf{r}) \quad (32)$$

and the coefficients of the expansion will be the annihilation operators:

$$a_k = \int d^3r u_k^*(\mathbf{r}) \psi(\mathbf{r}) \quad (33)$$

A similar expansion stands for the creation operator: a_k^\dagger . The usual interpretation is that a_k^\dagger describes the “creation of a particle” with wave function $u_k(\mathbf{r})$, while a_k describes the “annihilation of a particle”. These operators satisfy the commutation relations

$$\begin{aligned} [a_k, a_l]_{\mp} &= 0 \\ [a_k^\dagger, a_l^\dagger]_{\mp} &= 0 \\ [a_k, a_l^\dagger]_{\mp} &= \delta_{kl} \end{aligned} \quad (34)$$

$$N_k = a_k^\dagger a_k \quad (35)$$

It is possible to put things in a more familiar way by using the “[,]” symbol for bosonic commutation relations

$$[a_\alpha; a_\beta^\dagger] = a_\alpha a_\beta^\dagger - a_\beta^\dagger a_\alpha = \delta_{\alpha\beta} I \quad (36)$$

$$[a_\alpha^\dagger; a_\beta^\dagger] = 0 \quad (37)$$

$$[a_\alpha; a_\beta] = 0 \quad (38)$$

and the symbol $\{; \}$ for fermions (with C_α^\dagger and C_α playing the role of fermionic creation and annihilation operators respectively),

$$\{C_\alpha; C_\beta^\dagger\} = C_\alpha C_\beta^\dagger + C_\beta^\dagger C_\alpha = \delta_{\alpha\beta} I \quad (39)$$

$$\{C_\alpha^\dagger; C_\beta^\dagger\} = 0 \quad (40)$$

$$\{C_\alpha; C_\beta\} = 0 \quad (41)$$

Substitution of (32) in (31) yields:

$$H = \sum_{kl} a_k^\dagger T_{kl} a_l + \frac{1}{2} \sum_{klpq} a_k^\dagger a_l^\dagger V_{klpq} a_p a_q \quad (42)$$

where the matrix elements T_{kl} and V_{klpq} are given by:

$$\begin{aligned}
T_{kl} &= \int d^3r u_k^*(\mathbf{r}) \left[\left(-\frac{\hbar^2 \nabla^2}{2m} \right) + V_1(\mathbf{r}) \right] u_l(\mathbf{r}) \\
V_{klpq} &= \int d^3r \int d^3r' u_k^*(\mathbf{r}) u_l^*(\mathbf{r}') V_2(\mathbf{r}, \mathbf{r}') u_p(\mathbf{r}') u_q(\mathbf{r})
\end{aligned} \tag{43}$$

and similar expressions can be found for more general obserbables.

2.2 Coherent states of the electromagnetic field

Once that the second quantization procedure is outlined, we are ready to show a quantum state which can be realized in the laboratory. Let us consider electromagnetic Maxwell's equations (without density charges)

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \tag{44a}$$

$$\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \tag{44b}$$

$$\nabla \cdot \mathbf{E} = 0 \tag{44c}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{44d}$$

and then apply second quantization to them. This means that we must quantize its solutions, and this means to elevate fields $\mathbf{E}(x)$ and $\mathbf{B}(x)$ to the category of operators, call them $\hat{\mathbf{E}}(x)$ and $\hat{\mathbf{B}}(x)$. Now, after some algebra, it is possible to decompose $\hat{\mathbf{E}}(x)$ and $\hat{\mathbf{B}}(x)$ in terms of fourier expansion

$$\hat{\mathbf{E}}(x) = \sum_k (2\pi\hbar\omega_k)^{\frac{1}{2}} \{ a_k^\dagger(t) e_k^*(x) + a_k(t) e_k(x) \} \tag{45}$$

The hamiltonian of the m th mode of the field can be written in terms of the creation and annihilation operators a_k^\dagger and a_k

$$H_n = \hbar\omega (a_k^\dagger a_k + \frac{1}{2}) \tag{46}$$

and so, each a_m^\dagger (a_m) creates (annihilates) a photon in mode m . Then, a fock space state can be expressed as

$$|n_1, n_2, \dots, n_m, \dots\rangle = |n_1\rangle \otimes |n_2\rangle \otimes \dots \otimes |n_m\rangle \otimes \dots \tag{47}$$

with n_i the number of photons present in each mode of the field. If for simplicity we concentrate in only one frequency mode of the field, we can create any normalized superposition of states, and in particular, the famous *coherent state*

$$|z\rangle = \exp\left(-\frac{1}{2}|z|^2\right) \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{\frac{1}{2}}} |n\rangle \tag{48}$$

which can be realized in laboratory [53]. State (48) is clearly a superposition of different photon number states and thus is not an eigenstate of the particle number operator. It follows that,

according to the standard interpretation, it represents a physical system formed by an undefined number of photons. It is important to remark that there are -at least- two interpretations of (48)

- 1-Equation (48) represents an statistical mixture of states with definite particle number.
- 2-Equation (48) represents an state which has no definite particle number.

The orthodox interpretation of QM points in the direction of the second option and the first one is very difficult to sustain unless involved hypotheses are made [60]. Regardless the interpretational debate, it will suffice for us that *there exists at least one interpretation compatible with quantum mechanics in which particle number is undefined*. Thus, given that systems in states like (48) are predicted by QM and can indeed be reproduced in the laboratory, we are going to propose below a formalism in order to incorporate physical systems in such states in a set theoretical framework.

3 Preliminaries and primitive symbols

We will work with a variant of Zermelo-Frenkel (ZF) set theory [6] with physical things (PTs). We will denote this theory by ZF^* . The underlying logic of ZF^* is the classical first order predicate calculus with equality (identity). The primitive symbols of ZF^* are the following

- Those of classical first order predicate calculus using only identity and the membership symbol “ \in ”
- the unary predicate symbol “ $\mathcal{C}()$...” (such that “ $\mathcal{C}(x)$ ” reads “ x is a set”)
- a binary predicate symbol “ \sqsubset ” whose meaning will be clear below, when we give the general mereological axioms used in our framework

ZF^* concerns sets and PTs (which are not sets), and so, it is involved with a kind of mereology. PTs are meant to represent physical objects. Depending on the particular interpretation of our framework, PTs may represent fields, particles, strings or any collection of physical objects whose interpretation is compatible with the ontology intended for our framework. In particular, we will consider the system represented by a state such as the one of (48), *as formed by an undefined number of photons*.

Definitions of formulas, sentences (formulas without free variables), bound variables, free variables, etc., are the standard ones. As usual, we write “ $\exists_{\mathcal{C}}x(F(x))$ ” instead of “ $\exists x(\mathcal{C}(x) \wedge F(x))$ ” and “ $\forall_{\mathcal{C}}x(F(x))$ ” instead of “ $\forall x(\mathcal{C}(x) \rightarrow F(x))$ ”.

ZF^* possesses axioms of two different kinds: the ones concerned with sets and the ones concerned with PTs . Let us begin by listing the set theoretical axioms.

3.1 Set theoretical axioms

The following postulates constitute an adaptation of those of Zermelo-Frenkel set theory (see [6] for details).

Axiom 3.1 (Extensionality).

$$(\forall_{\mathcal{C}}x)(\forall_{\mathcal{C}}y)((\forall z)(z \in x \leftrightarrow z \in y) \rightarrow x = y)$$

Axiom 3.2 (Union).

$$(\forall x)(\forall y)(\exists_{\mathcal{C}}t)(\forall z)(z \in t \longleftrightarrow (z \in x \vee z \in y))$$

Axiom 3.3 (Power set).

$$(\forall_{\mathcal{C}}x)(\exists_{\mathcal{C}}y)(\forall_{\mathcal{C}}t)(t \in y \longleftrightarrow t \subseteq x)$$

If $F(x)$ is a formula, x , y and z are distinct variables and y does not occur free in $F(x)$, we have

Axiom 3.4 (Separation).

$$(\forall_{\mathcal{C}}z)(\exists_{\mathcal{C}}y)(\forall x)(x \in y \longleftrightarrow F(x) \wedge x \in z)$$

Axiom 3.5 (Empty set).

$$(\exists_{\mathcal{C}}t)(\forall x)(x \notin t)$$

Axiom 3.6 (Amalgamation).

$$(\forall_{\mathcal{C}}x)((\forall y)(y \in x \longrightarrow \mathcal{C}(y)) \longrightarrow (\exists_{\mathcal{C}}z)(\forall t)(t \in z \longleftrightarrow (\exists v)(v \in x \wedge t \in v)))$$

If $F(x, y)$ is a formula and the variables satisfy evident conditions we have:

Axiom 3.7 (Replacement).

$$(\forall x)(\exists!y)(F(x, y)) \longrightarrow (\forall_{\mathcal{C}}u)(\exists_{\mathcal{C}}v)(\forall y)(y \in v \longleftrightarrow (\exists x)(x \in u \wedge F(x, y)))$$

Axiom 3.8 (Infinity).

$$(\exists_{\mathcal{C}}z)(\emptyset \in z \wedge (\forall x)(x \in z \longrightarrow x \cup \{x\} \in z))$$

Axiom 3.9 (Choice).

$$\begin{aligned} &(\forall_{\mathcal{C}}x)\{(\forall y)(y \in x \longrightarrow \mathcal{C}(y)) \wedge (\forall y)(\forall z)(y \in x \wedge z \in x \longrightarrow (y \cap z = \emptyset \wedge y \neq \emptyset)) \\ &\longrightarrow (\exists_{\mathcal{C}}u)(\forall y)(\exists v)(y \in x \longrightarrow (y \cap u = \{v\}))\} \end{aligned}$$

Axiom 3.10 (Foundation).

$$(\forall x)(x \neq \emptyset \wedge (\forall y)(y \in x \longrightarrow \mathcal{C}(y))) \longrightarrow (\exists z)(z \in x \wedge z \cap x = \emptyset)$$

3.2 Axioms for PTs

Now we list the axioms for PTs. We will use small Greek letters for variables restricted to PTs. Informally, the symbol “ \sqsubset ” will express the “being part of” relation. Thus, “ $\alpha \sqsubset \beta$ ” means that “ α and β are PTs and α is a part of β ”. We start with some preliminary definitions.

Definition 3.11 (Disjointness).

$$\alpha|\beta := \neg\exists\gamma(\gamma \sqsubset \alpha \wedge \gamma \sqsubset \beta)$$

$\alpha|\beta$ is interpreted as “ α and β are PTs which share no part in common”. A possible definition of indistinguishability could be given as follows (though we will not use it in this work)

Definition 3.12 (Indiscernibility).

$$\alpha \equiv \beta := \alpha \sqsubset \beta \wedge \beta \sqsubset \alpha$$

$\alpha \equiv \beta$ means that α and β are indistinguishable, in the sense that they cannot be discerned by any physical means.

Definition 3.13 (PT).

$$T(x) := \neg \mathcal{C}(x)$$

$T(x)$ reads “ x is not a set”, and thus, it is a PT.

Definition 3.14 (Sum of parts).

$$\mathcal{S}(x, \alpha) := \mathcal{C}(x) \wedge \forall y(y \in x \longrightarrow T(y)) \longrightarrow \forall \gamma(\gamma | \alpha \longleftrightarrow \forall \beta(\beta \in x \longrightarrow \beta | \gamma))$$

The explanation of $\mathcal{S}(x, \alpha)$ is that if x is a set such that all its elements are PTs, then for every γ which satisfies being disjoint to α , then it will also be disjoint to any element β in x and viceversa. Intuitively, the only PT α which has this property is the *physical sum* of all the PTs belonging to x .

We now formulate a general axiomatic for PTs. These axioms may encompass a general class of entities, ranging from field quanta to non relativistic particles. But it is important to remark that all these entities need more specific axioms in order to be fully characterized; we are concentrating here in their general mereological properties.

We start by stating that every thing is a part of itself

Axiom 3.15.

$$(\forall \alpha)(\alpha \sqsubset \alpha)$$

It is reasonable to assume transitivity of the relationship “ \sqsubset ”

Axiom 3.16.

$$(\forall \alpha)(\forall \beta)(\forall \gamma)(\alpha \sqsubset \beta \wedge \beta \sqsubset \gamma \longrightarrow \alpha \sqsubset \gamma)$$

We will postulate that there exists the sum of any non empty set of PTs

Axiom 3.17.

$$(\forall x)(\exists \alpha)(\mathcal{S}(x, \alpha))$$

4 Things with undefined number of parts

We will use the following notation

Definition 4.1.

$$\exists\{x | F(x)\} := (\exists y)(\forall x)(x \in y \longleftrightarrow F(x))$$

and the following definition will allow us to present a possible solution to the problem posed in Section 1

Definition 4.2.

$$Cant(\alpha) := \exists\{\beta | \beta \sqsubset \alpha\}$$

If $Cant(\alpha)$ we will say that α is *cantorian*². The above definition says that if a PT α is cantorian, then, all parts of α form a set (and vice versa). Thus, it is possible to assign a cardinal to any Cantorian thing α by assigning a cardinal number to its set of parts in the usual way (using choice axiom 3.9). Notice that it is straightforward to show that if α is Cantorian, then there exists *only one* set satisfying the equality of definition 4.2.

For any x such that $\mathcal{C}(x)$, denote $\sharp(x)$ the cardinal assigned in the usual way using the ZF axiomatic (and we can use it for sets, because the axiomatic of ZF^* includes that of ZF). Thus we define

Definition 4.3. *If $Cant(\alpha)$, let z be the only set satisfying the equality of definition 4.2. Then we define the cardinal of α (abbreviated as $\sharp(\alpha)$) as*

$$\sharp(\alpha) := \sharp(z)$$

Any PT α will be cantorian or not. If α is not Cantorian (i.e., if $\neg(Cant(\alpha))$), then, there is no means for ensuring that its parts form a set using the above axioms. Because of this, there is no way in which we can assign to α a cardinal using ZF axioms, and from this point of view, it is reasonable to interpret a non Cantorian PT as having no cardinal. In this way, we find that the axiomatic framework presented in this work is useful to represent PTs with undefined number of constituents as the ones presented in section 2. But once this general solution is presented, new problems may be posed. We list them below:

1. We provided a general axiomatic for PTs. But it is clear that each theory and spatio-temporal setting will have its own and characteristic ontological features implying its particular axiomatic. Which should be the specific axioms for non relativistic quantum mechanics and relativistic quantum mechanics respectively?
2. How to represent a physical thing which is in a superposition state like the one represented by equation (48)?
3. How to represent a physical superposition in general?
4. Related to (1) and (2), how to represent entanglement?

In this work, we presented a possible solution for question 2. Systems formed of an undefined particle number are represented by non-cantorian things. But -up to now- our formalism does not distinguishes the state $a_1|n\rangle + a_2|m\rangle$ from $a'_1|n\rangle + a'_2|m\rangle$ (with $a'_1 \neq a_1$ or $a'_2 \neq a_2$). In future works, we will essay possible solutions for the problems posed above.

Notwithstanding, something can be said about superpositions using non-cantorian sets right now (thus providing a partial answer to question 3). The following construction, shows that non-cantorian sets possess unexpected properties, which are capable to yield non-standard mathematics and can represent physical situations at the same time. Suppose that α is such that $\neg(Cant(\alpha))$. Then, given a formula $F(x)$, it is impossible -with the above axioms- to grant the existence of the set

$$\alpha_F = \{\beta \sqsubset \alpha \mid F(\beta)\} \tag{49}$$

The separation axiom *cannot be applied*, because the parts of α do not conform necessarily a set! But in a standard set theory (like ZF), “properties” are usually expressed as the membership to given set. For example, if we want to state that the number 4 is even, we can express this by

²We use “Cantorian” in analogy with the system NF of Quine [61, 62]. But this should not lead to any confusion: the analogy is not too deep.

the formula $4 \in \{x \in \mathbb{N} \mid x = 2 \times y \wedge y \in \mathbb{N}\}$. But if we want to interpret our formula $F(x)$ as representing a physical property in ZF^* (defined by extension as the set of all PTs possessing that property), we will face a problem. We cannot grant the existence of the set formed by the parts of α possessing the property defined by $F(x)$. This is a direct consequence of $\neg(Cant(\alpha))$. This situation could be interpreted as follows: “if α is not Cantorian, we cannot assert that its parts possess the property defined by $F(x)$ or that they do not possess it”. This fact, does not constitutes a real problem for our framework, but an unexpected advantage: *this kind of undetermination in the possession of a property can be interpreted as being in a superposition state*. Indeed, a key feature of a quantum mechanical superposition is the lack of meaning in asserting or denying the possession of a given property.

When we face a superposition —say, in a system of spin $\frac{1}{2}$ — such as $\frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$, we are not capable of asserting that the system has spin up nor spin down: this is a key aspect of superpositions, captured by our framework. Thus, our framework is also capable of giving a precise logical meaning to superpositions (at least of a special kind). In order to make things clearer, think of α as formed by the photons of a state of the electro magnetic field such as (48). As it is a superposition in particle number, its energy is also undefined, and thus, the set of photons possessing a definite energy value will inherit the non-Cantorianity of α .

Taking into account the above discussions, it would be interesting to provide a definition of what should be considered classical and quantum PTs within our framework. We give definitions below trying to capture such notions.

Definition 4.4 (Irreducible Part).

$$\mathcal{I}(\alpha, \beta) := \alpha \sqsubset \beta \wedge (\forall \gamma)(\gamma \sqsubset \alpha \longrightarrow \gamma \equiv \alpha)$$

$\mathcal{I}(\alpha, \beta)$ will be interpreted as “ α is an irreducible part of β ”, and this means that α is a part of β and that any part of α will be indistinguishable of α itself. It is straightforward to show that if α is cantorian, then there exists the set of all irreducible parts (hint: use separation). We remark that this set may be the empty set. Now we will define the important notions of *classical part* and *quantum part* with respect to a well formed formula $F(x)$. If α is a PT and $F(x)$ is a formula, we define

Definition 4.5. $Cant_F(\alpha) := \exists\{\beta \sqsubset \alpha \mid F(\beta)\}$

If $Cant_F(\alpha)$ we will say that α has a cantorian subset of parts satisfying $F(x)$. If $\neg Cant_F(\alpha)$, we will interpret this as: “some parts of α are in a superposition state with respect to the property $F(x)$ ”. Thus, given a formula $F(x)$, we will say that

Definition 4.6 (Quantum Part).

$$\mathcal{QP}_F(\alpha) := \neg Cant_F(\alpha)$$

and interpret this as: “ α is quantal with respect to $F(x)$ ”.

Definition 4.7 (Classical Part).

$$\mathcal{CP}_F(\alpha) := Cant_F(\alpha)$$

and interpret this as: “ α is classical with respect to $F(x)$ ”.

We conclude this Section by adding a list of general remarks which could be useful to consider in further developments of a mereology involving quantum entities.

1. As remarked above, different axioms could be added to the above framework in order to capture different kinds of PTs. The specific form of these axioms will depend on the particular physical theory but also -and strongly- on the interpretation of that theory.
2. It should be clear that the spatio-temporal setting in which the theory is developed (v.g., Galilean space time for non-relativistic quantum mechanics and Minkowski space-time for QFT) have a crucial influence in the mereological properties of the corresponding physical objects. This implies that, in order to develop a more specific framework, axioms containing specific space time notions should be added to the axiomatic presented in this work.
3. We may represent a general physical system as a triplet $\langle P, M, S \rangle$, where P is a set representing PTs, M is the corresponding space-time differential manifold of the theory and S is a mathematical structure involving mathematical objects, some of which are built with the help of M . For example, non-relativistic quantum mechanics may be represented as a set, endowed with Galilean manifold and the axiomatic of von Neumann written in the mathematical language of functional analysis. A unitary transformation will thus be a mathematical concept linked to the space-time notion of Galilean symmetry transformation. It is important to remark that the explicit inclusion of the space time manifold, while necessary for experimental verification of the theory, does not implies necessarily that the entities involved has well defined spatio-temporal properties, as is the case in the orthodox interpretation of QM.

In future works, we will address these questions by developing a new system, namely Z^{**} , capable of incorporating all these features, and thus, providing a complete quantum mereology. The development of a quantum mereology is still an open problem, and the formal framework presented here is a concrete step in this direction.

5 Conclusions

In this work we presented a solution for what can be considered a generalization of the Manin's problem, namely, the problem of incorporating in a set theoretical framework the quantum mechanical notion of undefined particle number. Although our proposal is a valid solution for the problems posed in [46, 47, 48, 49, 50], a lot of questions arise and remain unsolved. In particular, it would be interesting to search for other axiomatic systems. Most remarkably, it would be very interesting to find a set theoretical analog of quantum superpositions in general, and in particular, of quantum entanglement.

Besides the general mereological framework presented in this work, it would be interesting to look for the specific implications that the spatio-temporal setting has for the mereological axiomatic capturing the properties of the physical systems of different theories. In particular, a quantum relativistic and non-relativistic mereology is lacking, and we think that the development of set theoretical frameworks like the one presented in this work could be useful for that purpose. In this work we provide a characterization of undefined particle number, which could be used in different –and perhaps, more sophisticated– frameworks. We have also shown that our framework is capable of describing quantum superpositions more general than those which appear in undefined particle number states.

Another interesting question to look at would be that of the implications for mathematics of systems like the one presented here. How would it be a mathematics not based on our every day concepts, but on quantum mechanics? Such a question was partially answered [12], but our system opens a new door to such a research program. In particular, the system presented above, constitutes a novel example of non-standard mathematics, which gives a precise logical meaning to the –up to now– intuitive notion of what physicists mean by “undefined particle number”.

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